Frame Properties of Beth Models

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"Either you're an intuitionist, or you're not"

Abstract

In this thesis I intend to present a comparison of Kripke and Beth models for Intuitionistic Logic, foremost to see how the two relate to each other and to explore frame properties of both. I will start off in chapter 2 by introducing the reader to Intuitionism, Intuitionistic Propopsition Logic and Intuitionistic Predicate Logic. In chapter 3 and 4 I will introduce the Kripke and Beth models and will see in chapter 5 if they can be translated into each other and by which means. In chapter 6 we look at what is known about frame properties of Kripke models, and we will see if we can find frame properties for Beth models as well. We will look at the intermediate logics KC and LC and see if we can find Beth models related to them.

Acknowledgements

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Also I would like to thank Sophie Zuiderduin and Jochem Bongaerts for lending their ears, when I was once again stuck.

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Chapter 1

Intuitionism

Intuitionism is a philosophy of mathematics. It was developed by L.E.J. Brouwer (1881–1966), a Dutch mathematician. The main idea of Intuitionism is that mathematics is a creation of the mind. This has major implications for what is concidered true, because many statements can no longer be upheld. Intuitionism shares many properties with constructivism, the philosophy of mathematics which claims that something can only exist if it can be constructed.

The believe that mathematics is a creation of the mind implies that something that has not yet been created (in the mind), is not yet true. Logic is no longer tenseless in this philosophy. A statement P might not be true at the moment, but as soon as one finds a proof for P it becomes true. Intuitionism therefore differs from classical mathematics like Platonism and Formalism, where truth is eternal and a truth is 'discovered', more than 'created'.

In Intuitionism the $\vee$-connective is interpreted much stronger than in classical logic. This has, for example, consequences for the *Principle of Excluded Middle* ($P \vee \neg P$), which longer holds in Intuitionistic reasoning. Concider for example the famous Riemann Hypothesis, a proposition for which there currently is neither a proof, not a proof of its negation. Since P can only be true if there is a proof of P, and the same goes for $\neg P$, we cannot conclude $P \vee \neg P$ at the moment. Only when we have either proven or disproven the Riemann Hypothesis we can conclude $P \vee \neg P$.

Brouwer's intention was to rebuild mathematics with this more demanding view. A proof from *Reductio ad Absurdum* just wasn't good enough. And although mainstream mathematics is still classical, intuitionism remains a topic of research until today.

1.1 Intuitionistic Logic

Although Brouwer himself was not particulary interested the use of logic – his proofs were written in natural language – his pupil A. Heyting has formalised a logic that is acceptable from the perspective of Intuitionism: Intuitionistic Logic. This logic is similar to classical logic, with the exception that the rule of excluded middle has been left out.

Intuitionistic Logic is also the logic that is used for most of the other branches of constructivism. And since the rules of Intuitionistic Logic are a subset of the rules of classical logic, the formulae that are true in Intuitionistic Logic are also a subset of those in classical logic. Intuitionistic Logic can therefore be seen as a stronger sort of logic, a sort where more is demanded of a proof.

- $\varphi \wedge \psi$ is true iff there is a proof of φ and a proof of ψ
- $\varphi \vee \psi$ is true iff there is a proof of φ or there is a proof of ψ
- $\varphi \to \psi$ is true iff there is a general method that transforms any proof of φ into a proof of ψ
- There is *no* proof of $\perp$
- $\forall x \varphi(x)$ is true iff there is a method that given an object a transforms it into a proof for $\varphi(a)$
- $\exists x \varphi(x)$ is true iff there is an object a with a proof of $\varphi(a)$

$\neg\varphi$ means $\varphi \to \perp$, and can therefore be read as: "Any proof of φ can be transformed into a proof of $\perp$, and since no proof of $\perp$ can exist, a proof of φ cannot exist".

These definitions can be used to form natural deduction-style introduction and elimination rules:

Introduction rules	Elimination rules
$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge\text{-I}$	$\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\text{-E} \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\text{-E}$
$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\text{-I} \qquad \dfrac{\psi}{\varphi \vee \psi}\ \vee\text{-I}$	$\dfrac{\varphi \vee \psi \quad \begin{matrix}\varphi^{(1)} \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}\psi^{(2)} \\ \vdots \\ \chi\end{matrix}}{\chi^{(1),(2)}}\ \vee\text{-E}$

$$\dfrac{\begin{array}{c}\varphi^{(1)}\\ \vdots \\ \psi\end{array}}{\varphi \to \psi^{(1)}} \to\text{-I} \qquad\qquad \dfrac{\varphi \to \psi \quad \varphi}{\psi} \to\text{-E}$$

$$\dfrac{\varphi(a)}{\exists x \varphi(x)}\ \exists\text{-I} \qquad\qquad \dfrac{\exists x \varphi(x) \qquad \begin{array}{c}\varphi(x)^{(1)}\\ \vdots \\ \psi\end{array}}{\psi^{(1)}}\ \exists\text{-E}$$

$$\dfrac{\begin{array}{c}\vdots \\ \varphi(a)\end{array}}{\forall x \varphi(x)}\ \forall\text{-I} \qquad\qquad \dfrac{\forall x \varphi(x)}{\varphi(x)}\ \forall\text{-E}$$

Any number on top represents an open assumption and a number below represents the cancelation of that open assumptions.

To this we add the intuitionistic *absurdity rule*, which states that *ex falso sequitur quodlibet* or: “from a contradiction we can deduce anything”:

$$\dfrac{\bot}{\varphi}\ \bot_i$$

These deduction rules together form the rules of **Intuitionistic Predicate Logic** or IQC[1]. To create **Classical Predicate Logic** or CQC one only needs to add one rule, which state that from the absurdity of the absurdity of a claim, we can deduce that claim:

$$\dfrac{\begin{array}{c}\neg\varphi\\ \vdots \\ \bot\end{array}}{\varphi}\ \bot_c$$

If we remove the ∀-I, ∀−-E, ∃-I and ∃-E rules from IQC and CQC we get **Intuitionistic Propositional Logic** (IPC) and **Classical Propositional Logic** (CPC) respectively.

For example, while we can not prove $\neg\neg\varphi \to \varphi$ in IQC, we can prove $\neg\neg(\neg\neg\varphi \to$

[1] IQC stands for Intuitionistic Quantifier Calculus. The yet to be defined abbreviations CQC, IPC and CPC stand for Classical Quantifier Calculus, Intuitionistic Proposition Calculus and Classical Proposition Calculus

φ).

$$
\begin{prooftree}
\AxiomC{$\neg(\neg\neg\varphi \to \varphi)^{(1)}$}
\AxiomC{$\neg\neg\varphi^{(2)}$}
\AxiomC{$\neg(\neg\neg\varphi \to \varphi)^{(1)}$}
\AxiomC{$\neg\neg\varphi^{(4)}$}
\AxiomC{$\varphi^{(3)}$}
\RightLabel{$\to$ -I}
\BinaryInfC{$\neg\neg\varphi \to \varphi^{(4)}$}
\RightLabel{$\to$ -E}
\BinaryInfC{$\perp$}
\RightLabel{$\to$ -I}
\UnaryInfC{$\neg\varphi^{(3)}$}
\RightLabel{$\to$ -E}
\BinaryInfC{$\perp$}
\RightLabel{$\perp_i$}
\UnaryInfC{φ}
\RightLabel{$\to$ -I}
\UnaryInfC{$\neg\neg\varphi \to \varphi^{(2)}$}
\RightLabel{$\to$ -E}
\BinaryInfC{$\perp$}
\RightLabel{$\to$ -I}
\UnaryInfC{$\neg\neg(\neg\neg\varphi \to \varphi)^{(1)}$}
\end{prooftree}
\qquad \heartsuit \qquad (1.1)
$$

Glivenko's theorem

What we observe in (1.1) is an interesting feature, apparently we can't prove that $\neg\neg\varphi$ implies φ, but we can prove that we can't prove the negation of it. This can be generalised to the statement that if something is true classically, the negation of it's negation is true intuitionistically. This should be the case, because if something is true, classically it's negation can not possibly be true Intuitionistically. This is known as Glivenco's theorem:

Theorem 1 (Glivenco's Theorem for propositional logic)

$$\mathsf{CPC} \vdash A \Leftrightarrow \mathsf{IPC} \vdash \neg\neg A$$

Proof That $\mathsf{IPC} \vdash \neg\neg A \Rightarrow \mathsf{CPC} \vdash A$ is easy to prove. Since any derivation in IPC also exists in CPC it follows that $\mathsf{IPC} \vdash \neg\neg A \Rightarrow \mathsf{CPC} \vdash \neg\neg A$ and since $\mathsf{CPC} \vdash \neg\neg A \to A$ it follows that $\mathsf{CPC} \vdash \neg\neg A \Rightarrow \mathsf{CPC} \vdash A$.

For the other direction ($\mathsf{CPC} \vdash A \Rightarrow \mathsf{IPC} \vdash \neg\neg A$) we will device a way of rewriting every proof of A in CPC into a proof of $\neg\neg A$ in IPC. Every classical inference rule with $\{\varphi, \psi \ldots\}$ as it's premises and χ as it's conclusion will be transformed to an inference rule from $\{\neg\neg\varphi, \neg\neg\psi, \ldots\}$ to $\neg\neg\chi$. Now for a given classical proof, we can replace all inference rules for our new inference rules and all premises and conclusions for the double negations of those formulae. If our new inference rules are intuitionistically valid, so will the derivation be. We will now give intuitionistic proofs for the validity of $\neg\neg(\neg\varphi \to \perp) \vdash_i \neg\neg\varphi$ (the double negation of the classical $\perp_c$-rule) (1.2) and for $\neg\neg(\varphi \to \psi), \neg\neg\varphi \vdash_i \neg\neg\psi$ (the double negation version of the $\rightarrow$-elimination rule) (1.3).

$$\dfrac{\dfrac{\neg\neg(\neg\varphi\rightarrow\bot)\qquad \dfrac{\dfrac{\neg\varphi^{(1)}\quad \neg\varphi\rightarrow\bot^{(2)}}{\bot}}{\neg(\neg\varphi\rightarrow\bot)^{(2)}}}{\bot}}{\neg\neg\varphi^{(1)}} \tag{1.2}$$

$$\dfrac{\dfrac{\neg\neg(\varphi\rightarrow\psi)\qquad \dfrac{\dfrac{\neg\neg\varphi\qquad \dfrac{\dfrac{\neg\psi^{(1)}\quad \dfrac{\varphi\rightarrow\psi^{(2)}\quad \varphi^{(3)}}{\psi}}{\bot}}{\neg\varphi^{(3)}}}{\bot}}{\neg(\varphi\rightarrow\psi)^{(2)}}}{\bot}}{\neg\neg\psi^{(1)}} \tag{1.3}$$

We leave the other inference rules as an exercise for the reader. ♡

For predicate logic this rule does not hold, but with a simple additional rule this can be fixed.

Definition The **Double Negation Shift** or **DNS** is the rule $\forall x\neg\neg\varphi(x)\rightarrow\neg\neg\forall x\varphi(x)$

Theorem 2 (Glivenko's Theorem for predicate logic)

$$\mathsf{CQC}\vdash A \Leftrightarrow \mathsf{IQC}+DNS\vdash\neg\neg A$$

Proof As with the proof for propositional logic, the implication of Intuitionistic Logic to classical logic is immediate.

