Utrecht University

Master Thesis Mathematics

The Recognition Principle for Grouplike $\mathcal{C}_n$-algebras

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Introduction

Recall that the loop space ΩX of a pointed space $(X, *)$ is defined as the space of continuous maps $\alpha : [0,1] \to X$ satisfying $\alpha(0) = * = \alpha(1)$. Similarly, given a pointed space $(X, *)$ one defines the n-fold loop space $\Omega^n X$ of X as the space of maps $\alpha : [0,1]^n \to X$ that send the boundary $\partial[0,1]^n$ of the n-cube to the base point $*$ of X. Of course, loop spaces are closely related to homotopy groups and are ubiquitous in homotopy theory. It should therefore come as no suprise that it is desirable to find some criterion for a space to be a loop space. More precisely, we would like to have

- a sufficient condition for a space X to be weakly homotopy equivalent to an n-fold loop space $\Omega^n Y$ for some Y
- a construction that provides us with such a space Y if it exists.

To do this one needs to exploit the structure loop spaces have. Recall that the loop space ΩX of X comes with a multiplication $*$ given by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

An important property of this multiplication is that it is associative up to homotopy, i.e. $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are homotopic. Similarly, $\Omega^n X$ comes with n multiplications defined by

$$\alpha *_k \beta(t_1, \cdots, t_n) = \begin{cases} \alpha(t_1, \cdots, 2t_k, \cdots, t_n) & \text{if } 0 \leq t_k \leq \frac{1}{2} \\ \beta(t_1, \cdots, 2t_k - 1, \cdots, t_n) & \text{if } \frac{1}{2} \leq t_k \leq 1 \end{cases}$$

Again, these multiplications are associative up to homotopy. These multiplications also distribute over one another, i.e. $(\alpha *_i \beta) *_j (\gamma *_i \delta) = (\alpha *_j \beta) *_i (\gamma *_j \delta)$. The first step to find a criterion as described above is to find a systematic way to describe n-ary operations (and hence multiplications). The way to do this is by using operads. An operad $\mathcal{C}$ consists of spaces $\mathcal{C}(n)$ the points of which are thought of as n-ary operations. An operad comes with certain data that relates these n-ary operations in a systematic way. To relate operads to loop spaces we need to consider a special kind of operads called the operads of little n-cubes $\mathcal{C}_n$. A little n-cube is a linear embedding $f : (0,1)^n \to (0,1)^n$ with parallel axes, i.e. $f = f_1 \times \cdots \times f_n$ where each f_i is a linear function $(0,1) \to (0,1)$. The space $\mathcal{C}_n(k)$ consists of k-tuples of little n-cubes with disjoint images. It turns out that an n-fold loop space is what is called a $\mathcal{C}_n$-space. This means that there are canonical maps

$$\theta_{n,k} : \mathcal{C}_n(k) \times (\Omega^n X)^k \to \Omega^n X$$

given by

$$\theta_{n,k}(f, x)(t) = \begin{cases} x_i(s) & \text{if } f_i(s) = t \\ * & \text{otherwise} \end{cases}$$

satisfying certain conditions. Moreover, May showed (in [7]) that a partial converse of this statement also holds: if X is a connected $\mathcal{C}_n$-space then X is an n-fold loop space up to weak homotopy equivalence. This partial converse is what May calls the Recognition Principle. Moreover, the proof of this princple actually provides a construction for the n-fold delooping of X, i.e. it provides a space Y such that X and $\Omega^n Y$ are weakly equivalent. The idea behind this construction is as follows. As was said, a $\mathcal{C}_n$-space X comes with canonical maps

$$\mathcal{C}_n(k) \times X^k \to X$$

We can then construct a functor from $\mathcal{C}_n$ to C_n in such a way that these canonical maps correspond exactly to a single map $C_n X \to X$. On the one hand, this functor is closely related to loop spaces in that there is a special kind of natural transformation $\alpha_n : C_n \to \Omega^n \Sigma^n$ where Σ^n denotes the functor obtained by applying the reduced suspension n times. On the other hand, by iteratively applying C_n to X we obtain a simplicial space $B_*(\Sigma, C_n, X)$ defined by

$$B_k(\Sigma, C_n, X) = \Sigma(C_n)^k X$$

It turns out that the geometric realization $|B_*(\Sigma, C_n, X)|$ is exactly the n-fold delooping of X in the sense that X is weakly homotopy equivalent to $\Omega^n |B_*(\Sigma, C_n, X)|$ provided that X is connected.
The condition that X is connected is stronger than what is needed for the Recognition Principle. Recall that an H-space X is called grouplike if $\pi_0(X)$ is a group. It turns out that the Recognition Principle remains valid for grouplike $\mathcal{C}_n$-spaces. This version of the principle finds its origins in the work of May [8], Cohen [1] and McDuff and Segal [10]. Though large parts of May's proof of the original Recognition Principle carry over the absence of the connectedness condition presents technical difficulties. Luckily, the above cited works provide the tools to solve these difficulties.

In chapter 1 our central objects of study, operads, will be introduced and we will prove some elementary closure properties of the category of operads. We will show that every operad determines a more simple mathematical object called a monad and we will prove that spaces with an action of an operad correspond exactly to algebras over the monad associated to the operad in question.
In the second chapter we will focus on some specific operads. We will introduce A_∞-operads which are needed for studying 1-fold loop spaces. A large part of this chapter is dedicated to the operads of little n-cubes $\mathcal{C}_n$ which are the operads we are interested in.
The whole third chapter is devoted to the proof of the so-called Approximation Theorem (Theorem 3.1). We will first construct morphisms of monads $\alpha_n : C_n \to \Omega^n \Sigma^n$. The Approximation Theorem then states under what conditions the map $\alpha_n : C_n X \to \Omega^n \Sigma^n X$ is a weak homotopy equivalence. This theorem will be one of the main steps in proving the Recognition Princple.
Chapter 4 deals with simplicial spaces and geometric realization. We first take a slight detour through simplicial complexes to provide some basic geometric intuition. Then we will introduce simplicial spaces and discuss some basic simplicial homotopy theory. Because we want to use simplicial spaces to construct the n-fold delooping of a space we need to go from simplicial spaces to topological spaces. The usual way of doing this is by a construction called geometric realization. After we have discussed this construction we will see how various homotopy properties of simplicial spaces are related to the corresponding homotopy properties of their geometric realization. In the last section of chapter 4

we will investigate how geometric realization relates to the simplicial suspension functor, the simplicial loop space functor and the simplicial functor associated to any operad.
In chapter 5 the double bar construction will be discussed. This construction provides us with a simplicial space whose geometric realization will be the required delooping of the space we started out with. Chapter 6 is devoted to proving the Recognition Principle for connected spaces. Finally, in chapter 7 we consider homological group completions and we will extend the Recognition Principle to grouplike spaces.

1 Operads and Monads

1.1 Operads

In this section we will introduce the notion of an operad which will be one of the central notions of this thesis. Furthermore, we will derive some basic closure properties of the category of operads.
Recall that a topological space X is called compactly generated if it satisfies the following condition: a subspace A of X is closed in X if and only if for any compact subspace K of X we have that $A \cap K$ is closed in K. We denote by **CG** the category of compactly generated Hausdorff spaces and continuous maps between such spaces. Since NDR pairs will be used a lot we recall the definition. Results about NDR pairs will be established whenever they are needed.

Definition 1.1. Let (X, A) be a pair in **CG**. Then (X, A) is called an NDR pair if there exists a map $u : X \to I = [0, 1]$ such that $A = u^{-1}(0)$ and a homotopy $h : I \times X \to X$ such that $h(0, x) = x$ for all $x \in X$, $h(t, a) = a$ for all $a \in A$ and all t and $h(1, x) \in A$ for all $x \in u^{-1}[0, 1)$. In that case (h, u) is called a representation of (X, A) as an NDR pair.

We denote by $\mathbf{CG}_*$ the category of based compactly generated Hausdorff spaces and based maps. We will always require that the base point $*$ is non-degenerate, i.e. we require that $(X, *)$ is an NDR pair.

Definition 1.2. An operad $\mathcal{C}$ consists of spaces $\mathcal{C}(n) \in \mathbf{CG}$ for $n \in \mathbb{N}$ where $\mathcal{C}(0)$ is a single point $*$ together with

1. continuous maps $\gamma : \mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \to \mathcal{C}(j)$ where $j = \sum_{n=1}^{k} j_n$ satisfying for all $c \in \mathcal{C}(k), d_n \in \mathcal{C}(j_n), e_m \in \mathcal{C}(i_m)$
$$\gamma(\gamma(c; d_1, \cdots, d_k); e_1, \cdots, e_j) = \gamma(c; f_1, \cdots, f_k)$$
where
$$f_n = \gamma(d_n; e_{j_1+\cdots+j_{n-1}+1}, \cdots, e_{j_1+\cdots j_n})$$
and $f_n = *$ if $j_n = 0$.

2. an element $1 \in \mathcal{C}(1)$ such that for $d \in \mathcal{C}(j)$ we have $\gamma(1; d) = d$ and for $c \in \mathcal{C}(k)$ we have $\gamma(c; 1^k) = c$ where $1^k = (1, \cdots, 1) \in \mathcal{C}(1)^k$.

3. a right operation of the symmetric group S_j on $\mathcal{C}(j)$ such that for all $c \in \mathcal{C}(k), d_n \in \mathcal{C}(j_n), \sigma \in S_k, \tau_n \in S_{j_n}$ we have
$$\gamma(c\sigma; d_1, \cdots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \cdots, d_{\sigma^{-1}(k)})\sigma(j_1, \cdots, j_k)$$
and
$$\gamma(c; d_1\tau_1, \cdots, d_k\tau_k) = \gamma(c; d_1, \cdots, d_k)(\tau_1 \oplus \cdots \oplus \tau_k)$$
where $\sigma(j_1, \cdots, j_k)$ is that permutation of j letters which permutes the k blocks of the partition $j_1, \cdots, j_k$ of j as σ permutes k letters and $\tau_1 \oplus \cdots \oplus \tau_k$ is the image of $(\tau_1, \cdots, \tau_k)$ under the inclusion $S_{j_1} \times \cdots \times S_{j_k} \to S_j$.

It will be convenient for some proofs to use the following notation. Write

$$\langle e \rangle_n = (e_{j_1+\cdots+j_{n-1}+1}, \cdots, e_{j_1+\cdots j_n})$$

so that the first condition states

$$\gamma(\gamma(c; d_1, \cdots, d_k); e_1, \cdots, e_j) = \gamma(c; \gamma(d_1; \langle e \rangle_1), \cdots, \gamma(d_k; \langle e \rangle_k))$$

Definition 1.3. A morphism of operads $\phi : \mathcal{C} \to \mathcal{C}'$ consists of S_j-intertwining maps $\phi_j : \mathcal{C}(j) \to \mathcal{C}'(j)$ (i.e. $\phi_j(c\sigma) = \phi_j(c)\sigma$ for all $c \in \mathcal{C}(j)$ and $\sigma \in S_j$) such that $\phi_1(1) = 1$ and the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \arrow[r, "\gamma"] \arrow[d, "\phi_k \times \phi_{j_1} \times \cdots \times \phi_{j_k}"'] & \mathcal{C}(j) \arrow[d, "\phi_j"] \\
\mathcal{C}'(k) \times \mathcal{C}'(j_1) \times \cdots \times \mathcal{C}'(j_k) \arrow[r, "\gamma'"'] & \mathcal{C}'(j)
\end{tikzcd}$$

An operad $\mathcal{C}$ is called discrete if each $\mathcal{C}(j)$ is a discrete space and is called S-free if each S_j acts freely on $\mathcal{C}(j)$.

Example 1.1. Let $X \in \mathbf{CG}_*$. The endomorphism operad $\mathcal{E}_X$ of X is defined as follows. Let $\mathcal{E}_X(j)$ denote the space of based maps $X^j \to X$, $\mathcal{E}_X(0)$ consists solely of the inclusion $X^0 = * \to X$. We define

1. for $f \in \mathcal{E}_X(k)$ and $g_n \in \mathcal{E}_X(j_n)$:

$$\gamma(f; g_1, \cdots, g_k) = f \circ (g_1 \times \cdots \times g_k)$$

2. the identity element $1 \in \mathcal{E}_X(1)$ is defined as the identity map $1_X : X \to X$

3. for $f \in \mathcal{E}_X(j), \sigma \in S_j$ and $y \in X^j$ we put $(f\sigma)(y) = f(\sigma y)$ where S_j acts on X^j as $\sigma(x_1, \cdots, x_j) = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(j)})$.

A morphism of operads $\theta : \mathcal{C} \to \mathcal{E}_X$ is also called an action of $\mathcal{C}$ on X and in that case (X, θ) is said to be a $\mathcal{C}$-space. A morphism of $\mathcal{C}$-spaces $(X, \theta), (X', \theta')$ is a based map $f : X \to X'$ such that for all $c \in \mathcal{C}(j)$ the following diagram commutes

$$\begin{tikzcd}
X^j \arrow[r, "f^j"] \arrow[d, "\theta_j(c)"'] & (X')^j \arrow[d, "\theta'_j(c)"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}$$

We denote by $\mathcal{C}[\mathbf{CG}_*]$ the category of $\mathcal{C}$-spaces.
The following result provides a useful reformulation of the notion of an action of an operad on a space.

Proposition 1.1. Let $\mathcal{C}$ be an operad and (X, θ) a $\mathcal{C}$-space. Then $\theta : \mathcal{C} \to \mathcal{E}_X$ determines and is determined by maps $\theta_j : \mathcal{C}(j) \times X^j \to X$ for $j \geq 0$ with $\theta_0 : * \to X$ such that

1. each diagram

$$\begin{tikzcd}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \times X^j \arrow[rr, "\gamma\times 1"] \arrow[dd, "1\times u"'] & & \mathcal{C}(j) \times X^j \arrow[dr, "\theta_j"] & \\
& & & X \\
\mathcal{C}(k) \times \mathcal{C}j_1) \times X^{j_1} \times \cdots \times \mathcal{C}(j_k) \times X^{j_k} \arrow[rr, "1\times\theta_{j_1}\times\cdots\times\theta_{j_k}"'] & & \mathcal{C}(k) \times X^k \arrow[ur, "\theta_k"'] &
\end{tikzcd}$$

commutes where $j = \Sigma j_n$ and $u : \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \times X^j \to \mathcal{C}(j_1) \times X^{j_1} \times \cdots \times \mathcal{C}(j_k) \times X^{j_k}$ is the homeomorphism interchanging the factors as indicated.

2. for every $x \in X$ we have $\theta_1(1; x) = x$.

3. for every $c \in \mathcal{C}(j), \sigma \in S_j$ and $y \in X^j$ we have $\theta_j(c\sigma; y) = \theta_j(c; \sigma y)$.

Proof. The correspondence between the action and the maps is simply the adjunction

$$\mathrm{Hom}(\mathcal{C}(j) \times X^j, X) \cong \mathrm{Hom}(\mathcal{C}(j), X^{X^j})$$

The verification of the properties is trivial. □

Using this reformulation a morphism of $\mathcal{C}$-spaces (X, θ) and (X', θ') is a map $f : X \to X'$ in $\mathbf{CG}_*$ such that

$$\begin{tikzcd}
\mathcal{C}(j) \times X^j \arrow[r, "\theta_j"] \arrow[d, "1\times f^j"'] & X \arrow[d, "f"] \\
\mathcal{C}(j) \times (X')^j \arrow[r, "\theta'_j"'] & X'
\end{tikzcd}$$

commutes for every j.
The remainder of this section will be devoted to establishing some basic results about the category of operads.

Lemma 1.1. Let (X, θ) be a $\mathcal{C}$-space and (Y, A) an NDR-pair in $\mathbf{CG}$. Define $X^{(Y,A)}$ to be the space of maps $(Y, A) \to (X, *)$ with base-point the trivial map. Define $\theta_k^{(Y,A)} : \mathcal{C}(k) \times (X^{(Y,A)})^k \to X^{(Y,A)}$ by

$$\theta_k^{(Y,A)}(c; f_1, \cdots, f_k)(y) = \theta_k(c; f_1(y), \cdots, f_k(y))$$

Then $(X^{(Y,A)}, \theta^{(Y,A)})$ is a $\mathcal{C}$-space.

Proof. We prove this result by checking that the three properties from Proposition 1.1 indeed hold. For the first property we need to check that

$$\theta_j^{(Y,A)} \circ (\gamma \times 1) = \theta_k^{(Y,A)} \circ (1 \times \theta_{j_1}^{(Y,A)} \times \cdots \times \theta_{j_k}^{(Y,A)}) \circ (1 \times u)$$

where u is the shuffle homeomorphism from Proposition 1.1. So let $c \in \mathcal{C}(k)$, $x_i \in \mathcal{C}(j_i)$, $f_i \in X^{(Y,A)}$ and $y \in Y$, then evaluating the left hand side gives

$$\begin{aligned}\theta_j^{(Y,A)} \circ (\gamma \times 1)(c; x_1, \cdots, x_k, f_1, \cdots, f_j)(y) &= \theta_j^{(Y,A)}(\gamma(c; x_1, \cdots, x_k), f_1, \cdots, f_j)(y) \\ &= \theta_j(\gamma(c; x_1, \cdots, x_k), f_1(y), \cdots, f_j(y))\end{aligned}$$

and the right hand side gives

$$\begin{aligned}&\theta_k^{(Y,A)} \circ (1 \times \theta_{j_1}^{(Y,A)} \times \cdots \times \theta_{j_k}^{(Y,A)}) \circ (1 \times u)(c, x_1, \cdots, x_k, f_1, \cdots, f_j)(y) = \\ &\quad \theta_k^{(Y,A)} \circ (1 \times \theta_{j_1}^{(Y,A)} \times \cdots \times \theta_{j_k}^{(Y,A)})(c, x_1, \langle f \rangle_1, \cdots, x_k, \langle f \rangle_k)(y) \\ &\quad = \theta_k(c; \theta_{j_1}^{(Y,A)}(x_1, \langle f \rangle_1)(y), \cdots, \theta_{j_k}^{(Y,A)}(x_k, \langle f \rangle_k)(y)) \\ &\quad = \theta_k(c; \theta_{j_1}(x_1, \langle f \rangle_1(y)), \cdots, \theta_{j_k}(x_k, \langle f \rangle_k(y))) \\ &\quad = \theta_k(c; \theta_{j_1}(x_1, \langle f(y) \rangle_1), \cdots, \theta_{j_k}(x_k, \langle f(y) \rangle_k))\end{aligned}$$

Since the θ_j define a $\mathcal{C}$-morphism it follows that these two are equal by 1.1.
It is immediately clear that the second property is satisfied.
Finally, let $c \in \mathcal{C}(j)$, $f_1, \cdots, f_j \in X^{(Y,A)}$, $y \in Y$ and $\sigma \in S_j$ then

$$\begin{aligned}\theta_j^{(Y,A)}(c\sigma, f_1, \cdots, f_j)(y) &= \theta_j(c\sigma; f_1(y), \cdots, f_j(y)) \\ &= \theta_j(c; f_{\sigma^{-1}(1)}(y), \cdots, f_{\sigma^{-1}(j)}(y)) \\ &= \theta_j^{(Y,A)}(c; f_{\sigma^{-1}(1)}, \cdots, f_{\sigma^{-1}(j)})(y)\end{aligned}$$

so we see that the third property is also satisfied. □

Put $\Omega\theta = \theta^{(I,\partial I)}$ and $P\theta = \theta^{(I,0)}$ then the lemma implies that $(\Omega X, \Omega\theta)$ and $(PX, P\theta)$ are $\mathcal{C}$-spaces for any $\mathcal{C}$-space (X, θ). Furthermore, it follows that the inclusion $i : \Omega X \to PX$ and end-point projection $p : PX \to X$ are morphisms of $\mathcal{C}$-spaces.

Lemma 1.2. Let θ_k denote the trivial map $\mathcal{C}(k) \to *$ then $(*, \theta) \in \mathcal{C}[\mathbf{CG}_*]$. Moreover, for any $(X, \theta) \in \mathcal{C}[\mathbf{CG}_*]$ the unique maps $* \to X$ and $X \to *$ in $\mathbf{CG}_*$ are morphisms of $\mathcal{C}$-spaces.

Proof. It is trivial that the properties from Proposition 1.1 are indeed satisfied. □

Given maps $f : X \to B$ and $g : Y \to B$ we denote their pullback (or fibred product) in $\mathbf{CG}_*$ by $X \times_B Y$. For $\mathcal{C}$-morphisms $f : (X, \theta) \to (B, \theta'')$ and $g : (Y, \theta') \to (B, \theta'')$ define a new map $\theta \times_B \theta'$ by

$$\begin{aligned}(\theta \times_B \theta')_j &: \mathcal{C}(j) \times (X \times_B Y)^j \to X \times_B Y \\ (\theta \times_B \theta')_j(c; (x_1, y_1), \cdots, (x_j, y_j)) &= (\theta_j(c; x_1, \cdots, x_j), \theta'_j(c; y_1, \cdots, y_j))\end{aligned}$$

Lemma 1.3. Given $\mathcal{C}$-morphisms $f : (X, \theta) \to (B, \theta'')$ and $g : (Y, \theta') \to (B, \theta'')$ their pullback (or fibred product) in $\mathcal{C}[\mathbf{CG}_*]$ is given by $(X \times_B Y, \theta \times_B \theta')$.

Proof. It is clear (using Proposition 1.1) that $(X \times_B Y, \theta \times_B \theta')$ is indeed a $\mathcal{C}$-space. Therefore, we only need to show that it is in fact the pullback in $\mathcal{C}[\mathbf{CG}_*]$.

So let (P, χ) be a $\mathcal{C}$-space such that

$$\begin{tikzcd} P \arrow[r, "v"] \arrow[d, "u"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. We need to show that there exists a unique $\mathcal{C}$-morphism $h : P \to X \times_B Y$ such that

$$\begin{tikzcd} P \arrow[dr, "h"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\ & X \times_B Y \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. Define $h(p) = (u(p), v(p))$ then it is clear that the diagram commutes. We first need to show that h is indeed a $\mathcal{C}$-morphism, i.e. we need to show that

$$\begin{tikzcd} \mathcal{C}(j) \times P^j \arrow[r, "\chi_j"] \arrow[d, "1 \times h^j"'] & P \arrow[d, "h"] \\ \mathcal{C}(j) \times (X \times_B Y)^j \arrow[r, "\theta_j \times_B \theta'_j"'] & X \times_B Y \end{tikzcd}$$

commutes. So let $c \in \mathcal{C}(j), p_1, \cdots, p_j \in P$ then on the one hand

$$\begin{aligned}(h \circ \chi_j)(c; p_1, \cdots, p_j) &= h(\chi_j(c; p_1, \cdots, p_j)) \\ &= (u(\chi_j(c; p_1, \cdots, p_j)), v(\chi_j(c; p_1, \cdots, p_j)))\end{aligned}$$

and on the other hand

$$\begin{aligned}(\theta_j \times_B \theta'_j) \circ (1 \times h^j)(c; p_1, \cdots, p_j) &= (\theta_j \times_B \theta'_j)(c; u(p_1), v(p_1), \cdots, u(p_j), v(p_j)) \\ &= (\theta_j(c; u(p_1), \cdots, u(p_j)), \theta'_j(c; v(p_1), \cdots, v(p_j))\end{aligned}$$

Since u and v are (by assumption) $\mathcal{C}$-morphism, these two compositions are equal.
Now, for uniqueness. Suppose we find another such $\mathcal{C}$-morphism h'. Since h' maps into subset of the product we can write $h'(p) = (h'_1(p), h'_2(p))$. Therefore, $h'_1(p) = \pi_1 h(p) = u(p)$ and $h'_2(p) = \pi_2 h(p) = v(p)$. Consequently, $h' = h$ and we are done. $\square$

Corollary 1.1. Let (X, θ) and (Y, θ') be $\mathcal{C}$-spaces then

1. $(X \times Y, \theta \times \theta')$ is the product of (X, θ) and (Y, θ') in $\mathcal{C}[\mathbf{CG}_*]$
2. the diagonal map $\Delta : X \to X \times X$ is a $\mathcal{C}$-morphism.

From the previous results it follows that any $\mathcal{C}$-morphism can be replaced by a fibration in $\mathcal{C}[\mathbf{CG}_*]$ as is made precise in the following result.

Corollary 1.2. Suppose $f : (X, \theta) \to (Y, \theta')$ is a $\mathcal{C}$-morphism then there exists a $\mathcal{C}$-space $(\tilde{X}, \tilde{\theta})$ and maps $i : X \to \tilde{X}$, $r : \tilde{X} \to X$ and $\tilde{f} : \tilde{X} \to Y$ such that

1. i is an inclusion, r is a retraction and $\tilde{f}$ is a fibration
2. the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & \tilde{X} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

3. i, r and $\tilde{f}$ are $\mathcal{C}$-morphisms

1.2 Intermezzo: Some Point-Set Topology

Before we can proceed with our discussion we need to make a slight detour and establish some results about compactly generated topological spaces and NDR-pairs. Most of the results on NDR pairs come from [12] and the appendix of [7]. First we need a definition.

Definition 1.4. A map $f : X \to Y$ is called proclusive if $f(X) = Y$ and for every $A \subseteq Y$ we have that if $f^{-1}(A)$ is open in X then A is open in Y .

Lemma 1.4. Suppose $f : X \to Y$ is proclusive, X Hausdorff and compactly generated and Y Hausdorff then Y is compactly generated

Proof. Assume that $A \subset Y$ hits each compact set in a closed set. We need to prove that A is closed. Assume $C \subset X$ is compact. Since C is compact, $f(C)$ is compact hence $A \cap f(C)$ is closed. Therefore, $f^{-1}(A \cap f(C))$ is closed and thus $f^{-1}(A \cap f(C)) \cap C$ is closed. But $f^{-1}(A \cap f(C)) \cap C$ is just $f^{-1}(A) \cap C$. So, $f^{-1}(A)$ hits C in a closed set. Since C was an arbitrary compact set in X, $f^{-1}(A)$ hits each compact set in X in a closed set. But X was (by assumption) compactly generated so this can only happen in $f^{-1}(A)$ is closed. But then A itself must be closed since f is proclusive. Consequently, Y is also compactly generated. ☐

Lemma 1.5. Suppose we have spaces $X_0 \subset X_1 \subset \cdots$ such that X_k is closed in X_{k+1} and each X_k is compactly generated and Hausdorff. Put $X = \cup_{k \geq 0} X_k$ and endow X with the topology of the union. If X is Hausdorff then X is compactly generated.

Proof. First note that each X_n is in fact imbedded into X. Now, assume that $A \subset X$ hits each compact subset of X in a closed set. Let n be a natural number and C compact in X_n. Since X_n is imbedded in X it follows that C is compact in X. Thus, $A \cap C$ is closed in X. But this is just $(A \cap X_n) \cap C$. Since this is closed in X it is also closed in X_n. But this implies that $A \cap X_n$ is closed since X_n was assumed to be compactly generated Hausdorff. Since n was arbitrary it follows that A must be closed in X, i.e. X itself is compactly generated. ☐

Lemma 1.6. Let X_n and X be as in the previous lemma. Suppose that C is compact in X then C is contained in X_n for some n.

Proof. Suppose that A is not contained in any of the X_n. Then we can find $x_n \in A \cap (X \setminus X_n)$. Let $B_m = \{x_n | n \geq m\}$. Then $\{B_m\}$ is a decreasing sequence with empty intersection. Note that $T_m \cap X_n$ is a finite set hence a closed set. Therefore T_m is also closed in X. Clearly, the $X \setminus T_m$ then form an open cover of X hence of A. By construction, A is not covered by a finite subcover hence A is not compact. □

Lemma 1.7. Let X_n and X be as before and assume each (X_{n+1}, X_n) is an NDR pair then X is Hausdorff and each (X, X_n) is an NDR pair.

Proof. Let $x, y \in X$ be distinct. Pick m such that for all $n \geq m$ we have $x, y \in X_n$. Since X_m is Hausdorff there are open (in X_m) sets U_m and V_m such that $x \in U_m$, $y \in V_m$ and $U_m \cap V_m = \emptyset$. We will now construct a sequence (U_n, V_n) for $n \geq m$ by induction such that U_n and V_n are open in X_n, $x \in U_n$, $y \in V_n$, $U_n \cap V_n = \emptyset$, $U_{n+1} \cap X_n = U_n$ and $V_{n+1} \cap X_n = V_n$. For the base step we can clearly take the U_m and V_m we obtained above. Assume that U_n and V_n are constructed for $m \leq n \leq p$. We will construct U_{p+1} and V_{p+1}. Since (X_{p+1}, X_p) is an NDR pair we know there exists an open neighborhood $W \subset X_{p+1}$ of X_p and a retract r of W into X_p. Define $U_{p+1} = r^{-1}(U_p)$ and $V_{p+1} = r^{-1}(V_p)$. Clearly, these sets are as required. Put $U = \cup_{n \geq m} U_n$ and $V = \cup_{n \geq m} V_n$. Since $U \cap X_n = U_n$ and $V \cap X_n = V_n$ it follows that U and V are open. Obviously, they separate x and y and thus X is Hausdorff. By Lemma 1.5 it follows that X is compactly generated.
Since each (X_{m+1}, X_m) is an NDR pair we know that for each m there exists a retract

$$r_m : I \times X_{m+1} \to 0 \times X_{m+1} \cup I \times X_m$$

Clearly, these can be extended to retracts

$$s_m : 0 \times X \cup I \times X_{m+1} \to 0 \times X \cup I \times X_m$$

by simply defining $S_m(0, x) = (0, x)$. We can now define a map

$$s : I \times X \to 0 \times X \cup I \times X_n$$

to be $s_n \circ \cdots \circ s_{m-1}$ when restricted to $0 \times X \cup I \times X_m$ for $m > n$. To see that s is continuous let A be closed in $0 \times X \cup I \times X_n$. Then $s^{-1}(A) \cap (0 \times X \cup I \times X_m)$ is closed for each $m > n$. Suppose C is compact in $I \times X$ then its projection D in X must be compact hence by Lemma 1.6 D is contained in some X_m. But then C is contained in $I \times X_m$. Since $S^{-1}(A) \cap (I \times X_m)$ is closed it follows that $s^{-1}(A) \cap C$ is closed. Since this holds for any compact C and $I \times X$ is compactly generated it follows that $s^{-1}(A)$ must itself be closed and hence s is continuous. Since s is clearly also a retract it follows that (X, X_n) is an NDR pair. □

Lemma 1.8. Let (h, u) and (j, v) represent, respectively, (X, A) and (Y, B) as NDR pairs. Then

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

is represented as a NDR pair by (k, w) where

$$w(x, y) = \min(u(x), v(y))$$

and

$$k(t,x,y) = \begin{cases} (h(t,x), j(\frac{u(x)}{v(y)}t, y)) & \text{if } v(y) \geq u(x) \\ (h(\frac{v(y)}{u(x)}t, x), j(t,y)) & \text{if } u(x) \geq v(y) \end{cases}$$

Proof. There are four things we need to check, namely

1. $X \times B \cup A \times Y = w^{-1}(0)$
2. for $(x,y) \in X \times Y$ we have $k(0,x,y) = (x,y)$
3. for all $(t,a,b) \in I \times (X \times B \cup A \times Y)$ we have $k(t,a,b) = (a,b)$
4. for all $(x,y) \in w^{-1}[0,1)$ we have $k(1,x,y) \in X \times B \cup A \times Y$

For the first property we note that $w(x,y) = 0$ if and only if $u(x) = 0$ or $v(y) = 0$, the first case is equivalent with $x \in A$ and the second with $y \in B$. Now, for the second condition. Let $(x,y) \in X \times Y$ then

$$k(0,x,y) = (h(0,x), j(0,y)) = (x,y)$$

Thirdly, suppose $(a,b) \in X \times B \cup A \times Y$. Without loss of generality we may assume that $(a,b) \in X \times B$ as the other case is entirely similar. Since $(a,b) \in X \times B$, $b \in B$ and thus $v(b) = 0$ hence $u(a) \geq v(b)$. Therefore

$$k(t,a,b) = (h(\frac{v(b)}{u(a)}t, a), j(t,b)) = (h(0,a), b) = (a,b)$$

Lastly, assume $w(x,y) < 1$. Without loss of generality we may assume $u(x) \leq v(y)$. Since $w(x,y) = \min(u(x), v(y)) < 1$ it follows that $u(x) < 1$. Therefore

$$h(1,x,y) = (h(1,x), j(\frac{u(x)}{v(y)}, b)) \in A \times Y$$

since $h(1,x) \in A$ because $u(x) < 1$. The case that $v(y) \leq u(x)$ is similar and will be omitted. □

Lemma 1.9. Suppose (h,u) represents (X,A) as an NDR pair. Then (h_k, u_k) represents

$$(X,A)^k = (X^k, \cup_{i=1}^k X^{i-1} \times A \times X^{k-i})$$

as a S_k-equivariant NDR pair, where

$$u_k(x_1, \cdots, x_k) = \min(u(x_1), \cdots, u(x_k))$$
$$h_k(t, x_1, \cdots, x_k) = (h(t_1, x_1), \cdots, h(t_k, x_k))$$

with

$$t_i = \begin{cases} t\min_{i \neq j} \frac{u(x_j)}{u(x_i)}, & \text{if some } u(x_j) < u(x_i), i \neq j \\ t, & \text{if all } u(x_j) \geq u(x_i), j \neq i \end{cases}$$

Proof. S_k-equivariance is clear. The rest of the proof is simply applying the previous lemma k times with $X = Y$. □

1.3 Monads

In this section we will see that an operad determines a more simple mathematical structure called a monad. We will also introduce the notion of an algebra over a monad and show that $\mathcal{C}$-spaces, where $\mathcal{C}$ is an operad, correspond to algebras over the monad associated to $\mathcal{C}$. First, we give the definition of a monad.