Also the proof from classical logic to Intuitionistic logic is similar. We construct inference rules for the double negations. These are the same as those for propositional logic, plus the rules for the introduction and elimination of the quantifiers. We will give the introduction and elimination rules (resp. (1.4) and (1.5)) for the $\forall$-quantifier as an example. The rest is left as an exercise for the reader.

$$\dfrac{\dfrac{\neg\neg\varphi(a)}{\forall x\neg\neg\varphi(x)}\qquad \forall x\neg\neg\varphi(x)\rightarrow\neg\neg\forall x\varphi(x)}{\neg\neg\forall x\varphi(x)} \tag{1.4}$$

$$
\begin{prooftree}
\AxiomC{$\neg\neg\forall x\varphi(x)$}
\AxiomC{$\neg\varphi(a)^{(1)}$}
\AxiomC{$\forall x\varphi(x)^{(2)}$}
\UnaryInfC{$\varphi(a)$}
\BinaryInfC{$\bot$}
\UnaryInfC{$\neg\forall x\varphi(x)^{(2)}$}
\BinaryInfC{$\bot$}
\UnaryInfC{$\neg\neg\varphi(a)^{(1)}$}
\end{prooftree}
\tag{1.5}
$$

♡

Chapter 2

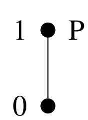
Kripke models

To show that a formula φ is provable in IQC we now have the natural deduction-style deduction method. But to show that φ is not provable in IQC, we need something else.

Remember that in CQC counterexamples are very simple. They consist of a set of elements and a set of atomic formulae that are true, and assume the rest of the atomic formulae to be false. For example, to show that the statement $\forall x(A(x) \vee B(x)) \to (\forall x A(x) \vee \forall x B(x))$ is not provable, we can give the following counter model: $\{a, b\}$ with $A(a)$ and $B(b)$. This is less trivial in IQC, where we must have a counter model for e.g. $\varphi \vee \neg\varphi$. For this purpose we can use Kripke models.

At first we will look at Kripke models for IPC. A Kripke model for IPC is a set K of possible worlds that represent certain stages of knowledge. We can move between those worlds according to a relation $\preccurlyeq$. Since known information cannot be made unknown (knowledge is monotone), such a relation must be a partial order and for all knowledge in world k it must be so that for all k' that follow k at least that same knowledge is known.

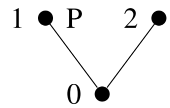
Concider the following Kripke model:



(2.1)

We will always visualise a Kripke model from the bottom up. When two nodes are connected by a line, the higher node comes after the lower node in the partial order. The bottom node 0 is the root node, which can be seen as the starting situation. In 0 we do not know if P is true or not. As we move up we gain more information. In 1 we do know P. So in 0 we cannot with certainty say that we will never find out P, which would be equivalent to knowing $\neg P$. And since we know neither P nor $\neg P$ we cannot conclude $P \vee \neg P$.

Now consider the following Kripke model:



(2.2)

Here, there are two successors of node 0. In node 1 we have again P, but in node 2 we now have $\neg P$. This makes (2.2) different from (2.1), in the sense that in (2.1) $\neg P$ could never be true. Therefore in (2.1) one can conclude $\neg\neg P$, also implying $\neg P \vee \neg\neg P$, another tautology in classical logic. But in (2.2) this is not the case. One cannot conclude either $\neg P$ or $\neg\neg P$ in 0, making this a countermodel to $\neg P \vee \neg\neg P$.

Definition A binary relation R is a **partial order** over a set iff:

- R is reflexive, i.e. aRa,
- R is antisymmetric, i.e. if aRb and bRa then $a = b$,
- R is transitive, i.e. if aRb and bRc then aRc

Definition A **frame** $\mathscr{F}$ is a tupel $\langle K, \preccurlyeq \rangle$ where

1. K is a non-empty set of nodes,
2. $\preccurlyeq$ is a partial order on those nodes,

In the visualisation of frames we will only draw the immediate successor relation S. The $\succcurlyeq$-relation is the transitive reflexive closure [1] over S. Naturally $p \preccurlyeq q \Leftrightarrow q \succcurlyeq p$.

We will now proceed with the formal definition of a Kripke model for propositional logic.

Definition A **Kripke model** $\mathscr{K}$ for propositional logic is a tupel $\langle K, \preccurlyeq, \Vdash \rangle$, where

1. $\langle K, \preccurlyeq \rangle$ is a frame,

[1]A transitive closure R^+ over a relation R on a set X is the minimal transitive relation on X that contains R. A reflexive closure $R^=$ over a relation R on a set X is the minimal reflexive relation on X that contains R.

2. $\Vdash$ is a relation between a node and a formula and is for atomic formula P defined such that:

   - $\forall k, k' \in K(k \Vdash P \text{ and } k \preccurlyeq k' \Rightarrow k' \Vdash P)$

   i.e. $\Vdash$ is upwards closed for atomic formulae.

For compound formulae the $\Vdash$-relation is defined as follows:

- $k \Vdash \varphi \wedge \psi$ iff $k \Vdash \varphi$ and $k \Vdash \psi$
- $k \Vdash \varphi \vee \psi$ iff $k \Vdash \varphi$ or $k \Vdash \psi$
- $k \Vdash \varphi \rightarrow \psi$ iff $\forall k' \succcurlyeq k$ (if $k' \Vdash \varphi$ then $k' \Vdash \psi$)
- $k \Vdash \neg\varphi$ iff $\forall k' \succcurlyeq k\,(k' \nVdash \varphi)$

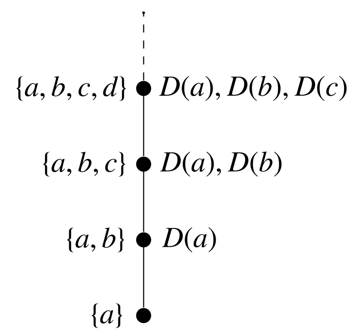
The $\Vdash$-relation defined above is pronounced as '*k forces P*', and means P is true in k.

Looking at (2.2) we have according to the definition: $\mathscr{K} = \langle K, \preccurlyeq, \Vdash\rangle$ with:

1. $K = \{k_0, k_1, k_2\}$
2. $\preccurlyeq = \{k_0 \preccurlyeq k_1, k_0 \preccurlyeq k_2\}$
3. $\Vdash = \{k_1 \Vdash P\}$

If we want to extend the Kripke models to IQC, we will add to each node a set of objects, known at that node. The atomic propositional variables are replaced by prime formulae of the form $R^n(d_1, \ldots, d_n)$, meaning R is a n-ary predicate and $d_1, \ldots, d_n$ are parameters (e.g. $Gives^3(john, pierre, jacket)$, meaning that John gives Pierre the jacket), or $R(\vec{d})$ for short.

As an example we will take an intuitionistic counter model for the Drinkers paradox. The Drinkers paradox is the statement that in every non-empty café there exists a person in the café for who it is true that if he drinks everybody drinks, or in formula form $\exists x(P(x) \rightarrow \forall y(P(y)))$. This statement is true in CQC, for if there is someone who does not drink, we can take him to make the antecedent false, and if there is no such person the consequent is true. In Intuitionistic Logic this is not the case as is illustrated by the following Kripke model.



(2.3)

In each node we know one person of which we can not say if he drinks, so the consequent – that it is known that everybody drinks – is never true. But there is never a person for which we can say that he doesn't drink –thus making the antecedent false –, for every person that we do not know if he drinks, eventually ends up drinking. Note that the bottom two nodes are sufficient to prove this point. In the root node we only know a, but we cannot say that if he drinks everybody drinks, because as we will see in the next node, he does drink, but not everybody drinks.

Definition A **Kripke model** $\mathscr{K}$ for predicate logic is a tupel $\langle K, \preccurlyeq, D, \Vdash \rangle$, where

1. $\langle K, \preccurlyeq \rangle$ is a frame,
2. D is the domain function that assigns to every node k a non-empty set $D(k)$ of elements such that $\forall k, k' \in K(k \preccurlyeq k' \rightarrow D(k) \subseteq D(k'))$, i.e. D is monotone.
3. $\Vdash$ is a relation between a node k and a formula and is for prime formula $R^n(d_1, \ldots, d_n)$ defined such that:
   - $k \Vdash R(\vec{d}) \Rightarrow d \in D(k)$ for all $d \in \vec{d}$
   - $\forall k, k' \in K(k \Vdash R^n(d_1, \ldots, d_n)$ and $k \preccurlyeq k' \Rightarrow k' \Vdash R^n(d_1, \ldots, d_n))$

   i.e. $\Vdash$ is upwards persistent for prime formulae.

For compound formulae the $\Vdash$-relation is defined as follows:

- $k \Vdash \varphi \wedge \psi$ iff $k \Vdash \varphi$ and $k \Vdash \psi$
- $k \Vdash \varphi \vee \psi$ iff $k \Vdash \varphi$ or $k \Vdash \psi$
- $k \Vdash \varphi \rightarrow \psi$ iff $\forall k' \geqslant k$(if $k' \Vdash \varphi$ then $k' \Vdash \psi$)
- $k \Vdash \neg\varphi$ iff $\forall k' \geqslant k(k' \nVdash \varphi)$
- $k \Vdash \forall x(\varphi(x))$ iff $\forall k' \geqslant k(\forall d \in D(k')(\varphi(d)))$
- $k \Vdash \exists x(\varphi(x))$ iff there is a $d \in D(k)$ for which it is the case that $k \Vdash \varphi(d)$

Rules (1) defines the frame, rule (2) assigns to every node in the frame a domain, which is monotone, and (3) assigns to every node in the frame a set of formulae which are true in that given node. Formulae that are true, remain true and every object in a formula that is true at a node is also in the domain of that node. This is because we can not know things about objects that we do not know.

The rules for the connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$ and $\exists$ can be read as follows:

- $\varphi \wedge \psi$ is true in a node if both φ and ψ are true
- $\varphi \vee \psi$ is true in a node if at least one of them is true (and as we know from Intuitionistic Logic, this means, that we also know which one that is)
- $\varphi \rightarrow \psi$ is true in a node, if we know that in every successive node if φ is true, then ψ must be true too
- $\neg\varphi$ is true in a node, if we know that there is no successive node where φ is true, i.e. φ will never be true.
- $\forall x(\varphi(x))$ is true in a node, if we know for every element d we now know, and ever will know, that $\varphi(d)$ is true
- $\exists x\varphi(x)$ is true in a node, if we know at least one element d for which $\varphi(d)$ is true

For prime formulae it is by definition that $\Vdash$ is upwards persistent. But for compound formulae it is not specifically in the definition. However this is still true.

Theorem 3 *The $\Vdash$-relation is upwards persistent for all formulae.*

Proof by induction. For prime formulae upwards persistence follows by definition. For compound formulae it follows by induction. We will give one induction step as an example.

Suppose $\Vdash$ is upwards persistent for φ and ψ and suppose $k \Vdash \varphi \vee \psi$ and $k \preccurlyeq k'$. Then either k forces φ or k forces ψ, and since $\Vdash$ is upwards persistent for both, either $k' \Vdash \varphi$ or $k' \Vdash \psi$. Therefore $k \Vdash \varphi \vee \psi$.

We leave the proofs for the other connectives as an exercise for the reader. ♡

Definition Let Γ be a set of formulae, or a **theory**. If any node k that forces Γ must also force a formula φ, then we write $\Gamma \Vdash \varphi$, i.e. $\Gamma \Vdash \varphi \Leftrightarrow (k \Vdash \Gamma \Rightarrow k \Vdash \varphi)$.