Definition 1.5. Let $\mathbf{C}$ be a category. A monad in $\mathbf{C}$ is a triple (T, μ, η) where $T : \mathbf{C} \to \mathbf{C}$ is a functor and $\mu : T^2 \to T$ and $\eta : 1 \to T$ are natural transformations such that the following diagrams commute for all $X \in \mathbf{C}$

$$\begin{tikzcd} TX \arrow[r, "T(\eta_X)"] \arrow[dr, "1"'] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "\eta_{TX}"'] \arrow[dl, "1"] \\ & TX & \end{tikzcd} \qquad \begin{tikzcd} T^3X \arrow[r, "\mu_{TX}"] \arrow[d, "T(\mu_X)"'] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

A morphism $\phi : (T, \mu, \eta) \to (T', \mu', \eta')$ of monads is a natural transformation $\phi : T \to T'$ such that for every $X \in \mathbf{C}$ the following diagrams commute

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ TX \arrow[rr, "\phi"'] & & T'X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\phi^2_X"] \arrow[d, "\mu_X"'] & (T')^2X \arrow[d, "\mu'_X"] \\ TX \arrow[r, "\phi_X"'] & T'X \end{tikzcd}$$

where ϕ^2 is defined via the commutative diagrams

$$\begin{tikzcd} T^2X \arrow[r, "T(\phi_X)"] \arrow[d, "\phi_{TX}"'] \arrow[dr, "\phi^2_X"] & TT'X \arrow[d, "\phi_{T'X}"] \\ T'TX \arrow[r, "T'(\phi_X)"'] & (T')^2X \end{tikzcd}$$

Definition 1.6. Let $\mathbf{C}$ be a category and (T, μ, η) a monad. An algebra (X, ξ) over T is an object $X \in \mathbf{C}$ together with a map $\xi : TX \to X$ such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "1"'] & TX \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "T(\xi)"'] & TX \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"'] & X \end{tikzcd}$$

A morhpism $f : (X, \xi) \to (X', \xi')$ of algebras over T is a map $f : X \to X'$ in $\mathbf{C}$ such that

$$\begin{tikzcd} TX \arrow[r, "T(f)"] \arrow[d, "\xi"'] & TX' \arrow[d, "\xi'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

commutes.

The category of algebras over T will be denoted by $T[\mathbf{C}]$.

Our next goal is to construct a monad associated to a given operad. This construction will yield a functor from the category of operads to the category of monads in $\mathbf{CG}_*$. We will first introduce some additional notation to handle base points. Let $\mathcal{C}$ be an operad. We define maps $\sigma_i : \mathcal{C}(j) \to \mathcal{C}(j-1)$ for $1 \leq i \leq j-1$ as follows. For $c \in \mathcal{C}(j)$ we put $\sigma_i(c) = \gamma(c; s_i)$ where

$$s_i = 1^i \times * \times 1^{j-i-1} \in \mathcal{C}(1)^i \times \mathcal{C}(0) \times \mathcal{C}(1)^{j-i-1}$$

In particular, when we are working in the endomorphism operad $\mathcal{E}_X$ of $X \in \mathbf{CG}_*$ we obtain, for $f : X^j \to X$ and $y \in X^{j-1}$, $(\sigma_i f)(y) = f(s_i y)$ where

$$s_i(x_1, \cdots, x_{j-1}) = (x_1, \cdots, x_i, *, x_{i+1}, \cdots, x_{j-1})$$

Now, let $\mathcal{C}$ be an operad. We want to construct a monad (T, μ, η) associated to $\mathcal{C}$. Let $X \in \mathbf{CG}_*$ and consider the equivalence relation $\sim$ on $\coprod_{k \geq 0} \mathcal{C}(k) \times X^k$ generated by

1. for c $\in \mathcal{C}(k)$, $0 \leq i < k$ and $y \in X^{k-1}$: $(\sigma_i c, y) \sim (c, s_i y)$
2. for c $\in \mathcal{C}(k)$, $\sigma \in S_k$ and $y \in X^k$: $(c\sigma, y) \sim (c, \sigma y)$.

We then put $TX = \coprod_{k \geq 0} \mathcal{C}(k) \times X^k / \sim$. Consider

$$f_n : \coprod_{k=0}^{n} \mathcal{C}(k) \times X^k \to TX$$

and let $F_n = \mathrm{im}(f_n)$. We endow F_n with the quotient topology. This makes F_{n-1} into a closed subspace of F_n. Now, give TX the topology of the union of the F_n. That is to say, $U \subset TX$ is open if and only if $U \cap F_n$ is open for all n. Notice that F_0 is just a single point and thus TX is naturally a pointed space with F_0 as base point.
To define the multiplication μ and unit η of the monad we proceed as follows. For $(c, y) \in \mathcal{C}(n) \times X^n$ put $[c, y] = f_n(c, y)$. Define the multiplication μ, for $c \in \mathcal{C}(k)$, $d_i \in \mathcal{C}(j_i)$ and $y_i \in X^{j_i}$, by

$$\mu : T^2X \to TX$$
$$\mu[c, [d_1, y_1], \cdots, [d_k, y_k]] = [\gamma(c; d_1 \cdots, d_k); y_1, \cdots, y_k]$$

For the unit η we put

$$\eta : X \to TX$$
$$\eta(x) = [1, x]$$

for $x \in X$. Using the properties of operads we see that μ is well-defined and satisfies $\mu \circ \mu = \mu \circ T(\mu)$ and $\mu \circ T(\eta) = 1 = \mu \circ \eta$. For a based map $f : X \to X'$ we define

$$T(f) : TX \to TX'$$
$$Tf[c; y] = [c; f^k(y)]$$

for $[c; y] \in F_k$.
Furthermore, if $\phi : \mathcal{C} \to \mathcal{C}'$ is a morphism of operads we define an associated morphism of monads, which we will also denote by ϕ, by

$$\phi : TX \to T'X$$
$$\phi[c, y] = [\phi_k(c); y]$$

where $[c, y] \in F_k$.
It is clear that the above construction gives a functor from the category of operads to the category of monads. What is not clear however is the fact that TX is in fact in $\mathbf{CG}_*$ whenever X is. This will be part of Theorem 1.1 below. Before we can prove this theorem we need another definition.

Definition 1.7. Let $W \in \mathbf{CG}$ and suppose G is a subgroup of S_k acting on the right on W. For $X \in \mathbf{CG}_*$ the left action of S_k on X^k induces a left action of G on the k-fold smash product $X^{[k]}$. Define the equivariant half-smash product $e[W, G, X]$ by

$$e[W, G, X] = W \times X^{[k]} / \sim$$

where $\sim$ is defined by

1. for $w, w' \in W : (w, *) \sim (w', *)$
2. for $w \in W, \sigma \in G$ and $y \in X^{[k]} : (w\sigma, y) \sim (w, \sigma y)$

Now, for the promised theorem.

Theorem 1.1. Let $\mathcal{C}$ be an operad and $X \in \mathbf{CG}_*$ then

1. (F_n, F_{n-1}) is an NDR-pair for $n \geq 1$. Moreover, $TX \in \mathbf{CG}_*$
2. the quotient F_n/F_{n-1} is homeomorphic to $e[\mathcal{C}(n), S_n, X]$
3. $T : \mathbf{CG}_* \to \mathbf{CG}_*$ is a functor that preserves both homotopies and limits

Proof. We clearly have that

$$F_n \setminus F_{n-1} = \mathcal{C}(n) \times_{S_j} (X \setminus *)^n$$

It is easily verified that the F_n are Hausdorff. Recall that $F_n = \mathrm{im}(f_n)$ where

$$f_n : \coprod_{k=0}^{n} \mathcal{C}(k) \times X^k \to TX$$

Clearly, f_n is proclusive as a map into F_n. Since $\coprod_{k=0}^{n} \mathcal{C}(k)$ is compactly generated Hausdorff it follows that F_n is compactly generated by Lemma 1.4. Since $(X, *)$ is an NDR pair we know by Lemma 1.9 that we can find a representation (h_n, u_n) of $(X, *)^n$ as an S_n-equivariant NDR pair. Define

$$g_n : I \times F_n \to F_n$$
$$g_n(t, z) = z \quad \text{if} \quad z \in F_{n-1}$$
$$g_n(t, z) = [c, h_n(t, y)] \quad \text{if} \quad z = [c, y], (c, y) \in \mathcal{C}(n) \times (X \setminus *)^n$$

and

$$\begin{aligned} v_n &: F_n \to I \\ v_n(z) &= 0 \;\text{ if }\; z \in F_{n-1} \\ v_n(z) &= u_n(y) \;\text{ if }\; z = [c,y], (c,y) \in \mathcal{C}(n) \times (X \setminus *)^n \end{aligned}$$

It is clear that (g_n, v_n) represents (F_n, F_{n_1}) as an NDR pair. Therefore, it follows by Lemma 1.5 and Lemma 1.7 that $TX \in \mathbf{CG}$ and each (TX, F_n) is an NDR pair. From this it follows that $TX \in \mathbf{CG}_*$. It is now straightforward that (ii) holds. It is clear that (iii) holds. □

Our primary interest is in the relation between $\mathcal{C}$-spaces and T-algebras where T is the monad associated to the operad $\mathcal{C}$. For this we have the following result.

Theorem 1.2. Let $\mathcal{C}$ be an operad and T the corresponding monad, then there is a one-to-one correspondence between $\mathcal{C}$-actions $\theta : \mathcal{C} \to \mathcal{E}_X$ and T-algebra structure maps $\xi : TX \to X$. Moreover, this correspondence defines an isomorphism between the category of $\mathcal{C}$-spaces and the category of T-algebras.

Proof. Define maps π_k as the composites

$$\mathcal{C}(k) \times X^k \to F_k \to TX$$

We define the desired correspondence by letting a $\mathcal{C}$-action θ correspond to a structure map ξ if and only if

$$\begin{tikzcd} \mathcal{C}(k) \times X^k \arrow[rr, "\pi_k"] \arrow[dr, "\theta_k"'] & & TX \arrow[dl, "\xi"] \\ & X & \end{tikzcd}$$

commutes for every k. To see that this indeed defines a one-to-one correspondence we first note by construction of TX that a map $\xi : TX \to X$ determines and is determined by maps $\theta_k : \mathcal{C}(k) \times X^k \to X$ satisfying

1. $\theta_{k-1}(\sigma_i c, y) = \theta_k(c, s_i y)$
2. $\theta_k(c\sigma, y) = \theta_k(c, \sigma y)$

Now, $\sigma_i y = \gamma(c; s_i)$ where $s_i = 1^i \times * \times 1^{k-i-1}$ and thus the maps θ_k from the $\mathcal{C}$-action do indeed satisfy these relations. Conversely, suppose we are given a structure map $\xi : TX \to X$. By definition of such a map we then have $\xi \circ \mu = \xi \circ T(\xi)$ where μ is the multiplication of the monad. Using the fact that $\xi \circ \pi_k = \theta_k$ it then follows that each diagram

$$\begin{tikzcd} \mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \times X^j \arrow[r, "\gamma \times 1"] \arrow[d, "1 \times \phi"'] & \mathcal{C}(j) \times X^j \arrow[dr, "\theta_j"] & \\ & & X \\ \mathcal{C}(k) \times \mathcal{C}(j_1) \times X^{j_1} \times \cdots \times \mathcal{C}(j_k) \times X^{j_k} \arrow[r, "1 \times \theta_{j_1} \times \cdots \times \theta_{j_k}"'] & \mathcal{C}(k) \times X^k \arrow[ur, "\theta_k"'] & \end{tikzcd}$$

commutes. Moreover, since $\xi \circ \eta = 1$ it clearly follows that for every $x \in X$ we have $\theta_1(1, x) = x$. But these two statements just say that the θ_k define an action of $\mathcal{C}$. So, to sum up, a map $\xi : TX \to X$ is a T-algebra structure map if and only if the corresponding maps θ_k define a $\mathcal{C}$-action on X.
To see that this correspondence defines an isomorphism of categories we observe if (X, ξ) and (X', ξ') are T-algebras and $f : X \to X'$ is a map in $\mathbf{CG}_*$ then $f \circ \xi = \xi' \circ T(f)$ if and only if for each k we have $f \circ \theta_k = \theta'_k \circ (1 \times f^k)$. $\square$

Recall the following standard result from category theory.

Lemma 1.10. Let $\mathbf{C}$ be any category and (T, μ, η) a monad in $\mathbf{C}$ then for any object $X \in \mathbf{C}$ it follows that (TX, μ) is a T-algebra. Moreover, let $\xi : TX \to X$ denote the algebra structure map then

$$\begin{gathered} \phi : \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{T[\mathbf{C}]}((TX\mu), (Y, \xi)) \\ \phi(f) = \xi \circ T(f) \end{gathered}$$

is a natural isomorphism with inverse $\phi^{-1}(g) = g \circ \eta$.

Therefore, the theorem implies that TX is the free $\mathcal{C}$-space generated by X.

2 Examples of Operads

2.1 A_∞-operads and E_∞-operads

The purpose of this section is to investigate two special types of operads called A_∞-operads and E_∞-operads. It turns out that these notions are closely related to the notions of (respectively) a topological monoid and of a commutative topological monoid.

Definition 2.1. Let S_j be the symmetric group on j letters and $e_j \in S_j$ be the identity permutation. Define an operad $\mathcal{M}$ as follows. Put $\mathcal{M}(j) = S_j$ for $j \geq 0$ and $\mathcal{M}(0) = \{e_0\}$. The data are defined as follows. Put

$$\gamma(e_k; e_{j_1}, \cdots, e_{j_k}) = e_j \quad \text{where} \quad j = \Sigma j_s$$

and extend the domain of γ to $S_k \times S_{j_1} \times \cdots \times S_{j_k}$ by using the equivariance formulas from Definition 1.2. We define the identity element 1 to be $e_1 \in \mathcal{M}(1)$.

Clearly, $\mathcal{M}$ is a discrete operad.

Definition 2.2. Define an operad $\mathcal{N}$ as follows. For each j we define $\mathcal{N}(j) = \{f_j\}$, i.e. it is just a single point. The identity elements is defined as $1 = f_1$ and the S_j action is just the trivial action. Lastly, we put

$$\gamma(f_k; f_{j_1}, \cdots, f_{j_k}) = f_j \quad \text{where} \quad j = \Sigma j_s$$

Again, it is obvious that $\mathcal{N}$ is indeed a discrete operad.

Proposition 2.1. Let $\mathcal{C}$ be an operad with each $\mathcal{C}(j)$ non-empty. Then every $\mathcal{N}$-space is a $\mathcal{C}$-space.

Proof. The unique maps $\phi_j : \mathcal{C}(j) \to \mathcal{N}(j)$ obviously define a morphism of operads ϕ. Let (X, θ) be a $\mathcal{N}$-space then $\theta : \mathcal{N} \to \xi_X$ is a morphism of operads. Hence we have a morphism of operads $\theta \circ \phi : \mathcal{C} \to \xi_X$ hence $(X, \theta \circ \phi)$ is a $\mathcal{C}$-space. □

Notice that if G is a topological monoid in $\mathbf{CG}_*$ with identity element $*$ then G is determined by and determines a morphism of operads $\theta : \mathcal{M} \to \xi_G$ by defining $\theta_j(e_j) : G^j \to G$ to be the j-fold product and extending the domain of definition using the equivariance formulas. In particular we see that G is commutative if and only if θ factors through $\mathcal{N}$. The following result is easily proven. Here M and N denote the monads associated to $\mathcal{M}$ and $\mathcal{N}$ respectively.

Proposition 2.2.

1. The category $\mathcal{M}[\mathbf{CG}_*] = M[\mathbf{CG}_*]$ is isomorphic to the category of topological monoids and for $X \in \mathbf{CG}_*$ we have that MX is the free topological monoid on X subject to the relation $* = 1$.
2. The category $\mathcal{N}[\mathbf{CG}_*] = N[\mathbf{CG}_*]$ is isomorphic to the category of commutative topological monoids and for $X \in \mathbf{CG}_*$ we have that NX is the free commutative topological monoid on X subject to the relation $* = 1$.

So, a $\mathcal{M}$-space is just a M-algebra which is just a topological monoid and a $\mathcal{N}$-space is just a N-algebra which is just a commutative topological monoid.

Let $\mathcal{C}$ be any operad. The set of path components of $\mathcal{C}(j)$ is denote by $\pi_0\mathcal{C}(j)$. Define a map

$$\delta_j : \mathcal{C}(j) \to \pi_0\mathcal{C}(j)$$
$$\delta_j(c) = [c]$$

where $[c]$ is the path component containing c. It is easy to verify that the data of $\mathcal{C}$ uniquely determine data for $\pi_0\mathcal{C}$. Clearly, with this date $\pi_0\mathcal{C}$ is a discrete operad and δ a morphism of operads. In fact, π_0 defines a functor from the category of operads to the category of discrete operads. Lastly, we note that if $\mathcal{D}$ is a discrete operad and $\epsilon : \mathcal{C} \to \mathcal{D}$ a morphism of operads then we can write $\epsilon = \pi_0(\epsilon) \circ \delta$ since $\pi_0\mathcal{D} = \mathcal{D}$.

Definition 2.3. Let $\mathcal{D}$ be a discrete operad. An operad over $\mathcal{D}$ is a pair $(\mathcal{C}, \epsilon)$ where $\mathcal{C}$ is an operad an $\epsilon : \mathcal{C} \to \mathcal{D}$ is a morphism of operads such that $\pi_0(\epsilon) : \pi_0\mathcal{C} \to \mathcal{D}$ is an isomorphism of operads. In that case ϵ is called the augmentation of $\mathcal{C}$. A morphism $\phi : (\mathcal{C}, \epsilon) \to (\mathcal{C}', \epsilon')$ of operads over $\mathcal{D}$ is a morphism of operads $\psi : \mathcal{C} \to \mathcal{C}'$ such that

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\phi"] \arrow[dr, "\epsilon"'] & & \mathcal{C}' \arrow[dl, "\epsilon'"] \\
& \mathcal{D} &
\end{tikzcd}$$

commutes.

Recall that a space X is called n-connected if X is path connected and for every $1 \leq k \leq n$ we have that $\pi_k(X) = 0$.

Definition 2.4.
1. An operad $\mathcal{C}$ is called locally n-connected if each $\mathcal{C}(j)$ is n-connected.

2. A morphism of operads $\phi : \mathcal{C} \to \mathcal{C}'$ is called a local equivalence if each $\phi_j : \mathcal{C}(j) \to \mathcal{C}'(j)$ is a homotopy equivalence.

3. A morphism of operads $\phi : \mathcal{C} \to \mathcal{C}'$ is called a local S-equivalence if each $\phi_j : \mathcal{C}(j) \to \mathcal{C}'(j)$ is an S-equivariant homotopy equivalence, that is if the homotopies are S-equivariant.

The reader can easily verify the following proposition.

Proposition 2.3. Let $\mathcal{C}$ be an operad then

1. $\mathcal{C}$ can be augmented over $\mathcal{N}$ if and only if $\mathcal{C}$ is locally connected. In that case, the augmentation is unique.

2. $\mathcal{C}$ can be augmented over $\mathcal{M}$ if and only if $\pi_0\mathcal{C}(j)$ is isomorphic to S_j for all j. The augmentation is then given by a coherent choice of isomorphisms.

We have the following useful result.

Theorem 2.1. Let $\phi : \mathcal{C} \to \mathcal{C}'$ be a morphism of operads over $\mathcal{M}$ or $\mathcal{N}$. Suppose that either ϕ is a local S-equivalence or that ϕ is a local equivalence and $\mathcal{C}$ and $\mathcal{C}'$ are S-free. Then the corresponding maps $\phi : TX \to T'X$ are weak homotopy equivalences whenever X is connected.

Proof. Since T and T' are monads, TX and $T'X$ are endowed with a multiplication and unit. It is readily verified that this makes TX and $T'X$ into H-spaces. Therefore, since both TX and $T'X$ are connected, it is sufficient to prove that ϕ induces an isomorphism in integral homology. By Theorem 1.1 and the five lemma it is sufficient to show that each $e[\mathcal{C}(j), S_j, X] \to e[\mathcal{C}'(j), S_j, X]$ (which are determined by the ϕ_j) induced isomorphisms in homology. Clearly, if ϕ is a local S-equivalence then these maps will be homotopy equivalences and we are done. So assume that ϕ is not a local S-equivalence. Then ϕ is a local equivalence and $\mathcal{C}$ and $\mathcal{C}'$ are S-free. Since each $\mathcal{C}(j)$ is S_j-free it is clear that

$$\mathcal{C}(j) \times X^{[j]} \to \mathcal{C}(j) \times_{S_j} X^{[j]}$$

is a covering map. Therefore, this map gives a spectral sequence that converges from $E_2 = H_*(S_j; H_*(\mathcal{C}(j) \times X^{[j]}))$ to $H_*(\mathcal{C}(j) \times_{S_j} X^{[j]})$. Since $\mathcal{C}(j)$ and $\mathcal{C}'(j)$ are S_j-free and ϕ_j is a homotopy equivalence it follows that ϕ_j induces an isomorphism on the E_2 page. Therefore, ϕ_j induces an isomorphism on $H_*(\mathcal{C}(j) \times_{S_j} X^{[j]})$ hence on $H_*(e[\mathcal{C}(j), S_j, X])$. $\square$

Definition 2.5.

1. an A_∞ operad is an S-free operad $\mathcal{C}$ over $\mathcal{M}$ such that $\epsilon : \mathcal{C} \to \mathcal{M}$ is a local S-equivalence. An A_∞-space (X, θ) is a $\mathcal{C}$-space over any A_∞ operad $\mathcal{C}$.
2. an E_∞ operad is an S-free operad $\mathcal{C}$ over $\mathcal{N}$ such that $\epsilon : \mathcal{C} \to \mathcal{N}$ is a local equivalence. An E_∞-space (also called a homotopy everything space) (X, θ) is a $\mathcal{C}$-space over any E_∞ operad $\mathcal{C}$.

The following lemma is just a reformulation of the definitions.

Lemma 2.1.

1. An operad $\mathcal{C}$ is an E_∞ operad if and only if each $\mathcal{C}(j)$ is S_j-free and contractible. In particular, $\mathcal{N}$ is *not* an E_∞ operad.
2. An operad $\mathcal{C}$ is an A_∞ operad if and only if each $\pi_0\mathcal{C}(j)$ is isomorphic to S_j and each $\mathcal{C}(j)$ has contractible components. In particular, $\mathcal{M}$ *is* an A_∞ operad.

Note that from (i) it follows that if $\mathcal{C}$ is an E_∞ operad then $\mathcal{C}(j)/S_j$ is a classifying space for S_j.

Lemma 2.2. Any operad $\mathcal{C}$ over $\mathcal{M}$ is S-free and any local equivalence $\phi : \mathcal{C} \to \mathcal{C}'$ is an S-equivalence.

Proof. Let $\sigma \in S_j$. Then σ must act on $\mathcal{C}(j)$ by permuting components carrying $\epsilon^{-1}(\tau)$ homeomorphically to $\epsilon^{-1}(\tau\sigma)$. Therefore, if σ fixes some point of $\mathcal{C}(j)$ it fixes the corresponding component which can only happen if σ is the identity. For the second part we may assume that $\epsilon' \circ \phi = \epsilon$ since otherwise we can simply redefine ϵ in that way. In that case, ϕ_j restricts to a homotopy equivalence $\epsilon_j^{-1}(e_j) \to (\epsilon'_j)^{-1}(e_j)$. The associated homotopies can be extended to the other components of $\mathcal{C}(j)$ and $\mathcal{C}'(j)$. From this the result follows. $\square$

For the next results we first need some definitions.

Definition 2.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be operads. Define a new operad $\mathcal{C} \times \mathcal{C}'$ as follows. For each j we put $(\mathcal{C} \times \mathcal{C}')(j) = \mathcal{C}(j) \times \mathcal{C}'(j)$. The data are define as

1. $(\gamma \times \gamma')(c \times c'; d_1 \times d'_1, \cdots, d_k \times d'_k) = \gamma(c; d_1, \cdots, d_k) \times \gamma(c'; d'_1, \cdots, d'_k)$ for $c \times c' \in \mathcal{C}(k) \times \mathcal{C}'(k)$ and $d_s \times d'_s \in \mathcal{C}(j_s) \times \mathcal{C}'(j_s)$
2. $1 = 1 \times 1 \in \mathcal{C}(1) \times \mathcal{C}'(1)$
3. $(c \times c')\sigma = c\sigma \times c'\sigma$ for $c \times c' \in \mathcal{C}(j) \times \mathcal{C}'(j)$ and $\sigma \in S_j$

The monad corresponding to $\mathcal{C} \times \mathcal{C}'$ is denoted by $T \times T'$.

We point out that $T \times T'$ is just notation. We do not claim that it is the product in the category of monads. It is easily verified that this definition indeed gives an operad.

Definition 2.7. Let $(\mathcal{C}, \epsilon)$ and $(\mathcal{C}', \epsilon')$ be operads over $\mathcal{M}$. We define a new operad $\mathcal{C}\nabla\mathcal{C}'$ by letting $\mathcal{C}\nabla\mathcal{C}$ be the fibred product of of ϵ and ϵ' in the category of operads. The required augmentation $\epsilon\nabla\epsilon'$ is then given via the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C}\nabla\mathcal{C}' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\epsilon\nabla\epsilon'"] & \mathcal{C}' \arrow[d, "\epsilon'"] \\
\mathcal{C} \arrow[r, "\epsilon"'] & \mathcal{M}
\end{tikzcd}$$

where π_i is the projection on the i-th factor. The monad corresponding to $\mathcal{C}\nabla\mathcal{C}'$ will be denoted by $T\nabla T'$.

Theorem 2.2.
1. Let $\mathcal{C}$ be an A_∞ operad and $\mathcal{C}'$ an operad over $\mathcal{M}$ then $\pi_2 : \mathcal{C}\nabla\mathcal{C}' \to \mathcal{C}'$ is a local S-equivalence.
2. Let $\mathcal{C}$ be an E_∞ operad and $\mathcal{C}'$ an S-free operad then $\pi_2 : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'$ is a local equivalence of S-free operads.

Proof.
1. This follows from Lemma 2.2 since $\epsilon_j^{-1}(\sigma)$ is contractible for each $\sigma \in S_j$ hence $\pi_2 : \epsilon_j^{-1}(\sigma) \times (\epsilon'_j)^{-1}(\sigma) \to (\epsilon'_j)^{-1}$ is a homotopy equivalence.
2. This is immediate.

$\square$

Corollary 2.1. Suppose $\mathcal{C}$ is an E_∞ operad. Then $\pi_2 : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is a local S-equivalence and thus $\mathcal{C} \times \mathcal{M}$ is an A_∞ operad. Moreover, if (X, θ) is a $\mathcal{C}$-space, then $(X, \theta \circ \pi_1)$ is a $\mathcal{C} \times \mathcal{M}$-space where $\pi_1 : \mathcal{C} \times \mathcal{M} \to \mathcal{C}$ is the projection. In particular, any E_∞-space is an A_∞-space.

2.2 The Little Cubes Operads

In this section we will introduce the operads of little n-cubes which are the operads we are mainly interested in. Some basic properties about these operads will be established and we will relate them to the so-called configuration spaces of $\mathbb{R}^n$.
To simplify notation we will write ${}^kX$ for the k-fold disjoint of X with itself.

Definition 2.8. Let I^n denote the unit n-cube and J^n its interior. A little n-cube is a linear embedding $f : J^n \to J^n$ of the form $f = f_1 \times \cdots \times f_n$ where, for each i, f_i is of the form $f_i(t) = (y_i - x_i)t + x_i$ with $0 \le x_i < y_i \le 1$. Define

$$\mathcal{C}_n(k) = \{(c_1, \cdots, c_k) | c_i \text{ is a little } n\text{-cube and } \operatorname{im}(c_i) \cap \operatorname{im}(c_j) = \emptyset \text{ if } i \neq j\}$$

Notice that we can consider $(c_1, \cdots, c_k)$ as a map ${}^k J^n \to J^n$ and thus $\mathcal{C}(k)$ can be topologized a subspace of all continuous maps ${}^k J^n \to J^n$. We will write $\mathcal{C}(0) = <>$ where $<>$ is regarded as the unique embedding $\emptyset \to J^n$. The data of the n-cubes operad are defined by

1. for $c \in \mathcal{C}_n(k)$ and $d_i \in \mathcal{C}_n(j_i)$:

$$\gamma(c; d_1, \cdots, d_k) = c \circ (d_1 + \cdots + d_k) : {}^{j_1} J^n + \cdots + {}^{j_k} J^n \to J^n$$

2. $1 \in \mathcal{C}_n(1)$ is just the identity $1 : J^n \to J^n$

3. for $\sigma \in S_k : (c_1, \cdots, c_k)\sigma = (c_{\sigma^{-1}(1)}, \cdots, c_{\sigma^{-1}(k)})$

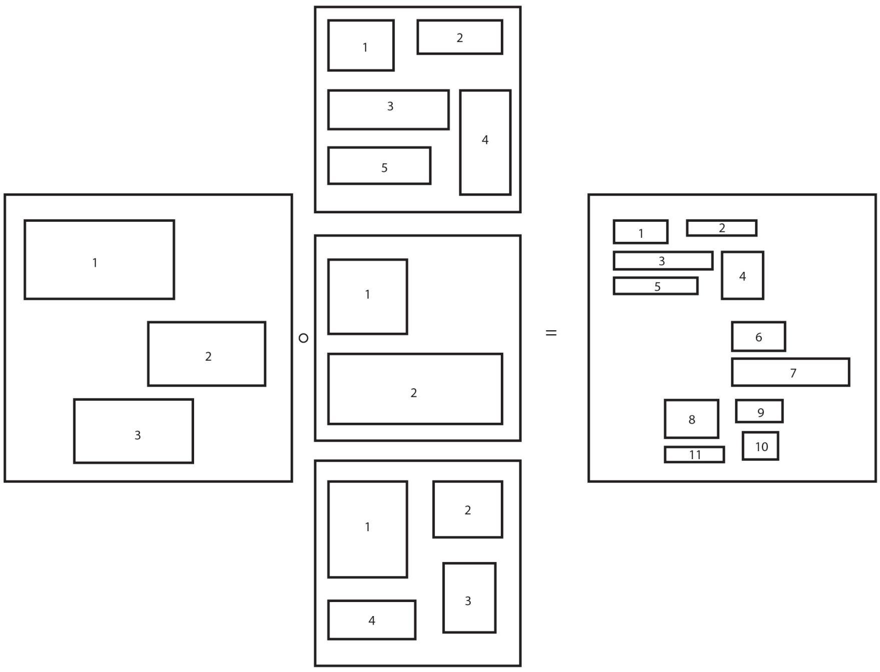

Figure 1: Example of composition in $\mathcal{C}_2$

Notice that

$$\sigma_i(c_1, \cdots, c_k) = (c_1, \cdots, c_i, c_{i+2}, \cdots, c_k)$$

It is trivial that the conditions demanded from the data of an operad do in fact hold. Furthermore, $(c_1, \cdots, c_k)\sigma = (c_1, \cdots, c_k)$ if and only if $\sigma = 1$ since the c_i have disjoint images. Therefore, the S_k-action is free.

In many situations we want to compare $\mathcal{C}_n$ with $\mathcal{C}_{n+1}$ and the way we do this is, of course, by a morphism of operads $\mathcal{C}_n \to \mathcal{C}_{n+1}$ introduced in the following definition.

Definition 2.9. Define a morphism of operads $\sigma_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$ as follows. Let $1 : J \to J$ denote the identity and define for each k

$$\sigma_{n,k} : (c_1, \cdots, c_k) = (c_1 \times 1, \cdots, c_k \times 1)$$

It is easily verified that the resulting map σ_n is indeed a morphism of operads. We would like to investigate the homotopy type of the $\mathcal{C}_n(k)$. We will do this by relating them to configuration spaces of $\mathbb{R}^n$. First, however, we will have a closer look at the topology of the $\mathcal{C}_n(k)$.
Suppose $c = (c_1, \cdots, c_k) \in \mathcal{C}_n(k)$. Then c determines and is determined by the point $c(a, b) \in J^{2nk}$ defined as $c(a, b) = (c_1(a), c_1(b), \cdots, c_k(a), c_k(b))$ where $a = (\frac{1}{4}, \cdots, \frac{1}{4})$ and $b = (\frac{3}{4}, \cdots, \frac{3}{4})$. Let $\mathcal{T}$ denote the topology of $\mathcal{C}_n(k)$ by regarding it as a subspace of J^{2nk} in this fashion and let $\mathcal{S}$ denote the topology on $\mathcal{C}_n(k)$ as defined in Definition 2.8. Then we have the following lemma.

Lemma 2.3. The topologies $\mathcal{T}$ and $\mathcal{S}$ are the same.

Proof. Let U be open in J^n and C compact in ${}^kJ^n$ and define

$$A(C, U) = \{c \in \mathcal{C}_n(k) | c(C) \subset U\}$$

then $A(C, U)$ is $\mathcal{S}$-open. We will write a_i for the point $a = (\frac{1}{4}, \cdots, \frac{1}{4})$ in the i-th domain cube of ${}^kJ^n$ and b_i for the point $b = (\frac{3}{4}, \cdots, \frac{3}{4})$ in the i-th domain cube. For $i = 1, \cdots, k$ let U_i and V_i be open in J^n then we can write

$$\mathcal{C}_n(k) \cap (U_1 \times V_1 \times \cdots \times U_k \times V_k) = \cap_{i=1}^k (A(a_i, U_i) \cap A(b_i, V_i))$$

from which it obviously follows that any $\mathcal{T}$-open set is $\mathcal{S}$ open.
For the converse, consider $A(C, U)$. Without loss of generality we may assume that U is the image of some open little cube g and that C is contained in a single domain cube J_i^n of ${}^kJ^n$. Let $C' \subset J_i^n$ be the image of the smallest closed little cube f containing C. By linearity, $A(C, U) = A(C', U)$. Let $c = (c_1, \cdots, c_k)$. Since f is a little cube it can be written as a product of linear maps $f_j : J \to J$ and similarly for g and $c_i \circ f$. Then we see that $c \in A(C', U)$ if and only if

$$(c_i f)_j(0) > g_j(0) \text{ and } (c_i f)_j(1) < g_j(1)$$

from which it follows that $A(C', U)$ is $\mathcal{T}$-open and thus we are done. $\square$

It is now time to briefly discuss configuration spaces.

Definition 2.10. Let M be a manifold of dimension n. The k-th configuration space of M is

$$F(M, k) = \{(x_1, \cdots, x_k) \in M^n | x_r \neq x_s \text{ if } r \neq s\}$$

with the subspace topology inherited from M^n.

We mention that $F(M, k)$ is itself a manifold of dimension nk and that (clearly) $F(M, 1) = M$. The configuration space $F(M, k)$ can be endowed with a right S_k-action in the following way. For $(x_1, \cdots, x_k) \in F(M, k)$ and $\sigma \in S_k$ we define

$$(x_1, \cdots, x_k)\sigma = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(k)})$$

Since the x_i are distinct it is clear that this action is free. We will write $B(M,k)$ for the orbit space $F(M,k)/S_k$.
Let $n \geq 2$ and let M be a manifold of dimension n. Put $Y_0 = \emptyset$ and $Y_r = \{y_1, \cdots, y_r\}$ for $1 \leq r \leq k$ where the $y_i \in M$ are distinct points. Define a map by

$$\begin{aligned} \pi_r : F(M \setminus Y_r, k-r) &\to M \setminus Y_r \\ \pi_r(x_1, \cdots, x_{k-r}) &= x_1 \quad \text{for} \quad 0 \leq r \leq j-1 \end{aligned}$$

then we have the following theorem.

Theorem 2.3. The map π_r is a fibration. The fibre over the point y_{r+1} is $F(M \setminus Y_{r+1}, k-r-1)$. Moreover, for $r \geq 1$ the map π_r admits a section.

Proof. A proof of this theorem can be found in [2]. □

Let S^n denote the n-sphere then we get the following corollary.

Corollary 2.2.

1. for $n \geq 3$ we have

$$\pi_i F(\mathbb{R}^n, k) = \sum_{r=1}^{k-1} \pi_i(\bigvee_{s=1}^{r} S^{n-1})$$

2. $\pi_i F(\mathbb{R}^2, k) = 0$ if $i \neq 1$

3. $\pi_1 F(\mathbb{R}^2, k)$ is constructed from the free groups $\pi_1(\bigvee_{s=1}^{r} S^1)$ by successive split extensions.

Proof. Using the fact that $\mathbb{R}^n \setminus Y_r$ is homotopy equivalent to $\bigvee_{s=1}^{r} S^{n-1}$ this follows easily from the theorem. □

For the case that $n = 1$ we have the following lemma.

Lemma 2.4. $\pi_0 F(\mathbb{R}, k)$ is isomorphic to S_k. Moreover, the components of $\pi_0 F(\mathbb{R}, k)$ are contractible.

Proof. Define

$$F_0 = \{(x_1, \cdots, x_k) | x_1 < \cdots < x_k\} \subset F(\mathbb{R}, k)$$

Since F_0 is homeomorphic to the interior of a simplex it is contractible. It is easily seen that F_0 is one component of $F(\mathbb{R}, k)$. Define

$$\begin{aligned} \phi : F_0 \times S_k &\to F(\mathbb{R}, k) \\ ((x_1, \cdots, x_k), \sigma) &\mapsto (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(k)}) \end{aligned}$$

It is readily verified that ϕ defines a homeomorphism which proves the result. □

Before proving the main result of this section we need a definition.

Definition 2.11. Let $c = (c_1, \cdots, c_k)$ be an element of $\mathcal{C}_n(k)$, then c_i can be written as $c_i = c_{i1} \times \cdots \times c_{in}$ where

$$\begin{aligned} c_{ij} &: J \to J \\ c_{ij}(t) &= (y_{ij} - x_{ij})t + x_{ij} \end{aligned}$$

for suitable x_{ij} and y_{ij}. It is said that c is equidiameter of diameter d for all i and j we have that $y_{ij} - x_{ij} = d$. Note that this just means that each c_i is a cube and that all the cubes are of the same size.

Now, for the promised theorem.

Theorem 2.4. For $n, k \geq 1$, $\mathcal{C}_n(k)$ is S_k-equivariantly homotopy equivalent to $F(\mathbb{R}^n, k)$.

Proof. Let $n, k \geq 1$. It is clear that it will be sufficient to show the result for J^n instead of $\mathbb{R}^n$. Let $\gamma = (\frac{1}{2}, \cdots, \frac{1}{2})$ and define

$$g : \mathcal{C}_n(k) \to F(J^n, k)$$
$$g(c_1, \cdots, c_k) = (c_1(\gamma), \cdots, c_k(\gamma))$$

For each $b \in F(J^n, k)$ we can find an equidiameter $c \in \mathcal{C}_n(k)$ satisfying $g(c) = b$. Now define

$$f : F(J^n, k) \to \mathcal{C}_n(k)$$
$$f(b) = c$$

where $c \in \mathcal{C}_n(k)$ is the unique element such that $g(c) = b$ and c is equidiameter with maximal diameter subject to the condition that $g(c) = b$. Clearly, $g \circ f = 1$ and f and g are S_k-equivariant. Lemma 2.3 guarantees the continuity of both f and g.
Let c be as in the definition of f and let d be the diameter of $f(g(c))$. Define a homotopy

$$h : \mathcal{C}_n(k) \times I \to \mathcal{C}_n(k)$$
$$h(c, u) = (c_{11}(u, \cdots, c_{1n}(u), \cdots, c_{k1}(u), \cdots, c_{kn}(u))$$

where

$$c_{ij}(u)(t) = ((1-u)(y_{ij} - x_{ij}) + ud)t + \frac{1}{2}(uy_{ij} + (2-u)x_{ij} - ud)$$

Notice that h just expands or contracts each interval c_{ij} from its middle point to an interval of length d. Clearly, h is both continuous and S_k-equivariant. The reader can easily convince himself that $h(c, 0) = c$ and $h(c, 1) = f(g(c))$. Hence $F(J^n, k)$ is S_k-equivariantly homotopy equivalent to $\mathcal{C}_n(k)$. $\square$

It is worthwhile to point out that $F(J^n, k)$ is in fact a strong S_k-equivariant deformation retract of $\mathcal{C}_n(k)$ since the homotopy h from the proof also has the property that $h(f(b), u) = f(b)$. The central consequence of the theorem is the following corollary.

Corollary 2.3.

1. $\mathcal{C}_1$ is an A_∞ operad
2. for $n > 1$, $\mathcal{C}_n$ is a locally $(n-2)$-connected S-free operad over $\mathcal{N}$

Now, define $\sigma'_{n-1,k} : \mathcal{C}_{n-1}(k) \to \mathcal{C}_n(k)$ by mapping a little $(n-1)$-cube f to $1 \times f$. The following lemma will be needed in the next chapter.

Lemma 2.5. $\sigma'_{n-1,k}$ is S_k-equivariantly homotopic to $\sigma_{n-1,k}$ where $\sigma_{n-1,k}$ is defined as in Definition 2.9.

Proof. Put $\sigma_{n-1}(x) = (x, \frac{1}{2})$ and $\sigma'_{n-1}(x) = (\frac{1}{2}, x)$. It is clearly sufficient to show that $\sigma_{n-1} \simeq \sigma'_{n-1}$ as maps $F(J^{n-1}, k) \to F(J^n, k)$. Define for $(s, x) \in J \times J^{n-1}$

$$\tau : F(J^n, k) \to F(J^n, k)$$
$$\tau(s, x) = (x, s)$$

and

$$\tau' : F(J^n, k) \to F(J^n, k)$$
$$\tau'(s, x) = \begin{cases} (s, x), & \text{if } n \text{ is odd} \\ (1 - s, x), & \text{if } n \text{ is even} \end{cases}$$

then we clearly have

$$\tau \circ \sigma'_{n-1} = \sigma_{n-1} \qquad\qquad \tau' \circ \sigma'_{n-1} = \sigma'_{n-1}$$

Therefore, it is sufficient to show that τ and τ' are S_k-equivariantly homotopic. Let $e_0 = (1, 0, \cdots, 0)$ and define $\psi^n : (I^n, \partial I^n) \to (S^n, e_0)$ as follows. The center $(\frac{1}{2}, \cdots, \frac{1}{2})$ is mapped to $e_0^* = (-1, 0, \cdots, 0)$. Each point on the boundary ∂I^n is mapped to e_0. Let $\overline{c_0 x}$ denote the line segment between c_0 and a given point x in the interior of I^n. The line segment $\overline{c_0 x}$ is mapped, in the obvious way, by ψ^n onto the semi-circle connecting e_0^* and e_0 whose tangent vector at e_0^* us parallel to the vector $(-1) \times \overline{c_0 x}$. It is left to the reader to chech that ψ^n defines a relative homeomorphism and that as maps $S^n \to S^n$

$$\psi_n \tau \psi_n^{-1}(s_1, \cdots, s_{n+1}) = (s_1, s_3, s_4, \cdots, s_{n+1}, s_2)$$
$$\psi_n \tau' \psi_n^{-1}(s_1, \cdots, s_{n+1}) = (s_1, (-1)^{n-1} s_2, s_3, \cdots, s_{n+1})$$

Since both maps have degree $(-1)^{n-1}$ they lie in the same component of $O(n)$. So, let $k : I \to O(n)$ be a path connecting them and define

$$h_t = \psi^n k(t) \psi_n^{-1} : J^n \to J^n$$

Then clearly, $h_0 = \tau$ and $h_1 = \tau'$. Now, each h_t is a homeomorphism and therefore the product homotopy $h_t^k : (J^n)^k \to (J^n)^k$ restricts to a S_k-equivariant homotopy between τ and τ' on $F(J^n, k)$. $\square$

3 The Approximation Theorem

Our primary goal will be to prove the following theorem.

Theorem 3.1 (The Approximation Theorem). Let $n \geq 1$ and let $\mathcal{C}_n$ be the operad of little n-cubes and C_n the associated monad, then there exists a morphism of monads

$$\alpha_n : C_n \to \Omega^n \Sigma^n$$

where Ω denotes the loop space functor and Σ the reduced suspension functor. Moreover, if X is connected then

$$\alpha_n : C_n X \to \Omega^n \Sigma^n X$$

is a weak homotopy equivalence, i.e. it induces isomorphisms on all homotopy groups.

3.1 The morphisms α_n

In this section we will first investigate loop spaces more closely. This will allow us to construct the α_n and show that this indeed defines a morphism of monads. The subsequent two sections will be devoted to proving the remainder of the approximation theorem.
For $n \geq 1$ we will first consider the category of n-fold loop sequences $\mathbf{L}_n$.

Definition 3.1. 1. The objects of $\mathbf{L}_n$ are sequences $\{X_k | 0 \leq k \leq n\}$ with $X_k \in \mathbf{CG}_*$ such that $X_k = \Omega X_{k+1}$. The morphisms of $\mathbf{L}_n$ are sequences $\{f_k | 0 \leq k \leq n\}$ with f_k maps in $\mathbf{CG}_*$ such that $f_k = \Omega(f_{k+1})$.

2. Consider the forgetful functor

$$U_n : \mathbf{L}_n \to \mathbf{CG}_*$$
$$U_n(\{X_k\}) = X_0$$

An n-fold loop space (map) is a space (map) in the image of U_n

Note we can write an n-fold loop sequence in the form $\{\Omega^{n-k} X\}$ and then $U_n(\{\Omega^{n-k} X\}) = \Omega^n$. The reason why the $\mathbf{L}_n$ are used is because otherwise we would not get a well-defined category. This is so because X is not uniquely determined by ΩX. For example, let $X = \{0\}$ and $Y = \{0, 1\}$. Then ΩX and ΩY both consists solely of the unique map mapping S^1 onto 0 since the maps preserve base points. Therefore, $\Omega X = \Omega Y$ but of course X is not homotopy equivalent to Y. Using $\mathbf{L}_n$ we record with which space we started and thus do not run into this problem. Consequently, from now on we will use Ω^n both to refer to the space itself and to refer to the whole sequence. Naturality statements will be implicitly be understood as referring to the objects of $\mathbf{L}_n$.
We view $\Omega^n X$ as the space of maps $(S^n, *) \to (X, *)$ where S^n is identified with $I^n / \partial I^n$. Let $X \in \mathbf{CG}_*$, $c = (c_1, \cdots, c_k) \in \mathcal{C}_n(k)$ and $x = (x_1, \cdots, x_k) \in (\Omega^n X)^k$ and define for $t \in S^n$

$$\theta_{n,k} : \mathcal{C}_n(k) \times (\Omega^n X)^k \to \Omega^n X$$
$$\theta_{n,k}(c, x)(t) = \begin{cases} x_i(s), & \text{if } c_i(s) = t \\ *, & \text{if } t \notin \mathrm{im}(c) \end{cases}$$

We have the following theorem.

Theorem 3.2. 1. The $\theta_{n,k}$ define an action θ_n of $\mathcal{C}_n$ on $\Omega^n X$ for every $X \in \mathbf{CG}_*$.

2. Suppose $X = \Omega Y$. Then $\theta_n = \theta_{n+1} \circ \sigma_n$ where $\sigma : \mathcal{C}_n \to \mathcal{C}_{n+1}$ is the morphism defined in Definition 2.9 and θ_{n+1} is the action of $\mathcal{C}_{n+1}$ on $\Omega^{n+1} X$.

3. The θ_n are natural on maps in $\mathcal{L}_n$ in the following sense: define $W_n : \mathbf{L}_n \to C_n[\mathbf{CG}_*]$ by $W_n X = (U_n X, \theta_n)$ on objects where $\theta_n : C_n U_n X \to U_n X$ is the C_n-algebra structure map corresponding to the $\theta_{n,k}$ and by $W_n(f) = U_n(F)$ on morphisms. Then W_n is a functor from n-fold loop sequences to C_n-algebras.

Proof. 1. It is straightforward to verify that the diagrams of Lemma 1.1 indeed commute hence the $\theta_{n,k}$ indeed define an action.

2. Since $X = \Omega Y$ it follows that $\Omega^n X = \Omega^{n+1} Y$ via the correspondence between ϕ and ψ defined using

$$\phi(u)(t) = \psi(u,t) \quad \text{for} \quad (u,t) \in I^n \times I$$

If f is a little n-cube then $\sigma_n(f) = f \times 1$. Consequently, it follows that $\theta_n = \theta_{n+1}\sigma_n$.

3. This is trivial.

$\square$

It should come as now surprise that the idea is to use the natural $\mathcal{C}_n$-action θ_n to construct the desired morphism of monads $\alpha_n : C_n \to \Omega^n \Sigma^n$. For this construction we exploit the adjunction (where each Hom is in $\mathbf{CG}_*$ unless otherwise indicated)

$$\begin{gathered} \phi : \mathrm{Hom}(X, \Omega Y) \to \mathrm{Hom}(\Sigma X, Y) \\ \phi(f)([x,t]) = f(x)(t) \end{gathered} \tag{1}$$

By iteration one obtains

$$\phi^n : \mathrm{Hom}(X, \Omega^n Y) \to \mathrm{Hom}(\Sigma X, Y) \tag{2}$$

Now, define $Q_n X = \{\Omega^{n-i}\Sigma^n X | 0 \le i \le n\} \in \mathbf{L}_n$. Note that $U_n Q_n X = \Omega^n \Sigma^n X$. Therefore,

$$\mathrm{Hom}(\Sigma^n X, Y) = \mathrm{Hom}_n(Q_n X, \{\Omega^{n-i} Y\})$$

since a morphism $\{g_i | 0 \le i \le n\}$ is determined by g_n. Therefore, from (2) we obtain the adjunction

$$\phi_n : \mathrm{Hom}(X, U_n\{\Omega^{n-i}Y\}) \to \mathrm{Hom}_{\mathcal{L}_n}(Q_n X, \{\Omega^{n-i} Y\}) \tag{3}$$

By (3) it follows that $Q_n X$ is the free n-fold loop sequence. Whenever we call $\Omega^n \Sigma^n X$ the free n-fold loop space this is what is implicitly meant. To make the passage to monads we need the following standard lemma from category theory.

Lemma 3.1. Let $\phi : \mathrm{Hom}_{\mathbf{C}}(X, UY) \to \mathrm{Hom}_{\mathbf{L}}$ be an adjunction between functors $U : \mathbf{L} \to \mathbf{C}$ and $Q : \mathbf{C} \to \mathbf{L}$. For $X \in \mathbf{C}$ we put

$$\eta = \phi^{-1}(1_{QX}) : X \to UQX$$

and

$$\mu = U\phi(1_{UQX}) : UQUQX \to UQX$$

Then (UQ, μ, η) is a monad. Moreover, if we define, for $Y \in \mathbf{L}$,

$$\xi = U\phi(1_{UY}) : UQUY \to UY$$

then (UY, ξ) is an UQ-algebra and $\xi : UQU \to U$ is a natural transformation. Consequently, there is a well-defined functor $V : \mathbf{L} \to UQ[\mathbf{C}]$ given by $VY = (Y, \xi)$ on objects and $V(g) = U(g)$ on morphisms.

Proof. It is straightforward to verify that the diagrams in Definition 1.5 and Definition 1.6 are indeed commutative. Therefore, (UQ, μ, η) is a monad and (UY, ξ) is an UQ-algebra. From this the other assertions are clear. $\square$

The lemma implies that our ϕ_n give monads $(\Omega^n\Sigma^n, \mu_n, \eta_n)$ with

$$\eta_n = \phi^{-n}(1_{\Sigma^n X})$$

$$\mu_n = \Omega^n\phi^n(1_{\Omega^n\Sigma^n X})$$

and $\Omega^n\Sigma^n$-algebras (Y, ξ) with

$$\xi_n = \Omega^n\phi^n(1_{\Omega^n Y})$$

The lemma also gives functors $V_n : \mathbf{L}_n \to \Omega^n\Sigma^n[\mathbf{CG}_*]$ defined by $V_nY = (U_nY, \xi_n)$ on objects.
Let (C_n, μ_n, η_n) be the monad associated to $\mathcal{C}_n$. Let $\alpha_n : C_nX \to \Omega^n\Sigma^n X$ be the composite

$$C_nX \xrightarrow{C_n\eta_n} C_n\Omega^n\Sigma^n X \xrightarrow{\theta_n} \Omega^n\Sigma^n X$$

Furthermore, define

$$\sigma_n = \Omega^n\phi^{-1}(1_{\Sigma^{n+1}X} : \Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1}X$$

It is easily seen that σ_n defines a morphism of operads. We then we have the following theorem.

Theorem 3.3. 1. $\alpha_n : C_n \to \Omega^n\Sigma^n$ is a morphism of monads and the following diagram of functors commutes

$$\begin{tikzcd} & \mathbf{L}_n \arrow[dl, "V_n"'] \arrow[dr, "W_n"] & \\ \Omega^n\Sigma^n[\mathbf{CG}_*] \arrow[rr, "\alpha_n^*"'] & & C_n[\mathbf{CG}_*] \end{tikzcd}$$

where $\alpha_n^*(Y, \xi) = (Y, \xi \circ \alpha_n)$.

2. The following diagrams of monads commute

$$\begin{tikzcd}
C_n \arrow[r, "\alpha_n"] \arrow[d, "\sigma_n"'] & \Omega^n\Sigma^n \arrow[d, "\sigma_n"] \\
C_{n+1} \arrow[r, "\alpha_{n+1}"'] & \Omega^{n+1}\Sigma^{n+1}
\end{tikzcd}$$

where the left σ_n is the map of Definition 2.9.

Proof. Since θ_n acts naturally on μ_n, ξ_n and σ_n it follows (from the definition of natural transformation and of monad) that the following four diagrams commute for all $X \in \mathbf{CG}_*$.

$$\begin{tikzcd}
& C_nX \arrow[d, "C_n\eta_n"'] \arrow[dr, "\alpha_n"] & \\
X \arrow[ur, "\eta_n"] \arrow[dr, "\eta_n"'] & C_n\Omega^n\Sigma^nX \arrow[r, "\theta_n"] & \Omega^n\Sigma^nX \\
& \Omega^n\Sigma^nX \arrow[u, "\eta_n"] \arrow[ur, "1"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
C_nC_nX \arrow[r, "C_nC_n\eta_n"] \arrow[d, "\mu_n"'] & C_nC_n\Omega^n\Sigma^nX \arrow[r, "C_n\theta_n"] \arrow[d, "\mu_n"'] & C_n\Omega^n\Sigma^nX \arrow[r, shift left, "C_n\eta_n"] \arrow[d, "\theta_n"'] & C_n\Omega^n\Sigma^n\Omega^n\Sigma^nX \arrow[l, shift left, "C_n\mu_n"] \arrow[d, "\theta_n"] \\
C_nX \arrow[r, "C_n\eta_n"'] & C_n\Omega^n\Sigma^nX \arrow[r, "\theta_n"'] & \Omega^n\Sigma^nX & \Omega^n\Sigma^n\Omega^n\Sigma^n \arrow[l, "\mu_n"']
\end{tikzcd}$$

$$\begin{tikzcd}
C_n\Omega^nX \arrow[r, "C_n\eta_n"] \arrow[dr, "1"'] & C_n\Omega^n\Sigma^n\Omega^nX \arrow[r, "\theta_n"] \arrow[d, "C_n\xi_n"] & \Omega^n\Sigma^n\Omega^nX \arrow[d, "\xi_n"] \\
& C_n\Omega^nX \arrow[r, "\theta_n"'] & \Omega^nX
\end{tikzcd}$$

$$\begin{tikzcd}
C_nX \arrow[rr, "C_n\eta_n"] \arrow[dd, "\sigma_n"'] & & C_n\Omega^n\Sigma^nX \arrow[rr, "\theta_n"] \arrow[dl, "\sigma_n"'] \arrow[dr, "C_n\sigma_n"] & & \Omega^n\Sigma^nX \arrow[dd, "\sigma_n"] \\
& C_{n+1}\Omega^n\Sigma^nX \arrow[dr, "C_{n+1}\sigma_n"] & & C_{n+1}\Omega^{n+1}\Sigma^{n+1}X \arrow[dl, "\sigma_n"'] \arrow[dr, "\theta_n"] & \\
C_{n+1}X \arrow[ur, "C_{n+1}\eta_n"] \arrow[rr, "C_{n+1}\eta_{n+1}"'] & & C_{n+1}\Omega^{n+1}\Sigma^{n+1}X \arrow[rr, "\theta_{n+1}"'] & & \Omega^{n+1}\Sigma^{n+1}X
\end{tikzcd}$$

The first diagram implies that $\alpha_n\eta_n = \eta_n$ and the second diagram that $\mu_n\alpha_n^2 = \alpha_n\mu_n$. From the third diagram it follows that $\xi_n\alpha_n = \theta_n$. So by these three we obtain that $\alpha^*V_n = W_n$. The last diagram implies that $\sigma_n\alpha_n = \alpha_{n+1}\sigma_n$. $\square$

The last part of this section is dedicated to constructing a morphism of monads $C_n \to \Omega^iC_{n-i}\Sigma^i$. The following standard lemma from category theory implies that we only need to consider the case when $i = 1$.

Lemma 3.2. Suppose

$$\phi : \mathrm{Hom}_{\mathbf{C}}(X, \Lambda Y) \to \mathrm{Hom}_{\mathbf{C}}(\Sigma X, Y)$$

is an adjunction and let (T, μ, η) be a monad in $\mathbf{C}$. Then

1. $(\Lambda T\Sigma, \tilde{\mu}, \tilde{\eta})$ is a monad where $\tilde{\mu}$ is defined as the composition

$$\Lambda T\Sigma\Lambda T\Sigma X \xrightarrow{\Lambda T\phi(1)} \Lambda TT\Sigma X \xrightarrow{\Lambda\mu\Sigma} \Lambda T\Sigma X$$

and $\tilde{\eta}$ is defined as the composite

$$X \xrightarrow{\phi^{-1}(1)} \Lambda\Sigma X \xrightarrow{\Lambda\eta\Sigma} \Lambda T\Sigma X$$

Now, for the construction of $C_n \to \Omega^i C_{n-i}\Sigma^i$.

Theorem 3.4. For all $n > 1$ there exists a morphism of monads $\beta_n : C_n \to \Omega C_{n-1}\Sigma$ satisfying $\alpha_n = (\Omega\alpha_n\Sigma)\beta$. Consequently, α factors as the composite of morphisms of monads

$$C_n \to \Omega C_{n-1}\Sigma \to \cdots \to \Omega_{n-1}C_1\Sigma^{n-1} \to \Omega^n\Sigma^n$$

Proof. Let $c = (c_1, \cdots, c_k) \in \mathcal{C}_n(k)$, $x = (x_1, \cdots, x_k) \in X^k$ and $t \in I$. For any j we can write $c_j = a_j \times b_j$ where $a_j : J \to J$ and $b_j : J^{n-1} \to J^{n-1}$. Let $r_1, \cdots, r_i$, ordered according to size, denote those indices j for which $t \in a_j(J)$. The little n-cubes c_j have disjoint images and thus the little $(n-1)$-cubes b_j for $j = r_1, r_2, \cdots, r_i$ have disjoint images. Therefore, we can define

$$\beta_n[c, x](t) = \begin{cases} *, & \text{if } t \notin \cup_{j=1}^{k} a_j(J) \\ [(b_{r_1}, \cdots, b_{r_i}), [x_{r_1}, s_1], \cdots, [x_{r_i}, s_i]], & \text{if } \forall 0 \le j \le i a_{r_j}(s_j) = t \\ & \text{and } \forall r \notin \{r_1, \cdots, r_i\} : t \notin a_r(J) \end{cases}$$

It is easy to see that β_n is well-defined and continuous. Let $v \in S^{n-1}$ then by definition of β_n and Theorem 3.2 we can write

$$(\Omega\alpha_{n-1}\Sigma)\beta_n[c, x](t, v) = \begin{cases} [x_j, s, u], & \text{if } c_j(s, u) = (t, v) \\ *, & \text{if } (t, v) \notin \mathrm{im}(c) \end{cases}$$

But this is just α_n. Since β_n and $\Omega\alpha_{n-1}\Sigma$ are inclusions and α_n and $\Omega\alpha_{n-1}\Sigma$ are morphisms of monads it follows that β_n is a morphism of monads. □

3.2 The Approximation Theorem

The main goal of this section is to prove the following theorem (where p denotes the usual path space fibration)

Theorem 3.5. Let $X \in \mathbf{CG}_*$ and $n \ge 1$. Then

1. there exists a space $E_nX \supset C_nX$ and maps $\pi_n : E_nX \to C_{n-1}\Sigma X$ and $\tilde{\alpha}_n : E_nX \to P\Omega^{n-1}\Sigma^n X$ such that the following diagram commutes

$$\begin{tikzcd} C_nX \arrow[r, "\text{incl.}"] \arrow[d, "\alpha_n"] & E_nX \arrow[r, "\pi_n"] \arrow[d, "\tilde{\alpha}_n"'] & C_{n-1}\Sigma X \arrow[d, "\alpha_{n-1}"] \\ \Omega^n\Sigma^n X \arrow[r, "\text{incl.}"'] & P\Omega^{n-1}\Sigma^n X \arrow[r, "p"'] & \Omega^{n-1}\Sigma^n X \end{tikzcd}$$

where (if $n = 1$) $C_0\Sigma X = \Sigma X$ and α_0 is just the identity. Morever, $E_n X$ is contractible.

2. furthermore, if X is connected then π_n is a quasi-fibration with fiber $C_n X$ and (thus) α_n is a weak homotopy equivalence.

The notion of a quasi-fibration is introduced in Definition 3.3 below. Since we have already seen that α_n is a morphism of monads the second part clearly implies the Approximation Theorem (Theorem 3.1).
First we will construct $E_n X$. We will do this by constructing a functor E_n from pairs (X, A) to spaces and then we will let $E_n X = E_n(CX, X)$ where CX is the cone on X.
Let (X, A) be a pair in $\mathbf{CG}_*$, i.e. A is closed in X and contains $*$. Let $f : J^n \to J^n$ be a little n-cube. Then we can write $f = f' \times f''$ where $f' : J \to J$ and $f'' : J^{n-1} \to J^{n-1}$ if $n > 1$. If $n = 1$ we just take $f = f'$. Let $\epsilon_n(k, X, A)$ denote the subspace of $\mathcal{C}_n(k) \times X^k$ consisting of all $((c_1, \cdots, c_k), x_1, \cdots, x_k)$ satisfying the following condition

- if $x_j \notin A$ then the intersection in J^n of $(c'_j(0), 1) \times c''_j(J^{n-1})$ with $c_i(J^n)$ is empty if $i \neq j$.

The following figure provides a picture for $n = k = 2$.

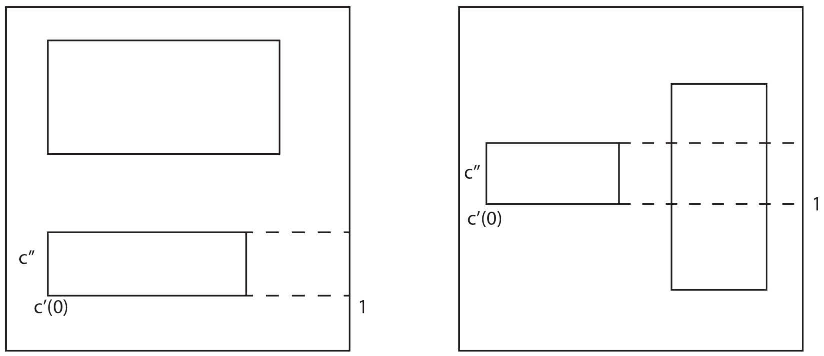

Figure 2: Example of a 2-cube that satifies the condition (left) and one that does not (right).

Recall the equivalence relation $\sim$ on $\coprod_{k \geq 0} \mathcal{C}_n(k) \times X^k$ generated by

1. for c $\in \mathcal{C}_n(k)$, $0 \leq i < k$ and $y \in X^{k-1}$: $(\sigma_i c, y) \sim (c, s_i y)$
2. for c $\in \mathcal{C}_n(k)$, $\sigma \in S_k$ and $y \in X^k$: $(c\sigma, y) \sim (c, \sigma y)$.

where $\sigma_i(c) = \gamma(c; s_i)$ with γ the composition map of the operad and $s_i(x_1, \cdots, x_{k-1}) = (x_1, \cdots, x_i, *, x_{i+1}, \cdots, x_{k-1})$. This equivalence relation gives a new equivalence relation also denoted by $\sim$ when restricted to $\coprod_{k \geq 0} \epsilon_n(k, X, A)$. We define

$$E_n(X, A) = \coprod_{k \geq 0} \epsilon_n(k, X, A) / \sim$$

We endow $E_n(X, A)$ with the subspace topology of $C_n X$. Now, $E_n(X, A)$ is closed in $C_n X$ since A is closed in X. Therefore, $E_n(X, A)$ is in $\mathbf{CG}$. Define a filtration on $E_n(X, A)$ by

$$F_k E_n(X, A) = \begin{cases} E_n(X, A) \cap F_k, & \text{if } k > 0 \\ *, & \text{if } k = 0 \end{cases}$$

where F_k is the image of $\coprod_{j=0}^{k} \mathcal{C}_n(j) \times X^j$ in $C_n X$ as defined in section 1.2. This filtration makes $E_n(X, A)$ into a filtered space. Since $\mathcal{C}_n(k) \subset \epsilon(k, X, A)$ it follows that $C_n A \subset E_n(X, A)$.
Because we want E_n to be a functor we should also specify how it acts on a map $f : (X, A) \to (X', A')$ of pairs of spaces. But this is easy. We already know C_n is a functor and thus we simply let $E_n(f) : E_n(X, A) \to E_n(X', A')$ be the restriction of $C_n(f) : C_n X \to C_n X'$ to $E_n(X, A)$.

Lemma 3.3. Let $X \in \mathbf{CG}_*$ and $[c, x] \in E_n(X, *)$ where $c = (c_1, \cdots, c_k)$ and $x \in (X \setminus *)^k$. Then

1. if $n = 1$ then $k = 1$
2. if $n > 1$ then $c'' = (c_1'', \cdots, c_k'') \in \mathcal{C}_{n-1}(k)$

Proof. Let $i, j \leq k$ with $i \neq j$ be given. Without loss of generality we may assume $c_i'(0) \leq c_j'(0)$. Pick $t \in c_j'(J)$, then $n = 1$ forces that $t \in (c_i(0), 1) \cap c_j(J)$ contradicting the definition of $\epsilon_1(k, X, A)$. Therefore, $i \neq j$ could not possibly hold, i.e. $k = 1$. Now, let $n > 1$ and pick $v \in c_i''(J^{n-1}) \cap c_j''(J^{n-1})$. Then we get that $(t, v) \in c_j(J^n)$ and $t \in (c_i'(0), 1)$ again contradiction the definition of $\epsilon_n(k, X, A)$. Therefore, the images of c_i'' and c_j'' must be disjoint and thus $c'' \in \mathcal{C}_{n-1}(k)$. □

Corollary 3.1. Define $v_n : E_n(X, *) \to C_{n-1} X$ in the following way on points other than $*$

$$v_n[c, x] = \begin{cases} x & \text{if } n = 0 \\ [c'', x] & \text{if } n > 1 \end{cases}$$

then v_n is a natural surjective map of based spaces.

Suppose $\pi : (X, A) \to (Y, *)$ is a map of pairs in $\mathbf{CG}_*$ and write π_n for the composition

$$E_n(X, A) \xrightarrow{E_n(\pi)} E_n(Y, *) \xrightarrow{v_n} C_{n-1} Y$$

then we have the following lemma.