Theorem 4 *Any Kripke model is intuitionistically sound, that is* $\Gamma \vdash \varphi \Rightarrow \Gamma \Vdash \varphi$

Proof By induction. Suppose that for φ and ψ soundness is proven. Now given a node k for which $k \Vdash \varphi \rightarrow \psi$, and $k \Vdash \varphi$. Then by the Kripke rule for $\rightarrow$, for all $k' \succcurlyeq k$ (so including k) it is the case that if $k \Vdash \varphi$ then $k \Vdash \psi$. So $k \Vdash \psi$.

The soundness proofs for the other derivation rules are of a similar nature, and are left as an exercise. ♡

2.1 Finitary Kripke models for IPC

We'll now prove that for any infinite Kripke model proving that $\Gamma \nVdash \varphi$ there is finite Kripke model also proving $\Gamma \nVdash \varphi$. First we'll define the formula Δ, which defines for each node a set of subformulae of φ that are true in that node. Then from a infinite tree we'll create a new tree, which uses Δ as an equivalence relation. After that we'll prove for the constructed tree, that it is finite and that it proves $\Gamma \nVdash \varphi$.

Definition Let φ be a theory and let $Sub(\psi)$ the function that maps a formula ψ onto the set of all it's subformulae. Then let Δ_φ be the function that maps a theory $\Gamma \equiv \{\psi_0, \psi_1, \ldots, \psi_n\}$ onto the set of subformulae overlapping with the subformulae of φ, that is:

$$\Delta_\varphi(\Gamma) \equiv \bigcup_{i=0}^{n} (Sub(\varphi) \cap Sub(\psi_i))$$

If a node k forces exactly the set of formulae Γ then we define the function $\Delta_\varphi(k)$ to mean the same as $\Delta_\varphi(\Gamma)$.

Definition Let $\mathscr{K}$ be a Kripke model $\langle K, \preccurlyeq, \Vdash \rangle$ and φ is a formula. Then let the **φ-filtration** of $\mathscr{K}$ be the Kripke model $\mathscr{K}' \equiv \langle K', \preccurlyeq', \Vdash' \rangle$ constructed by the following transformation. First define the Δ-equivalence on nodes as follows: $k \overset{\Delta}{\equiv} k'$ iff $\Delta_\varphi(k) \equiv \Delta_\varphi(k')$. To avoid confusion the nodes in the filtration will be referred to with $[k]$'s.

- K' consists of all non-empty equivalence classes on K, that is any $[k] \in K'$ is the set $\langle k_0, k_1, \ldots \rangle$ where for all k_i, k_j in $[k]$, it is the case that $k_i \overset{\Delta}{\equiv} k_j$.
- $[k] \preccurlyeq' [k]'$ iff $\Delta_\varphi([k]) \subseteq \Delta_\varphi([k]')$, where $\Delta_\varphi([k])$ for a $[k] \in K'$ maps to the same set that the nodes $k \in [k]$ map.
- $[k] \Vdash' P$ where P is an atomic formula iff $P' \in \Delta_\varphi([k])$.

Lemma 5 *Let $\mathscr{K}' \equiv \langle K', \preccurlyeq', \Vdash' \rangle$ be the φ-filtration of $\mathscr{K} \equiv \langle K, \preccurlyeq, \Vdash \rangle$. Then there is a node $k \in \mathscr{K} \nVdash \varphi$ iff there is a node $[k] \in \mathscr{K}' \nVdash \varphi$.*

Proof There is a surjection from K to K', mapping $k \in K$ to $[k] \in K'$ iff $\Delta_\varphi(k) \equiv \Delta_\varphi([k])$ (again we'll use $[k]$ to denote nodes from K', to avoid confusion). For each formula $\psi \in Sub(\varphi)$, it is the case that $k \Vdash \psi$ iff $[k] \Vdash \psi$. We will show this by induction.

- For the atomic formulae this is by definition.
- Suppose it is proven for φ and ψ. Then if $k \nVdash \varphi \wedge \psi$, it is either the case that $k \nVdash \varphi$ or $k \nVdash \psi$, which means it is proven that either $[k] \nVdash' \varphi$ or $[k] \nVdash' \psi$ from which follows $[k] \nVdash \varphi \wedge \psi$. The reverse is similar.
- Suppose it is proven for φ and ψ. Then if $k \nVdash \varphi \vee \psi$, it is both the case that $k \nVdash \varphi$ and $k \nVdash \psi$, which means it is proven that both $[k] \nVdash' \varphi$ and $[k] \nVdash' \psi$ from which follows $[k] \nVdash \varphi \vee \psi$. The reverse is similar.
- Suppose it is proven for φ and ψ. Take a node $k \in K$ and a node $[k] \in K'$ where k is mapped to. Then if $k \nVdash \varphi \rightarrow \psi$ then there is a node $k' \succcurlyeq k \Vdash \varphi$ and $k' \nVdash \psi$. This k' is mapped to $[k]' \in K'$. And since for the relation is proven for φ and for ψ it is known that $[k] \Vdash' \varphi$ and $[k] \nVdash' \psi$. And since $\Delta_\varphi(k') \supseteq \Delta_\varphi(k)$ it is known that $\Delta_\varphi([k]') \supseteq \Delta_\varphi([k])$ and therefore $[k]' \succcurlyeq [k]$. Thus $[k] \nVdash' \varphi \rightarrow \psi$.

  For the reverse we take $k \in K \Vdash \varphi \rightarrow \psi$. k is mapped to $[k] \in K'$. If $k \Vdash \varphi$ then $k \Vdash \psi$, and since this is proven for φ and ψ, it is the case that if $[k] \Vdash' \varphi$

then $[k] \Vdash' \psi$. Now to show that there are no nodes $[k]' \geqslant [k]$ for which $[k]' \Vdash' \varphi$ and $[k]' \nVdash' \psi$,we take a node after $[k]'' \geqslant [k]$. For this node we know there is a node $k'' \in K$ which maps to $[k]''$. Since $[k]''$ is a successor of $[k]$, and by definition $\varphi \to \psi \in \Delta_\varphi([k])$, it is the case that $\varphi \to \psi \in \Delta_\varphi([k]'')$ and by the reasoning above, if $[k]'' \Vdash' \varphi$ then $[k]'' \Vdash' \psi$.

- $\neg\varphi$ is the same as $\varphi \to \bot$ therefore the induction step above can be used to prove it for $\neg\varphi$.

Since φ is it's own subformula, it is proven that if there is a node $k \in K \nVdash \varphi$ then there is a node $[k] \in K' \nVdash' \varphi$. ♡

Lemma 6 *Let $\mathscr{K}$' be the φ-filtration of $\mathscr{K}$. Then $\mathscr{K}$' is finite.*

Proof Since there are finitely many subformulae of φ there can only be finitely many unique sets of subformulae. These sets define the possible equivalence classes and for each such a class there is at most one node in $\mathscr{K}$'. Therefore there are finitely many nodes in $\mathscr{K}$'. ♡

Theorem 7 *Let $\mathscr{K}$ be an infinite countermodel proving $\Gamma \nvdash \varphi$, then there is a finite countermodel $\mathscr{K}$' proving $\Gamma \nvdash \varphi$.*

Proof Let $\mathscr{K}$' be the φ-filtration of $\mathscr{K}$. Let n be the node in $\mathscr{K}$, which contains Γ but does not force φ (i.e. the node that proves $\Gamma \nvdash \varphi$). Then by lemma 5 there is a node $[k]$ in $\mathscr{K}$', where k is mapped to which does not force φ too. Now we extend that node $[k]$ with all formulae in Γ. It is easy to see that all relevant formulae which might change things for φ are already used in this node and all successive nodes, so these will not affect φ. The node now contains all formulae in Γ but does not force φ. And since $\mathscr{K}$' is finite (by theorem 6), we now have a finite countermodel proving $\Gamma \vdash \varphi$. ♡

2.2 Completeness of Kripke semantics

Definition Let Γ be a theory (set of formulae). Then Γ is **saturated** if:

- if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$
- if $\varphi \vee \psi \in \Gamma$ then either $\varphi \in \Gamma$ or $\psi \in \Gamma$

Lemma 8 *For any consistent theory* Γ*, then there is a saturated consistent* $\Gamma' \supseteq \Gamma$ *which can be constructed from* Γ.

Proof Given is the theory Γ. The saturated theory Γ^ω is constructed as follows.

Let Γ^0 be the smallest set such that $\Gamma^0 \supseteq \Gamma$ where $(\Gamma^0 \vdash \varphi) \Rightarrow (\varphi \in \Gamma^0)$. Let the formulae in the theory be ordered on the length of the formulae[2]. Now by induction we define Γ^k. Given is Γ^{k-1}. Let $\Gamma^k \supseteq \Gamma^{k-1}$. Let φ be the first formula in Γ^{k-1} such that $\varphi \equiv \psi \vee \chi$, but neither $\psi \in \Gamma^{k-1}$ nor $\chi \in \Gamma^{k-1}$, then:

- if $\Gamma^{k-1} \cup \{\psi\} \nvdash \bot$, then $\psi \in \Gamma^k$
- otherwise $\chi \in \Gamma^k$

Let $\Gamma^k \vdash \varphi \Rightarrow \varphi \in \Gamma^k$ and Γ^k be ordered on the length of the formulae.

If no such φ exists, then $\Gamma^{k-1} \equiv \Gamma^k$. Γ^ω is the result of the infinite iteration. This Γ^ω is the saturated theory.

Given that there are finitely many propositional variables in Γ, Γ^ω will be saturated. This can be proven as follows. Given a Γ^k, and a formula $\varphi \equiv \psi \vee \chi \in \Gamma^k$, for which neither ψ nor χ are in Γ^k. Then there are only finitely many formulae that can come before φ. If any of these formulae will be handled before this, that is, if there is a disjunction ψ without any of the disjuctive formulae, then handling this formula will add at least one formula to the formulae smaller then φ, namely one of the disjunctive formulae of ψ. Since there are only finitely many formulae that can come before φ then there are also only finitely many steps that can be made before φ. Thus each formula in Γ^k that needs to be handled, will be handled in a finite amount of time. Therefore in Γ^ω there will be no disjunction, for which neither disjunctive formula is the case. ♡

Theorem 9 *Kripke models for* IPC *are complete, that is* $\Gamma \vdash \varphi$ *iff for all nodes on all Kripke models for* IPC $\Gamma \Vdash \varphi$.

Proof In theorem 4 we have proven that if $\Gamma \vdash \varphi$ then $\Gamma \Vdash \varphi$. Now we'll prove that if $\Gamma \nvdash \varphi$ then $\Gamma \nVdash \varphi$, or there is a node k concievable, that $k \Vdash \Gamma$ and $k \nVdash \varphi$. We will do this by induction. Let Γ be saturated:

- Let $\Gamma \nvdash P$ where P is an atomic formula. Then $P \notin \Gamma$, and thus if $k \Vdash \Gamma$ then $k \nVdash P$

[2] This will be done in such a way that the shortest formula will be at the start. Furthermore, the atomic formulae will be written as $p, p', p'', \ldots$, i.e. the n^{th} atomic formula has a length of n

- Let it be proven that $\Gamma \vdash \varphi \Leftrightarrow \Gamma \Vdash \varphi$ and $\Gamma \vdash \psi \Leftrightarrow \Gamma \Vdash \psi$. Now suppose $\Gamma \nvdash \varphi \wedge \psi$. Then $(\Gamma \nvdash \varphi \wedge \psi) \Leftrightarrow (\Gamma \nvdash \varphi$ or $\Gamma \nvdash \psi) \Leftrightarrow (\Gamma \nVdash \varphi$ or $\Gamma \nVdash \psi) \Leftrightarrow (\Gamma \nVdash \varphi \wedge \psi)$.