Lemma 3.4. Suppose the following diagram of pairs of spaces commutes

$$\begin{tikzcd} (X, A) \arrow[r, "\pi"] \arrow[d, "f"'] & (Y, *) \arrow[d, "g"] \\ (X', A') \arrow[r, "\pi'"'] & (Y', *) \end{tikzcd}$$

then the following diagram also commutes

$$\begin{tikzcd} E_n(X,A) \arrow[r, "\pi_n"] \arrow[d, "E_n(f)"'] & C_{n-1}Y \arrow[d, "C_{n-1}(g)"] \\ E_n(X',A') \arrow[r, "\pi'_n"'] & C_{n-1}Y' \end{tikzcd}$$

Proof. Since E_n is a functor and v_n a natural transformation it follows that the following diagram commutes

$$\begin{tikzcd} E_n(X,A) \arrow[r, "E_n(\pi)"] \arrow[d, "E_n(f)"'] & E_n(Y,*) \arrow[r, "v_n"] \arrow[d, "E_n(g)"'] & C_{n-1}Y \arrow[d, "C_{n-1}(g)"] \\ E_n(X',A') \arrow[r, "E_n(\pi')"'] & E_n(Y',*) \arrow[r, "v_n"'] & C_{n-1}Y' \end{tikzcd}$$

Since the composition along the top row is by definition π_n and the composition along the bottom row π'_n the result follows. $\square$

Recall that the cone functor CX is defined by

$$CX = X \times I/(* \times I \cup X \times 0)$$

We have an embedding

$$\begin{aligned} X &\to CX \\ x &\mapsto [x,1] \end{aligned}$$

The reduced suspension ΣX can then be identified with CX/X and we write $\pi : CX \to \Sigma X$ for the natural projection map. Lastly, we define for $[x,s] \in CX, t \in I$ and $v \in S^{n-1}$.

$$\begin{aligned} \tilde{\eta}_n &: CX \to P\Omega^{n-1}\Sigma^n X \\ \tilde{\eta}_n[x,s](t)(v) &= [x, st, v] \end{aligned}$$

Lemma 3.5. Let $X \in \mathbf{CG}_*$ and $n \geq 1$ then the following diagram commutes

$$\begin{tikzcd} C_nX \arrow[r, "\text{incl.}"] \arrow[d, "C_n(\eta_n)"'] & C_n(CX,X) \arrow[r, "\pi_n"] \arrow[d, "E_n(\tilde{\eta}_n)"'] & C_{n-1}\Sigma X \arrow[d, "C_{n-1}(\eta_{n-1})"] \\ C_n\Omega^n\Sigma^n \arrow[r, "\text{incl.}"'] & E_n(P\Omega^{n-1}\Sigma^n X, \Omega^n\Sigma^n X) \arrow[r, "p_n"'] & C_{n-1}\Omega^{n-1}\Sigma^n X \end{tikzcd}$$

Proof. From the definitions of the maps involved it is immediate that

$$\begin{tikzcd} X \arrow[r, "\text{incl.}"] \arrow[d, "\eta_n"'] & CX \arrow[r, "\pi"] \arrow[d, "\tilde{\eta}_n"'] & \Sigma X \arrow[d, "\eta_{n-1}"] \\ \Omega^n\Sigma^n X \arrow[r, "\text{incl.}"'] & P\Omega^{n-1}\Sigma^n X \arrow[r, "p"'] & \Omega^{n-1}\Sigma^n X \end{tikzcd}$$

commutes. The result now follows from Lemma 3.4. $\square$

We will need two more lemmas in order to prove the Approximation Theorem. First we define a map

$$\tilde{\theta}_{n,k} : \epsilon_n(k, P\Omega^{n-1}X, \Omega^n X) \to P\Omega^{n-1}X$$

as follows. Let $(c, x) \in \epsilon_n(k, P\Omega^{n-1}X, \Omega^n X)$ and write $c = (c_1, \cdots, c_k)$ and $x = (x_1, \cdots, x_k)$, then, for $t \in I$ and $v \in S^{n-1}$, we define

$$\tilde{\theta}_{n,k}(c, x)(t, v) = \begin{cases} x_i(s)(u), & \text{if } c_i(s, u) = (t, v) \\ x_i(1)(u) & \text{if } t \geq c_i'(1), c_i''(u) = v, x_i \notin \Omega^n X \\ *, \text{otherwise} & \end{cases}$$

From the definition of $\epsilon_n(k, P\Omega^{n-1}X, \Omega^n X)$ it follows that if $i \neq j$ and $x_j \notin \Omega^n X$ then no element of $c_i(J^n)$ can be written as (t, v) with $t \geq c_j'(1)$ and $v \in c_j''(J^{n-1})$. Therefore, the first and second part of the definition have disjoint domains. Since $y_j(s)(u) = *$ for $u \in \partial I^{n-1}$ it thus follows that $\tilde{\theta}_{n,l}$ is continuous.

Lemma 3.6. Let $X \in \mathbf{CG}_*$ and $\tilde{\theta}_{n,k}$ as above, then the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}_n(k) \times (\Omega^n X)^k \arrow[r, "\theta_{n,k}"] \arrow[d, "\text{incl.}"'] & \Omega^n X \arrow[d, "\text{incl.}"] \\
\epsilon_n(k, P\Omega^{n-1}X, \Omega^n \arrow[r, "\tilde{\theta}_{n,k}"] & P\Omega^{n-1}X \arrow[dd, "p"] \\
\mathcal{C}_{n-1}(k) \times (P\Omega^{n-1}X)^k \arrow[u, "\sigma'_{n-1,k} \times 1^k"] \arrow[ur, "P\theta_{n-1,k}"'] \arrow[d, "1 \times p^k"'] & \\
\mathcal{C}_{n-1}(k) \times (\Omega^{n-1}X)^k \arrow[r, "\theta_{n-1,k}"'] & \Omega^{n-1}X
\end{tikzcd}$$

Proof. Since $\theta_{n,k}$ and $\tilde{\theta}_{n,k}$ coincide on $\mathcal{C}_n(k) \times (\Omega^n X)^k$ it is clear that the top square commutes. Using Lemma 1.1 and the subsequent discussion on P if follows that the bottom square commutes. Since $\sigma'_{n-1,k}$ is given by $f \mapsto 1 \times f$ on little n-cubes f, it follows that the triangle commutes. Therefore, the whole diagram commutes and we are done. $\square$

Lastly, we have the following lemma.

Lemma 3.7. Let $X \in \mathbf{CG}_*$ then the maps $\tilde{\theta}_{n,k} : \epsilon_n(k, P\Omega^{n-1}X, \Omega^n X) \to P\Omega^{n-1}X$ induce a map $\tilde{\theta}_n : E_n(P\Omega^{n-1}X, \Omega^n X) \to P\Omega^{n-1}X$ such that

$$\begin{tikzcd}
C_n\Omega^n X \arrow[r, "\text{incl.}"] \arrow[d, "\theta_n"'] & E_n(P\Omega^{n-1}X, \Omega^n X) \arrow[r, "p_n"] \arrow[d, "\tilde{\theta}_n"'] & C_{n-1}\Omega^{n-1}X \arrow[d, "\theta_{n-1}"] \\
\Omega^n X \arrow[r, "\text{incl.}"'] & P\Omega^{n-1}X \arrow[r, "p"'] & \Omega^{n-1}X
\end{tikzcd}$$

commutes.

Proof. In the notation we used when constructing a monad associated to an operad we have

$$\begin{aligned}\tilde{\theta}_{n,k}(c\sigma, x) &= \tilde{\theta}_{n,k}(c, \sigma x)\\ \tilde{\theta}_{n,k-1}(\sigma_i c, x) &= \tilde{\theta}_{n,k}(c, s_i x)\end{aligned}$$

proving that $\tilde{\theta}_n$ is well-defined. By Lemma 3.2 it follows that θ_n and $\tilde{\theta}$ coincide on $C_n\Omega^n X$. We have

$$p\tilde{\theta}_n[c, x](v) = \begin{cases} x_i(1)(u) & \text{if } c_i''(u) = v, y_i \notin \Omega^n X \\ *, & \text{otherwise} \end{cases}$$

By definition $p_n = v_n \circ E_n(p)$ and thus by Corollary 3.1 and Theorem 3.2 it follows that $p\tilde{\theta}_n = \theta_{n-1}p_n$. Therefore, the diagram commutes and we are done. $\square$

From Lemma 3.5 and Lemma 3.7 it obviously follows that the diagram in Theorem 3.5 commutes. The next section is devoted to establishing the statement that π_n is a quasi-fibration. We will finish up this section a corollary of Theorem 3.5. This corollary is immediate from the theorem and Proposition 2.1 and Theorem 2.2 and thus the proof will be omitted.

Corollary 3.2. Let $X \in \mathbf{CG}_*$ be connected and let $\mathcal{C}$ be an A_∞ operad then the following maps are all weak homotopy equivalences.

$$MX \xleftarrow{\epsilon} TX \xleftarrow{\pi_1} (T\nabla T_1)X \xrightarrow{\pi_2} T_1X \xrightarrow{\alpha_1} \Omega\Sigma X$$

3.3 Quasi-fibrations

In this section we will introduce quasi-fibrations so as to finish up the proof of Theorem 3.5. Before taking a closer look at quasi-fibrations we will first need some results concerning NDR pairs.

Lemma 3.8. Suppose (B, A) and (X, B) are NDR pairs, then there exists a representation (h, u) of (X, A) as an NDR pair such that $h(I \times B) \subset B$.

Proof. Let (j, v) and (k, w) be representations of, respectively, (B, A) and (X, B) as NDR pairs. Put

$$\begin{aligned}f &: I \times B \to I\\ f(t, b) &= (1 - t)w(b) + tv(b)\end{aligned}$$

From the fact that $B \to X$ is a cofibration it follows that there exist maps $\tilde{j} : I \times X \to X$ and $\tilde{f} : I \times X \to X$ such that

$$\begin{tikzcd} 0 \times B \arrow[rr] \arrow[dd] & & I \times B \arrow[dl, "j"'] \arrow[dd] \\ & X & \\ 0 \times X \arrow[ur, "1"] \arrow[rr] & & I \times X \arrow[ul, "\tilde{f}"'] \end{tikzcd} \qquad \begin{tikzcd} 0 \times B \arrow[rr] \arrow[dd] & & I \times B \arrow[dl, "f"'] \arrow[dd] \\ & I & \\ 0 \times X \arrow[ur, "w"] \arrow[rr] & & I \times X \arrow[ul, "\tilde{f}"'] \end{tikzcd}$$

commute. Now, put

$$u(x) = \max(\tilde{f}(1, k(1, x)), w(x))$$

and

$$h(t, x) = \begin{cases} k(2t, x), & \text{if } 0 \leq t \leq \frac{1}{2} \\ \tilde{j}(2t-1, k(1, x)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$\square$

Lemma 3.9. Let A_i be subspaces of X such that (X, A_i) is an NDR pair represented by (h_i, u_i) for $i = 1, \cdots, n$. Suppose that, for $i < j$, $h_j(I \times A_i) \subset A_i$ and, for $i < j$, $t \in I$ and $x \in X$, that if $u_j(x) < 1$ then $u_j h_i(t, x) < 1$. Then (j, v) represents $X, A_1 \cup \cdots \cup A_n)$ as an NDR pair where

$$v(x) = \min(u_1(x), \cdots, u_n(x))$$

and

$$j(t, x) = h_n(t_n, h_{n-1}(t_{n-1}, \cdots, h_1(t_1, x) \cdots))$$

with

$$t_i = \begin{cases} t \cdot \min_{i \neq j} \frac{u_j(x)}{u_i(x)}, & \text{if some } u_j(x) < u_i(x) \\ t, & \text{if all } u_j(x) \geq u_i(x) \end{cases}$$

Recall that a path-connected space X is called aspherical if $\pi_n(X) = 0$ for all $n \geq 2$.

Theorem 3.6. Let (X, A) be an NDR pair in $\mathbf{CG}_*$, then

1. $(F_k E_n(X, A), F_{k-1} E_n(X, A))$ is an NDR pair whenever $k \geq 1$
2. if X is contractible then $E_n(X, A)$ is a spherical and $E_n(X, A)$ is contractible if either X is compact or $X = CA$ or $n = 1$

Proof. For the first statement we first apply Lemma 3.8 to $(X, A, *)$ to obtain a representation (h, u) of $(X, *)$ as an NDR pair with $h(I \times A) \subset A$. The representation (h, u) gives, by Lemma 1.9, a representation (h_j, u_j) of $(X, *)^k$ as an S_k-equivariant NDR pair. In Proposition 1.1 we have constructed a representation $(\tilde{h}_k, \tilde{u}_k)$ of (F_k, F_{k-1}). Since any coordinate in A remains in A throughout the whole homotopy it follows that $(\tilde{h}_k, \tilde{u}_k)$ restricts to a representation of $(F_k E_n(X, A), F_{k-1} E_n(X, A))$ as an NDR pair.
Now, for the second statement. Let $g : I \times X \to X$ be a contracting homotopy for X, i.e. we have

$$g(0, x) = x \qquad g(t, *) = * \qquad g(1, x) = *$$

Let $c = (c_1, \cdots, c_k) \in \mathcal{C}_n(k)$ and put

$$\lambda(c) = \min_{1 \leq j \leq k} (c'_j(1) - c'_j(0))$$

and define

$$v_i(c) = 2 \max_{j \neq i} \frac{c'_j(1) - c'_i(1)}{\lambda(c)}$$

Now, define a homotopy

$$G : I \times (F_k E_n(X,A) \setminus F_{k-1}E_n(X,A)) \to F_k E_n(X,A)$$
$$G(t,[c,x_1,\cdots,x_k]) = [c, g(t_1,x_1),\cdots,g(t_k,x_k)]$$

where

$$t_i = \begin{cases} t, & \text{if } v_i(c) \leq 0 \\ t(1-v_i(c)), & \text{if } 0 \leq v_i(c) \leq 1 \\ 0, & \text{if } v_i(c) \geq 1 \end{cases}$$

Note that if $v_i(c) \leq 0$ then $(c_i'(0),1) \times c_i''(J^{n-1})) \cap c_j(J^n) = \emptyset$ for all $i \neq j$. Therefore, G is well-defined. It is clear the G starts at the identity and ends at $F_{k-1}E_n(X,A)$ since for every c there exists an i such that $v_i(c) \leq 0$. If X is compact then it follows that we can find $\epsilon > 0$ satisfying

$$g(I \times u^{-1}[0,\epsilon]) \subset u^{-1}[0,1)$$

On the other hand, if $X = TA$ and (j,v) represents $(A,*)$ as an NDR-pair then $u[a,s] = v(a)s$, $h(t,[a,s]) = [j(t,a),s]$ and $g(t,[a,s]) = [a,s-st]$. In that case any $\epsilon < 1$ wil satisfy

$$g(I \times u^{-1}[0,\epsilon]) \subset u^{-1}[0,1)$$

Now, define a homotopy

$$H : I \times F_k E_n(X,A) \to F_k E_n(X,A)$$

as follows. For $z \in F_{k-1}E_n(X,A)$ we simply put $H(t,z) = z$ and for $[c,y] \in F_k E_n(X,A) \setminus F_{k-1}E_n(X,A)$ we set

$$H(t,[c,y]) = \begin{cases} G(t,[c,y]), & \text{if } u_k(y) \geq \frac{\epsilon}{2} \\ G(\frac{2tu_k(y)}{\epsilon},[c,y]), & \text{if } u_k(y) \leq \frac{\epsilon}{2} \end{cases}$$

It is clear that this homotopy deforms $F_k E_n(X,A)$ into $\tilde{u}_k^{-1}[0,1)$ which is in turn deformed into $F_{k-1}E_n(X,A)$. Since $F_0 E_n(X,A) = *$ it follows that each $F_k E_n(X,A)$ is contractible by applying this process k times. Therefore, by the proof of Lemma 1.7, it follows that $E_n(X,A)$ itself is contractible.
Now, suppose X is compact and let $f : S^m \to E_n(X,A)$ then f maps S^m into some $F_k E_n(Y, A \cap Y)$ for some compact subset Y of X. Therefore, if ϵ is such that $g(I \times u^{-1}[0,\epsilon]) \subset u^{-1}[0,1)$ then H deforms $F_k E_n(Y, A \cap Y)$ into $\tilde{u}^{-1}[0,1)$ which implies that f is null homotopic and thus $E_n(X,A)$ is aspherical.
Now, assume $n = 1$. Then any point of $E_1(X,A)$ can be written as $[c,y]$ where the intervals c_i of $c \in \mathcal{C}_1(k)$ are arranged according to the size of their left endpoint (which makes sense since they are disjoint). Using the retracting homotopy for $(X,*)^k$ obtained from h_{k-1} on X^{k-1} and g on X can by Lemma 1.8 be used to deform $F_k E_1(X,A)$ into $F_{k-1}E_1(X,A)$. Since $F_0 E_1(X,A) = *$ it follows that $E_n(X,A)$ is contractible. $\square$

We will now investigate quasi-fibrations. First we recall the definition of relative homotopy groups.

Definition 3.2. Let (X, A) be a pair in of spaces and $*$ a point in A. We consider I^{n-1} as a face of I^n via the embedding $x \mapsto (x, 0)$ and we define K^{n-1} to be the closure of $\partial I^n \setminus I^{n-1}$, i.e. K^n is the union of the faces of I^n without I^{n-1}. We define the relative homotopy groups $\pi_n(X, A, *)$ to be the group of homotopy classes of maps $(I^n, \partial I^n, K^{n-1}) \to (X, A, *)$ where the homotopies are required to go through maps of the same form.

Definition 3.3. Let $p : E \to B$ be a map of spaces and U a subset of B then

1. p is called a quasi-fibration if p is surjective and

$$p_* : \pi(E, p^{-1}(x), y) \to \pi_i(B, x)$$

   is a group isomorphism for all $x \in B$, $y \in p^{-1}(x)$ and $i \geq 0$

2. U is called distinguished if $p : p^{-1}(U) \to U$ is a quasi-fibration

The following lemma gives a way to recognize quasi-fibrations

Lemma 3.10. Let B be a filtered space with filtration $\{F_k(B)\}$ and $p : E \to B$ a map. Then every $F_k(B)$ is distinguished. Moreover, if

1. $F_0(B)$ and every open subset of $F_k(B) \setminus F_{k-1}(B)$ for $k > 0$ is distinguished
2. for $j > 0$ and every open $U \subset F_k(B)$ containing $F_{k-1}(B)$ there exist homotopies $h_t : U \to U$ and $H_t : p^{-1}(U) \to p^{-1}(U)$ satisfying
   (a) $h_0 = 1$, $h_t(F_{k-1}(B)) \subset F_{k-1}(B)$ and $h_1(U) \subset F_{k-1}(B)$
   (b) $H_0 = 1$ and $p \circ H_t = h_t \circ p$
   (c) $H_1 : p^{-1}(x) \to p^{-1}(h_t(x))$ is a weak homotopy equivalence for all $x \in U$

then p is a quasi-fibration.

The notion of a strong NDR pair will be necessary in the coming theorem.

Definition 3.4. Suppose that (X, A) is an NDR pair represented by (h, u). Then (X, A) is called a strong NDR pair if $u(h(t, x)) < 1$ whenever $u(x) < 1$.

The following theorem will finish up the proof of the Approximation Theorem.

Theorem 3.7. Let (X, A) be a strong NDR pair in $\mathbf{CG}_*$ with A connected. Let $\pi : X \to X/A$ be the natural projection map. Then $\pi_n : E_n(X, A) \to T_{n-1}(X/A)$ is a quasi-fibration with fibre T_nA.

Proof. First note that if $n = 1$ then $T_0(X/A) = X/A$, $F_0(X/A) = *$ and $F_1(X/A)$. In this way the proof below goes through for $n = 1$ as well and thus need not be considered separately.
It is clear that $F_0T_{n-1}(X/A) = *$ is distinguished. Our first goal is the show that any open $V \subset F_kT_{n-1}(X/A) \setminus F_{k-1}T_{n-1}(X/A)$ is distinguished. Using the equivalence relation from the definition of $E_n(X, A)$ and the definition of π_n it follows that we can write any element $y \in \pi_n^{-1}(V)$ in the form $y = [(c, d), x, a]$ for some $c = (c_1, \cdots, c_k) \in \mathcal{C}_n(k)$, $d = (d_1, \cdots, d_l) \in \mathcal{C}_n(l)$, $x \in (X \setminus A)^k$ and $a \in A^l$. Observe that if $c_i = c_i' \times c_i''$ with $c_i' : J \to J$ and $c_i'' : J^{n-1} \to J^{n-1}$

then $((c_i'(0),1)\times c_i''(J^{n-1}))\cap d_j(J^n)=\emptyset$. Moreover, $\pi_n(y)[c'',\pi^k(x)]\in V$ where $c''=(c_1'',\cdots,c_k'')$ as usual. For such y we put

$$\begin{gathered} q:\pi_n^{-1}(V)\to T_nA \\ q(y)=[d,a] \end{gathered}$$

One can readily verify that q is continuous and well-defined. To show that V is distinguished it is clearly sufficient to show that $\pi_n\times q:\pi_n^{-1}(V)\to V\times T_nA$ is a fibre homotopy equivalence. To do this we first define morphisms of operads in the following way. First define maps

$$g^+:J\to J \qquad\qquad g^-:J\to J$$
$$g^+(s)=\frac{1}{2}(1+s) \qquad\qquad g^-(s)=\frac{1}{2}s$$

Note that $g^+(J)=(\frac{1}{2},1)$ and $g^-(J)=(0,\frac{1}{2})$. Now, for little $(n-1)$-cubes f and little n-cubes f', put

$$\sigma^+:\mathcal{C}_{n-1}\to\mathcal{C}_n \qquad\qquad \tau^-:\mathcal{C}_n\to\mathcal{C}_n$$
$$\sigma^+(f)=g^+\times f \qquad\qquad \tau^-(f)=(g^-\times 1^{n-1})\circ f$$

It is easy to verify that these are morphisms of operads. Now, set

$$\begin{gathered} w:V\times T_nA\to\pi_n^{-1}(V) \\ w([c'',\pi^k(x),[d,a]])=[(\sigma^+(c''),\tau^-(d),x,a] \end{gathered}$$

where $c''\in\mathcal{C}_{n-1}(k)$,$x\in(X\setminus A)^l$, $d\in\mathcal{C}_n(l)$ and $a\in A^l$. It is clear that w is continuous and fibrewise over V. Let τ^- also denote the morphism of monads associated to the morphism of operads $\tau^-:\mathcal{C}_n\to\mathcal{C}_n$, then we see that $(\pi_n\times q)\circ w$ is equal to τ^-. Let g_t^- be the homotopy defined by $g_t^-(s)=s-\frac{1}{2}st$. Then τ^- is homotopic to the identity via $F':f\mapsto(g_t^-\times 1^{n-1})\circ f$ for little n-cubes f. Therefore, $(\pi_n\times q)\circ w$ is fibre homotopic to the identity. We still need to show that $w\circ(\pi_n\times q)$ is fibre homotopic to the identity. Let $y\in\pi_n^{-1}(V)$ and write $y=[(c,d),x,a]$ as above. Then

$$w(\pi_n\times q)(y)=[(\sigma^+(c''),\tau^-(d)),x,a]$$

Let F'' denote the homotopy from c to $\sigma^+(c'')$ by linearly deforming each c_i'' into g^+. Now put

$$F(t)=\begin{cases}F'(2t), & \text{if } t\le\frac{1}{2}\\ F''(2t-1), & \text{if } t\ge\frac{1}{2}\end{cases}$$

Then F is a fibre-wise homotopy between the identity and $w\circ(\pi_n\times q)$. Therefore, $\pi_n\times q$ is a fibre homotopy equivalence and thus V is distinguished.
We would like to apply Lemma 3.10 but to do this we need a neighborhood U of $F_{k-1}T_{n-1}(X/A)$ in $F_kT_{n-1}(X/A)$ such that the conditions of the second part of that lemma are satisfied. For this, first let (l,v) be a representation of (X,A) as a strong NDR pair and define $B=v^{-1}[0,1)$. We have that $l(I\times B)\subset B$. Put

$$U'=\{[c'',\pi(x_1),\cdots,\pi(x_k)]|\exists i:x_i\in B\}$$

and define $U = U' \cup F_{k-1}T_{n-1}(X/A)$. The representation (l, v) induces via π are representation (h, u) of $(X/A, *)$ as an NDR pair. By Lemma 1.9 we also have representations (h_k, u_k) and (l_k, v_k) of, respectively, $(X/A, *)^k$ and $(X, A)^k$ as NDR pairs. From Proposition 1.1 we know that $(F_kT_{n-1}X/A, F_{k-1}T_{n-1}X/A)$ is an NDR pair, so let $(\tilde{h}_k, \tilde{u}_k)$ be a representation of it. Note that $\tilde{u}_k(x) < 1$ if and only if $x \in U$ and thus $\tilde{h}_k$ restricts to a strong deformation retract $\tilde{h}_k : I \times U \to U$ of U onto $F_{k-1}T_{n-1}X/A$. Put

$$F^{k-1}E_n(X, A) = \pi_n^{-1}(F_{k-1}T_{n-1}X/A)$$

and define $\tilde{l}^k : I \times \pi_n^{-1}(U) \to \pi_n^{-1}(U)$ by

$$\tilde{l}^k(t, y) = \begin{cases} y, & \text{if } y \in F^{k-1}E_n(X, A) \\ [(c, d), l_k(t, x), a], & \text{if } y \in \pi_n^{-1}(U) \setminus F^{k-1}E_n(X, A) \end{cases}$$

Since $l(t, a) = a$ for $a \in A$ it follows that $\tilde{l}^l$ is well-defined and since $\tilde{l}^k$ covers $\tilde{h}_k$ it follows that $\tilde{l}^k$ is a strong deformation retract of $\pi_n^{-1}(U)$ onto $F^{k-1}E_n(X, A)$. From Lemma 3.10 it follows that to complete the proof of the first part of the theorem we only need to show that for all $x \in U$ we have that

$$\tilde{l}_1^k = \tilde{l}^k(1, -) : \pi_n^{-1}(x) \to \pi_n^{-1}(x')$$

is a homotopy equivalence where $x' = \tilde{h}_k(1, x)$. For $x \in F_{k-1}T_{n-1}(X/A)$ this is obvious since $\tilde{l}^k$ is constant on $F^{k-1}E_n(X, A)$. Let $x \in U \setminus F_{k-1}T_{k-1}(X/A)$ then we can write $x = [c'', \pi(x_1), \cdots, \pi(x_k)]$ where $c'' = (c''_1, \cdots, c''_k)$ and $x_i \in X \setminus A$. Write $l_k(1, (x_1, \cdots, x_k)) = (x'_1, \cdots, x'_k)$ then some x'_r is in A. Without loss of generality we may assume that, for some i, $x'_r \notin A$ for $r \leq i$ and $x_r \in A$ for $r > i$, since otherwise we can use permutation and the equivalence relation to swap the x'_r. Then we obtain

$$x' = \tilde{h}_k(1, x) = [(c''_1, \cdots, c''_i), \pi(x'_1), \cdots, \pi(x'_i)]$$

We have the following diagram

$$\begin{tikzcd} \pi_n^{-1}(x) \arrow[r, "\tilde{l}^k(1,-)"] \arrow[d, shift left, "\pi_n \times q"] & \pi_n^{-1}(x') \arrow[d, shift left, "\pi_n \times q"] \\ x \times T_nA \arrow[r, "\tilde{h}_k(1,-)\times 1"'] \arrow[u, shift left, "w"] & x' \times T_nA \arrow[u, shift left, "w"] \end{tikzcd}$$

where q and w are as before. We want to show that $\tilde{l}^k(1, -)$ is homotopic to $w \circ (\tilde{h}_k(1, -) \times 1) \circ (\pi_n \times q)$. For this it is sufficient that we construct a homotopy $H : I \times (x \times T_nA) \to \pi_n^{-1}(x')$ from $\tilde{l}^k(1, -) \circ w$ to $w \circ (\tilde{h}_k(1, -) \times 1)$ since we have already seen that $\pi_n \times q$ and w are homotopy inverse to each other. Choose, for $r > i$, paths $p_r : I \to A$ with $p_r(0) = x'_r$ and $p_r(1) = *$, which is possible since A is connected. We define

$$H(t, x, [d, a]) = [(\sigma^+(c''), \tau^-(d)), x'_1, \cdots, x'_i, p_{i+1}(t), \cdots, p_k(t), a]$$

It is clear that H is well-defined and one readily verifies that $H_0 = \tilde{l}^k(1, -) \circ w$ and $H_1 = w \circ (\tilde{h}_k(1, -) \times 1)$ and thus proving the theorem. $\square$

4 Simplicial Objects and Geometric Realization

4.1 Simplicial Complexes

The purpose of this section is to give a short description of simplicial complexes and develop geometric intuition about them. Though we will not be concerned with simplicial complexes in the coming chapters we will spend a lot of time with simplicial objects which can be seen as generalizations of simplicial complexes. Therefore, it will be useful to recall some of the basics of simplicial complexes. Our treatment follows the discussion in [3]. Recall that $n+1$ vectors $v_0, \cdots, v_n$ in some euclidean space are called geometrically independent if $v_1 - v_0, \cdots, v_n - v_0$ are linearly independent. We have the following definition

Definition 4.1. A geometric n-simplex in $\mathbb{R}^k$ is the convex set spanned by $n+1$ geometrically independent vector $v_0, \cdots, v_n$. In that case we denote the simplex by $[v_0, \cdots, v_n]$.

Consider the geometric n-simplex determined by $\{v_0, \cdots, v_n\}$. The v_i are called the vertices of the simplex. A face of the simplex is the convex set spanned by a subset $\{v_{i_0}, \cdots, v_{i_k}\} \subset \{v_0, \cdots, v_n\}$ of the vertices. Notice that a face $\{v_{i_0}, \cdots, v_{i_k}\}$ is itself a k-simplex.

Definition 4.2. A geometric simplicial complex X in $\mathbb{R}^k$ is a collection of simplices (not necessarily of the same dimension) satisfying

1. if $\sigma \in X$ is a simplex and τ is a face of σ then $\tau \in X$
2. if $\sigma_1, \sigma_2 \in X$ then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and of σ_2

Observe that the second condition basically means that a simplicial complex consists of simplices that are glued together along common faces. Notice that any simplex gives rise to a simplicial complex, namely the complex consisting of the simplex and all of its faces.

Up until now we have considered simplices and simplicial complexes as being embedded into some euclidean space. Still, it should be clear that a simplicial complex is up to homeomorphism determined by combinatorial data, namely the amount of vertices and which vertices span a simplex. The first step in our process of abstraction will be to look simply at the underlying combinatorial data. We do this by looking at the so-called skeleta of the simplicial complex. Let X be a simplicial complex and let $\{v_i\}_{i \in I}$ be its vertices. Define $X_0 = \{v_i\}_{i \in I}$. For $k > 0$ we define X^k as follows. Let $\{v_{i_0}, \cdots, v_{i_k}\} \subset X^0$, then $\{v_{i_0}, \cdots, v_{i_k}\} \in X^k$ if and only if $[v_{i_0}, \cdots, v_{i_k}]$ is a k-simplex of X. Of course, we still started out with a geometrical simplicial complex but this discussion points out how to define abstract simplicial complexes that are purely combinatorial objects. We have the following definition.

Definition 4.3. An abstract simplicial complex X consists of a set X^0 whose elements are called vertices together with sets X^k consisting of $(k+1)$-element subsets of X^0 satisfying that any $(i+1)$-element subset of an element of X^k is an element of X^i. The elements of X^k are called the k-simplices of X.

We observe that the last condition makes sure that every face of a simplex in the complex is itself a simplex in the complex. It is clear that an abstract

simplicial complex provides the same combinatorial data as a geometric simplicial complex, however in the abstract case we have no geometric data at all. So what is the relation between a geometric and an abstract simplicial complex? On the one hand it is clear that a geometric complex determines an abstract complex by simply forget about the precise embedding into euclidean space. The other direction may be less clear. Surely, an abstract complex does not uniquely determine an geometric complex but only one up to homeomorphism. One straightforward way to construct a geometric complex from an abstract one is as follows. Assign a point to each element of X^0. To each abstract simplex we assign the geometric simplex spanned by the appropriate vertices and we then glue these simplices using the quotient topology. Though this show that we can find a geometric complex associated to an abstract one the process is not particularly satisfying and we will not be using it. Instead, we will realize the abstract complex geometrically in a different way, appropriately called geometric realization, which will be discussed later.
Of course, whenever we introduce a mathematical objects we should also discuss its morphisms so here is the definition.

Definition 4.4. Let X and Y be geometric simplicial complexes. A simplicial map $f : X \to Y$ is map $f : X \to Y$ of spaces such that if $[v_{i_0}, \cdots, v_{i_k}]$ is a simplex of X then $f(v_{i_0}), \cdots, f(v_{i_n})$ spans a simplex of Y.

There are a few things to note about this definition. First off, a simplicial map must map vertices of X to vertices of Y. Moreover, it is allowed that $f(v_{i_l}) = f(v_{i_m})$ for some $l \neq m$. We also observe that if we have a map of 0-skeleta $f : X^0 \to Y^0$ then f determines a unique simplicial map as follows. For $x \in X$ write x in barycentric coordinates, i.e.

$$x = \sum_{i=1}^{k} t_i v_{j_i}$$

where the t_i satisfy $\sum_i t_1 = 1$ and define

$$f(x) = \sum_{i=1}^{k} t_i f(v_{j_i})$$

There are two examples of simplicial maps that are crucial in the further development of the theory so we will discuss them here.

Example 4.1. Let X be a simplicial complex and suppose $Y = [v_{i_0}, \cdots, v_{i_n}]$ is a simplex in X. Then Y is itself a simplicial complex and we have the inclusion map $i : Y \to X$. It is obvious that i is indeed a simplicial map.

Example 4.2. This example is a sort of converse to the previous one. Whereas the inclusion map includes a smaller complex into a bigger one we will now give a map that collapses a bigger complex onto a smaller one. For concreteness, let $X = [v_0, v_1, v_2]$ be a 2-simplex and let $Y = [v_0, v_1]$ be a face. Define a map $f : X \to Y$ by $f(v_0) = v_0$ and $f(v_1) = f(v_2) = v_1$. For points that are not a vertex we just interpolate using barycentric coordinates as above. Clearly, the procedure can be applied to any simplex.

It is clear that a simplicial map of geometric simplicial complexes determines a simplicial map of abstract simplicial complexes. Conversely, a simplicial map of abstract complexes defines a simplicial map of geometric simplicial complexes by specifying images of vertices in the domain which fully determines the map at the geometric level. The notion of simplicial map gives us the notion of simplicial homeomorphism, namely a homeomorphism f is a simplicial homeomorphism if both f and f^{-1} are simplicial maps. It is now clear that if Y and Y' are geometric complexes corresponding to the abstract complex X then Y and Y' are simplicially homeomorphic. Consequently, we will no longer explicitly distinguish between geometric and abstract simplicial complexes.