- Let it be proven that $\Gamma \vdash \varphi \Leftrightarrow \Gamma \Vdash \varphi$ and $\Gamma \vdash \psi \Leftrightarrow \Gamma \Vdash \psi$. Now suppose $\Gamma \nvdash \varphi \vee \psi$. Then $(\Gamma \nvdash \varphi \vee \psi) \Leftrightarrow (\Gamma \nvdash \varphi$ and $\Gamma \nvdash \psi) \Leftrightarrow (\Gamma \nVdash \varphi$ and $\Gamma \nVdash \psi) \Leftrightarrow (\Gamma \nVdash \varphi \wedge \psi)$.

- Let Γ' be the saturation of $\Gamma \cup \{\varphi\}$. Let it be proven that $\Gamma' \vdash \psi \Leftrightarrow \Gamma' \Vdash \psi$. Now suppose $\Gamma \nvdash \varphi \to \psi$. Then $(\Gamma \nvdash \varphi \to \psi) \Leftrightarrow (\Gamma' \nvdash \psi) \Leftrightarrow (\Gamma' \nVdash \psi) \Leftrightarrow (\Gamma \nVdash \varphi \to \psi)$. The last step can be seen as follows. Take a node $k \Vdash \Gamma$, then there can be a node $k' \geqslant k \Vdash \Gamma \cup \{\varphi\}$ for which $k' \nVdash \psi$.

- $\neg\varphi \equiv \varphi \to \bot$ thus as above.

With this, the $\Leftrightarrow$ relation is proven. $\Gamma \vdash \varphi$ iff $\Gamma \Vdash \varphi$. Kripke models for IPC are complete. ♡

The proof described above can be used to create a tableaux style method to create countermodels, but instead of having a single node with true statements and false statements, at each point there is a possibly incomplete Kripke model. A complete Kripke countermodel is a model on which no rules can be used anymore. A rule takes a node k, and a formula φ for which it is known that $k \Vdash \varphi$ or $k \nVdash \varphi$, and has not yet been simplified. For example:

- If a node $k \nVdash \varphi \vee \psi$ in a Kripke model $\mathscr{K}$, then this is followed by a Kripke model $\mathscr{K}$', which is a copy of $\mathscr{K}$, except that now $k \nVdash \varphi$ and $k \nVdash \psi$.

- If a node $k \Vdash \varphi \vee \psi$ in a Kripke model $\mathscr{K}$, then this is followed by two Kripke models $\mathscr{K}$' and $\mathscr{K}$", both being copies of $\mathscr{K}$, only with $\mathscr{K}$' having $k \Vdash \varphi$ and $\mathscr{K}$" having $k \Vdash \psi$.

- If a node $k \nVdash \varphi \to \psi$ in a Kripke model $\mathscr{K}$, then this is followed by a Kripke model $\mathscr{K}$', which is a copy of $\mathscr{K}$, except for a new node $k' \geqslant k \Vdash \varphi$ and $k' \nVdash \psi$. For all $k'' \leqslant k'$ if $k'' \Vdash \chi \to \xi$ that formula is reactivated again.

For IQC the completeness can be proven in a similar way as for IPC. First of all the saturation used in the proof for the completeness of IPC needs to be extended.

Definition Let Γ be a theory (set of formulae) and C be a possibly infinite set of elements $\{c_0, c_1, \dots\}$. Then Γ is ***C*-saturated** if:

- if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$
- if $\varphi \vee \psi \in \Gamma$ then either $\varphi \in \Gamma$ or $\psi \in \Gamma$
- if $\exists x \varphi(x) \in \Gamma$ then for a $c \in C$ $\varphi(c) \in \Gamma$

Theorem 10 *Let Γ be a theory in the language $\mathscr{L}$, φ be a formula in $\mathscr{L}$ and $\Gamma \nvdash \varphi$. Let C be a set of elements not in $\mathscr{L}$, and $\mathscr{L}(C)$ be the language $\mathscr{L}$ extended with C. Then there is a theory $\Gamma' \supset \Gamma$ in $\mathscr{L}(C)$ which is C-saturated.*

Proof The method will be roughly the same as the method described in the proof of theorem 8. The main difference is now that the first formula in $\varphi \in \Gamma^{k-1}$ for which

- $\varphi \equiv \psi \vee \chi$ and $\varphi \notin \Gamma^{k-1}$ and $\psi \notin \Gamma^{k-1}$, *or*
- $\varphi \equiv \exists x \psi(x)$ and there is no $c \in C$ for which $\psi(c) \in \Gamma^{k-1}$

is taken to calculate Γ^k, taking in the last case a $c_i \in C$ that is not yet used and adding $\psi(c_i)$ to Γ^k. ♡

Definition Let Γ be a theory in $\mathscr{L}$, and let $C_0, C_1, \dots$ be infinitely many infinite disjoint sets of elements not occurring in $\mathscr{L}$. Let a set C_n^* be the set $\bigcup_{i=0}^{n} C_i$. For Γ we define the **canonical model** $\mathscr{K}$ as follows.

$$\mathscr{K} = \langle K. \preccurlyeq, D, \Vdash \rangle$$

where

- K is the set of nodes, having $k_0 \in K$, with Γ^0 is the C_0^*-saturation with respect to Γ and $k_0 \Vdash \Gamma^0$. Furthermore if there is a node $k \in K \Vdash \Gamma^i$ where Γ^i is C_n^*-saturated, then for all $\Gamma^j \supset \Gamma^i$ where Γ^j is C_{n+1}^*-saturated, there is a node $k' \Vdash \Gamma^j$.
- For the k and k' described above, kSk' and $\preccurlyeq$ being the transitive reflexive closure over S.
- If a k is C_n^*-saturated, then $D(k)$ is $D(\mathscr{L}) \cup C_n^*$.

- If $k \Vdash \Gamma$ then for all formulae $\varphi \in \Gamma(k \Vdash \varphi)$.

Theorem 11 *Kripke models for* IQC *are complete, that is $\Gamma \vdash \varphi$ iff for all nodes on all Kripke models for* IQC *$\Gamma \Vdash \varphi$.*

Proof Let Γ be a C-saturated theory. Suppose $\Gamma \Vdash \varphi$. Now we construct the canonical model for Γ and prove that $\Gamma \vdash \varphi$. This we will do by induction on the length of φ.

- For atomic formulae this follows immediate because of the C-saturation.
- Suppose it is already proven that $\Gamma \vdash \varphi \Leftrightarrow \Gamma \Vdash \varphi$ and $\Gamma \vdash \psi \Leftrightarrow \Gamma \Vdash \psi$. Then $(\Gamma \vdash \varphi \wedge \psi) \Leftrightarrow (\Gamma \vdash \varphi$ and $\Gamma \vdash \psi) \Leftrightarrow (\Gamma \Vdash \varphi$ and $\Gamma \Vdash \psi) \Leftrightarrow (\Gamma \Vdash \varphi \wedge \psi)$.
- Suppose it is already proven that $\Gamma \vdash \varphi \Leftrightarrow \Gamma \Vdash \varphi$ and $\Gamma \vdash \psi \Leftrightarrow \Gamma \Vdash \psi$. Then $(\Gamma \vdash \varphi \vee \psi) \Leftrightarrow (\Gamma \vdash \varphi$ or $\Gamma \vdash \psi) \Leftrightarrow (\Gamma \Vdash \varphi$ or $\Gamma \Vdash \psi) \Leftrightarrow (\Gamma \Vdash \varphi \vee \psi)$.
- Suppose $\Gamma \Vdash \varphi \to \psi$ and $\Gamma \nvdash \varphi \to \psi$. Then $\Gamma \cup \{\varphi\} \nvdash \psi$, so by induction there is a node $\Gamma' \supseteq \Gamma$ for which $\Gamma' \Vdash \varphi$ but $\Gamma' \nVdash \psi$. This contradicts $\Gamma \Vdash \varphi \to \psi$, therefore $\Gamma \vdash \varphi \to \psi$.
- Since $\neg\varphi \equiv \varphi \to \perp$ this is proven above.
- Let $\Gamma \Vdash \exists x\varphi(x)$, then by the C-saturation of Γ there is a $c \in C(\Gamma \Vdash \varphi(c))$ from which follows, again by saturation $\varphi(c) \in \Gamma$. Therefore $\Gamma \vdash \varphi(c)$ and $\Gamma \vdash \exists x\varphi(x)$.
- Let Γ be a saturated theory in the language $\mathscr{L}$ and $\Gamma \Vdash \forall x\varphi(x)$. Suppose $\Gamma \nvdash \forall x\varphi(x)$, then there is a language $\mathscr{L}(C)$ with a $c \in C$ and a $\Gamma' \supseteq \Gamma \nvdash \varphi(c)$, with Γ' is C-saturated. Then $\Gamma' \nVdash \varphi(c)$ and therefore $\Gamma' \nVdash \forall x\varphi(x)$, which contradicts $\Gamma \Vdash \forall x\varphi(x)$. Therefore $\Gamma \vdash \forall x\varphi(x)$.

For any formula φ for which $\Gamma \Vdash \varphi$ it is the case that $\Gamma \vdash \varphi$. ♡

Chapter 3

Beth models

When building models to evaluate Intuitionistic Logic Kripke models are not the only tools to our disposal.

After Heyting and Kolmogorov created the BHK-interpretation of Intuitionism, stating that the statement A was the same as *proving* A in the early 1930's there was a gap of semantics. The simple semantics that was used in classical logic could not be matched by any Intuitionistic equivalent. In 1945 the first true semantics was created by S.C. Kleene.

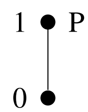
One year later E.W. Beth was appointed as the first Dutch professor of logic. A year after that Beth published a paper titled "Semantic Considerations on Intuitonistc Semantics". In it Beth underlines the notion of a spread as a critical notion. But only as late as 1955 Beth presented a sketch of what later became the Beth models n a lecture he gave in Paris. He continued developing his semantics trying to get a contructive completeness proof.

The choice of the use of trees was made by Beth because of their relation to tableaux. The use of choice sequences and barring are introduced. In 1956 Beth published a paper with a correct validity proof of his semantics and two proofs of completeness, one classical and one constructive. The classical proof is easily accepted but the constructive proof is not received as good. It was criticised as being unintelligible or flawed. Beth did not attent to those problems. But Beths method still maintained succesfull. [9]

For the Kripke's semantics for Intuitionistic Logic that are explored in the preceding chapter we still had to wait untill 1963 and are heavily influenced by Beth semantics. Beth models also use partially ordered nodes, with formulae and possibly domains of elements assigned to them. The interpretation however is slightly different from Kripke models. The major difference lies in the fact that in a node in Beth models a formula is considered true if under all possible future scenarios it will eventually become true.

What this means is that for example if we have a way of deciding in finite time if A or B is the case and we know that one of them will be the case, we can say $A \vee B$. In a Kripke model this is not allowed because we need to know either A or B at a node in order for $A \vee B$ to be true at that node. However, Beth models and Kripke models are equally strong when it comes to expressing logical formulae. For example, for the statement $A \vee \neg A$ to be true, it is needed to have a way of deciding between those two, in finite time.

Let's for example take the very basic Kripke model:



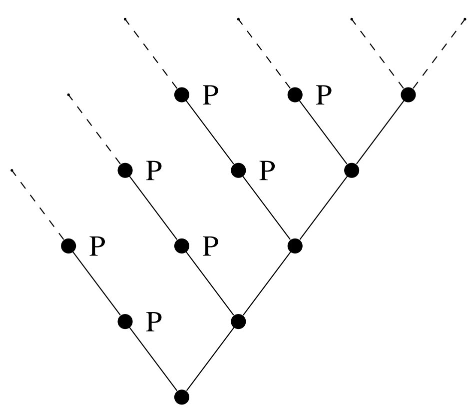
(3.1)

We can read this as a Beth model (the visualisation of Beth models is similar to Kripke models). In 1 P is true, just like in a Kripke model. But in 0 it is known that eventually we will find that P is true, therefore in 0 P must also be the case. So while (3.1) is a good Kripke counter model for $(P \vee \neg P)$, it is not a good Beth counter model. In Beth's view the notion of a spread was a fundamental ingredient.

Definition A **spread** is a tree where every node has at least one successor.

This has to do with the notion of an **absolutely free choice sequence** or a.f.c.s. An a.f.c.s. is a sequence chosen from a certain spread where the only restrictions on that sequence can be the restrictions defining the spread.

Now we know what not to expect from Beth models, let's build a countermodel for $P \vee \neg P$:



(3.2)

(3.2) is a infinite Beth model, where, as long as P is not yet known, one can always find P and if P is known, one can never ‘unknow’ it. But, at no point, when P is not known, one is forced to know P. One can always stay ignorant about P, or there is a a.f.c.s. where P will never be known.

Before we come up with a formal definition for Beth models we first need some notations and definitions.

A path is one possible way of moving through a model, that can only start at a node with no predecessors and can only end at a node with no successors.

Definition A **path** α in a frame $\langle K, \preccurlyeq \rangle$ (we will henceforth refer to paths with lowercase Greek letters) is an ordered set $\langle k_1, k_2, \ldots, k_n \rangle$ (possibly infinite, like $\langle k_1, k_2, \ldots \rangle$ or $\langle \ldots, k_{n-1}, k_n, k_{n+1}, \ldots \rangle$) of nodes from K for which:

- $\forall x, y \in \alpha$ if $z \in K$ and $x \preccurlyeq z \preccurlyeq y$ then $z \in \alpha$
- $\forall x \in \alpha$ if $\exists y \in K (y \prec x)$ then $\exists z \in \alpha (z \prec x)$
- $\forall x \in \alpha$ if $\exists y \in K (y \succ x)$ then $\exists z \in \alpha (z \succ x)$
- $\forall x, y \in \alpha$ either $x \preccurlyeq y$ or $y \preccurlyeq x$
- $\forall x, y \in \alpha$ if $x \prec y$ then y comes after x in α, i.e. $\alpha = \langle \ldots, x, \ldots, y, \ldots \rangle$

A path in a rooted tree is a set of nodes $\langle k_1, \ldots, k_n \rangle$ where k_1 is the root node and k_n is a leaf. If the path is infinite, no last node exists. For every node k in the path, all nodes predecessing it, are also in the path before k, given that the model is a tree.

If a node k occurs in a path α, or in other words α goes through k we write this as $k \in \alpha$ or $\alpha \ni k$.

Now for a formal definition of Beth models, let’s start with Beth models for propositional logic:

Definition A **Beth model** $\mathscr{B}$ for propositional logic is a tupel $\langle K, \preccurlyeq, \Vdash \rangle$, where

1. $\langle K, \preccurlyeq \rangle$ is a frame,
2. $\Vdash$ is a relation between a node k and a formula, and is for atomic formula P defined such that:
    - $k \Vdash P$ if $\forall \alpha \ni k (\exists k' \in \alpha (k' \Vdash P))$

For compound formulae, the $\Vdash$-relation is defined such that:

- $k \Vdash \varphi \wedge \psi$ iff $k \Vdash \varphi$ and $k \Vdash \psi$
- $k \Vdash \varphi \vee \psi$ iff $\forall \alpha \ni k(\exists k' \in \alpha(k' \Vdash \varphi$ or $k' \Vdash \psi))$
- $k \Vdash \varphi \rightarrow \psi$ iff $\forall k' \geqslant k($if $k' \Vdash \varphi$ then $k' \Vdash \psi)$
- $k \Vdash \neg\varphi$ iff $\forall k' \geqslant k(k' \nVdash \varphi)$

As with Kripke models, Beth models are easily extended to predicate logic.

Definition A **Beth model** $\mathscr{B}$ for predicate logic is a tupel $\langle K, \preccurlyeq, D, \Vdash\rangle$, where

1. $\langle K, \preccurlyeq\rangle$ is a frame,
2. D is the domain function that assigns to every node k a non-empty set $D(k)$ of elements such that $\forall k, k' \in K(k \preccurlyeq k' \rightarrow D(k) \subseteq D(k'))$, i.e. D is upwards closed
3. $\Vdash$ is a relation between a node k and a formula, and is for prime formula $R^n(d_1, \ldots, d_n)$ defined such that:
   - $k \Vdash R^n(d_1, \ldots, d_n)$ if $\forall \alpha \ni k(\exists k' \in \alpha(k' \Vdash R^n(d_1, \ldots, d_n)))$
   - $k \Vdash R^n(d_1, \ldots, d_n) \Rightarrow d_i \in D(k)$ for $1 \leq i \leq n$

For compound formulae, the $\Vdash$-relation is defined such that:

- $k \Vdash \varphi \wedge \psi$ iff $k \Vdash \varphi$ and $k \Vdash \psi$
- $k \Vdash \varphi \vee \psi$ iff $\forall \alpha \ni k(\exists k' \in \alpha(k' \Vdash \varphi$ or $k' \Vdash \psi))$
- $k \Vdash \varphi \rightarrow \psi$ iff $\forall k' \geqslant k($if $k' \Vdash \varphi$ then $k' \Vdash \psi)$
- $k \Vdash \neg\varphi$ iff $\forall k' \geqslant k(k' \nVdash \varphi)$
- $k \Vdash \forall x(\varphi(x))$ iff $\forall k' \geqslant k(\forall d \in D(k')(\varphi(d)))$
- $k \Vdash \exists x(\varphi(x))$ iff $\forall \alpha \ni k(\exists k' \in \alpha(\exists d \in D(k')($for which it is the case that $k' \Vdash \varphi(d)))$

Rule (1) defines the frame, rule (2) assigns to the nodes a set of known elements ensuring monotocity, rule (3) assigns to each node a set of predicates, making sure that no things are known about unknown objects. Also (3) ensures that if a prime formula bars node k, it is true in k.

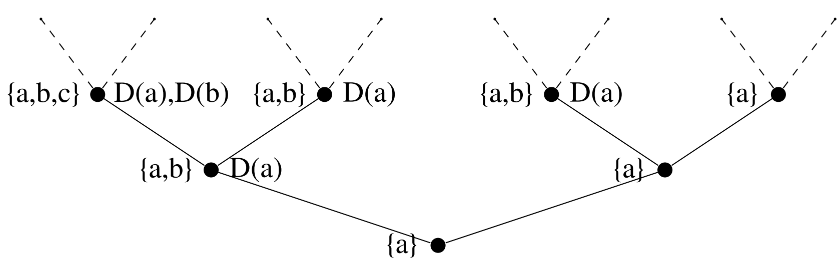
Definition A set of nodes $\vec{k}$ **bars** a node k iff each path α through k has at least one node in $\vec{k}$. If all nodes in $\vec{k}$ force P and $\vec{k}$ bars k, we say P bars k.

So if we take the set of $\vec{k} = \{k \in K | k \Vdash P\}$ then $\vec{k}$ bars k iff from k one will eventually end up in $\vec{k}$. In a Beth model, this means that due to (3) $k \Vdash P$, for we will eventually but without a doubt find out that P is the case.

The rules for the connectives can be read to mean the following:

- $\varphi \wedge \psi$ is true in k, if φ is true in k and ψ is true in k
- $\varphi \vee \psi$ is true in k, if $\{k|k \Vdash \varphi\} \cup \{k|k \Vdash \psi\}$ bars k, i.e. if eventually we will find out that φ or ψ is true, i.e. for every path through k we will reach a node k' in which either φ is true or ψ is true
- $\varphi \to \psi$ is true in k, if for every node $k' \geqslant k$ it is true that if φ is true, then ψ is also true.
- $\neg\varphi$ is true in k if there is no k' after k where φ is true
- $\forall x \varphi(x)$ is true in k if for every node $k' \geqslant k$ and for every $d \in D(k')$ $\varphi(d)$ is true
- $\exists x \varphi(x)$ is true in k if eventually we will come to a node where there is an element d for which $\varphi(d)$ is true.

If we would like to make a Beth counter model for the drinkers paradox, it would look something like this.



(3.3)

In each node there are two possible successor nodes. To the left is a node that adds one new element to the domain, and adds for the already known element d the formula $D(d)$. To the right is the successor node, where no new information is added.

The rules do not directly state that the $\Vdash$-relation is persistent for all formulae. This is however the case.

Theorem 12 *The $\Vdash$-relation is upwards persistent for all formulae.*

Proof by induction. For prime formulae the persistency follows by definition. For comound formulae it follows by induction. We give one induction step as an example.

Suppose $\Vdash$ is persistent for φ and ψ and suppose $k \Vdash \varphi \vee \psi$ and $k \leqslant k'$. Then k is barred by $\{k|k \Vdash \varphi\} \cup \{k|k \Vdash \psi\}$. Since k' comes after k it is barred by the same set. Therefore $k' \Vdash \varphi \vee \psi$.

The rest is left as an exercise for the reader. ♡

Theorem 13 *If a node k is barred by a set of nodes $\vec{k}$ where $\forall k \in \vec{k}(k \Vdash \varphi)$ then $k \Vdash \varphi$.*

Proof by induction. For prime formulae it follows by definition. For compound formulae it follows by induction. We give two of the induction steps as an example.

Suppose for φ and ψ this is already proven. Now suppose a node k is barred by K where $K \equiv \{k|k \Vdash \varphi \wedge \psi\}$. Then all nodes in K also force φ and ψ. Since K bars k and for φ and ψ it is already proven that barring forces that formula, $k \Vdash \varphi$ and $k \Vdash \psi$. Then $k \Vdash \varphi \wedge \psi$.