We would like to slightly change our notion of simplicial complex to obtain the notion of an ordered simplicial complex. Let X be a simplicial complex and assume that X^0 is totally ordered. We will agree to only write $[v_{i_o}, \cdots, v_{i_n}]$ for a simplex if the following holds: if $k < l$ then $v_{i_k} < v_{i_l}$. Notice that any collection $\{v_{i_0}, \cdots, v_{i_n}\}$ of vertices still determines at most one simplex. The only thing that has changed is that we have done away with a lot of labels for the simplices. For example, in the unordered cases the 1-simplex determined by $\{v_0, v_1\}$ was denoted by both $[v_0, v_1]$ and $[v_1, v_0]$. If we order the vertices by declaring $v_0 < v_1$ then only $[v_0, v_1]$ denotes the 1-simplex (and $[v_1, v_0]$ does not denote anything).
The basic example of an ordered simplicial complex is the ordered n-simplex together with its faces. The vertices of the ordered n-simplex $|\Delta^n|$ are usually denoted by $0, 1, \cdots, n$ so that $|\Delta^n| = [0, \cdots, n]$. Every k-dimensional face (k-face) of $|\Delta^n|$ is then of the form $[i_0, \cdots, i_k]$ where $0 \leq i_0 < i_1 < \cdots < i_k \leq n$.
The reader may wonder why it is useful to consider *ordered* simplicial complexes instead of unordered ones. Consider an arbitrary ordered n-simplex $[v_{i_0}, \cdots, v_{i_n}]$ in ordered simplicial complex X. As was discussed before, $[v_{i_0}, \cdots, v_{i_n}]$ can be considered as the image of the standard n-simplex under a simplicial map. Of course, this map is in general not unique and this is were the order comes in. Replace the standard n-simplex by the ordered n-simplex $|\Delta^n|$ then $[v_{i_0}, \cdots, v_{i_n}]$ can still be considered as the image under some simplicial map. But X is also ordered so we may demand that the map is order-preserving and clearly there is only one such map. So the point of using orders is that each simplex in a simplicial complex is the image of $|\Delta^n|$ under a *unique* order-preserving simplicial map and that the whole complex is made up out of such images.
Another important reason to use ordered simplicial complexes is that it gives us a systematic way to consider the faces of a simplex using the so-called face maps. Consider the standard ordered n-simplex $|\Delta^n|$. For $j = 0, \cdots, n$ define the j face map d_j by

$$d_j[0, \cdots, n] = [0, \cdots, \hat{j}, \cdots, n]$$

where $\hat{j}$ means that j is omitted. What the face map d_j does is taking the n-simplex and give back the $(n-1)$ faces that misses the vertex j. Let X be an ordered simplicial complex and $[v_{i_0}, \cdots, v_{i_n}]$ be an n-simplex then we define

$$d_j[v_{i_0}, \cdots, v_{i_n}] = [v_{i_0}, \cdots, \widehat{v_{i_j}}, \cdots, v_{i_n}]$$

In this way we get, for each n, maps $d_0, \cdots, d_n : X^n \to X^{n-1}$. We note that this only makes sense in the ordered context. When the vertices are not ordered

then, say, $[v_0, v_1]$ and $[v_1, v_0]$ define the same 1-simplex and thus d_0 and d_1 are not even well-defined.
The face maps are related to each other in various ways. One central relation is that $d_i d_j = d_{j-1} d_i$ for $i < j$ since on the standard simplex we have

$$d_i d_j[0, \cdots, n] = [0, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, n] = d_{j-1} d_i[0, \cdots, n]$$

Though the above approach works nicely in lots of situations it is not quite right for our purpose. To motivate our coming approach we will reconsider Example 4.2. In that example we considered the collapse map $f : X \to Y$ by $f(v_0) = v_0$ and $f(v_1) = f(v_2) = v_1$, where $X = [v_0, v_1, v_2]$ and $Y = [v_0, v_1]$. For our purpose it will be necessary to be able to distinguish the image of f from Y. Using the notation from above this is, however, not possible since both are equal to $[v_0, v_1]$. If we follow the above discussion we will throw away essential information about the image of f, namely that whereas Y is a proper 1-simplex the image of f is a 2-simplex hidden inside the 1 simplex. To adequately account for this we need degenerate simplices and degeneracy maps.
An obvious way to remedy this situation is to allow repeated vertices in our definition, i.e. the image of f will be denoted by $[v_0, v_1, v_1]$ and Y will still be denoted by $[v_0, v_1]$. In this notation all information can be easily read off: since there are 3 (not necessarily distinct) v_i the original simplex was of dimension 3 and since there are 2 distinct v_i our simplex is collapsed onto the 1 simplex given by those vertices, i.e. $[v_0, v_1]$. This leads to the following definition.

Definition 4.5. A simplex $[v_{i_0}, \cdots, v_{i_n}]$ is called degenerate if the v_{i_j} are not all distinct.

We point out that it still is required that the v_{i_j} form an increasing sequence.

Example 4.3. Let $|\Delta^1|$ be our standard 1-simplex. We would like to count the number of 1-simplices in it. First off, we have the simplex itself. Moreover, we have two degenerate simplices namely $[0, 0]$ and $[1, 1]$. To see that these are all note that the (degenerate or not) 1-simplices are given by $[n, m]$ with $n \leq m$. Therefore, there are three 1-simplices: $[0, 0]$, $[0, 1]$ and $[1, 1]$.

Example 4.4. Similarly, consider $|\Delta^2|$ and count its one simplices using the fact that they are given by $[n, m]$ with $n \leq m$. We get $[0, 0]$,$[0, 1]$,$[0, 2]$,$[1, 1]$,$[1, 2]$ and $[2, 2]$.

We point out that, any simplex contains degenerate simplices of arbitrary large dimension. For example, $|\Delta^1|$ contains the degenerate n-simplex $[0, \cdots, n]$. Though the introduction of degenerate simplices resolved our previous problem they are quite unwieldy: there are infinitely many of them and we do not have an easy way to refer to them. For this purpose we introduce the degeneracy maps. A degenerate simplex will then be a simplex that is in the image of some degeneracy map. Define maps $s_0, \cdots, s_n$ by

$$s_j[0, \cdots, n] = [0, \cdots, j, j, \cdots, n]$$

So, s_j takes $[0, \cdots, n]$ and gives back the (unique) degenerate $(n + 1)$-simplex in $|\Delta^n|$ that repeats only the j-th vertex. Geometrically, one should think of

$s_j|\Delta^n|$ as collapsing the standard $(n+1)$-simplex onto the standard n-simplex via

$$f_j(i) = \begin{cases} i, & \text{if } i \leq j \\ i-1 & \text{if } i \geq j+1 \end{cases}$$

To generalize s_j to simplicial complexes we put

$$s_j[v_{i_0}, \cdots, v_{i_n}] = [v_{i_0}, \cdots, v_{i_j}, v_{i_j}, \cdots, v_{i_n}]$$

where $[v_{i_0}, \cdots, v_{i_n}]$ is a (possible degenerate) n-simplex. It is clear that any degenerate simplex can be obtained from repeated application of degeneracy maps. Moreover, we have $s_i s_j = s_{j+1} s_i$ whenever $i \leq j$ since

$$s_i s_j[0, \cdots, n] = [0, \cdots, i, i, \cdots, j, j, \cdots, n] = s_{j+1} s_i[0, \cdots, n]$$

There are also relations between the face maps and degeneracy maps.

Lemma 4.1. The following relations hold

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{if } i < j \\ 1, & \text{if } i = j, j+1 \\ s_j d_{i-1}, & \text{if } i > j+1 \end{cases}$$

Proof. Let $i < j$ then

$$\begin{aligned} d_i s_j[0, \cdots, n] &= d_i[0, \cdots, j, j, \cdots, n] \\ &= [0, \cdots, \hat{i}, \cdots, j, j, \cdots, n] \\ &= s_{j-1}[0, \cdots, \hat{i}, \cdots, j, \cdots, n] \\ &= s_{j-1} d_i[0, \cdots, n] \end{aligned}$$

Now suppose that $i = j$ or $i = j+1$ then

$$\begin{aligned} d_i s_j[0, \cdots, n] &= d_i[0, \cdots, j, j, \cdots, n] \\ &= [0, \cdots, j, \cdots, n] \end{aligned}$$

Note that if $i = j$ then d_i deletes the j-th index (which is j) and that when $i = j+1$ d_i deletes the $(j+1)$-th index (which is also j). Lastly, suppose $i > j+1$ then

$$\begin{aligned} d_i s_j[0, \cdots, n] &= d_i[0, \cdots, j, j, \cdots, n] \\ &= [0, \cdots, j, j, \cdots, \widehat{i-1}, \cdots, n] \\ &= s_j[0, \cdots, \widehat{i-1}, \cdots, n] \\ &= s_j d_{i-1}[0, \cdots, n] \end{aligned}$$

$\square$

4.2 Simplicial Sets and Simplicial Objects

In this section we will introduce the notion of a simplicial set and more generally the notion of a simplicial object in any category. Our discussion will be centered around simplicial sets since they are somewhat more concrete and all theory remains the same when we replace the category **Set** with any other category. We will jump right in with the definition

Definition 4.6. A simplicial set X consists of a sequence $\{X_n\}_{n\in\mathbb{N}}$ of sets and, for $n \geq 0$ and $i = 1, \cdots, n$, maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying

$$d_i d_j = d_{j-1} d_i \quad \text{if} \quad i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{if } i < j \\ 1, & \text{if } i = j, j+1 \\ s_j d_{i-1}, & \text{if } i > j+1 \end{cases}$$

$$s_i d_j = s_{j+1} s_i \quad \text{if} \quad i \leq j$$

An element of X_n is called an n-simplex, the d_i are called the face maps and the s_i are called the degeneracy maps. A morphism $f : X \to Y$ of simplicial sets consists a maps $f : X_n \to Y_n$ such that, for all i and n, $d_i f_n = f_{n-1} d_i$ and $s_i f_n = f_{n+1} s_i$.

The definition is of course modelled on ordered simplicial complexes and hence it should come as no surprise that a simplicial complex is, almost, a simplicial set.

Example 4.5. Let X be an ordered simplicial complex and let $\{v_i\}_{i\in I}$ denote the collection of its vertices. We define the simplicial set $\tilde{X}$ associated to X as follows. The set $\tilde{X}_n$ consists non-decreasing sequences $[v_{i_0}, \cdots, v_{i_n}]$ such that $v_{i_k} \leq v_{i_l}$ and $\{v_{i_0}, \cdots, v_{i_n}\}$ spans a simplex of X. Note that the entries of $[v_{i_0}, \cdots, v_{i_n}]$ need not be distinct, i.e. it may be degenerate. On the other hand, the v_i in $\{v_{i_0}, \cdots, v_{i_n}\}$ are all distinct (since it is just a set). Therefore, we cannot expect $\{v_{i_0}, \cdots, v_{i_n}\}$ to span an n-simplex of X. Instead, we require it to span a k-simplex of X for some $k \leq n$. From this it is clear that $\tilde{X}$ is just X together with all its possible degenerate simplices.

Applying this example to $|\Delta^n|$ we obtain the standard n-simplex (as a simplcial set) Δ^n. We will consider the case for $n = 0$ and $n = 1$ in slightly more detail. First, consider Δ^0 then the collection of vertices is just consists only of 0. We want to know what its n-simplices are. By definition an element of Δ^0_n is of the form $[i_0, \cdots, i_n]$ where $i_0 \leq \cdots \leq i_n$ and $\{i_0, \cdots, i_n\}$ spans a simplex of $|\Delta^n|$. So, the $i_0, \cdots, i_n$ should form a non-decreasing sequence of zeroes. But of course there is only one such sequence. Consequently, Δ^0 has a unique simplex in dimension, namely $[0, \cdots, 0]$ ($n+1$ zeroes).
Now, consider the case $n = 1$. The vertices of $|\Delta^1|$ are $\{0, 1\}$. Now, an n-simplex of Δ^1 is $[i_0, \cdots, i_n]$ with $0 \leq i_0 \leq \cdots \leq 1$ and $\{i_0, \cdots, i_n\}$ should span a simplex of $|\Delta^1|$. Therefore, the sequence $i_0, \cdots, i_n$ should be a non-decreasing sequence of zeroes and ones. It is clear that there are precisely $n+2$ of those. For example, the collection of 2-simplices is $\{[0,0,0], [0,0,1], [0,1,1], [1,1,1]\}$.

Definition 4.7. Let X be a simplicial set. A simplex $x \in X_n$ is called degenerate if x is not of the form $s_i(y)$ for any $y \in X_{n-1}$ and any i.

We would like to point out that faces of degenerate simplices need not be degenerate. For example, let $x \in X_n$ be non-degenerate. Then $s_i(x)$ is degenerate but $d_i s_i(x) = x$ which is again non-degenerate.
There is a more concise categorical description of simplicial sets (or simplicial objects in general) available. Though we will not be making much use of this

description it is quite useful to gain a better understanding of simplicial sets. Let $[n]$ denote the ordered set

$$\{0, 1, \cdots, n\}$$

and recall that a map $f : [m] \to [n]$ is called order-preserving if for all $k, l \in [m]$ we have $f(k) \leq f(l)$ whenever $k \leq l$.

Definition 4.8. Define the simplex category Δ as follows. The objects of Δ are the finite ordered sets $[n]$ for $n \in \mathbb{N}$. The morphisms of Δ are order-preserving maps $f : [m] \to [n]$.

For $0 \leq i \leq n$ define the following maps

$$D_i : [n] \to [n+1]$$
$$D_i(j) = \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } j \geq i \end{cases}$$

and

$$S_i : [n+1] \to [n]$$
$$S_i(j) = \begin{cases} j, & \text{if } j \leq i \\ j-1, & \text{if } j > i \end{cases}$$

In fact, the D_i and S_i generate all morphisms of the category Δ since any order-preserving map can be written as a composition of the D_i and S_i.

Definition 4.9. A simplicial sets is a contravariant functor $X : \Delta \to \mathbf{Set}$. A morphism of simplicial sets $X, Y : \Delta \to \mathbf{Set}$ is a natural transformation $f : X \to Y$. The category of simplicial sets and these morphisms is denoted by $S\mathbf{Set}$.

To relate this definition to our previous one we proceed as follows. An object $[n] \in \Delta$ can be considered as the standard (ordered) n-simplex. The map D_i can then be seen as the inclusion of $[n]$ in $[n+1]$ as the i-th face of $[n+1]$. Therefore, the morphism corresponding to D_i in the opposite category Δ^{opp} should assign to the $(n+1)$-simplex its i-th face. But this is just the i-th face map d_i defined by

$$d_i(j) = \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } j \geq i \end{cases}$$

On the other hand, the map $S_i : [n+1] \to [n]$ can be seen as collapsing the $(n+1)$-simplex onto the n-simplex by identifying the i-th and $(i+1)$-th face. Consequently, in the opposite category the corresponding morphism should assign to the n-simplex the degenerate $(n+1)$-simplex in $[n]$ which repeats the i-th vertex. But this is just the i-th degeneracy map s_i defined by

$$s_i(j) = \begin{cases} j, & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

It is easy to verify the following relations

$$d_i d_j = d_{j-1} d_i \text{ if } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{if } i < j \\ 1, & \text{if } i = j, j+1 \\ s_j d_{i=1}, & \text{if } i \geq j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ if } i \leq j$$

Therefore, $\{X([n])\}_{n \in \mathbb{N}}$ together with with the face maps d_i and the degeneracy maps s_i form a simplicial set. Conversely, given a simplicial set $X_0, X_1, \cdots$ with d_i and s_i one can simply define $X : \Delta \to \mathbf{Set}$ by $X([n]) = X_n$, $X(D_i) = d_i$ and $X(S_i) = s_i$. For a general map f in Δ first write $f = f_1 \circ \cdots \circ f_n$ where f_i is a D_i or a S_i and then define $X(f) = X(f_1) \circ \cdots \circ X(f_n)$. Therefore, the two definitions are equivalent.
Now, consider a morphism $f : X \to Y$ of simplicial sets (i.e. a natural transformation $f : X \to Y$ of functors). Then for any n and i the following diagrams (writing $X(n) = X([n])$) commute

$$\begin{tikzcd} X(n) \arrow[r, "f(n)"] \arrow[d, "X(D_i)"'] & Y(n) \arrow[d, "Y(D_i)"] \\ X(n-1) \arrow[r, "f(n-1)"'] & Y(n-1) \end{tikzcd} \qquad \begin{tikzcd} X(n) \arrow[r, "f(n)"] \arrow[d, "X(S_i)"'] & Y(n) \arrow[d, "Y(S_i)"] \\ X(n+1) \arrow[r, "f(n+1)"'] & Y(n+1) \end{tikzcd}$$

But $X(D_i) = d_i$ and $X(S_i) = s_i$. The diagrams then state that $d_i f(n) = f(n-1) d_i$ and $s_i f(n) = f(n+1) s_i$ so this is just a morphism of simplicial sets in our old sense.
It is now time to introduce simplicial objects in a general category. The definition is as follows.

Definition 4.10. Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a contravariant functor $X : \Delta \to \mathcal{C}$. A simplicial morphism $f : X \to Y$ in $\mathcal{C}$ between simplicial objects $X, Y : \Delta \to \mathcal{C}$ in $\mathcal{C}$ is a natural transformation $f : X \to Y$ between these functors. The simplicial objects in $\mathcal{C}$ with these morphisms constitute a new category $S\mathcal{C}$ of simplicial objects in $\mathcal{C}$.

By similar considerations as above we have a more explicit description of simplicial objects.

Definition 4.11. Let $\mathcal{C}$ be a category then the category $S\mathcal{C}$ of simplicial objects in $\mathcal{C}$ is defined as follows. The objects of $S\mathcal{C}$ are sequences of objects $X_n \in \mathcal{C}$ together with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ in $\mathcal{C}$ for $i = 0, \cdots, n$ satisfying the following relations

$$d_i d_j = d_{j-1} d_i \text{ if } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{if } i < j \\ 1, & \text{if } i = j, j+1 \\ s_j d_{i=1}, & \text{if } i \geq j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ if } i \leq j$$

A morphism $f : X \to Y$ in $S\mathcal{C}$ consists of morphisms $f_n : X_n \to Y_n$ in $\mathcal{C}$ such that

$$d_i f_n = f_{n-1} d_i \qquad\qquad s_i f_n = f_{n+1} s_i$$

We will also need the notion of a homotopy between simplicial objects.

Definition 4.12. Suppose $f, g : X \to Y$ are morphisms in $S\mathcal{C}$ then a homotopy h between f and g consists of a sequence of morphisms $h_i : X_n \to Y_{n+1}$ in $\mathcal{C}$ for $i = 0, \cdots, n$ satisfying

$$d_0 h_0 = f_n \qquad\qquad d_{n+1} h_n = g_n$$

$$d_i h_j = \begin{cases} h_{j-1} d_i, & \text{if } i < j \\ d_j h_{j_1}, & \text{if } i = j > 0 \\ h_j d_{i-1}, & \text{if } i > j+1 \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j \end{cases}$$

The remainder of this section is devoted to collect some useful properties of the category $S\mathcal{C}$ and its relation to $\mathcal{C}$ that will be needed in later sections. Suppose that $X \in \mathcal{C}$ where $\mathcal{C}$ is any category. Then we can define an object $X_* \in S\mathcal{C}$ by setting $X_n = X$ and letting each d_i and s_i be the identity map 1_X. Now, suppose $f : X \to Y$ is a morphism in $\mathcal{C}$, then we define a morphism $f_* : X_* \to Y_*$ in $S\mathcal{C}$ by defining $f_n = f$. We have the following useful lemmas the proofs of which are trivial.

Lemma 4.2. Let $X \in \mathcal{C}$ and $Y \in S\mathcal{C}$, then a map $p : X \to Y_0$ determines and is determined by the map $\tau_*(p) : X_* \to Y$ defined as $\tau_n(p) = s_0^n p$. Moreover, if $f \in \mathcal{C}$ and $g \in S\mathcal{C}$ are morphisms such that

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] & Y_0 \arrow[d, "g_0"] \\ X' \arrow[r, "p'"'] & Y_0' \end{tikzcd}$$

commutes then

$$\begin{tikzcd} X_* \arrow[r, "\tau_*(p)"] \arrow[d, "f_*"'] & Y \arrow[d, "g"] \\ X_*' \arrow[r, "\tau_*(p')"'] & Y' \end{tikzcd}$$

also commutes.

Lemma 4.3. Let $X \in \mathcal{C}$ and $Y \in S\mathcal{C}$ and suppose $q : Y_0 \to X$ in $\mathcal{C}$ satisfies $qd_0 = qd_1$ where $d_0, d_1 : Y_1 \to Y_0$. Then q determines and is determined by

$\epsilon_*(q') : Y \to X_*$ defined by $\epsilon_n(q) = qd_0^n$. Moreover, if $f \in \mathcal{C}$ and $g \in S\mathcal{C}$ are morphisms such that

$$\begin{tikzcd} Y_0 \arrow[r, "q"] \arrow[d, "g_0"'] & X \arrow[d, "f"] \\ Y_0' \arrow[r, "q'"'] & X' \end{tikzcd}$$

where $qd_0 = qd_1$ and $q'd_0 = q'd_1$ then

$$\begin{tikzcd} Y \arrow[r, "\epsilon_*(q)"] \arrow[d, "g"'] & X_* \arrow[d, "f_*"] \\ Y' \arrow[r, "\epsilon_*(q')"'] & X_*' \end{tikzcd}$$

also commutes.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor then F induces a functor $F_* : S\mathcal{C} \to S\mathcal{D}$ which is given by $F_n X = F(X_n)$ on objects $X \in S\mathcal{C}$. The face and degeneracy maps are then given by $F(d_i)$ and $F(s_i)$. If $F, G : \mathcal{C} \to \mathcal{D}$ are functors and $\mu : F \to G$ is a natural transformation then μ induces a natural transformation $\mu_* : F_* \to G_*$ defined by $\mu_n = \mu$. We have the following theorem.

Theorem 4.1. If (T, μ, η) is a monad in $\mathcal{C}$ then (T_*, μ_*, η_*) is a monad in $S\mathcal{C}$. Moreover, the category $ST[\mathcal{C}]$ of simplicial T-algebras is isomorphic to the category $T_*[S\mathcal{C}]$ of T_*-algebras.

Proof. It is immediately clear that T_* indeed defines a monad. For the second statement we note that an object of $ST[\mathcal{C}]$ or of $T_*[S\mathcal{C}]$ is just an object $X \in S\mathcal{C}$ together with maps $\xi_n : TX_n \to X_n$ such that $(X_n, \xi_n) \in T[\mathcal{C}]$ and the following diagrams commute

$$\begin{tikzcd} TX_n \arrow[r, "\xi_n"] \arrow[d, "T(d_i)"'] & X_n \arrow[d, "d_i"] \\ TX_{n-1} \arrow[r, "\xi_{n-1}"'] & X_{n-1} \end{tikzcd} \qquad \begin{tikzcd} TX_n \arrow[r, "\xi_n"] \arrow[d, "T(s_i)"'] & X_n \arrow[d, "s_i"] \\ TX_{n+1} \arrow[r, "\xi_{n+1}"'] & X_{n+1} \end{tikzcd}$$

Therefore, the category of simplicial T-algebras and the category of T_*-algebras are isomorphic. □

4.3 Geometric Realization

In the previous section we introduced simplicial spaces. We would like to be able to associate a topological space to a simplicial space. The way to do this is by geometric realization which will be introduced in this section. First, consider the standard geometric n-simplex $|\Delta^n|$ given by

$$|\Delta^n| = \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

with associated maps

$$D_i : |\Delta^{n-1}| \to |\Delta^n|$$
$$D_i(t_0, \cdots, t_n) = (t_0, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n)$$

and

$$S_i : |\Delta^{n+1}| \to |\Delta^n|$$
$$S_i(t_0, \cdots, t_n) = (t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_n)$$

We then have the following definition

Definition 4.13. Let $X \in S\mathbf{CG}$. Then the geometric realization of X is defined as

$$|X| = \coprod_{n=0}^{\infty} X_n \times |\Delta^n| / \sim$$

where the equivalence relation $\sim$ is generated by

1. for $x \in X_{n+1}, y \in |\Delta^n|$: $(x, D_i(y)) \sim (d_i(x), y)$
2. for $x \in X_{n-1}, y \in |\Delta^n|$: $(x, S_i(y)) \sim (s_i(x), y)$.

We topologize $|X|$ as follows. Endow each $X_n \times |\Delta^n|$ with the product topology. Let $F_n|X|$ denote the image $\coprod_{k=0}^{n} X_k \times |\Delta^k|$ in $|X|$ and endow it with the quotient topology. One readily verifies that $F_k|X|$ is closed in $F_{k+1}|X|$ and thus we can equip $|X|$ with the topology of the union. The class of a point (x, y) will be denoted by $|x, y|$.
If $f : X \to Y$ is a map in $S\mathbf{CG}$ then we define $|f| : |X| \to |Y|$ by

$$|f||x, y| = |f(x), y|$$

It is useful to stop a moment and consider what the geometric realization actually does. If X is a simplicial set we think of the elements of X_n as n-simplicies. Therefore, in the realization we would expect to have an n-simplex for each element of X_n and, indeed, this is what $X_n \times |\Delta^n|$ gives us. Furthermore, we would expect that the realization will respect the faces and degeneracies of the simplicies involved. To see that the definition of the geometric realization actually does this first consider the first part of the equivalence relation for $x \in X_{n+1}, y \in |\Delta^n|$ we have

$$(x, D_i(y)) \sim (d_i(x), y)$$

First consider $(x, D_i(y))$. Here x is an $(n+1)$-simplex of X and $D_i(y)$ is a point on the i-th face of a geometric $(n+1)$-simplex. The right hand side $(d_i(x), y)$ consists of the i-th face of x together with the same point y now considered as point in an n-simplex. Therefore, the equivalence relation $(x, D_i(y)) \sim (d_i(x), y)$ takes the n-simplex in $X_n \times |\Delta^n|$ corresponding to $d_i(x)$ and glues it as the i-th face of the simplex corresponding to x in $X_{n+1} \times |\Delta^n|$. For any z and j with $d_i(x) = d_j(z)$ something similar is done. So, what this equivalence relation does is the following: if x and x' have a common face in X then the realization of those faces will be identified.

Consider the second part of the equivalence relation which says that for $x \in X_{n-1}, y \in |\Delta^n|$ we have

$$(x, S_i(y)) \sim (s_i(x), y)$$

One would expect that the definition will do away with degenerate simplices since if an n-simplex gets mapped degenerately onto a k-simplex then geometrically we just see the k-simplex. As we shall see this is precisely what the equivalence relation does. More precisely, $(s_i(x), y)$ consists of a degenerate n-simplex $s_i(x)$ together with a point $y \in |\Delta^n|$. The left hand side, $(x, S_i(y)$ consists of an n-simplex x together with a point $S_i(y)$ in the image of the collapse map S_i that collapses an n-simplex onto an $(n-1)$-simplex. The relation thus states that the point y gets glued to the point $S_i(y)$ in the image of the collapse map, i.e. in an $(n-1)$-simplex. To put it more concise: if $|\Delta^n|$ corresponds to a degenerate simplex then it gets collapsed to the $(n-1)$-simplex it is the degeneracy of.

Example 4.6. Consider the standard n-simplex $\Delta^n = [0, \cdots, n]$. Then the geometric realization $|[0, \cdots, n]|$ is just $|\Delta^n|$.

The rest of this section is devoted to establishing that geometric realization behaves nicely under different topological constructions.

Lemma 4.4. Let z be a degenerate simplex, then there exist an unique non-degenerate simplex x such that $z = s_{i_1} \cdots s_{i_k} x$ for some degeneracy maps s_{i_j}.

Proof. First we prove existence. Let z be degenerate. Then there exists x_1 and s_{i_1} such that $z = s_{i_1} x_1$ by definition of a degenerate simplex. Now, either x_1 is degenerate or it is not. Suppose that it is degenerate, then by the same argument we obtain s_{i_2} and x_2 such that $x_1 = s_{i_2} x_2$. By induction continue this process to obtain $z = s_{i_1} \cdots s_{i_k}$. This process stops at some finite k since the dimension of x_j is strictly less than x_{j-1}.
Now, we prove uniqueness. Suppose x_1 and x_2 are non-degenerate simplices satisfying $S_1 x_1 = S_2 x_2$ for some compositions of degeneracy maps S_i. We need to show $x_1 = x_2$. Write $S_1 = s_{i_1} \cdots s_{i_k}$ for some degeneracy maps s_{i_j} and define $D = d_{i_k} \cdots d_{i_1}$. Then

$$x_1 = DS_1 x_1 = DS_2 x_2$$

by the simplicial identities. But then, using the simplicial identities, we can write $x_1 = SD' x_2$ where S is a composition of degeneracy maps and D' of face maps. But, by assumption, x_1 is non-degenerate therefore S can only be the zero-fold composition, i.e. the identity. So, $x_1 = D' x_2$ hence x_1 is a face of x_2. But by the same argument one gets that x_2 is also a face of x_1 which can only happen if $x_1 = x_2$. $\square$

Lemma 4.5. Let $t \in |\Delta^n|$ then there exists a unique point s in the interior of $|\Delta^m|$ (for some $m \leq n$) such that $t = D_{i_1} \cdots D_{i_k} s$ for some D_{i_j}.

Proof. We may assume that t is in a face of $|\Delta^n|$ since otherwise t is itself interior. Therefore, we can write $t = D_{i_1} s_1$. Now, s_1 is either interior or in a face. In the first case we are done and in the second case we can continue inductively. Since the dimension of the face containing s_i less than that of the face containing s_{i-1} this process stops after finitely many steps. Clearly, the

last s_i must then be interior. So, we have shown existence.
Now, for uniqueness. Suppose $t = D_{i_1} \cdots D_{i_k} s_1 = D_{j_1} \cdots D_{j_l} s_2$. Since

$$S_{i_k} \cdots S_{i_1} D_{i_1} \cdots D_{i_k} s_1 = s_1$$

we get

$$s_1 = S_{i_k} \cdots S_{i_1} D_{j_1} \cdots D_{j_l} s_2$$

Using the simplicial identities we can rearrange this in the form $s_1 = DSs_2$, where D is a composition of D_{m_i} and S of S_{n_j}. But s_1 is interior hence D is the zero-fold composition. Therefore, $s_1 = Ss_2$, i.e. s_1 is a degeneracy of s_2. But by the same argument the converse holds too. So, s_1 and s_2 are degeneracies of each other which implies that they are the same. $\square$

A point $(x, t) \in X_n \times |\Delta^n|$ is called non-degenerate if x is non-degenerate and t is in the interior of $|\Delta^n|$.

Lemma 4.6. Let $X \in S\mathbf{CG}$ then each point of $\coprod_{n\geq 0} X_n \times |\Delta|$ is equivalent to a unique non-degenerate point.

Proof. Define $\lambda : \coprod_{n\geq 0} X_n \times |\Delta| \to \coprod_{n\geq 0} X_n \times |\Delta|$ by

$$\lambda(x, t) = (y, S_{i_1} \cdots S_{i_k} t)$$

where y is the unique non-degenerate point such that $x = s_{i_k} \cdots s_{i_1}$ and $0 \leq i_1 < \cdots < i_k y$. This is possible by Lemma 4.4. Similarly, we define $\rho : \coprod_{n\geq 0} X_n \times |\Delta| \to \coprod_{n\geq 0} X_n \times |\Delta|$ by

$$\rho(x, t) = (d_{i_1} \cdots d_{i_l} x, s)$$

where s is the unique interior point such that $t = D_{i_l} \cdots D_{i_1} s$ and $0 \leq i_1 < \cdots < i_l$. This is possible by Lemma 4.5. Therefore, $\lambda \circ \rho$ maps a point (x, t) onto a unique point (y, s) and one easily sees that (y, s) is non-degenerate. $\square$

The following definition will be helpful in what is to come.

Definition 4.14. Let $X \in S\mathbf{CG}$. Put

$$s(X_n) = \bigcup_{k=0}^{n} s_k(X_n)$$

Then X is called

1. proper if each (X_{n+1}, X_n) is a strong NDR pair
2. strictly proper if X is proper and for each $0 \leq k \leq n$ we have that $(X_{n+1}, s_k X_n)$ is an NDR pair via a homotopy $h : I \times X_{n+1} \to X_{n+1}$ such that
$$h(I \times \bigcup_{i=0}^{k-1} s_i X_n) \subset \bigcup_{i=0}^{n-1} s_i X_n$$

We have the following result.

Lemma 4.7. Let $X \in S\mathbf{CG}$ be proper and let $F_n|X|$ be as in the definition of geometric realization (Definition 4.13). Then each $(F_n|X|, F_{n-1}|X|)$ is an NDR pair and $|X| \in \mathbf{CG}$. Furthermore, $F_n|X|/F_{n-1}|X|$ is homeomorphic to $\Sigma^n(X_n/sX_{n-1})$.

Proof. Observe that

$$F_n|X| \setminus F_{n-1}|X| = (X \setminus s(X_{n-1})) \times (|\Delta^n| - \partial|\Delta^n|)$$

Now, if X is proper then

$$(X_n \times |\Delta^n|, X_n \times \partial|\Delta^n| \cup s(X_{n-1}) \times |\Delta^n|)$$

is an NDR-pair by Lemma 1.8. Therefore from Lemma 1.5 and 1.7 it follows that $|X| \in \mathbf{CG}$. The quotient map $X_n \to X_n/s(X_{n-1})$ together with any heomeomorphism of pairs $(|\Delta^n|, \partial|\Delta^n|) \to (I^n, \partial I^n)$ induces a map

$$F_n|X|/F_{n-1}|X| = (X_n \times \|\Delta^n|)/(X_n \times \partial|\Delta^n| \cup s(X_{n-1}) \times |\Delta^n|) \to \Sigma^n(X_n/s(X_{n-1}))$$

which is clearly bijective. The fact that the inverse map is also continuous follows from Theorem 4.4 of [12]. $\square$

This lemma has the following nice consequence.