Again suppose that for φ and ψ it is already proven. Now suppose a node k is barred by K where $K \equiv \{k|k \Vdash \varphi \to \psi\}$. Now take any node $k' \geqslant k$. If k' is a successor of a node in K or is a node in K, then $k \Vdash \varphi$ immediately. If k' is not in this group, then it is still barred by K. Now assume that $k' \Vdash \varphi$, then all nodes in K that are a successor of k' force φ and therefore ψ. This means that k' is barred by ψ and by the induction hypothesis, $k' \Vdash \psi$. Thus if k is barred by $\varphi \to \psi$, then for every node $k' \geqslant k$ if $k' \Vdash \varphi$ then $k' \Vdash \psi$, and so $k \Vdash \varphi \to \psi$. ♡

3.1 Completeness for Beth semantics

Theorem 14 *Any Beth model is intuitionistically sound, that is if $\Gamma \vdash \varphi$ then $\Gamma \Vdash \varphi$*

Proof By induction. Suppose that for φ and ψ soundness is proven. Now presume $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$ then $\Gamma \vdash \varphi \wedge \psi$. Since soundness is proven for φ and ψ, it is the case that $\Gamma \Vdash \varphi$ and $\Gamma \Vdash \psi$. Then by the rule for conjunction on Beth models, it is the case that $\Gamma \Vdash \varphi \wedge \psi$. Thus it is also proven for the $\varphi \wedge \psi$.

The other induction steps are left as an exercise for the reader. ♡

Theorem 15 *Beth models for* IQC *are complete, that is* $\Gamma \vdash \varphi$ *iff for all nodes on all Beth models for* IQC $\Gamma \Vdash \varphi$

Proof The left to right implication is proven in theorem 14.

To prove that if for all nodes on all Beth models $\Gamma \Vdash \varphi$ then $\Gamma \vdash \varphi$, we will prove that $\Gamma \nvdash \varphi$ iff there is a countermodel in which there is a node $\Gamma \nVdash \varphi$.

As we will see in section (4.2.1) there is a way of transforming a Kripke model into a Beth model. If $\Gamma \nvdash \varphi$ then as we have proven in the previous chapter there is a Kripke model $\mathscr{K}$ which is a countermodel proving $\Gamma \nVdash \varphi$. Since there is a transformation from Kripke models to Beth models we can transform $\mathscr{K}$ into a Beth model $\mathscr{B}$, which also proves $\Gamma \nVdash \varphi$. $\heartsuit$

Chapter 4

Transformations

4.1 Relation between Kripke and Beth models

We can define thruth in Beth models in terms of thruth in Kripke models.

Theorem 16 *$k \Vdash_{\mathscr{B}} \varphi$ iff $k \Vdash_{\mathscr{K}} \varphi$ or $\{k' | k' \Vdash_{\mathscr{B}} \varphi\}$ bars k*

Proof As we have proven in theorem 13, φ bars $k \Leftrightarrow k \Vdash_{\mathscr{B}} \varphi$, which proves the left to right implication. For the other direction it suffices to prove that if $k \Vdash_{\mathscr{K}} \varphi \Rightarrow k \Vdash_{\mathscr{B}} \varphi$. This can be done inductively.

- For atomic formulae, the Beth definition of $\Vdash$ is twofold. A node *must* force an atomic formula if it is forced by a preceding node in exactly the same way as in a Kripke model, or it is forced by a set of nodes that bar that node. The former forces that if it holds in a Kripke model, it must hold in a Beth model.

- For the $\wedge$, $\rightarrow$, $\forall$ and $\neg$ the rules for the composites are the same in Beth as in Kripke. As an example we will prove that for $\varphi \rightarrow \psi$ the induction step holds. Let's assume that theorem 16 has been proven for φ and ψ. Now suppose that for a node k in our model $k \Vdash_{\mathscr{K}} \varphi \rightarrow \psi$. Then in all nodes $k' \geqslant k$ it holds that $k' \Vdash_{\mathscr{K}} \psi$ or $k' \nVdash_{\mathscr{K}} \varphi$. For all those nodes $k' \Vdash_{\mathscr{K}} \psi$ it is already proven that $k' \Vdash_{\mathscr{B}} \psi$. For those nodes $k' \nVdash_{\mathscr{K}} \varphi$ it might be the case that in our $\mathscr{B}$-interpretation $k' \Vdash_{\mathscr{B}} \varphi$ because of barring. If that is the case then k' is barred by a set of nodes $\vec{k}$ that forces φ, but not from barring themselves. For those nodes $k'' \in \vec{k}$ it then holds that $k'' \Vdash_{\mathscr{K}} \varphi$ and since they are successors of k this leads to $k'' \Vdash_{\mathscr{K}} \psi$. This means that $k'' \Vdash_{\mathscr{B}} \psi$ and by the barring rule $k' \Vdash_{\mathscr{B}} \psi$.

The proof for the other connectives follows along the same line.

- For the $\vee$ and $\exists$ the Kripke definition is made in such a way, that for a formula to be true in k, k itself would be enough to bar it, in order for the formula in the Beth interpretation to be true. For example, $k \Vdash_{\mathscr{K}} \varphi \vee \psi$ iff $k \Vdash_{\mathscr{K}} \varphi$ or $k \Vdash_{\mathscr{K}} \psi$, both cases being enough to let k be the node to bar itself in order to let $k \Vdash_{\mathscr{B}} \varphi \vee \psi$

♡

In a way, a Kripke model *must* force a formula, if it were true in the past and a Beth model *must* force a formula, if it were true in the past, or it will certainly be true somewhere in the future.

Another way to define the difference between Beth models and Kripke models is as follows:

Theorem 17 *A Kripke model can be seen as a Beth model where a path can also stay in a node, i.e. not every node in a path has to be the successor of the last node, but can also be the same as the last node.*

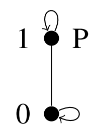
Proof All rules for Beth models lead to the rules of Kripke models when every node can be seen as its own successor in a path. E.g. for $\varphi \vee \psi$ the barring rule in Beth models implies that $\{k|k \Vdash \varphi\} \cup \{k|k \Vdash \psi\}$ bars k. Now there is a path α possible that stays in k indefinitely. In order to bar k, it must be the case that a node in α forces either φ or ψ. This must mean that k must force either φ or ψ, and therefore force it in precisely those cases that a Kripke model forces it. For all the other connectives the same reasoning can be applied. ♡

So in other words, a Kripke model is a reflexive Beth model, a Beth model where you can get ‘stuck’ in a node.

4.2 Transformation from Kripke to Beth

4.2.1 The simple transformation

Taking theorem 17 we can see that taking a Kripke model, we can easily read it as a Beth model, by applying the Beth rules to it, but defining a path in such a way that it can stay in a node. For example we can take model (2.1) and add the reflexive property.



(4.1)

We can unfold this structure into a tree, by taking each reflexive node, copying that node, including successors and relations on them, and adding it as a child. This will result in the Beth model (3.2).

Formally we can define this procedure as follows: take a Kripke model $\mathscr{K} = \langle K, \preccurlyeq, D, \Vdash \rangle$. We will transform it to a Beth model $\mathscr{B} = \langle B, \preccurlyeq', D', \Vdash' \rangle$.

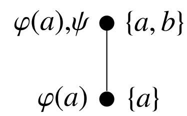
All nodes in K will be uniquely labeled. We will define $\mathscr{B}$ as follows.

- B is defined recursively.
  - For all $k \in K$, if k has no predecessors $\langle k \rangle \in B$
  - If $\langle k_1, k_2, \ldots, k_n \rangle$ is in B, then $\langle k_1, k_2, \ldots, k_n, k_n \rangle$ is also in B
  - For $\langle k_1, k_2, \ldots, k_n \rangle \in B$ and $k_m \in K$, if k_m is an immediate successor of k_n, then $\langle k_1, k_2, \ldots, k_n, k_m \rangle$ is also in B
- The $\preccurlyeq'$-relation is defined as follows: for $\vec{k}_a = \langle k_1, \ldots, k_{n-1} \rangle$ and $\vec{k}_b = \langle k_1, \ldots, k_{n-1}, k_n \rangle$, $\vec{k}_a S \vec{k}_b$, with S being the immediate successor function. $\preccurlyeq'$ is the transitive, reflexive closure over S.
- $D'(k)$ is a function that maps elements $k \in B$ onto $d \in D'$ as follows: if $\langle k_1, \ldots, k_n \rangle$ is a node in B, then $D'(\langle k_1, \ldots, k_n \rangle) = D(k_n)$
- $\Vdash$' is a predicate that is defined as follows: if $\langle k_1, \ldots, k_n \rangle$ is an node in B, then if $k_n \Vdash \phi$ then $\langle k_1, \ldots, k_n \rangle \Vdash' \phi$

4.2.2 The transformation to a constant domain

Definition A Kripke or Beth model has a **constant domain** (or **CD**) if each node has the same domain assigned to it, i.e. $D(k) \equiv D(k')$ for each k and k' in K.

It is not always possible to create a counter model for a certain formula in Kripke semantics using a constant domain. For example, the a countermodel for the formula $\forall x(\varphi(x) \vee \psi) \rightarrow (\forall x \varphi(x) \vee \psi)$ is not possible with a Kripke model with constant domain, for if a node k would force $\forall x(\varphi(x) \vee phi)$ either for all the elements x it would have to be the case that k forces $\varphi(x)$, or it must be the case that k forces ψ, to make the statement true for each element x for which k doesn't force $\varphi(k)$. A Kripke countermodel with a non-constant domain could look something like this:



(4.2)

The class of Beth models with a **CD** however is still complete. Kripke himself deviced a manner to transform a Kripke model $\mathscr{K}$ into a Beth model $\mathscr{B}$ in such a way that $\mathscr{B}$ has a constant domain $\mathscr{D}$. $\mathscr{D}$ is the set of all natural numbers $\mathbb{N}$, and there is a mapping from each node in the Beth model to the elements in the Kripke model.

Formally the transformation is as follows. Given is a Kripke model $\mathscr{K} = \langle K, \preccurlyeq, D, \Vdash\rangle$. We'll define the Beth model $\mathscr{B}$ as follows: $\mathscr{B} = \langle B, \preccurlyeq', D', \Vdash'\rangle$. B and $\preccurlyeq'$ are defined as before:

- B is defined as in (4.2.1).
- The $\preccurlyeq'$-relation is defined as in (4.2.1).

We split up the domain $\mathbb{N}$ into infinitely many infinite subsets $N_1, N_2, \ldots$ in such a way that:

- $\forall N_i, N_j(i \neq j \rightarrow N_i \cap N_j = \emptyset)$, i.e. no two distinct subsets have elements in common
- $\mathbb{N} = \bigcup(N_i$ where $i \in \mathbb{N})$, i.e. every element of $\mathbb{N}$ is in a subset
- $M_j = \bigcup\limits_{i \leq j} N_i$

Let the length of a node $\vec{k} = \langle k_1, k_2, \ldots, k_n\rangle$ be $lth(\vec{k}) = n$. For each node $\vec{k} = \langle k_1, k_2, \ldots, k_n\rangle$ in B we'll define the function $\phi_{\vec{k}}(x)$ in such a way that $\phi_{\vec{k}}$ maps for each k_i in $\vec{k}$ the subset N_i onto $D(k_i)$. This is done in such a way, that for $\vec{k} = \langle k_1, k_2, \ldots k_n\rangle$ and $\vec{k}' = \langle k_1, k_2, \ldots, k_n, k_{n+1}\rangle$ it is the case that $\psi_{\vec{k}}(m) = \psi_{\vec{k}'}(m)$ for all $m \in M_n$. For a node $\vec{k}$ all $m \notin M_{lth(\vec{k})}$ do not map to any nodes in D.