Theorem 4.2. Let $X \in S\mathbf{CG}$ be such that each X_n is a CW-complex and each d_i and S_i are cellular maps. Then $|X|$ is a CW-complex with one $(k+n)$-cell for each k-cell of $X_n \setminus sX_{n-1}$. Moreover, if $f : X \to X'$ is a morphism between two such objects such that each f_n is cellular, then $|f|$ is cellular.

Sometimes such X and f in $S\mathbf{CG}$ are also called cellular.
A crucial fact about geometric realization is that it respects product if we work in $S\mathbf{CG}$. Before we can prove this we need the definition.

Definition 4.15. Let $X, Y \in S\mathbf{CG}$. The product $X \times Y$ of X and Y is defined by

$$(X \times Y)_n = X_n \times Y_n$$

with face maps defined by $d_i(x, y) = (d_i x, d_i y)$ for $(x, y) \in (X \times Y)_n$ and degeneracy maps defined by $s_i(x, y) = (s_i x, s_i y)$.

The simplicial identities for $X \times Y$ follow immediately from those of X and Y hence it is clear that $X \times Y$ is indeed in $S\mathbf{CG}$. Now, for the promised theorem.

Theorem 4.3. Let $X, Y \in S\mathbf{CG}$, then

$$|\pi_1| \times |\pi_2| : |X \times Y| \to |X| \times |Y|$$

is a natural homeomorphism whose inverse ρ is commutative and associative. Furthermore, ρ is cellular whenever X and Y are cellular.

Proof. For the proof we will explicitly construct the map ρ. Let $u = (t_0, \cdots, t_n) \in |\Delta^n|$ and $v = (t'_0, \cdots, t'_m) \in |\Delta^m|$. Put, for $0 \leq k \leq m$, $u^k = \sum_{i=0}^{k} t_i$ and put, for $0 \leq l \leq n$, $v^l = \sum_{j=0}^{l} t'_j$. Let $w^0 \leq \cdots \leq w^{n+m+1}$ denote the sequence obtained by ordering the elements of $\{u^0, \cdots u^m, v^0, \cdots v^n\}$. Now define w by

$$w = (t''_0, \cdots, t''_{n+m})$$

where $t''_p = w^p - w^{p-1}$, $w^{-1} = 0$ and $w^{n+m} = 1$. It is clear that w is an element of $|\Delta^m| \times |\Delta^n|$. Pick sequences $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_n$ such that

1. for all p, q: $i_p \neq j_q$
2. for all p, q: $w^{i_p} \in \{u^0, \cdots, u^m\}$ and $w^{j_q} \in \{v^0, \cdots, v^n\}$.

It is easily seen that

$$u = S_{i_1} \cdots S_{i_m} w$$
$$v = S_{j_1} \cdots S_{j_n} w$$

We now define the map ρ on $x \in X_m$ and $y \in Y_m$ as

$$\rho(|x, u|, |y, v|) = |(s_{i_m} \cdots s_{i_1} x, s_{j_n} \cdots, s_{j_1} y), w|$$

From Lemma 4.6 it follows that ρ is well-defined and the inverse of $|\pi_1| \times |\pi_2|$. Commutativity and associativity follow from the commutativity and associativity of its inverse. Let $K(i, j)$ be the set of points in $|\Delta^m| \times |\Delta^n|$ that determine two given sequences $i = \{i_1, \cdots, i_m\}$ and $j = \{j_1, \cdots, j_n\}$ as above. Define $s^i = s_{i_m} \cdots s_{i_1}$ and $s^j = s_{j_n} \cdots s_{j_1}$. Furthermore, define maps

$$\alpha_{i,j} : K(i, j) \to |\Delta^{n+m}|$$

by $\alpha_{i,j}(u, v) = w$ with w as before. Then the following diagram commutes

$$\begin{tikzcd}
X_m \times Y_n \times |\Delta^m| \times |\Delta^n| \arrow[r, "1\times\tau\times 1"] & X_m \times |\Delta^m| \times Y_n \times |\Delta^n| \arrow[r, "q_X \times q_Y"] & F_m|X| \times F_n|Y| \arrow[d, "\rho"] \\
X_m \times Y_n \times K(i,j) \arrow[u, "\text{incl.}"] \arrow[r, "s^i \times s^j \times \alpha_{i,j}"'] & X_{n+m} \times Y_{n+m} \times |\Delta^{n+m}| \arrow[r, "q_{X\times Y}"'] & F_{n+m}|X \times Y|
\end{tikzcd}$$

where q_X, q_Y and $q_{X\times Y}$ are the quotient maps. From this it follows that ρ is continuous. Moreover, if X and Y are cellular then the diagram implies that ρ is a cellular map. □

Corollary 4.1. Suppose $f : X \to B$ and $g : Y \to B$ are morphisms in $S\mathbf{CG}$ then $|X \times_B Y|$ is naturally homeomorphic to $|X| \times_{|B|} |Y|$ where

$$(X \times_B Y)_n = \{(x, y)|f_n(x) = g_n(y)\}$$

is the fibre product in $S\mathbf{CG}$.

Corollary 4.2. If G is a simplicial topological monoid (group) then $|G|$ is a topological monoid (group). Furthermore, if G is commutative (abelian) then so is $|G|$.

We will now investigate the geometric realization of simplicial homotopies. First, we need a lemma.

Lemma 4.8. Let $X \in \mathbf{CG}$ then $|X_*|$ may be identified with X.

Proof. Recall that the simplicial space X_* is defined by $X_n = X$ and all the degeneracy maps and face maps are just the identity. In particular, all n-simplcies are degenerate if $n \neq 0$. Therefore, $X = F_0|X_*| = |X_*|$. □

Corollary 4.3. Suppose $h : I_* \times X \to Y$ is a map in $S\mathbf{CG}$. Assume that $h_t : X \to Y$ is defined by $h_{t,n}(x) = h(t, x)$ for $x \in X_n$ and $t = 0$ or $t = 1$. Then

$$I \times |X| \xrightarrow{\rho} |I_* \times X| \xrightarrow{|h|} |Y|$$

is a homotopy between $|h_0|$ and $|h_1|$.

Proof. By definition of ρ we have for all $s \in I$ that

$$|h|\rho(s, |x, u|) = |h(t, x)u|$$

and the result follows. □

The corollary strongly suggests that we could have defined a simplicial homotopy as being a simplicial map $h : I_* \times X \to Y$. We will now relate this to our definition of simplicial homotopies as given in Definition 4.12. First we have a definition

Definition 4.16. The fundamental n-simplex $\Delta[n]$ is defined to be the contravariant functor $\Delta \to S\mathbf{SET}$ given on objects $[m]$ by $\Delta[n]([m]) = \mathrm{Hom}_\Delta([m], [n])$ and on morphisms $\mu : [k] \to [l]$ by $\Delta[n](\mu)(\lambda) = \lambda \circ \mu$ for $\lambda : [l] \to [n]$. The face and degeneracy maps are given by $\Delta[n](D_i)$ and $\Delta[n](S_i)$ where Δ_i and S_i are the face and degeneracy maps in Δ.

By identifying $\lambda \in \mathrm{Hom}_\Delta([m], [n])$ with its image $\lambda([m])$ we see that the m-simplices of $\Delta[n]$ can be identified with sequences of integers $(a_0, \cdots, a_m)$ such that $0 \leq a_0 \leq \cdots a_m \leq n$ and

$$d_i(a_0, \cdots, a_m) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_m)$$
$$s_i(a_0, \cdots, a_m) = (a_0, \cdots, a_i, a_i, \cdots, a_m)$$

Define $[n]$ to be $(0, 1, \cdots n) \in \Delta[n]_n$. We let $\Delta^0[n]$ denote the subcomplex generated by $d_0[n]$ and $\Delta^1[n]$ the subcomplex generated by $\{d_i[n] | i \geq 1\}$.

Lemma 4.9. Let $f, g : X \to Y$ be morphisms of simplicial sets. Then f and g are homotopic if and only if there exists a simplicial map $H : \Delta[1] \times X \to Y$ such that, for all simplicies x of X, $H((0), x) = g(x)$ and $H((1), x) = f(x)$ where (0) is any simplex of $\Delta^1[1]$ and (1) is any simplex of $\Delta^0[1]$.

Proof. Let h be a homotopy $h : f \simeq g$. For $x \in X_k$ define $H((0), x) = g(x)$, $H((1), x) = f(x)$ and

$$H(s_{k-1}, \cdots, s_{i+1}, s_{i-1}, \cdots, s_0(1), x) = d_{i+1}h_i(x)$$

for $0 \leq i \leq k - 1$. It is easy to check that H is indeed a simplicial map.
Conversely, given such H we define

$$h_i(x) = H(s_k \cdots s_{i+1}s_{i-1} \cdots s_0(1), s_i x)$$

where $x \in X_n$ and $0 \leq i \leq n$. One readily verifies that h is indeed a simplicial homotopy. □

It is the following corollary we are mainly interested in.

Corollary 4.4. Suppose that h is a homotopy between simplicial maps $f, g : X \to Y$ in **SCG** as defined in Definition 4.12. Then h determines a homotopy $\tilde{h} : I \times |X| \to |Y|$ between $|f|$ and $|g|$.

Proof. Define a map $H : \Delta[1] \times X \to Y$ of simplicial sets by

$$H(s_{k-1}, \cdots, s_{i+1}, s_{i-1}, \cdots, s_0(1), x) = d_{i+1}h_i(x)$$

as in the lemma. Since the h_i and d_i are continuous this map is continuous and hence a map of simplicial spaces. Notice that $|\Delta[1]|$ is homeomorphic to I. Define $\tilde{h}$ to be the composite

$$I \times |X| \longrightarrow |\Delta[1]| \times |X| \xrightarrow{\rho} |\Delta[1] \times X| \xrightarrow{|H|} |Y|$$

then $\tilde{h}$ is a homotopy between $|f|$ and $|g|$. □

The next couple of results relate the connectivity of the X_n to the connectivity of $|X|$.

Lemma 4.10. Let $X \in S\mathbf{CG}$, then $\pi_0|X| = \pi_0(X_0)/\sim$ where $\sim$ is the equivalence relation generated by $[d_0x] \sim [d_1x]$ for $x \in X_1$. Here, $[y]$ denotes the path component of y.

Proof. The simplicial space X determines a simplicial set $\pi_0(X)$ with n simplices the path components of X_n. The lemma then states that $\pi_0|X| = \pi_0|\pi_0X|$. To prove this we proceed as follows. Let $(x, s) \in X_n \times |\Delta^n|$ where $n > 0$ and let $\alpha : I \to |\Delta^n|$ be a path connecting s to $d_0^n|\Delta^0|$. Then the path $\tilde{\alpha}(t) = |x, \alpha(s)|$ connects $|x, s|$ to some point in $X_0 = F_0|X_0|$. Suppose $x \in X_1$, then $\beta(t) = |x, (t, 1-t)|$ is a path connecting d_0x and d_1x and the result follows. □

Theorem 4.4. Let $n \geq 0$. Suppose X is a strictly proper simplicial space such that, for all $k \leq n$, X_n is $(n-k)$-connected. Then $|X|$ is n-connected.

Proof. For $n = 0$ this follows from the lemma. First we consider the case $n = 1$. Without loss of generality we may assume that X_k is connected for $k \geq 2$ since we can always delete components that do not intersect the simplicial subspace of X generated by X_0 and X_1 without changing $\pi_1|X|$. In that case, $|\Omega_* X|$ is weak homotopy equivalent to $\Omega|X|$ by Theorem 4.7 below. Since $|\Omega_* X|$ is connected it follows that $|X|$ is simply connected.
Now, let $n \geq 2$. Using the Hurewicz theorem it is sufficient to show that $\tilde{H}_i(X) = 0$ for $i \leq n$. To prove this it is sufficient to show that $\tilde{H}_i F_k|X| = 0$ for all $i \leq n$ and $k \geq 0$. We prove this by induction. The base step is clear since $X_0 = F_0|X|$ is n-connected by assumption. So, assume that $\tilde{H}_i(F_{k-1}|X|) = 0$ for $i \leq n$. We need to show that $\tilde{H}_i(F_k|X|) = 0$ for $i \leq n$. For this it is sufficient to show that $\tilde{H}_i(F_k|X|/F_{k-1}|X|) = 0$ for $i \leq n$ since $(F_k|X|, F_{k-1}|X|)$ is an NDR pair. But $F_k|X|/F_{k-1}|X| \cong \Sigma^k(X_k/s(X_{k-1}))$. Therefore, we only need to show that $\tilde{H}_i(X_k/s(X_{k-1}) = 0$ for $i \leq n-k$. Since X_k is by assumption $(n-k)$-connected and $(X_k, s(X_{k-1}))$ is an NDR pair it is sufficient to show that $\tilde{H}_i(s(X_{k-1})) = 0$ for $i < n-k$. We will show that

$$\tilde{H}_i(\bigcup_{j=0}^{m} s_j X_{k-1}) = 0$$

for $i \leq n - q + 1$ and $0 \leq m \leq k$. By the induction hypothesis for k we have

$$\tilde{H}_i(\bigcup_{j=0}^{m} s_j X_{k-2}) = 0$$

for $i \leq n - q + 2$ and $0 \leq m \leq k - 1$. For $0 \leq j < k$ we have that $s_j : X_{k-1} \to s_j X_{k-1}$ and $d_j : s_j X_{k-1} \to X_{k-1}$ are inverse homeomorphisms. From this it follows that

$$\tilde{H}_i(s_j X_{k-1}) = 0$$

for $i \leq n + 1 - q$. Assume by induction that $\tilde{H}_i(\bigcup_{j=0}^{m-1} s_j X_{k-1}) = 0$ for $i \leq n - q + 1$. Consider the excision map

$$(\bigcup_{j=0}^{m-1} s_j X_{k-1}, s_m X_{k-1} \cap \bigcup_{j=0}^{m-1} s_j X_{k-1}) \to (\bigcup_{j=0}^{m} s_j X_{k-1}, s_m X_{k-1})$$

This map is a map of NDR pairs and therefore we get a Mayer-Vietoris sequence

$$\cdots \longrightarrow H_i(\textstyle\bigcup_{j=0}^{m-1} s_j X_{k-1}) \oplus H_i(s_m X_{k-1}) \longrightarrow H_i(\textstyle\bigcup_{j=0}^{m} s_j X_{k-1})$$

$$H_i(\textstyle\bigcup_{j=0}^{m} s_j X_{k-1}) \longrightarrow H_{i-1}(s_m X_{k-1} \cap \textstyle\bigcup_{j=0}^{m-1} s_j X_{k-1}) \longrightarrow \cdots$$

If $s_k y = s_j z$ for $j < m$ then $y = d_{m+1} s_j z = s_j d_m z$ and since $s_m s_j = s_j s_{m-1}$ it follows that

$$s_m X_{k-1} \cap \bigcup_{j=0}^{m-1} s_j X_{m-1} = \bigcup_{j=0}^{m-1} s_m s_j X_{k-2}$$

Using the fact that

$$s_m : \bigcup_{j=0}^{m-1} s_j X_{m-2} \to \bigcup_{j=0}^{m-1} s_m s_j X_{k-2}$$

is a homeomorphism together with the induction hypothesis the Mayer-Vietoris sequence gives

$$\tilde{H}_i(\bigcup_{j=0}^{m} s_j X_{k-1}) = 0$$

for $i \leq n - k + 1$. □

Theorem 4.5. Suppose $f : X \to Y$ is a simplicial map between strictly proper simplicial spaces such that each f_n is a weak homotopy equivalence. Assume that either both $|X|$ and $|Y|$ are simply connected or that $|f|$ is an H-map of connnected H-spaces. Then $|f|$ is a weak homotopy equivalence.

Proof. Using the Whitehead Theorem it is sufficient to show that $|f|$ induces an isomorphism in integral homology. The proof is basically the same as that of the previous theorem. By induction one shows that each $F_n|f|$ is an isomorphism in homology. The same reduction as in the previous proof applies and thus we get the above Mayer-Vietoris sequence for both X and Y. The induction hypothesis guarantees that all but the middle arrow are isomorphisms. Therefore, the result follows with the five lemma. □

4.4 Geometric Realization of Σ_*, Ω_* and $\mathcal{C}_*$

In the discussion immediately preceding Theorem 4.1 we saw that a functor $F : \mathbf{C} \to \mathbf{D}$ induces a simplicial functor $F_* : S\mathcal{C} \to S\mathcal{D}$. In this section we will discuss how geometric realization relates to the functors Σ_*, Ω_* and $\mathcal{C}_*$. Suppose that $X \in S\mathbf{CG}_*$. We define the base point of $|X|$ to be $* \in X_0 = F_0|X|$. Note that if X is proper then it follows by Lemma 4.6 and Lemma 4.7 that the base point is non-degenerate and that $|X| \in \mathbf{CG}_*$. First we consider how geometric realization relates to the suspension functor. For this we have the following result.

Proposition 4.1. Let $X \in S\mathbf{CG}_*$. Then there exists a natural homeomorphism

$$\tau : |\Sigma_*| \to \Sigma|X|$$

Proof. Define $\tau|[x,t],s| = [|x,s|,t]$ for $x \in X_n$, $t \in I$ and $s \in |\Delta^n|$. The reader can readily verify that τ is well-defined and continuous. The inverse is given by $\tau^{-1}[|x,s|,t] = |[x,t],s|$ for $x \in X_n$, $t \in I$ and $s \in |\Delta^n|$ which is again easily seen to be well-defined and continuous. Obviously, τ and τ^{-1} are indeed mutually inverse. □

The relation between geometric realization and $\mathcal{C}_*$ turns out to be quite nice. We will first prove the following lemma.

Lemma 4.11. Let X and Y in $S\mathbf{CG}$ then

1. the map

$$\rho : |X| \times |Y| \to |X \times Y|$$

   is filtration preserving.

2. The diagonal map

$$\Delta : |X| \to |X| \times |X|$$

   is naturally homotopic to a filtration preserving map

Proof. 1. This follows immediately from the definition of ρ.

2. Let $t = (t_0, \cdots, t_n) \in |\delta^n|$. Define p to be the least integer such that $\sum_{i=0}^{p} t_1 > \frac{1}{2}$. Put

$$g_0(t) = d_n \cdots d_{p+1}(2t_0, \cdots, 2t_{p-1}, 1 - \sum_{i=0}^{p-1} 2t_i)$$

   and

$$g_1(t) = d_0^p(1 - \sum_{i=p+1}^{n} 2t_i, 2t_{p+1}, \cdots, 2t_n)$$

   The g_i induce a maps $G_i : |X| \to |X|$ that are homotopic to the identity. Therefore, Δ is homotopic to the filtration preserving map $(G_0 \times G_1) \circ \Delta$. □

Theorem 4.6. Let $\mathcal{C}$ be any operad. Denote by C the corresponding monad. Then for any $X \in S\mathbf{CG}_*$ there exists a natural homeomorphism

$$\nu : |C_*X| \to C|X|$$

making the diagrams

$$\begin{tikzcd} |X| \arrow[r, "|\eta_*|"] \arrow[dr, "\eta"'] & |C_*X| \arrow[d, "\nu"] \\ & C|X| \end{tikzcd} \qquad \begin{tikzcd} |C_*^2 \arrow[r, "\nu^2"] \arrow[d, "|\mu_*|"'] & C^2|X| \arrow[d, "\mu"] \\ |C_*X| \arrow[r, "\nu"'] & C|X| \end{tikzcd}$$

commute.

Proof. Let $|[c; x_1, \cdots, x_n], s] \in |C_*X|$ where $c \in \mathcal{C}(n)$, $x_i \in X_n$ and $s \in |\delta^n|$. Put

$$\nu[c; x_1, \cdots, x_n], s] = [c, |x_1, s|, \cdots, |x_n, s|]$$

It is clear that ν is compatible with the base point and equivariance identifications in the definition of CX_n. Note that

$$Cd_i[c, x_1, \cdots, x_n] = [c, d_ix_1, \cdots, d_ix_n]$$
$$Cs_i[c, x_1, \cdots, x_n] = [c, s_ix_1, \cdots, s_ix_n]$$

Therefore, ν is also compatible with the face and degeneracy maps from the definition of the realization functor. The map

$$\rho : |X| \times |X| \to |X \times X|$$

from Theorem 4.3 can be iterated to give

$$\rho^n : |X|^n \to |X^n|$$
$$\rho^n(|x_1, s_1|, \cdots, |x_n, s_n|) = |(y_1, \cdots, y_n)v|$$

in the notation of the theorem. Note that ρ^n is unambiguous since ρ is associative. Therefore, we can define ν^{-1} by

$$\nu^{-1}[c, |x_1, s_1|, \cdots |x_n, s_n|] = |[c, y_1, \cdots, y_n], v|$$

Since ρ is continuous so is ρ^n and therefore ν^{-1} is continuous. Since ρ is also commutative it follows that ν^{-1} is compatible with the equivariance identifications. Compatibility with the other identifications is obvious. The fact that ν and ν^{-1} are inverses follows immediately from Lemma 4.3. The fact that both diagrams are commutative can be easily verified using the definition of ν. $\square$

Corollary 4.5. Let $\mathcal{C}$ be any operad with associated monad C. If $(X, \xi) \in SC[\mathbf{CG}_*]$ then $(|X|, |\xi|\nu^{-1}) \in C[\mathbf{CG}_*]$. Therefore, geometric realization defines a functor $SC[\mathbf{CG}_*] \to C[\mathbf{CG}_*]$.

The remainder of this section will be devoted to investigating the relation between the loop space functor and geometric realization.

Theorem 4.7. Let $X \in S\mathbf{CG}_*$ then

1. $|P_*X|$ is contractible where P is the path space functor.
2. There exists natural maps $\gamma : |\Omega_* X| \to \Omega|X|$ and $\tilde{\gamma} : |P_*X| \to P|X|$ such that the diagram

$$\begin{tikzcd}
|\Omega_* X| \arrow[r, "\text{incl.}"] \arrow[d, "\gamma"'] & |P_*X| \arrow[r, "|p_*|"] \arrow[d, "\tilde{\gamma}"'] & |X| \\
\Omega|X| \arrow[r, "\text{incl.}"'] & P|X| \arrow[ur, "p"'] &
\end{tikzcd}$$

commutes where $p : PY \to Y$ is the end point evaluation map.
3. If X is proper and each X_n is connected then $|p_*|$ is a quasi-fibration with fibre $|\Omega_* X|$. Therefore, in that case $\gamma : |\Omega_* X| \to \Omega|X|$ is a weak homotopy equivalence.

Proof. 1. For $Y \in \mathbf{CG}_*$ we have the usual contracting homotopy $H : I \times PY \to PY$ given by $H(t, \alpha)(s) = \alpha((1-t)s)$. Applying this to every X_n we get a simplicial homotopy

$$I_* \times P_* X \to P_* X$$

that contracts P_*X. Applying Corollary 4.3 yields the result.

2. Let $\alpha \in PX_n$, $s \in |\Delta^n|$ and $t \in I$ and define

$$\tilde{\gamma}(|\alpha, s|)(t) = |\alpha(t), s|$$

It is clear that $\tilde{\gamma}$ is well-defined and continuous and satisfies $p\tilde{\gamma} = |p_*|$. Define γ to be the restriction of $\tilde{\gamma}$ to $|\Omega_* X|$ then obviously the diagram commutes.

3. This follows from Theorem 4.9 which we will prove shortly.

□

The next theorem relates the results obtained in this section in a nice way.

Theorem 4.8. Let $X \in S\mathbf{CG}_*$ then the n-fold composition $\gamma^n : |\Omega_*^n X| \to \Omega^n|X|$ of γ is a morphisms of C_n-algebras. Moreover, the following diagram commutes

$$\begin{tikzcd}
|C_{n*}X| \arrow[r, "\nu"] \arrow[d, "|\alpha_{n*}|"'] & C_n|X| \arrow[d, "\alpha_n"] \\
|\Omega_*^n \Sigma_*^n X| \arrow[r, "\Omega^n \tau^n \circ \gamma^n"'] & \Omega^n \Sigma^n |X|
\end{tikzcd}$$

Proof. First we prove that γ^n is a morphism of C_n-algebras. It is sufficient to show that

$$\begin{tikzcd}
C_n|\Omega_*^n X| \arrow[rr, "C_n(\gamma^n)"] \arrow[d, "\nu^{-1}"'] & & C_n\Omega^n|X| \arrow[d, "\theta_n"] \\
|C_{n*}\Omega_*^n X| \arrow[r, "|\theta_{n*}|"'] & |\Omega_*^n X| \arrow[r, "\gamma^n"'] & \Omega^n|X|
\end{tikzcd}$$

commutes. Note that, since ν is a natural homeomorphism, we can replace ν^{-1} by ν and show that the resulting diagram commutes. So let $y = |[c; f_1, \cdots, f_k], t] \in |C_{n*}\Omega^n_* X|$ where $c = (c_1, \cdots, c_k) \in \mathcal{C}_n(k)$, $f_i \in \Omega^n X_m$ and $t \in |\Delta^m|$ for some m. Now, let $s \in I^n$. First assume $s \notin \cup c_i(J^n)$ then

$$\gamma^n|\theta_{n*}|(y)(s) = * = \theta_n \circ C_n(\gamma^n) \circ \nu(y)(s)$$

On the other hand, if $s \in \cup c_i(J^n)$ then write $s = c_i(s')$ and then (using Theorem 3.2) we get

$$\begin{aligned}
\theta_n \circ C_n(\gamma^n) \circ \nu(y)(s) &= \theta_n[c; \gamma^n|f_1, t|, \cdots, \gamma^n|f_k, t|](s) \\
&= \gamma^n|f_i, u|(s') \\
&= |f_i(s'), t| \\
&= |\theta_n[c; f_1, \cdots, f_k](s), t| \\
&= \gamma^n|\theta_{n*}|(y)(s)
\end{aligned}$$

Therefore, γ^n is a morphism of C_n-algebras.
Now, for the commutativity of the diagram. Recall that $\alpha_n = \theta_n \circ C_n\eta_n$. Therefore, it is sufficient to show that

$$\begin{tikzcd}
{|C_{n*}X|} \arrow[r, "\nu"] \arrow[d, "{|C_{n*}\eta_{n*}|}"'] & C_n|X| \arrow[d, "{C_n|\eta_{n*}|}"'] \arrow[drr, "C_n\eta_n"] & & \\
{|C_{n*}\Omega^n_*\Sigma^n_*X|} \arrow[r, "\nu"] \arrow[d, "{|\theta_{n*}|}"'] & C_n|\Omega^n_*\Sigma^n_*X| \arrow[r, "C_n\gamma^n"] & C_n\Omega^n|\Sigma^n_*X| \arrow[r, "C_n\Omega^n\tau^n"'] \arrow[d, "\theta_n"'] & C_n\Omega^n\Sigma^n|X| \arrow[d, "\theta_n"] \\
{|\Omega^n_*\Sigma^n_*X|} \arrow[rr, "\gamma^n"'] & & \Omega^n|\Sigma^n_*X| \arrow[r, "\Omega^n\tau^n"'] & \Omega^n\Sigma^n|X|
\end{tikzcd}$$

commutes. But this is now straightforward. In the first part of the proof we have shown that the bottom left square commutes. The top square and triangle commute by the previous three results. The commutativity of the bottom right square is easily verified by direct calculation. $\square$

The remainder of this section will be concerned with the geometric realization of a Hurewicz fibration. First, we fix some notation. Let $B \in \mathbf{CG}$ and write ΠB for the space of paths $I \to B$. If $p : E \to B$ is any map in $\mathbf{CG}$ then we define

$$\Gamma(p) = \{(e, f) | p(e) = f(o)\} \subset E \times \Pi B$$

Define $\pi : \Pi E \to \Gamma(p)$ by $\pi(g) = (g(0), pg)$. Then p is a Hurewicz fibration if and only if there exists $\lambda : \Gamma(p) \to \Pi E$ with $\pi\lambda = 1_{\Gamma(p)}$.
Let $p : E \to B$ be a map in $S\mathbf{CG}$. Note that if $\pi_n = \pi : \Pi E_n \to \Gamma(p_n)$ then $\pi_* : \Pi_* E \to \Gamma_*(p)$ is a map in $S\mathbf{CG}$. For $f, g \in \Pi E_n$ with $f(1) = g(0)$ we write x_* and y_* for, respectively, the simplicial subspace of B generated by the n-simplex $x = f(0)$ and by $y = g(1)$. We have the following definition.

Definition 4.17. A map $p : E \to B$ in $S\mathbf{CG}$ is said to be a simplicial Hurewicz fibration if there exists a map $\lambda_* : \Gamma_*(p) \to \Pi_* E$ such that

1. $\pi_*\lambda_* = 1_{\Gamma_*(p)}$

2. For any $f, g \in \Pi E_n$ with $f(1) = g(0)$ there exists a simplicial homotopy $H : I_* \times p^{-1}(x_*) \to p^{-1}(y_*)$ such that for any i-simplex e of $p^{-1}(x_*)$ satisfying $p_i(e) = \gamma x$, where γ is any composite of face and degeneracy operators, we have

$$H_i(0, e) = \lambda_i(\lambda_i(e, \gamma f), \gamma g)(1)$$
$$H_i(1, e) = \lambda_i(e, \gamma(gf))(1)$$

Let e denote an i-simplex of $p^{-1}(x_*)$ with $p(e) = \gamma x$ as above then the following lemma is easily seen to hold.

Lemma 4.12. 1. If $h : I \to \Pi B_n$ satisfies $h(t)(0) = x$ and $h(t)(1) = y$ for every $t \in I$ then

$$H_i(t, e) = \lambda_i(e, \gamma h(t))(1)$$

defines a simplicial homotopy $H : I_* \times p^{-1}(x_*) \to p^{-1}(y_*)$.

2. If $c_x : I \to B_n$ is the constant path at $x \in B_n$ then

$$H_i(t, e) = \lambda_i(e, \gamma c_x)(t)$$

defines a simplicial homotopy $H : I_* \times p^{-1}(x_*) \to p^{-1}(x_*)$ which starts at the identity $1_{p^{-1}(x_*)}$.

The following lemma will be needed where P denotes the path space functor.

Lemma 4.13. Let $X \in S\mathbf{CG}_*$ then $p : P_*X \to X$ is a simplicial Hurewicz fibration.

Proof. Let $r : I \times I \to I \times 1 \cup 0 \times I$ be a retration satisfying

$$r(s, 0) = (0, 0)$$

and

$$r(1, t) = \begin{cases} (0, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ (2t - 1, 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

If $Y \in \mathbf{CG}_*$ and $p : PY \to Y$ define $\lambda : \Gamma(p) \to \Pi PY$ by

$$\lambda(e, f)(s)(t) = \begin{cases} e(u), & \text{if } r(s, t) = (0, u) \\ f(v), & \text{if } r(s, t) = (v, 1) \end{cases}$$

where $e \in PY$, $f \in \Pi Y$ and $e(1) = f(0)$. Clearly, λ is a lifting function for p. Note that $\lambda(e, f)(1) = fe$ is the usual concatenation of paths. Consequently, if $f, g \in \Pi Y$ satisfy $f(1) = g(0)$ then

$$\lambda(\lambda(e, f)(1), g)(1) = g(fe)$$

and

$$\lambda(e, gf)(1) = (gf)e$$

Put $\lambda_n = \lambda$ for each n. By naturality of λ it follows that λ_* is a simplcial map and $\pi_*\lambda_* = 1_{\Gamma(p)}$. The second condition from the definition of a simplicial Hurewicz fibration is also satisfied since there are evident homotopies for each fixed γ that are easily seen to fit together into the desired homotopy. $\square$

Theorem 4.9. Suppose $p : E \to B$ is a simplicial Hurewicz fibration in $S\mathbf{CG}_*$ and put $F = p^{-1}(*)$. If B is a proper simplicial space with each B_n connected then $|p| : |E| \to |B|$ is a quasi-fibration with fiber $|F|$.