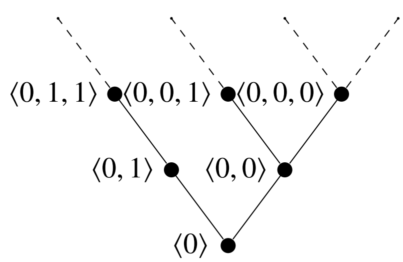
We define the $\Vdash$'-relation for atomic formulae on a node $\vec{k} = \langle k_1, \ldots, k_n \rangle$ as follows:

$$\vec{k} \Vdash' P(d_1, d_2, \ldots, d_m) \text{ if } k_n \Vdash P(\phi_{\vec{k}}(d_1), \phi_{\vec{k}}(d_2), \ldots, \phi_{\vec{k}}(d_m))$$

Now for the countermodel for $\forall x(\varphi(x) \vee \psi) \rightarrow (\forall x \varphi(x) \vee \psi)$, we take the Kripke countermodel (4.2) and use the above rewrite method. First of the disjunct subsets N will be defined by taking for N_1 all the odd numbers, and for each other subset N_i, take all elements from N_{i-1} and multiply them by 2.[1]

- $N_1 = \{1, 3, 5, 7, \ldots\}$
- $N_2 = \{2, 6, 10, 14, \ldots\}$
- $N_3 = \{4, 12, 20, 28, \ldots\}$
- $N_4 = \{8, 24, 40, 56, \ldots\}$
- etc.

In the model (4.2) the root will be called 0, and the child node 1. The Beth model will look like this:



(4.3)

Each node will of course have $\mathbb{N}$ at its domain. For a node $\vec{k} = \langle k_1, k_2, \ldots, k_n \rangle$:

- if k_i is 0, $\phi_{\vec{k}}$ maps all elements from N_i to a.
- if k_i is 1, $\phi_{\vec{k}}$ maps the elements from N_i alternating to a and b.

If $\phi_{\vec{k}}(m)$ maps m to a, then $\vec{k} \Vdash \varphi(m)$. If $\vec{k}$ contains 1, then $\vec{k} \Vdash \psi$.

Now we verify if it is a countermodel for $\forall x(\varphi(x) \vee \psi) \rightarrow (\forall x \varphi(x) \vee \psi)$. First of all, to verify that $\forall x(\varphi(x) \vee \psi)$, take an element $n \in \mathbb{N}$. By definition there is a N_i for which $n \in N_i$. This means that at each node $\langle k_1, \ldots, k_i \rangle$ (thus having

[1] There are of course millions of easy ways of taking infinitely many infinite disjunct subsets from $\mathbb{N}$, but I like this one.

length i) it is either the case that either $k_i = 0$ or $k_i = 1$. In the former case n is mapped to a and thus $\varphi(n)$, in the latter ψ is the case. In either way, $\varphi(n) \vee \psi$. Since the set of all nodes at height i bar the root node, $\varphi(n) \vee \psi$ is true in the root node for any n. Therefore the antecedent is true. Now for the consequent; there is a path $\langle 0, 0, 0, \dots \rangle$, where a 1 is never encountered. In that path ψ will never be the case, and so ψ does not bar the root node, so the root node does not force ψ. Furthermore, in the node $\langle 0, 1 \rangle$ at least one element n is mapped to b (which was not mapped to any element in D in the root node), and it will be mapped to b in all following nodes. The formula $\varphi(n)$ will therefore never be true after that node. Again, the root node does not force $\varphi(n)$ and consequently does not force $\forall x \varphi(x)$. Since it forces neither ψ, nor $\forall x \varphi(x)$ it does not force $\forall x \varphi(x) \vee \psi$, the consequent is not forced and the implication is not true.

Theorem 18 *Let $\mathscr{K}$ be a Kripke model, and $\mathscr{B}$ be a Beth model that is the result of Kripke's own transformation described above. Let $\vec{k} = \langle k_1, \dots, k_n \rangle$ be a node in $\mathscr{B}$. Then for all e where $\phi_{\vec{k}}(e)$ maps to an element in $D(k_n)$*

$$\vec{k} \Vdash' \varphi(e_1, \dots, e_m) \Leftrightarrow k_n \Vdash \varphi(\phi_{\vec{k}}(e_1), \dots, \phi_{\vec{k}}(e_m))$$

Proof For atomic formulae the right to left implication is by definition. There are only three ways which can make a node n force an atomic formula φ in our Beth model. Two of them come from the behaviour of a Beth model, namely if a node before n forces φ, and if a set of nodes that force φ bar n. The third way arises from the transformation itself.

To prove the left to right implication it suffices to prove that the Beth model behaviour does not matter. For predecessors, if given a node $\vec{k} = \langle k_0, \dots, k_n, \dots, k_m \rangle \in B$ and a predecessor $\vec{k}' = \langle k_0, \dots, k_n \rangle \in B$ that forces $P(e_1, \dots, e_q)$, it must be the case that $k_n \Vdash P(\phi_{\vec{k}'}(e_1), \dots, \phi_{\vec{k}'}(e_q)$. k_n is either a predecessor of k_m or it is the same node. Therefore $k_n \Vdash P(\phi_{\vec{k}'}(e_1), \dots, \phi_{\vec{k}'(e_q)}$. $\phi_{\vec{k}}$ maps all elements $e_1, \dots, e_q$ from the formula to exactly the same elements as $\phi_{\vec{k}'}$ by definition. Therefore $\vec{k}$ forces $P(e_1, \dots, e_q)$ too.

It is easy to see that barring does not matter either, since for a node $\vec{k} = \langle k_0, \dots, k_n \rangle$ there is a path consisting of all the nodes $\langle k_0, \dots, k_n \ \dots, k_n \rangle$. All nodes in this path will force the same formulae, for the elements for which $\phi_{\vec{k}}$.

For composite formulae the theorem is proven by induction. For example for a formula $\forall x \varphi(x)$ it is proven as follows: suppose the theorem is proven for a formula $\varphi(x)$. Let $k_n \in K \Vdash \forall x \varphi(x)$ and let $\vec{k} = \langle k_0, \dots, k_n \rangle$. Now let e be an element in $D'(\vec{k})$. If $e \in M_n$ then $\phi_{\vec{k}}(e)$ maps to an element $e' \in D(k_n)$. Since

$k_n \Vdash \forall x\varphi(x)$ it follows that $k_n \Vdash \varphi(e')$. Therefore $\vec{k} \Vdash' \varphi(e)$. If however $e \notin M_n$ then $\phi_{\vec{k}}(e)$ does not map to an element in $D(k_n)$. However there is a height m at which $e \in M_n$. For each node $\vec{k}' = \langle k_1, \dots, k_m \rangle$ at that height e is mapped to an element $e' \in D(k_m)$. Since $k_m \geqslant k_n$ it follows that $k_m \Vdash \varphi(e')$ and $\vec{k}' \Vdash \varphi e$. Therefore $\varphi(e)$ bars $\vec{k}$. We have proven that for any $e \in D'(\vec{k})$, $\varphi(e)$ holds, thus $k_n \Vdash \forall x\varphi(x) \Rightarrow \vec{k} \Vdash' \forall x\varphi(x)$.

Now let $k_n \in K$, let $\vec{k} = \langle k_1, \dots, k_n \rangle \in B$ and let $\vec{k} \Vdash' \forall x\varphi(x)$. Take an element $e \in D'(\vec{k})$. If $\phi_{\vec{k}}(e)$ maps to an element $e' \in D(k_n)$, then $\vec{k} \Vdash' \varphi(e)$ can only be the case if $k_n \Vdash \varphi(e')$. If $\phi_{\vec{k}}(e)$ does not map to an element $e' \in D(k_n)$, then $\vec{k} \Vdash \varphi(e)$ because it is barred. There is a m for which $e \in N_m$. For all nodes $\vec{k}' = \langle k_0, \dots, k_m \rangle$ at height m, e is mapped to an element in k_m. For that node the same reasoning follows. $\vec{k}' \Vdash' \varphi(e)$ can only be if $k_m \Vdash \varphi(\phi_{\vec{k}'}(e))$. So for each element e' in each new node k_m that can possibly come up after k_n it is proven that $k_m \Vdash \varphi(e')$. Therefore $k_n \Vdash \forall x\varphi(x)$. This proves that $\vec{k} \Vdash' \forall x\varphi(x) \Leftrightarrow k_n \Vdash \forall x\varphi(x)$.

The other steps are left as an exercise for the reader. ♡

4.3 Transformation from Beth to Kripke

While a transformation from Kripke models to Beth models is pretty easy, the converse is not. As can be seen in Lòpez-Escobar (1981) [6] it is not even possible to make a natural transformation from Beth models to Kripke models. However, for some Beth models it is possible to transform them into Kripke models.

Definition A propositional Beth model is **strong** if

- for all nodes k it is the case that if $k \Vdash \varphi \vee \psi$, then $k \Vdash \varphi$ or $k \Vdash \psi$.
- for all nodes k it is the case that if $k \Vdash \exists x\varphi(x)$, then there is an element $e \in D(k)$ for which $k \Vdash \varphi(e)$

In such a strong Beth model a transformation is automatic.

Each node in this Beth model is saturated and therefore forces exactly the same formulae as when it is interpreted as a Kripke model.

Even more, when transforming from Kripke to Beth using the simple method, when the resulting Beth model is interpreted as a Kripke model, it is equivalent to the original Kripke model.

Theorem 19 *Given a Kripke model $\mathscr{K}$ and a Beth model $\mathscr{B}$, which is the simple transformation of $\mathscr{K}$. Then given a node $k_n \Vdash_{\mathscr{K}} \varphi$ iff $\langle k_0, k_1, \dots, k_n \rangle \Vdash_{\mathscr{B}} \varphi$.*

Proof Take a node k_n in $\mathscr{K}$ and $b \equiv \langle k_0, k_1, \ldots, k_n \rangle$ in $\mathscr{B}$. By definition both nodes force the same atomic formulae. That the nodes force the rest of the formulae follows by induction on the length of the formula:

- Suppose both k_n and b force φ and ψ, then both k_n as b force $\varphi \wedge \psi$ by the definition of Beth and Kripke models. The inverse (if k_n and b both do not force both formulae φ and ψ) follows along the same lines.

- Suppose both k_n and b force φ or ψ, then both k_n and b force $\varphi \vee \psi$ by the definition of Beth and Kripke models. Suppose k_n and b both do not force φ nor ψ. Then k_n does not force $\varphi \vee \psi$. For b there is a path through b that goes through all nodes $\langle k_0, k_1, \ldots, k_n, \ldots, k_n \rangle$. Since we could prove by induction that b does not force φ and ψ we can prove the same for all nodes $\langle k_0, k_1, \ldots, k_n, \ldots, k_n \rangle$. Therefore there is a path through b in which $\varphi \vee \psi$ meaning that b is not barred by $\varphi \vee \psi$ and thus b does not force $\varphi \vee \psi$.

- Suppose that for all nodes k_n in $\mathscr{K}$ and for all associated nodes $\langle k_0, \ldots, k_n \rangle$ in $\mathscr{B}$ it is proven that $k_n \Vdash \varphi$ iff $\langle k_0, \ldots, k_n \rangle \Vdash \varphi$ and that $k_n \Vdash \psi$ iff $\langle k_0, \ldots, k_n \rangle \Vdash \psi$. Now suppose a node k_n in $\mathscr{K}$ forces $\varphi \to \psi$. Then of course all nodes $k_m \geqslant k_n \Vdash \varphi \to \psi$. So all nodes $\langle k_0, \ldots, k_n, \ldots, k_m \rangle$ force $\varphi \to \psi$ and therefore $\langle k_0, \ldots, k_n \rangle \Vdash \varphi \to \psi$. The proof that $\langle k_0, \ldots, k_n \rangle \Vdash \varphi \to \psi$ then $k_n \Vdash \varphi \to \psi$ follows along the same lines.