Proof. First consider the restriction of $S_{j_1} \cdots S_{j_r} : |\Delta^{q+r}| \to |\Delta^q|$ to the inverse image of $|\Delta^q| \setminus \partial|\Delta^q|$ and inductively define lifting functions of these restrictions as follows. Let $(u, f) \in \Gamma(S_j)$ and $f(s) = (t_0(s), \cdots, t_q(s))$. Since $S_j(u) = f(0)$ there exists some $a \in I$ such that

$$u = (t_0(0), \cdots, t_{j-1}(0), at_j(0), (1-a)t_j(0), t_{j+1}(0), \cdots, t_q(0))$$

Define

$$\begin{aligned} &\gamma_j : \Gamma(S_j) \to \Pi(|\Delta^{q+1})| \\ &\gamma_j(u, f)(s) = (t_0(s), \cdots, t_{j-1}(s), at_j(s), (1-a)t_j(s), t_{j+1}(s), \cdots, t_q(s)) \end{aligned}$$

Inductively we put

$$\begin{aligned} \gamma_{j_r \cdots j_1} &: \Gamma(S_{j_1} \cdots S_{j_r}) \to \Pi(|\Delta^{q+r}|) \\ &\gamma_{j_r \cdots j_1}(u, f) = \gamma_{j_r}(u, \gamma_{j_{r-1} \cdots j_1}(S_{j_r} u, f)) \end{aligned}$$

By direct inspection $\gamma_j(u, f)(o) = u$ and $S_j \gamma_j(u, f) = f$ from which it follows that $\pi\gamma_j = 1$. For $i < j$ we know that $S_i S_j = S_{j-1} S_i$ and therefore $\gamma_{j,i} = \gamma_{i,j-1}$. Consequently, we have that

$$S_j \gamma_{j_r \cdots j_1}(u, f) = \gamma_{i_{r-1} \cdots i_1}(S_j u, f) \tag{4}$$

whenever $s_j s_{i_{r-1}} \cdots s_{i_1} = s_{j_r} \cdots s_{j_1}$. Furthermore, in the particular case that $a = 0$ (resp. $a = 1$) we have $\gamma_j(u, f)(s) \in \operatorname{im}(D_j)$ (resp. $\in \operatorname{im}(D_{j+1})$) from which it follows that

$$D_j \gamma_{j_r \cdots j_1}(u, f) = \gamma_{i_{r+1} \cdots i_1}(D_j u, f) \tag{5}$$

whenever $ds_{i_{r+1}} \cdots s_{i_1} = s_{j_r} \cdots s_{j_1}$.
We would like to use Lemma 3.10 to conclude that $|p|$ is a quasi-fibration. So let $V \subset F_q|B| \setminus F_{q-1}|B|$ be open where $F_{-1}|B| = \emptyset$. To verify the first condition of Lemma 3.10 we need to show that $|p| : |p|^{-1}(V) \to V$ is a quasi-fibration. Clearly, it is sufficient to show that it is even a Hurewicz fibration. To show this we need to find a lifting map $\tilde{\lambda}_q : \Gamma(|p|) \to \Pi(|p|^{-1}(V))$. Using Lemma 4.6 we see that $\Pi(V) \subset \Pi(B_q \setminus B_{q-1}) \times \Pi(|\Delta^q| \setminus \partial|\Delta^q|)$. Pick $(|e, w|, (f', f'')) \in \Gamma(|p|)$ where $(e, w) \in E_{q+r} \times |\Delta^{q+r}|$ is non-degenerate, $f' : I \to B_q \setminus sB_{q-1}$ and $f'' : I \to |\Delta^q| \setminus \partial|\Delta^q|$. Then, as we have seen in the proof of Lemma 4.6,

$$p_{q+r}(e) = s_{j_r} \cdots s_{j_1} f'(0)$$

where $S_{j_1} \cdots S_{j_r} w = f''(0)$. We now define $\tilde{\lambda}_q$ by

$$\tilde{\lambda}_q(|e, w|, (f', f'')) = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} f')(t), \gamma_{j_r \cdots j_1}(w, f'')(t)|$$

where λ is defined as in the proof of Lemma 4.6. Since λ_* is simplicial it follows by 4 and 5 that $\tilde{\lambda}_q$ is compatibel with the equivalence relation that defines geometric realization. It is easy to see that $\pi\tilde{\lambda}_q = 1$ and therefore the first

condition of Lemma 3.10 is satisfied.
We will now show that the second condition of Lemma 3.10 is also satisfied. Let $q > 0$ be fixed. Since (B_q, sB_{q-1}) and $(|\Delta^q|, \partial|\Delta^q|)$ are strong NDR pairs it follows by Lemma 1.8 that they induce a representation (k, v) of $(B_q \times |\Delta^q|, sB_{q-1} \times |\Delta^q| \cup B_q \times \partial|\Delta^q|)$ as a strong NDR pair. Let $U \subset F_q|B|$ be the union of $F_{q-1}|B|$ with the image of $v^{-1}[0,1)$ under the obvious map $B_q \times |\Delta^q| \to F_q|B|$. Define a homotopy $h_t : U \to U$ by $h_t(x) = x$ if $x \in F_{q-1}|B|$. If $(b, u) \in B_q \times |\Delta^q|$ and $v(b, u) < 1$ we put $h_t|b, u| = |k_t(b, u)|$. One easily verifies that h_t is a strong deformation retract of U onto $F_{q-1}|B|$.
We first lift h. Let $(e, w) \in E_{m+r} \times |\Delta^{m+r}|$ be a non-degenerate point such that $|e, w| \in |p|^{-1}(U)$ where $p_{m+r}(e) = s_{j_r} \cdots s_{j_1} b$ and $u = S_{j_1} \cdots S_{j_r}$ determine a non-degenerate representative of $|p|(|e, w|)$. Note that $m \leq q$ we will define a lift seperately for $m < q$ and $m = q$. If $m < q$ let $c_b : I \to B_m$ be the constant path at b and define

$$H(t, |e, w|)|\lambda_{m+r}(e, s_{j_r} \cdots s_{j_1} c_b)(t), w|$$

If $q = m$ let $f' : I \to B_q$ and $f'' : I|\Delta^q|$ be the following paths: $f'(t) = \pi_1 k_t(b, u)$ and $\pi_2 k_t(b, u)$ where the π_i are the projections of $B_q \times |\Delta^q|$ onto its factors. We define

$$H(t, |e, w|) = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} f')(t), \gamma_{j_r \cdots j_1}(w, f'')(t)|$$

To see that $\gamma_{j_r \cdots j_1}(w, f'')$ makes sense note that even though f'' may hit the boundary of $|\Delta^q|$ it will, in that case, just be the constant path a $f''(0)$. It is readily verified that H covers h and deforms $|p|^{-1}(U)$ into $|p|^{-1}F_{q-1}|B|$. Therefore, to verify the second condition of Lemma 3.10 the only thing left is to verify that for every $x \in U$ the map $H_1 : |p|^{-1}(x) \to |p|^{-1}h_1(x)$ is a weak homotopy equivalence.
If $x \in F_{q-1}|B|$ then $h_1(x) = x$ and it follows that H is a homotopy between the idenity and $H_1 : |p|^{-1}(x) \to |p|^{-1}(x)$. Now, assume that $x \notin F_{q-1}|B|$. Using the same notation as when we defined H, let $x = |b, u| = |f'(0), f''(0)|$. We then have $h_1(x) = |f'(1), f''(1)|$. Let $g : I \to B_q$ be a path from $f'(0)$ to $*$ and set $g' = g \circ (f')^{-1} : I \to B_q$ where $(f')^{-1}(t) = f'(1-t)$ is the inverse path. Clearly, g' is a path from $f'(1)$ to $*$.
Our first goal is to show that for any path $f : I \to B_q$ with $f(0) = b$ and $f(1) = *$ and any $u \in |\Delta^q|$ there exists a homotopy equivalence

$$\tilde{f}(u) : |p|^{-1}|b, u| \to |F|$$

Fix such a path f and let $\Delta[q]$ be the fundamental q-simplex introduced in Definition 4.16 regarded as a discrete simplicial space, i.e. with all faces and degeneracies added. Let $\bar{b} : \Delta[q] \to B$ be the unique simplicial map satisfying $\bar{b}(i_q) = b$ where i_q is the acutal fundamental q-simplex in $\Delta[q]$. Let $E(b)$ be the fibered product $E \times_B \Delta[q]$ of p and $\bar{b}$. Define a map

$$\begin{aligned} &f_* : E(b) \to F \times \Delta[q] \\ &f_i(e, \gamma i_q) = (\lambda_i(e, \gamma f)(1), \gamma i_q) \end{aligned}$$

where $e \in E_i$ is such that $p_i(e) = \gamma b = \gamma \bar{b}(i_q)$ for some composition γ of face and degeneracy maps. Define an inverse by

$$\begin{aligned} f_*^{-1} &: F \times \Delta[q] \to E(b) \\ f_i^{-1}&(e, \gamma i_q) = (\lambda_i(e, \gamma f^{-1})(1), \gamma i_q) \end{aligned}$$

where $e \in F_i$ and $\gamma i_q \in \Delta_i[q]$. It readily follows that f_* and f_*^{-1} are inverse fiber homotopy equivalences over $\Delta[q]$. Consequently, by Corollary 4.1, we get a fiber homotopy equivalence

$$|E| \times_{|B|} |\Delta^q| \xrightarrow{\zeta} |E(b)| \xrightarrow{|f_*|} |F \times \Delta[q]| \xrightarrow{|p_1|\times|p_2|} |F||\Delta^q|$$

over $|\Delta[q]| = |\Delta^q|$. For $u \in |\Delta^q|$ we can identify $p_2^{-1}(u)$ and $|p|^{-1}|b, u|$ and therefore by restriction the above map gives the desired homotopy equivalence

$$\tilde{f}(u) : |p|^{-1}|b, u| \to |F|$$

Lastly, we will show that the following diagram is homotopy commutative

$$\begin{tikzcd}
{|p|^{-1}(x) = |p|^{-1}|f'(0), f''(0)|} \arrow[rr, "H_1"] \arrow[dr, "\tilde{g}(f''(0))"'] & & {|p|^{-1}|f'(1), f''(1)| = |p|^{-1}h_1(x)} \arrow[dl, "\tilde{g}'(f''(1))"] \\
& {|F|} &
\end{tikzcd}$$

Since both vertical arrows are homotopy equivalences by the previous step of the proof certainly H_1 will then be a weak homotopy equivalence and thus the proof will be done. To show homotopy commutativity of the diagram we proceed as follows. Let $|e, w| \in |p|^{-1}(x)$ be as in the definition of H above then we get

$$\tilde{g}(f''(0))|e, w| = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} g)(1), w|$$

and

$$\tilde{g}'(f''(1)) \circ H_1|e, w| = |\lambda_{q+r}(\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} f')(1), s_{j_r} \cdots s_{j_1} g')(1), \gamma_{j_r \cdots j_1}(w, f'')(1)|$$

From the definition of a simplcial Hurewicz fibration together with the fact that $g' = g(f')^{-1}$ it follows that $g'(f''(1)) \circ H_1$ is homotopic to $l : |p|^{-1}(x) \to |F|$ defined by

$$l|e, w| = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} g)(1), \gamma_{j_r \cdots j_1}(w, f'')(1)|$$

So define a homotopy $L : I \times |p|^{-1}(x) \to |F|$ by

$$L(t, |e, w|) = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} g(1), \gamma_{j_r \cdots j_1}(w, f'')(t)|$$

Then clearly $L(0, -) = \tilde{g}(f''(0))$ and $L(1, -) = l$ and we are done. $\square$

5 The Double Bar Construction

In this section we will introduce a general construction called the double bar construction. This construction is functorial and takes values in simplicial spaces. In the next chapter we will see that this construction allows us to construct under certain conditions the n-fold delooping of a $\mathcal{C}_n$-space.

Definition 5.1. Let $\mathbf{C}$ be a category and assume (T, μ, η) is a monad in $\mathbf{C}$. A T-functor (F, λ) in a category $\mathbf{D}$ consists of a functor $F : \mathbf{C} \to \mathbf{D}$ together with a natural transformation $\lambda : FT \to F$ such that

$$\begin{tikzcd} F \arrow[r, "F(\eta)"] \arrow[dr, "1"'] & FT \arrow[d, "\lambda"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} FTT \arrow[r, "F(\mu)"] \arrow[d, "\lambda"'] & FT \arrow[d, "\lambda"] \\ FT \arrow[r, "\lambda"'] & T \end{tikzcd}$$

commute.
A morphism $\pi : (F, \lambda) \to (F', \lambda')$ of T functors in $\mathbf{D}$ is a natural transformation $\pi : F \to F'$ such that

$$\begin{tikzcd} FT \arrow[r, "\pi"] \arrow[d, "\lambda"'] & F'T \arrow[d, "\lambda'"] \\ F \arrow[r, "\pi"'] & F' \end{tikzcd}$$

commutes.

We will be mainly interested in the following four examples of T-functors.

Example 5.1.

1. Suppose (T, μ, η) is a monad in $\mathbf{C}$, then (T, μ) itself is a T-functor in $\mathbf{C}$. Observe that since (TX, μ) is a T-algebra and $\mu : T^2X \to TX$ a morphism of T-algebras we can also consider (T, μ) as a T-functor in $T[\mathbf{C}]$.

2. Let $\alpha : (T, \mu, \eta) \to (S, \nu, \zeta)$ be a morphism of monads in $\mathbf{C}$ and assume (F, λ) is a S-functor. Define

$$\alpha^*(F, \lambda) = (F, \lambda \circ F\alpha)$$

Consider

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "F\zeta"] \arrow[ddr, "1"'] & FT \arrow[d, "F\alpha"] \\ & FS \arrow[d, "\lambda"] \\ & F \end{tikzcd}$$

Here the top triangle commutes since α is a morphism of monads and F is a functor. The bottom triangle commutes since (F, λ) is a S-functor.

Therefore, the whole diagram commutes. Moreover, consider

$$\begin{tikzcd}
FTT \arrow[rr, "F\mu"] \arrow[d, "F\alpha"'] & & FT \arrow[d, "F\alpha"] \\
FST \arrow[r, "FS\alpha"] \arrow[d, "\lambda"'] & FSS \arrow[r, "F\nu"] \arrow[d, "\lambda"'] & FS \arrow[d, "\lambda"] \\
FT \arrow[r, "F\alpha"'] & FS \arrow[r, "\lambda"'] & F
\end{tikzcd}$$

Here the top rectangle commutes since α is a morphism of monads and the right bottom square commutes since (F, λ) is a S-functor. The bottom left square commutes since λ is in particular a natural transformation. Therefore, the whole diagram commutes. Consequently, $\alpha^*(F, \lambda) = (F, \lambda \circ F\alpha)$ is T-functor in $\mathbf{C}$.
We can combine this example with the previous one in the following way. We then get that $(S, \nu \circ S\alpha)$ is a T-functor in $S[\mathbf{C}]$. Since S can be composed with α^* we also see that $(S, \nu \circ S\alpha)$ is a T-functor in $T[\mathbf{C}]$. In that case, $\alpha : (T, \mu) \to (S, \nu \circ S\alpha)$ is a morphism of T-functors in $T[\mathbf{C}]$.

3. Let
$$\phi : \mathrm{Hom}_{\mathcal{C}}(X, \Lambda Y) \to \mathrm{Hom}_{\mathcal{D}}(\Sigma X, Y)$$
be an adjunction between functors $\Lambda : \mathbf{D} \to \mathbf{C}$ and $\Sigma : \mathbf{C} \to \mathbf{D}$. By Lemma 3.1 we know that we get a monad $(\Lambda\Sigma, \nu, \zeta)$. It obviously follows that $(\Sigma, \phi(1_{\Lambda\Sigma}))$ is a $\Lambda\Sigma$-functor in $\mathbf{D}$.

4. Let $(\Lambda\Sigma, \nu, \zeta)$ be the monad obtained in the previous example and assume that $\alpha : (T, \mu, \eta) \to (\Lambda\Sigma, \nu, \zeta)$ is a morphism of monads. Observe that $\phi(\alpha) = \phi(1) \circ \Sigma\alpha : \Sigma T \to \Sigma$. Combining the previous two examples we immediately see that $(\Sigma, \phi(\alpha))$ is a T-functor in $\mathbf{D}$ and $\alpha : (T, \mu) \to (\Lambda\Sigma, \Lambda\phi(\alpha))$ is a morphism of T-functors in $T[\mathbf{C}]$.

We will now introduce the so-called double bar construction which will be crucial in what is to come.

Definition 5.2. 1. Define a category $\mathbf{B}(\mathbf{C}, \mathbf{D})$ as follows. The objects of $\mathbf{B}(\mathbf{C}, \mathbf{D})$ are triples $((F, \lambda), (T, \mu, \eta), (X, \xi))$ where T is a monad, F is a T-functor in $\mathbf{D}$ and X is a T-algebra. We will denote such a triple $((F, \lambda), (T, \mu, \eta), (X, \xi))$ by (F, T, X). The morphisms $(\pi, \psi, f) : (F, T, X) \to (F', T', X')$ of $\mathbf{B}(\mathbf{C}, \mathbf{D})$ are triples (π, ψ, f) where $\psi : T \to T'$ is a morphism of monads, $\pi : F \to \psi^* F$ is a morphism of T-functors in $\mathbf{D}$ and $f : X \to \psi^* X'$ is a morphism of T-algebras.

2. Define a functor $B_* : \mathbf{B}(\mathbf{C}, \mathbf{D}) \to S\mathbf{D}$ as follows. On objects (F, T, X) we define $B_*(F, T, X)$ by
$$B_n(F, T, X) = FT^n X$$
with face maps
$$d_i = \begin{cases} \lambda, & \text{if } i = 0 \\ FT^{i-1}\mu, & \text{if } 0 < i < n \\ FT^{n-1}\xi, & \text{if } i = n \end{cases}$$

and degeneracy maps

$$s_i = FT^i\eta$$

for $i = 0, \cdots, n$. On morphisms (π, ψ, f) we define $B_*(\pi, \psi, f)$ by

$$B_n(\pi, \psi, f) = \pi\psi^n f : FT^n X \to F'(T')^n X$$

where $\pi\psi^n : FT^n \to F'(T')^n$ is a natural transformation and $\pi\psi^n f$ is defined via the commutative diagram

$$\begin{tikzcd} FT^nX \arrow[r, "FT^nf"] \arrow[d, "\pi\psi^n"'] \arrow[dr, "\pi\psi^n f"] & FT^nX' \arrow[d, "\pi\psi^n"] \\ F'(T')^nX \arrow[r, "F'(T')^nf"'] & F'(T')^nX' \end{tikzcd}$$

The following lemma is immediate from the definition.

Lemma 5.1. Let (F, λ) be a T-functor in $\mathbf{D}$ and let $G : \mathbf{D} \to \mathbf{E}$ be any functor. Then

1. $(GF, G\lambda)$ is a T-functor in $\mathbf{E}$
2. for any T-algebra X we have

$$B_*(GF, T, X) = G_*B_*(F, T, X)$$

The following two propositions will be needed later.

Proposition 5.1. Suppose (T, μ, η) is a monad in $\mathbf{C}$ and (X, ξ) a T-algebra. Then X_* is a strong deformation retract of $B_*(T, T, X)$.

Proof. Recall the definition of the maps τ_* and ϵ_* from respectively Lemma 4.2 and Lemma 4.3. We have that $\epsilon_*(\xi) : B_*(T, T, X) \to X_*$ is a morphism of simplicial T-algebras and $\tau_*(\eta) : X_* \to B_*(T, T, X)$ is a morphism in $S\mathbf{C}$ such that $\epsilon_*(\xi) \circ \tau_*(\eta) = 1_{X_*}$. Define, for $i = 0, \cdots, n$,

$$h_i : B_n(T, T, X) \to B_{n+1}(T, T, X)$$

by

$$h_i = s_0^i \eta d_0^i : T^{n+1}X \to T^{n+2}X$$

It is readily verified that h is a homotopy in $S\mathbf{C}$ from the identity on $B_*(T, T, X)$ to $\tau_*(\eta) \circ \epsilon_*(\xi)$ and $h_i \circ \tau_n(\eta) = \tau_{n+1}(\eta)$. But this just says that X_* is a strong deformation retract of $B_*(T, T, X)$. □

Proposition 5.2. Let (T, μ, η) be a monad in $\mathbf{C}$, (F, λ) a T-functor in $\mathbf{D}$ and $Y \in \mathbf{C}$. Then $(FY)_* = F_*Y_*$ is a strong deformation retract of $B_*(F, T, TY)$ in $S\mathbf{D}$.

Proof. Again, using the definitions of τ_* and ϵ_* from Lemmas 4.2 and 4.3 it follows that $\epsilon_*(\lambda) : B_*(F, T, TY) \to F_*Y_*$ and $\tau_*(F\eta) : F_*Y_* \to B_*(F, T, TY)$ are morphisms in $S\mathbf{D}$ such that $\epsilon_*(\lambda) \circ \tau_*(F\eta) = 1_{F_*Y_*}$. Consider

$$h_i : B_n(F, T, TY) \to B_{n+1}(F, T, TY)$$

defined by

$$h_i = s_n \cdots s_{i+1} \circ FT^{i+1}\eta \circ d_{i+1} \cdots d_n : FT^{n+1}Y \to FT^{n+2}Y$$

Then h is a homotopy from $\tau_*(F\eta) \circ \epsilon_*(\lambda)$ to the identity on $B_*(F, T, TY)$ in $S\mathbf{D}$ and $h_i \circ t_n(F\eta) = \tau_{n+1}(F\eta)$. But this just says that $(FY)_* = F_*Y_*$ is a strong deformation retract of $B_*(F, T, TY)$ in $S\mathbf{D}$. $\square$

Our interests lie in the application of these result to Example 5.1. This yields the following two theorems where α^* is suppressed in the notation for clarity.

Theorem 5.1. Let $\alpha : (T_1, \mu, \eta) \to (T_2, \nu, \zeta)$ be a morphism of monads in a category $\mathbf{C}$ then

1. for any T_1-algebra (X, ξ) we have that $B_*(T_2, T_1, X)$ is a T_2-algebra and

$$X_* \xleftarrow{\epsilon_*(\xi)} B_*(T_1, T_1, X) \xrightarrow{B_*(\alpha,1,1)} B_*(T_2, T_1, X)$$

are natural morphisms in $T[S\mathbf{C}]$. Moreover, $\epsilon_*(\xi)$ is a strong deformation retract in $S\mathbf{C}$ with right inverse $\tau_*(\eta)$ satisfying $B_*(\alpha, 1, 1) \circ \tau_*\eta = \tau_*(\zeta)$

2. for any T_2-algebra (X, ξ') the morphism $\epsilon_*(\xi') : B_*(T_2, T_1, X) \to X_*$ in $T_2[S\mathbf{C}]$ is natural and satisfies $\epsilon_*(\xi') \circ \tau_*(\zeta) = 1_{X_*}$ and $\epsilon_*(\xi') \circ B_*(\alpha, 1, 1) = \epsilon_*(\xi'\alpha)$

3. for any $Y \in \mathbf{C}$ the map

$$\epsilon_*(\nu \circ T_2\alpha) : B_*(T_2, T_1, T_1Y) \to (T_2)_*Y_*$$

is a natural strong deformation retract in $T_2[S\mathbf{C}]$ with right inverse $\tau_*(T_2\eta)$.

Proof.

1. The first statement is clear. The fact that $\epsilon_*(\xi)$ is a strong deformation retract is just Proposition 5.1. In the proof of that proposition we also saw that $\tau_*(\eta)$ is the right inverse of $\epsilon_*(\xi)$. Since α is a morphism of monads we have that $\alpha \circ \eta = \zeta$ and therefore it follows that $B_*(\alpha, 1, 1) \circ \tau_*\eta = \tau_*(\zeta)$.

2. It is clear that $\epsilon_*(\xi')$ is natural. Since $\xi' \circ \zeta = 1_X$ it follows that $\epsilon_*(\xi') \circ \tau_*(\zeta) = 1_{X_*}$. The fact that $\epsilon_*(\xi') \circ B_*(\alpha, 1, 1) = \epsilon_*(\xi'\alpha)$ easily follows from the definitions of ϵ_* and B_*.

3. This follows from Proposition 5.2.

$\square$

Theorem 5.2. Let

$$\phi : \mathrm{Hom}_{\mathbf{D}}(X, \Lambda Y) \to \mathrm{Hom}_{\mathbf{C}}(\Sigma X, Y)$$

be an adjunction and $\Lambda\Sigma$ the resulting monad. Suppose that $\alpha : T \to \Lambda\Sigma$ is a morphism of monads, then

1. for any T-algebra (X, ξ) we have

$$B_*(\Lambda\Sigma, T, X) = \Lambda_* B_*(\Sigma, T, X)$$

2. for any $Y \in \mathbf{C}$ we have that $(\Lambda Y, \Lambda\phi(1))$ is a $\Lambda\Sigma$-algebra and we have a natural map in $S\mathbf{C}$

$$\epsilon_*\phi(1) : B_*(\Sigma, T, \Lambda Y) \to Y_*$$

satisfying $\epsilon_*(\Lambda\phi(1)) = \Lambda_*\epsilon_*\phi(1)$

3. for any $Y \in \mathbf{D}$ there is a natural strong deformation retract

$$\epsilon_*\phi(\alpha) : B_*(\Sigma, T, TY) \to \Sigma_* Y$$

with right inverse $\tau_*(\Sigma\eta)$.

Proof. 1. Since $B_n(\Lambda\Sigma, T, X) = \Lambda\Sigma T^n X = \Lambda B_n(\Sigma, T, X)$ this is clear.

2. The first part is clear. The fact that $\epsilon_*(\Lambda\phi(1)) = \Lambda_*\epsilon_*\phi(1)$ is immediately clear from the definition of ϵ_* which can be found in the proof of Lemma 4.3.

3. This follows from Proposition 5.2.

□

6 Admissible Functors and The Recognition Principle

6.1 Admissible Functors

In this section will introduce the notion of an admissible functor. These functors behave nicely with respect to NDR pairs. It turns out that the functors we are interested in, the suspension and loop space functor and the monad associated to an operad, are all admissible. We have the following definition.

Definition 6.1. A functor $F : \mathbf{CG}_* \to \mathbf{CG}_*$ is called admissible if any representation (h, u) of (X, A) as an NDR pair induces a representation (Fh, Fu) of (FX, FA) as an NDR pair such that

1. $(Fh)_t = F(h_t)$ for all t
2. Suppose $g : X \to X$ satisfies $ug(x) < 1$ if $u(x) < 1$. Then $Fu : FX \to I$ satisfies $(Fu)(Fg)(y) < 1$ if $Fu(y) < 1$ where $y \in FX$.

We point out that in the induced representation (Fh, Fu) it is *not* the case that $Fu = F(u)$. In the following examples we already know that the first property of admissible functors hold. Therefore, we will only check the second property.

Example 6.1.
1. The reduced suspension functor Σ is admissible. Put

$$(\Sigma u)[x, t] = u(x)$$

Suppose $g : X \to X$ satisfies that $u(g(x)) < 1$ whenever $u(x) < 1$. Let $[x, t] \in \Sigma X$ be such that $\Sigma u[x, t] < 1$. We need to show that $\Sigma u \Sigma g[x, t] < 1$. Since $\Sigma u[x, t] = u(x)$ we have that $u(x) < 1$ hence $u(g(x)) < 1$. Therefore,

$$\Sigma u \Sigma g[x, t] = \Sigma u[g(x), t] = u(g(x)) < 1$$

2. The loop space functor Ω is admissible. Put, for $f \in \Omega X$,

$$(\Omega u)(f) = \max_{t \in I} u(f(t))$$

Suppose $g : X \to X$ satisfies that $u(g(x)) < 1$ whenever $u(x) < 1$. Let $f \in \Omega X$ be such that $\Omega u(f) < 1$. We need to show that $\Omega u \Omega g(f) < 1$. Since $\Omega u(f) < 1$ we have $\max_{t \in I} u(f(t)) < 1$ hence for every t we have $u(f(t)) < 1$ and therefore $u(gf(t)) < 1$ for all t. Consequently, $\max_{t \in I} u(gf(t)) < 1$ and therefore

$$\Omega u \Omega g(f) = \Omega u(gf) = \max_{t \in I} u(gf(t)) < 1$$

3. If $\mathcal{C}$ is any operad and C is the associated monad then C is admissible. Put, for $c \in \mathcal{C}(n)$ and $x_i \in X$,

$$(Cu)[c; x_1, \cdots, x_n] = \max_i u(x_i)$$

Suppose $g : X \to X$ satisfies that $u(g(x)) < 1$ whenever $u(x) < 1$. Let $[c; x_1, \cdots, x_n] \in CX$ be such that $Cu[c; x_1, \cdots, x_n] < 1$. We need to show that $CuCg([c; x_1, \cdots, x_n]) < 1$. Since $Cu[c; x_1, \cdots, x_n] < 1$ we have $\max_i u(x_i) < 1$. Therefore, $u(x_i) < 1$ for every i and thus $u(g(x_i)) < 1$ for every i. Consequently, $\max_i u(g(x_i)) < 1$ and thus

$$CuCg([c; x_1, \cdots, x_n]) = Cu[c; g(x_1), \cdots, g(x_n)] = \max_i u(g(x_i)) < 1$$

Definition 6.2.

1. Let $(X, *)$ be a pair in $\mathbf{CG}$. Put $X' = X \vee I$ where $0 \in I$ is the base point. The base point of X' is defined to be $1 \in I$. Then $(X', 1)$ is represented as an NDR pair in the followiong way. Define $u(x) = 1$ for $x \in X$ and for $t \in I$ we set

$$u(t) = \begin{cases} 1 & \text{if } t \leq \frac{1}{2} \\ 2 - 2t & \text{if } t \geq \frac{1}{2} \end{cases}$$

Furthermore, we define $h(s, x) = x$ for $x \in X$ and for $t \in I$ we set

$$h(s, t) = \begin{cases} t + st & \text{if } t \leq \frac{1}{2} \\ s + t - st & \text{if } t \geq \frac{1}{2} \end{cases}$$

We denote by $\iota : X \to X$ and $\rho = h_1 : X' \to X$ the obvious inclusion and retraction. Now, let $f : (X, *) \to (Y, *)$ be a based map and set $f' = f \vee 1 : X' \to Y$. It follows that $(M_{f'}, X')$ is a strong NDR pair and that $(M_{f'}, Y')$ is a DR pair.

2. Suppose G is a topological monoid with idenity e then G' is a topological monoid with idenity 1 and multiplication defined by $gt = g = tg$ for all $g \in G$ and $t \in I$. On I (or G) the multiplication of G' is simply the multiplication on I (or G). It follows that $\rho : G' \to G$ is a morphism of monoids.

3. Let $\mathcal{C}$ be any operad and denote its identity by $e \in \mathcal{C}(1)$. We define a new operad $\mathcal{C}'$ and a morphism of operads $\rho : \mathcal{C}' \to \mathcal{C}$ as follows. For $n > 1$ we let $\mathcal{C}'(n) = \mathcal{C}(n)$ as an S_n-space and we let ρ_n be the identity. For $n = 1$ we define $\mathcal{C}'(1) = \mathcal{C}(1)'$ as a monoid under the map γ' defined by

$$\begin{tikzcd} \mathcal{C}'(k) \times \mathcal{C}'(i_1) \times \cdots \times \mathcal{C}'(i_k) \arrow[r, "\gamma'"] \arrow[d, "\rho_k \times \rho_{i_1} \times \cdots \times \rho_{i_k}"'] & \mathcal{C}'(n) \\ \mathcal{C}(k) \times \mathcal{C}(i_1) \times \cdots \times \mathcal{C}(i_k) \arrow[r, "\gamma"'] & \mathcal{C}(n) \arrow[u, "\text{incl.}"'] \end{tikzcd}$$

where ρ_1 is defined as in the second part, $n = i_1 + \cdots i_k \neq 1$ or $k \neq 1$. Lastly, we sert $\mathcal{C}'(0) = * = \mathcal{C}(0)$.

In the above situation we say that X' is obtained from X by adding a whisker and X' is the whiskered space associated to X or the whiskered space of X. The terminology for monoids and operads is similar.

Lemma 6.1. *Let $\mathcal{C}$ be an operad and let $\mathcal{C}'$ be its whiskered operad. Let C and C' denote the respective associated monads. Let $X \in \mathbf{CG}_*$ and set $\eta = \eta_X : X \to CX$ then there exists a natural homeomorphism $\chi : M_\eta \to C'X$,* where M_η denotes the mapping cylinder of η. such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\eta'"'] & M_\eta \arrow[d, "r"] \arrow[dl, "\chi"'] \\ C'X \arrow[r, "\rho"'] & CX \end{tikzcd}$$

where i and r are the inclusion and retraction belonging to the mapping cylinder.

Proof. Since $CX \subset M_\eta$ we can define $\chi : CX \to C'X$ to be the inclusion. For $(x,t) \in X \times I$ we set $\chi(x,t) = [t,x]$ where the t on the right hand side is considered to be in $I \subset \mathcal{C}(1)'$. Note that χ is well-defined since

$$(x,0) = \eta(x) = [e,x] \in M_\eta$$

and

$$[0,x] = [e,x] \in C'X$$

It is easy to verify that χ is in fact a homeomorphism and that the diagram commutes. □

The main purpose of this section is the proof the following two results.

Theorem 6.1. *Let $\mathcal{C}$ be an operad, $\mathcal{C}'$ its whiskered operad and C' the monad corresponding to $\mathcal{C}$. Let X be a C'-algebra and F a C'-functor. If $(X,*)$ is a strong NDR pair and if F is an admissible functor then $B_*(F,C',X)$ is a strictly proper simplicial space.*

Proof. Let (h,u) be a representation of $(X,*)$ as a strong NDR pair then it induces a representation (Ch,Cu) of $(CX,*)$ as a strong NDR pair. Since $Ch_t \circ \eta = \eta \circ h_t$ and $Cu \circ \eta = u$ it follows that (M_η, X) is a strong NDR pair. Since (by the previous lemma) (M_η, X) and $(C'X, \eta' X)$ are homeomorphic it follows that (h,u) induces a representation of $(C'X, \eta' X)$ as a strong NDR pair. Define

$$Y = B_{n+1}(F,C',X) = F(C')^{n+1}X$$

and let $A_i = \mathrm{im}(s_i)$ where

$$s_i = F(C')^i \eta'$$

where $\eta' : (C')^{n-i}X \to (C')^{n-i+1}X$. Now, (h,u) determines a representation $((C')^{n-i}h, (C')^{n-i}u)$ of $((C')^{n-i}X,*)$ as a strong NDR pair and in the beginning of the proof we showed that we get a representation, call it (k_i, w_i), of $((C')^{n-i+1}X, \eta'(C')^{n-i}X)$ as a strong NDR pair. Since F and C' are admissible functors $F(C')^i$ is also admissible. Therefore, $(h_i,u_i) = (F(C')^i k_i, F(C')^i w_i)$ is a representation of (Y, A_i) as a strong NDR pair. Since $F(C')^i\eta$ is a natural transformation we obtain, for $i < j$ and $t \in I$, a commutative diagram

$$\begin{tikzcd} F(C')^{n+1}X \arrow[r, "F(C')^j k_{jt}"] & F(C')^{n+1}X \\ F(C')^n \arrow[u, "F(C')^i\eta'"] \arrow[r, "F(C')^{j-1}k_{jt}"'] & F(C')^n X \arrow[u, "F(C')^i\eta'"'] \end{tikzcd}$$

Consequently, for $i < j$ we have $h_j(I \times A_i) \subset A_i$. To show that $B_*(F, C', X)$ is strictly proper it is sufficient to show that it is proper by Definition 4.14. For this it is sufficient (by Lemma 3.9) to show that the following statement holds: for $i < j$,$t \in I$ we have that $u_j y < 1$ if $u_j h_i(t, y) < 1$. Using the definition of an admissible functor we see that this holds if the following is true: we have $((C')^{j-i} w_j) k_i(t, x) < 1$ if $(C')^{j-i} w_j(x) < 1$ for $i < j, tinI$ and $x \in (C')^{n-i+1} X$. Now, the corresponding statement for (h, u) holds since it represents $(X, *)$ as a strong NDR pair. Consequently, the fact $((C')^{j-i} w_j) k_i(t, x) < 1$ if $(C')^{j-i} w_j(x) < 1$ is an easy consequence from the way k_i and $(C')^{j-i} w_j$ are constructed from h and u. $\square$

The last result show that the assumption that $(X, *)$ is a strong NDR pair is no serious restriction.

Proposition 6.1. Let (X, θ) be a $\mathcal{C}$-space. Then there is an action θ' of $\mathcal{C}$ on X' such that $\rho : X' \to X$ is a morphism of $\mathcal{C}$-spaces.

Proof. Let $\iota : X \to X'$ be the inclusion and define $\theta'_k : \mathcal{C}(k) \times (X')^k \to X'$ by

$$\theta'_k(c; x_1, \cdots, x_k) = \begin{cases} \iota\theta_k(c, \rho x_1, \cdots, \rho x_k) & \text{if some } x_i \notin I \setminus 0 \\ x_1 \cdots x_k & \text{if all } x_i \in I \end{cases}$$

Note that $x_1 \cdots x_k$ is just multiplication in I. Both definition agree on their common domain since they both yield $0 = *$. It is straightforward to verify that θ' is indeed an action and that ρ becomes a morphism of $\mathcal{C}$-spaces. $\square$

6.2 The Recognition Principle

In this section we will prove the Recognition Principle for connected $\mathcal{C}_n$-spaces. Furthermore, we will see that the n-fold delooping of a $\mathcal{C}_n$-space is unique up to weak equivalence. Lastly, we will prove corresponding results for connected A_∞-spaces.

Before we get to the Recognition Principle we set up some notation. If T is a monad in $\mathbf{CG}_*$, X a T-algebra and F a T-functor then we can apply the double bar construction to obtain a simplcial space $B_*(F, T, X)$. We will be interested in the realization of such simplicial spaces and we will write

$$B(F, T, X) = |B_*(F, T, X)|$$

Similarly we will write

$$B(\pi, \psi, f) = |B_*(\pi, \psi, f)|$$

where $(\pi, \psi, f) : (F, T, X) \to (F', T', X')$. In the same vein we will write $\tau(\zeta) = |\tau_*(\zeta)| : Y \to B(F, T, X)$ for any map $\zeta : Y \to FX$ where τ is defined as in Lemma 4.2 and we will write $\epsilon(\pi) = |\epsilon_*(\pi)| : B(F, T, X) \to Y$ for any map $\pi : FX \to Y$ where ϵ is defined as in Lemma 4.3.

Theorem 6.2. Let $\pi : \mathcal{D} \to \mathcal{C}_n$ be a local equivalence of S-free operads and denote by $\pi : D \to C_n$ the associated morphism of monads. Let (X, ξ) be a D-algebra and consider

$$X \xleftarrow{\epsilon(\xi)} B(D, D, X) \xrightarrow{B(\alpha_n\pi, 1, 1)} B(\Omega^n\Sigma^n, D, X) \xrightarrow{\gamma^n} \Omega^n B(\Sigma, D, X)$$

Then the following statements hold

1. $\epsilon(\xi)$ is a strong deformation retract with right inverse $\tau(\zeta)$ where $\zeta : X \to DX$ is the unit of D

2. if X is path connected then $B(\alpha_n\pi, 1, 1)$ is a weak homotopy equivalence

3. γ^n is a weak homotopy equivalence for any X

4. The composition $\gamma^n \circ B(\alpha_n\pi, 1, 1) \circ \tau(\zeta) : X \to \Omega^n B(\Sigma^n, D, X)$ coincides with the adjoint of $\tau(1) : \Sigma^n X \to B(\Sigma^n, D, X)$

5. if X is m-connected then $B(\Sigma^n, D, X)$ is $(m+n)$-connected

6. for any $Y \in \mathbf{CG}_*$ the following diagram commutes

$$\begin{tikzcd}
B(D, D, \Omega^n Y) \arrow[r, "B(\alpha_n\pi,1,1)"] \arrow[d, "\epsilon(\theta_n\pi)"'] & B(\Omega^n\Sigma^n, D, \Omega^n Y) \arrow[d, "\gamma^n"] \arrow[dl, "\epsilon(\xi_n)"] \\
\Omega^n Y & \Omega^n B(\Sigma^n, D, \Omega^n Y) \arrow[l, "\Omega^n\epsilon\phi^n(1)"]
\end{tikzcd}$$

   Moreover, in the above diagram $\Omega\epsilon\phi^n(1)$ is a retraction with right inverse $\phi^{-n}\tau(1)$. Furthermore, if Y is n-connected then $\epsilon\phi^n(1) : B(\Sigma^n, D, \Omega^n Y) \to Y$ is a weak homotopy equivalence.

7. $\epsilon\phi^n(\alpha_n\pi) : B(\Sigma^n, D, DY) \to \Sigma^n Y$ is a strong deformation retract with right inverse $\tau(\Sigma^n\zeta)$ for any $Y \in \mathbf{CG}_*$.

Proof.

1. By Theorem 5.1 this holds on the level of the simplicial space and therefore (by Corollary 4.4) it holds on the level of the space.

2. Assume that X is connected then by the Approximation Theorem together with Theorem 2.1 it follows that each $\alpha_n\pi : D^{k+1}X \to \Omega^n\Sigma^n D^k X$ is a weak homotopy equivalence. Therefore, the result follows by Theorem 4.5.

3. For $i < n$ we have that $\Omega^i\Sigma^n D^k X$ is connected and thus we can apply Theorem 4.7 to $\Omega^i\Sigma^n D^k X$ which gives the result.

4. This is immediate from the definitions of the involved maps.

5. This follows from Theorem 4.4.

6. Since $\xi_n\alpha_n = \theta_n$ and ϵ is natural it follows that the upper triangle commutes. Furthermore, we have $\xi_n = \Omega^n\phi^n(1)$ and thus by Theorem 5.2 we have $\epsilon_*\Omega^n\phi^n(1) = \Omega^n_*\epsilon_*\phi^n(1)$. Consequently, the lower triangle commutes since γ^n is natural and reduces to the idenity on $\Omega^n Y = |\Omega^n_* Y_*|$. The commutativity of the diagram implies that $\epsilon\phi^n(1)$ is a weak homotopy equivalence whenever Y is n-connected.

7. First apply Theorem 5.2 to see that the result holds on the level of simplicial spaces. Therefore, the result follows from Corollary 4.4.

□

The space $B(\Sigma^n, D, X)$ is sometimes called the n-fold delooping of X. The point of the following corollary is that the n-fold delooping is unique up the weak equivalence.

Corollary 6.1. Let $\pi : \mathcal{D} \to \mathcal{C}_n$ be a local equivalence of S-free operads and let $\pi : D \to C_n$ denote the associated morphism of monads. Let (X, ξ) be a D-algebra. If

$$(X, \xi) \xleftarrow{f} (X', \xi') \xrightarrow{g} (\Omega^n Y, \theta_n \pi)$$

is a weak homotopy equivalence of connected D-algebras where Y is n-connected then

$$B(\Sigma^n, D, X) \xleftarrow{B(1,1,f)} B(\Sigma^n, D, X') \xrightarrow{\epsilon\phi^n(g)} Y$$

is a weak homotopy equivalence between Y and $B(\Sigma^n, D, X)$.

Proof. Since ϵ is natural it follows that

$$\epsilon\phi^n(g) = \epsilon\phi^n(1) \circ B(1, 1, g)$$

The theorem implies that $\epsilon\phi^n(1)$ is a weak homotopy equivalence since Y is assumed to be n-connected. From the Approximation Theorem it follows that for every k the maps $\Sigma^n D^k f$ and $\Sigma^n F^k g$ are weak homotopy equivalences since the functor $\Sigma^n(\Omega^n\Sigma^n)^k$ preserves weak homotopy equivalences. Therefore, $B(1, 1, f)$ and $B(1, 1, g)$ are, by Corollary 4.4, weak homotopy equivalences. $\square$

Before we move on to non-connected spaces we will investigate A_∞-spaces. Note this expands upon the case $n = 1$ in the Recognition Principle since the notion of an A_∞-algebra is broader than the notion of a C_1-space. The proof is largely the same as that of the Recognition Principle (Theorem 6.2).

Theorem 6.3. Let $\mathcal{C}$ be an A_∞-operad and let $\delta : \mathcal{C} \to \mathcal{M}$ be its augmentation. We write $\delta : C \to M$ for the induced morphism of monads. Let (X, θ) be a C-algebra and consider the following diagram of morphisms of C-algebras

$$X \xleftarrow{\epsilon(\theta)} B(C, C, X) \xrightarrow{B(\delta,1,1)} B(M, C, X)$$

then

1. $\epsilon(\theta)$ is a strong deformation retract with right inverse $\tau(\eta)$ where η is the unit of C

2. if X is connected then $B(\delta, 1, 1)$ is a weak homotopy equivalence

3. $B(M, C, X)$ has the natural structure of a topological monoid

4. if (G, ϕ) is an M-algebra (i.e. a topological monoid) then $\epsilon(\phi) : B(M, C, G) \to G$ is a morphism of monoids and the following diagram commutes

$$\begin{tikzcd} B(C,C,G) \arrow[r, "B(\delta,1,1)"] \arrow[d, "\epsilon(\phi)"'] & B(M,C,G) \arrow[d, "B(1,\delta,1)"] \arrow[dl, "\epsilon(\phi)"'] \\ G & B(M,M,G) \arrow[l, "\epsilon(\phi)"] \end{tikzcd}$$

Therefore, the diagonal $\epsilon(\phi)$ is a weak homotopy equivalence if X is connected.

5. Let $Y \in \mathbf{CG}_*$ and let $\nu : M^2 \to M$ be the multiplication of the monad. Then
$$\epsilon(\nu \circ M\delta) : B(M, C, CY) \to MY$$
is a strong deformation retract of topological monoids (in the sense that the deformation consists of morphisms of monoids h_t) with right inverse $\tau(M\eta)$.

Proof. 1. The proof is similar to that of Theorem 6.2.1.

2. Since Y is connected it follows (by Theorem 2.1) that $\delta : CX \to MX$ is a weak homotopy equivalence. The rest of the proof is the same as Theorem 6.2.2.

3. For any Y, MY is a topological monoid. Therefore, MC^kX is a topological monoid and thus the result follows from Corollary 4.2.

4. The proof is similar to Theorem 6.2.6

5. Both statements hold on the level of simplicial spaces by Theorem 5.1. We know that geometric realization preserves homotopies (Corollary 4.4). Furthermore, the realization of a (map of) simplicial monoid(s) is a (map of) monoid(s) by Corollary 4.2. Therefore, the result follows.

□

Again, the delooping of X is unique up to weak equivalence

Corollary 6.2. Let $\mathcal{C}$ be an A_∞-operad and let $\delta : \mathcal{C} \to \mathcal{M}$ be its augmentation. Write $\delta : C \to M$ for the induced morphism of monads. If
$$(X, \theta) \xleftarrow{f} (X', \theta') \xrightarrow{g} (G, \phi\delta)$$
is a weak homotopy equivalence of connected C-algebras where (G, ϕ) is an M-algebra then
$$B(M, C, X) \xleftarrow{B(1,1,f)} B(M, C, X') \xrightarrow{B(1,1,g)} B(M, C, G) \xrightarrow{\epsilon(\phi)} G$$
is a weak homotopy equivalence of topological monoids.

7 The Recognition Principle for Grouplike Spaces

7.1 Homological Group Completions

We proved the Recognition Principle for connected $\mathcal{C}_n$-spaces in the previous section. To extend this result to the grouplike case we will need the notion of a homological group completion which will be introduced in this section. The main ideas behind this section and the next come from May who used these techniques to prove the Recognition Principle for grouplike E_∞-algebras (see [8]). We will follow Stelzer's outline (found in [13]) for adapting these techniques to grouplike C_n-algebras. First, recall the Group Completion Theorem.

Theorem 7.1. Let M be a topological monoid and assume that $\pi_0(M)$ is in the centre of $H_*(M)$ then the obvious map $f : M \to \Omega BM$, where BM denotes the classifying space, induces an isomorphism

$$f_* : H_*(M)[\pi_0^{-1}] \to H_*(\Omega BM)$$

where $H_*(M)[\pi_0^{-1}]$ denotes the localization of $H_*(M)$ at $\pi_0 = \pi_0(M)$.

Proof. See [10]. $\square$

The map $f : M \to \Omega BM$ is called the group completion of M. The main idea of the proof of the Recognition Principle for grouplike spaces is to exploit this theorem. Of course, we cannot do this directly since a C_n-algebra is, in general, not a topological monoid. For this case we will consider so-called homological group completions. First, we recall the notion of a grouplike space.

Definition 7.1. An H-space (X, μ) is called grouplike if $(\pi_0(X), \pi_0(\mu), [*])$ is a group where $[*]$ denotes the path component of basepoint $*$.

Definition 7.2. An H-space (X, μ) is called admissible if

1. μ is homotopy associative
2. for every x the maps $\mu(x, -)$ and $\mu(-, x)$ are homotopic relative basepoint

Definition 7.3. An H-map $g : X \to Y$ between admissible H-spaces is called a homological group completion if

1. Y is grouplike
2. for any commutative coefficient ring k the unique morphism of k-algebras

$$\overline{g}_* : H_*(X; k)[\pi^{-1}(X)] \to H_*(Y; k)$$

is an isomorphism. Here $H_*(X; k)[\pi^{-1}(X)]$ denotes the localization of $H_*(X; k)$ at $\pi_0(X)$.

Theorem 7.2. Let $\alpha_n : C_n \to \Omega^n\Sigma^n$ be the morphism of monads from Theorem 3.3 then the induced map

$$\alpha_n : C_n X \to \Omega^n \Sigma^n X$$

is a homological group completion if $n > 1$.

The proof of this theorem would take us too far afield and has thus been omitted. The reader is referred to [1] and [11] for a proof of this theorem.

Definition 7.4. Let $\mathcal{C}$ and $\mathcal{D}$ be operads. Let $\gamma_{\mathcal{C}}$ (resp. $\gamma_{\mathcal{D}}$) denote the composition map of $\mathcal{C}$ (resp. $\mathcal{D}$) and let c (resp. d) be the unique element in $\mathcal{C}(0)$ (resp. $\mathcal{D}(0)$). Then the Boardman-Vogt tensor product $\mathcal{C} \otimes \mathcal{D}$ is defined as follows. We put $(\mathcal{C} \otimes \mathcal{D})(0) = \{(c, d)\}$. For each $f \in \mathcal{C}(n)$ there is a generator $f \otimes d$ and for each $g \in \mathcal{D}(n)$ there is a generator $c \otimes g$. The generators are subject to the following relations

1. $\gamma_{\mathcal{C}\otimes\mathcal{D}}(f \otimes d; f_1 \otimes d, \cdots, f_n \otimes d) = \gamma_{\mathcal{C}}(f; f_1, \cdots, f_n) \otimes d$
2. $(f \otimes d)\sigma = f\sigma \otimes d$
3. $\gamma_{\mathcal{C}\otimes\mathcal{D}}(c \otimes g; c \otimes g_1, \cdots, c \otimes g_n) = c \otimes \gamma_{\mathcal{D}}(g; g_1, \cdots, g_n)$
4. $(c \otimes g)\sigma = c \otimes g\sigma$
5. $\gamma_{\mathcal{C}\otimes\mathcal{D}}(f \otimes d; c \otimes g, \cdots, f \otimes g) = \gamma_{\mathcal{C}\otimes\mathcal{D}}(c \otimes g; f \otimes d, \cdots, f \otimes d)$.

The topology on $(\mathcal{C} \otimes \mathcal{D})(n)$ is defined as follows. Let $\sim$ denote the equivalence relation generated by (1) - (5). The topology on $(\mathcal{C} \otimes \mathcal{D})(n)$ is then the quotient topology on

$$\coprod_{i+j=n} \mathcal{C}(i) \times \mathcal{D}(j)/\sim$$

Theorem 7.3. There exist

1. a functor $G : C_n[\mathbf{CG}_*] \to \mathbf{CG}_*$ such that GX is a topological monoid
2. a natural transformation $g : U \to G$, where $U : C_n[\mathbf{CG}_*] \to \mathbf{CG}_*$ is the forgetfull functor, such that $g : X \to GX$ is a homological group completion.

Proof. First, assume that $n > 1$. Let $\mathcal{M}$ be the operad whose algebras are topological monoids which was defined in Definition 2.1. Let $f : \mathcal{C}_n \to \mathcal{M}\otimes\mathcal{C}_{n-1}$ be the obvious map. Define G by

$$GX = \Omega_M BB(M \otimes C_{n-1}, C_n, X)$$

where Ω_M denotes the Moore loop space functor, B the classifying space and $A \otimes C_{n-1}$ the monad corresponding to the operad $\mathcal{M} \otimes \mathcal{C}_{n-1}$.
Define g as the composition

$$X \xrightarrow{\tau} B(C_n, C_n, X) \xrightarrow{B(f,1,1)} B(M \otimes C_{n-1}, C_n, X) \xrightarrow{\overline{k}} \Omega_M BB(M \otimes C_{n-1}, C_n, X)$$

where $\overline{k}$ is the natural inclusion. It is readily verified that g is indeed a natural transformation. It follows from Proposition 5.1 and Corollary 4.4 that τ is a homotopy equivalence and a map of $\mathcal{C}_n$ spaces. Therefore, it is an H-map. Moreover, since $B(M \otimes C_{n-1}, C_n, X)$ is an $\mathcal{M} \otimes \mathcal{C}_{n-1}$-space it is a monoid (in the category of $\mathcal{C}_{n-1}$ spaces.
We still need to prove the second assertion. First, we show that $B(f, 1, 1)$ is actually a homotopy equivalence. Now, $B(f, 1, 1)$ is the geometric realization of

$$B_*(C_n, C_n, X) \xrightarrow{B_*(f,1,1)} B_*(M \otimes C_{n-1}, C_n, X)$$

which is a simplicial map between proper simplicial spaces (Lemma 6.1) since $M \otimes C_{n-1}$ is admissible (Example 6.1). Now, every $B(f,1,1)_n$ is a homotopy equivalence and therefore, by Corollary 4.4 it follows that $B(f,1,1)$ itself is a homotopy equivalence. By the group completion theorem (Theorem 7.1) the map $\overline{k}$ is a group completion. Consequently, the composition is a homological group completion since we can always replace Ω by Ω_M.

Now, suppose that $n = 1$. Consider the obvious operad map $f : \mathcal{C}_1 \to \mathcal{M}$. It is easy to see that this map is a local equivalence. In this case we set

$$GX = \Omega_M BB(M, C_1, X)$$

and we define g as the composition

$$X \xrightarrow{\tau} B(C_1, C_1, X) \xrightarrow{B(f,1,1)} B(M, C_1, X) \xrightarrow{\overline{k}} \Omega_M BB(M, C_1, X)$$

The rest of the proof is the same as the case $n > 1$ since the fact that f is a local equivalence clearly implies that each $B_n(f,1,1)$ is a homotopy equivalence. Therefore, g is again a homological group completion. $\square$

Theorem 7.4. Let X be a C_n-algebra then

$$X \xrightarrow{\tau} B(C_n, C_n, X) \xrightarrow{B(\alpha_n,1,1)} B(\Omega^n\Sigma^n, C_n, X)$$

is a homological group completion for $n > 1$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[d, "g"'] & B(C_n, C_n, X) \arrow[l, "\epsilon\xi"'] \arrow[r, "B(\alpha_n,1,1)"] \arrow[d, "B(gC_n,1,1)"'] & B(\Omega^n\Sigma^n, C_n, X) \arrow[d, "B(g\Omega^n\Sigma^n,1,1)"] \\
GX & B(GC_n, C_, X) \arrow[l, "|G_*\epsilon_*(\xi)|"] \arrow[r, "B(G\alpha_n,1,1)"'] & B(G\Omega^n\Sigma^n, C_n, X)
\end{tikzcd}$$

Note that $|G_*\epsilon_*(\xi)|$ can be inserted in the diagram since $B_*(GC_n, C_n, X) = G_*B_*(C_n, C_n, X)$. In fact, this map is a homotopy equivalence since we can apply G_* to the simplicial homotopy equivalence $\epsilon_*(\xi) : B_*(C_n, C_n, X) \to X$ and apply Corollary 4.4. Now, since g is a homological group completion and $\epsilon(\xi)$ and $|G_*\epsilon_*(\xi)|$ are homotopy equivalences it follows that $B(gC_n, 1, 1)$ is a homological group completion.

Now, $\alpha_n : C_nY \to \Omega^n\Sigma^nY$ is a homological group completion . Therefore, for every Y

$$G\alpha_n : GC_nY \to G\Omega^n\Sigma^nY$$

induces an isomorphism in homology since GC_nY is a monoid. Since g is a homological group completion and $\Omega^n\Sigma^nY$ can be replaced by the monoid $\Omega^n_M\Sigma^nY$ it follows that

$$g\Omega^n\Sigma^n : \Omega^n\Sigma^nY \to G\Omega^n\Sigma^nY$$

induces an isomorphism in homology. Therefore, passing to homology the right hand square gives for any commutative coefficient ring k

$$\begin{tikzcd}
H_*(B(C_n, C_n, X); k)[\pi_0^{-1}] \arrow[r] \arrow[d, "\cong"'] & H_*(B(\Omega^n\Sigma^n, C_n, X); k) \arrow[d, "\cong"] \\
H_*(B(GC_n, C_n, X); k) \arrow[r, "\cong"'] & H_*(B(G\Omega^n\Sigma^n, C_n, X); k)
\end{tikzcd}$$

where $\pi_0 = \pi_0(B(C_n, C_n, X))$ and the top arrow is induced from $B(\alpha_n, 1, 1)$. Therefore, $B(\alpha_n, 1, 1)$ is a homological group completion. □

7.2 The Recognition Principle Revisited

In this last section we will extend the Recoginition Princple to grouplike H-spaces. In the previous section we have seen that

$$B(\alpha_n, 1, 1) : B(C_n, C_n, X) \to B(\Omega^n\Sigma^n, C_n, X)$$

is a homological group completion for $n > 1$. In this section we will see that this implies that $B(\alpha_n, 1, 1)$ is in fact a weak homotopy equivalence.

Definition 7.5. A topological space X is said to be simple if X is path connected, the fundamental group of X is abelian and acts trivially on all the higher homotopy groups.

Lemma 7.1. Let X be a H-space then $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all n.

Proof. Let

$$\tau : \pi_1(X) \times \pi_n(X) \to \pi_n(X)$$

denote the action of the fundamental group. Let $\gamma \in \pi_1(X)$ and $\alpha \in \pi_n(X)$. Then γ can be represented by a map $u : I \to X$ and α by a map $f : S^n \to X$. Define a homotopy

$$\begin{aligned} H &: I \times S^n \to X \\ (t, y) &\mapsto \mu(u(t), f(y)) \end{aligned}$$

where μ is the product on X. Then H is a free homotopy between f and itself along u. Therefore, $\tau(\gamma, \alpha) = \alpha$ and we are done. □

The following lemma is a well-known result and is sometimes called the dual Whitehead Theorem. For a proof see [9].

Lemma 7.2. Suppose that $f : X \to Y$ is a map between simple spaces that induces isomorphisms

$$f_* : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$$

for every n. Then f is a weak homotopy equivalence.

Corollary 7.1. Suppose (X, θ) is a grouplike C_n-algebra then

$$B(\alpha_n, 1, 1) : B(C_n, C_n, X) \to B(\Omega^n\Sigma^n, C_n, X)$$

is a weak homotopy equivalence for $n > 1$.

Proof. Since $B(\alpha_n, 1, 1)$ is a homological group completion the induced map

$$B(\alpha_n, 1, 1)_* : H_n(B(C_n, C_n, X); \mathbb{Z})[\pi_0^{-1}] \to H_n(B(\Omega^n\Sigma^n, C_n, X); \mathbb{Z})$$

is an isomorphism for every n where π_0 denotes $\pi_0 B(C_n, C_n, X)$. Since X is grouplike $B(C_n, C_n, X)$ is grouplike and therefore

$$H_n(B(C_n, C_n, X); \mathbb{Z})[\pi_0^{-1}] = H_n(B(C_n, C_n, X); \mathbb{Z})$$

since elements of π_0 are already invertible. Therefore, we have isomorphisms

$$B(\alpha_n, 1, 1)_* : H_n(B(C_n, C_n, X); \mathbb{Z}) \to H_n(B(\Omega^n\Sigma^n, C_n, X); \mathbb{Z})$$

Now, for any $c \in C_n(2)$ we have that $\theta_2(c)$ defines a multiplication on X making $(X, \theta_2(C)$ into an H-space. From this it follows that both $B(C_n, C_n, X)$ and $B(\Omega^n\Sigma^n, C_n, X)$ are H-spaces under the induced multiplication. Consequently, by Lemma 7.1 the action of the fundamental group of $B(C_n, C_n, X)$ (resp. $B(\Omega^n\Sigma^n, C_n, X)$) on the higher homotopy groups of $B(C_n, C_n, X)$ (resp. $B(\Omega^n\Sigma^n, C_n, X)$) is trivial. This implies that for any $k > 0$ we have that $\pi_k(B(C_n, C_n, X), z_0)$ and $\pi_k(B(C_n, C_n, X), z_1)$ are canonically isomorphic for all points $z_0, z_1 \in B(C_n, C_n, X)$. Of course, the same conclusion holds for $B(\Omega^n\Sigma^n, C_n, X)$. Let $[B(C_n, C_n, X)]$ and $[B(\Omega^n\Sigma^n, C_n, X)]$ denote the path components of the base point. Then it follows that the map $B(\alpha_n, 1, 1)$ is a weak homotopy equivalence if the restriction

$$B(\alpha_n, 1, 1) : [B(C_n, C_n, X)] \to [B(\Omega^n\Sigma^n, C_n, X)]$$

is a weak homotopy equivalence. The restriction still induces isomorphisms on all homology groups and $[B(C_n, C_n, X)]$ and $[B(\Omega^n\Sigma^n, C_n, X)]$ are clearly simple spaces. Therefore, Lemma 7.2 implies that the restriction

$$B(\alpha_n, 1, 1) : [B(C_n, C_n, X)] \to [B(\Omega^n\Sigma^n, C_n, X)]$$

is indeed a weak homotopy equivalence and we are done. □

Finally, we have the Recognition Principle for grouplike $\mathcal{C}_n$-spaces.

Corollary 7.2. Let X be a grouplike $\mathcal{C}_n$-space, then X is of the homotopy type of an n-fold loop space.

Proof. First let $n > 1$ and consider

$$X \xrightarrow{\tau} B(C_n, C_n, X) \xrightarrow{B(\alpha_n,1,1)} B(\Omega^n\Sigma^n, C_n, X)$$

We have seen (Proposition 5.1 and Corollary 4.4) that τ is a strong deformation retract. By the previous theorem $B(\alpha_n, 1, 1)$ is a weak homotopy equivalence and therefore the composition is a weak homotopy equivalence. Moreover, by the third part of Theorem 6.2 we know that

$$\gamma^n : B(\Omega^n\Sigma^n, C_n, X) \to \Omega^n B(\Sigma^n, C_n, X)$$

is a weak homotopy equivalence for any X. Consequently, X and $\Omega^n B(\Sigma^n, C_n, X)$ are of the same homotopy type.
Now, suppose that $n = 1$. Here the case is slightly different since α_1 need not be a homological group completion. However, by Theorem 7.3 we have that

$$X \xrightarrow{\tau} B(C_1, C_1, X) \xrightarrow{B(f,1,1)} B(A, C_1, X) \xrightarrow{\overline{k}} \Omega_M BB(A, C_1, X)$$

is a homological group completion. Therefore, X is weakly homotopy equivalent to $\Omega_M BB(A, C_1, X)$ which is homotopy equivalent to $\Omega BB(A, C_1, X)$. So, in this case X is indeed a loop space. □

8 Concluding Remarks

In this last section we will compare our approach to proving the Recognition Principle with the approach taken by Lurie in [6]. Lurie's proof is part of larger discussion about ∞-categories so let us first consider the definition.

Definition 8.1. An ∞-category is a simplicial set $\mathcal{X}$ satisfying the following condition

- for $0 < i < n$ every map of simplicial sets $f_0 : \Lambda_i^n \to \mathcal{X}$ can be extended to a map $f : \Delta^n \to \mathcal{X}$.

where Λ_i^n is the (n, i)-horn of Δ^n.

We recall that to every category there corresponds a simplicial set that is obtained by taking the nerve of the category. It is well-known fact that a simplicial set corresponds to small category if every f_0 can be *uniquely* extended to a map f. When we consider ∞-categories there are some complications when we try to construct corresponding categories. Let $\mathcal{X}$ be an ∞-category. We define the objects of $\mathcal{X}$ to be the 0-simplices $\mathcal{X}_0$ and we define the morphisms of $\mathcal{X}$ to be the 1-simplices $\mathcal{X}_1$. The source map s and target map t are defined by the face maps of $\mathcal{X}$ by putting

$$s = d_1 : \mathcal{X}_1 \to \mathcal{X}_0$$
$$t = d_0 : \mathcal{X}_1 \to \mathcal{X}_0$$

and the identity is defined by

$$1 = s_0 : \mathcal{X}_0 \to \mathcal{X}_1$$

For a morphism $f \in \mathcal{X}$ we write $f : x \to y$ if $s(f) = x$ and $t(f) = y$. We run into problems when we try to define compositions. Suppose $f, g \in \mathcal{X}_1$ are morphisms, say $f : x \to y$ and $g : y \to z$. This defines a map $\Lambda_1^2 \to \mathcal{X}$. Since $\mathcal{X}$ is an ∞-category this map can be extended to a map $\sigma : \Delta^2 \to \mathcal{X}$. The situation is summed up nicely by the diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "d_1\sigma"'] & & z \end{tikzcd}$$

An obvious choice for $g \circ f$ would be $d_1\sigma$. However, σ is in general not unique. This can be remedied by only considering compositions up to homotopy.

Definition 8.2. Let $\mathcal{X}$ be an ∞-category and let $f, g : x \to y$ be two morphisms. Then f and g are called homotopic if there exists a 2-simplex $\sigma : \Delta^2 \to \mathcal{X}$ with boundary $\partial\sigma = (g, f, 1_x)$ as in

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "1_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

The notion of homotopy allows to do two things. Firstly, one can obtain an actual category by taking the quotient under the homotopy relation. We are not really interested in this approach but the interested reader is referred to [4] and [5]. For us the important observation is that an ∞-category is something in which one can do homotopy theory. In particular, one can define (see section 6.5.1 of [5]) the homotopy groups $\pi_n(x)$ of an object x of an ∞-category $\mathcal{X}$. It should come as no suprise that an object x is called grouplike in case $\pi_0(x)$ is a group. In analogy to a space being k-connected an object x of $\mathcal{X}$ is called k-connective if $\pi_i(x)$ is trivial for $i < k$.
This observation allows us to explain Lurie's approach: provide a Recognition Principle in the context of ∞-categories. For technical reasons Lurie restricts himself to a special class of ∞-categories called ∞-topoi. The idea behind ∞-topoi is that these are ∞-categories that behave like the category of topological spaces in some sense and indeed the category of spaces is the prototypical example of an ∞-topos. Sadly, we cannot go into the details of ∞-topoi so the reader is referred to Chapter 6 of [5] for a treatment of them.
In ordinary homotopy theory one is usually interested in pointed spaces and we would like to have a similar notion for ∞-topoi. Suppose $\mathcal{X}$ is an ∞-topos and suppose 1 is a terminal object in $\mathcal{X}$. A pointed object is a morphism $X_* : 1 \to X$ in $\mathcal{X}$. The pointed objects of $\mathcal{X}$ give rise to a new category $\mathcal{X}_*$ as follows. Let Δ^1 denote the category with two objects and a unique non-idenity morphism and consider the functor category $\mathrm{Fun}(\Delta^1, \mathcal{X})$. Then $\mathcal{X}_*$ is defined as the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{X})$ spanned by the pointed objects of $\mathcal{X}_*$.
We recall that if X is a $\mathcal{C}_n$-space then it comes with a multiplication induced from the action of the operad. Therefore, it should come as no suprise that we will be interested in objects that have some kind of multiplicative structure. We denote by Δ the usual simplex category of ordinals and order preserving maps and by $N(\Delta)$ we denote its nerve. We will write $[n] = \{0, \cdots, n\}$. Let $\mathcal{X}$ be an ∞-category. A monoid object in $\mathcal{C}$ is a functor $X : N(\Delta)^{\mathrm{opp}} \to \mathcal{X}$ with the property that the maps $X([n]) \to X(\{i, i+1\})$ exhibits $X([n])$ as a product $X(\{0,1\}) \times \cdots \times X(\{n-1, n\})$. We denote by $\mathrm{Mon}(\mathcal{X})$ the full subcategory of $\mathrm{Fun}(N(\Delta^{\mathrm{opp}}, \mathcal{X})$ spanned by the monoid objects in $\mathcal{X}$.
Given an ∞-cateogry $\mathcal{X}$ we denote by $\mathrm{Mon}^{\mathrm{gr}}_{\mathcal{C}_k}(\mathcal{X})$ the ∞-category of grouplike $\mathcal{C}_k$-monoid objects in $\mathcal{X}$. Here $\mathcal{C}_k$ denotes the ∞-operad of little k-cubes which is the analogue of our $\mathcal{C}_k$ in the context of ∞-categories. For a precise definition see Definition 5.1.0.2 of [6].

We are now in a position to state Lurie's generalization of the Recognition Principle.

Theorem 8.1. Let $k > 0$, let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}_*^{\geq k}$ denote the full subcategory of $\mathcal{X}$ spanned by those pointed spaces which are k-connective. Then there exists a canonical equivalence of ∞-categories

$$\alpha : \mathcal{X}_*^{\geq k} \simeq \mathrm{Mon}^{\mathrm{gr}}_{\mathcal{C}_k}(\mathcal{X})$$

Lurie's theorem provides a criterion for recognizing 'loop spaces' in any ∞-topos instead of just the ∞-topos of topological spaces. As such this theorem is applicable in much more general settings. However, an advantage of our approach is that it gives an explicit construction of the delooping of a space. Given a grouplike $\mathcal{C}_n$-space X we have seen that it has an n-fold delooping

$B(\Sigma, C_n, X)$ which can (in principle) be expliticly computed. For this reason our proof is still of great value for those who are interested in the delooping of a space.

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