♡

This means that if a Kripke countermodel is transformed into a Beth countermodel, this model can then be interpreted as a Kripke model which is again a countermodel for the same formula.

However, if the model is not strong, then there is at least one node that cannot exist in a Kripke model, since all nodes in a Kripke model need to be saturated.

Chapter 5

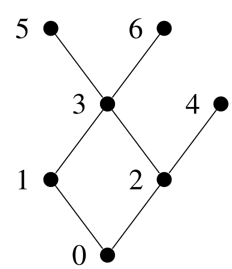
Frame properties

5.1 General frame properties

Although frames are defined as nodes with a partial order defined over them, they are commonly seen as trees. We will now look at several types of non-tree frames and how to rewrite them to trees.

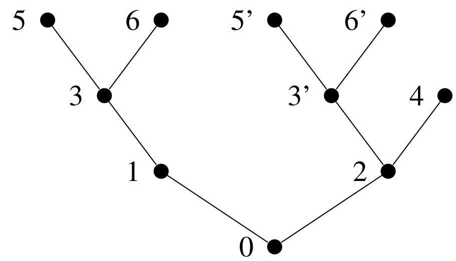
Definition Rewriting a frame $\mathscr{F}$ to another frame $\mathscr{F}$' is constructing $\mathscr{F}$' in such a way that for every model M on $\mathscr{F}$, there can be constructed a model M' on $\mathscr{F}$' in such a way that for every node $k \in M$ there is a $k' \in M'$, where $k \Vdash \varphi \Leftrightarrow k' \Vdash \varphi$.

The simplest type of frames to rewrite is the type where there is only one root. Consider for example the following frame $\mathscr{F}$:



(5.1)

Every path in $\mathscr{F}$ starts at 0. We can simply construct a tree, by taking every node with multiple immediate predecessors and copying that node, and all it's successors (including successor relations). For (5.1) this wil result in the following frame $\mathscr{F}$’:

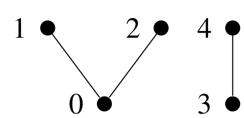


(5.2)

Definition Splitting is the rewriting step that takes a frame $\mathscr{F} \equiv \langle K, \preccurlyeq \rangle$, and two nodes $k_i, k_j \in K$, that have the same immediate successor k_s and rewrites it to $\mathscr{F}' \equiv \langle K', \preccurlyeq' \rangle$, where:

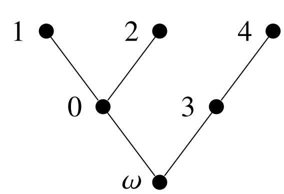
- For all $k_n \in K$ where $k_n \not\succcurlyeq k_s$ there is a $k'_n \in K'$.
- For all $k_m \in K$ where $k_m \succcurlyeq k_s$ there are two nodes $k'_{m,i}, k'_{j,m} \in K'$
- If for $k_n, k_m \in K$ where $k_n \not\succcurlyeq k_s$ and $k_m \not\succcurlyeq k_s$ it is the case that $k_n S k_m$ then for the corresponding nodes $k'_n, k'_m \in K'$ it is the case that $k'_n S k'_m$.
- If for $k_n, k_m \in K$ where $k_n \succcurlyeq k_s$ and $k_m \succcurlyeq k_s$ it is the case that $k_n S k_m$ then for the corresponding nodes $k'_{n,i}, k'_{m,i}, k_{n,j}, k'_{m,j} \in K'$ it is the case that $k'_{n,i} S k'_{m,i}$ and $k'_{n,j} S k'_{m,j}$.
- $k_i S k_{s,i}$ and $k_j S k_{s,j}$.

Consider the following frame:



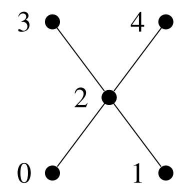
(5.3)

To rewrite a frame consisting of multiple trees (i.e. a forest) into a tree, one simply adds one node, which is the immediate predecessor of the root nodes of the all the trees. The new root node shall only be forcing those atomic formulae that all its direct successors force.



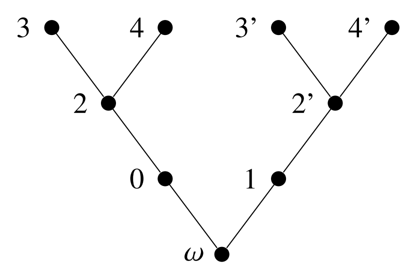
(5.4)

Now consider the next frame:



(5.5)

The same tactics as in the case of the forest can be applied to this frame, adding a root node to the frame, making it the direct predecessor of all nodes without predecessors. This creates a frame which has one root node, and for which we already have a method of creating a tree.



(5.6)

Definition Rerooting is the rewriting step that takes a frame $\mathscr{F} \equiv \langle K, \preccurlyeq \rangle$ where there are no nodes k that have no predecessors k' without predecessors, and rewrites it to $\mathscr{F}' \equiv \langle K', \preccurlyeq' \rangle$ where:

- For all nodes $k_n \in K$ there is a node $k'_n \in K'$.
- If for $k_n, k_m \in K$ it is the case that $k_n S k_m$, then for the corresponding nodes $k'_n, k'_m \in K'$ it is the case that $k'_n S k'_m$.
- There is a node ω in K' that does not correspond to any $k \in K$.
- If $k_n \in K$ has no predecessors, then for the corresponding node $k'_n \in K'$ it is the case that $\omega S k'_n$

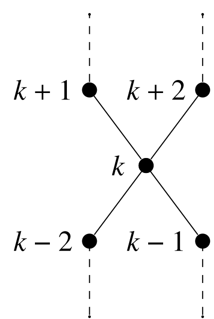
We can use this stacked applying of strategies to rewrite the most common frames.

Theorem 20 *Given that a frame $\mathscr{F}$ contains no infinite descending chains we can rewrite it to a tree $\mathscr{T}$ using only splitting and rerooting.*

Proof If $\mathscr{F}$ contains more than one node that has no predecessors, to create $\mathscr{T}$ we first reroot the frame. After that we will split every point in the tree where two nodes have the same immediate successor. We will do this row[1] by row starting at the first row above ω. The splits will happen on all nodes in that row that have multiple predecessors, and will be executed simultaneously.

Rerooting makes sure that there is only one root node, and the splitting after that does not add new root nodes to the model. Furthermore, it is easy to see that every node in the frame that has multiple immediate predecessors is split at some point so there are no two nodes k and k' anymore for which there is a third node k'' with $k \preccurlyeq k''$ and $k' \preccurlyeq k''$ but neither $k \preccurlyeq k'$ nor $k' \preccurlyeq k$. ♡

A more problematic frame is a frame that does have infinite descending chains. On a frame like this the method above does not suffice to make it into a tree. An example of this is the following:



(5.7)

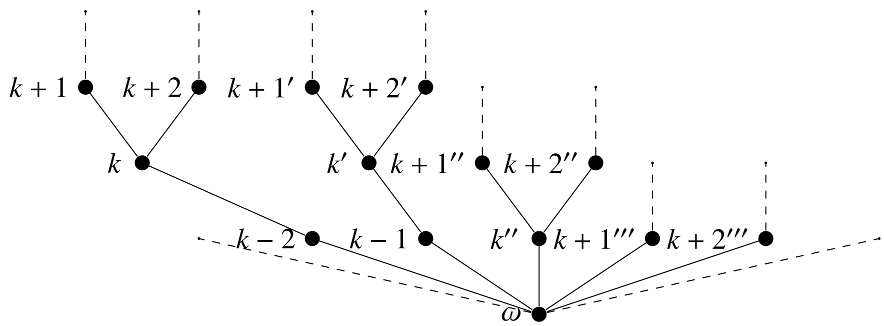
There are no nodes without predecessors. We can of course split the frame in two, but end up with two unrooted trees, which cannot be reconciled in the ways described above.

Frames like this will not be discussed in the sections to come. The following will therefore only be for completeness sake and is not necessary for later chapters.

To rewrite this frame we'll however add a new root node again. For each node in the frame we copy that node including all successors, and make the root node

[1] A row here will be defined the set of all nodes that can be reached in n steps from ω

its immediate predecessor.



(5.8)

This strategy can be used for any frame, but might unnecessarily lead to infinitely branching trees (for example with frames with infinitely many nodes, which are exactly those frames for which we want to use this method). But this shows that all frames can be rewritten into trees.

Definition Grafting is the rewriting method that takes a frame $\mathscr{F} \equiv \langle K, \preccurlyeq \rangle$ and rewrites it into a tree $\mathscr{T}^R \equiv \langle K^R, \preccurlyeq^R \rangle$ using the following method.

K^R will at least contain the node ω, corresponding to an empty sequence. All other nodes will correspond to subpaths in $\mathscr{F}$, that is a node $\vec{k} \in K^R$ is a finite sequence $\langle k_0, k_1, \ldots, k_n \rangle$ of nodes in K, such that

- if k_i and k_j are in $\vec{k}$ then $k_i \preccurlyeq k_j$ or $k_j \preccurlyeq k_i$
- if k_i and k_j are in $\vec{k}$ then all nodes k_m for which $k_i \preccurlyeq k_m \preccurlyeq k_j$ are also in $\vec{k}$

Then let the $\preccurlyeq^R$ relation be the following relation: if $\vec{k}_i$ is prefix of $\vec{k}_j$ then $\vec{k}_i \preccurlyeq^R \vec{k}_j$.

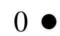
This method can with a small adjustment be used to rewrite dense models, but those models are beyond the scope of this thesis.

5.2 Frame properties of Kripke models

> **N.B.** *In this and following sections we'll restrict ourselves to frames that model partial orders that have no infinite descending chains.*

Definition If any model on a certain frame $\mathscr{F}$ forces a formula φ – that is to say any node in the model forces φ – then $\mathscr{F}$ is said to force φ. In such a case φ is a **frame property** of $\mathscr{F}$.

For example, the simplest frame that exists is of course the following:

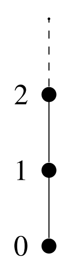


(5.9)

A model on a frame with only one node is exactly like a classical model, because everything that is not yet true now will never be true. This means that any model on (5.9) naturally forces $\neg\neg\varphi \to \varphi$. The entire idea of Kripke and Beth models stems from an extension of the classical notion of counter models. It is easy to see that every maximal node (i.e. a node without a successor) in a Kripke model behaves like a classical node.

Definition We will call a node k in a model **classical**, if every formula that can be classically derived from the formulae in k, is also true in k, or $\Gamma \subseteq k \vdash_c \varphi \Rightarrow k \Vdash \varphi$. A model $\mathscr{K}$ is classical if all nodes in $\mathscr{K}$ are classical. A frame is classical if all possible models on that frame are classical. This definition extends to Beth models as well.

Consider the following frame:



(5.10)

Theorem 21 *A Kripke model $\mathscr{K}$ on a frame in which every node has only one direct successor forces* $(\varphi \to \psi) \vee (\psi \to \varphi)$.

Proof If there is a node $k \in \mathscr{K}$ that forces φ but not ψ, then every node forces $\psi \to \varphi$, for there cannot be a node k' anymore where φ is not the case and ψ is the case, since the former demands that $k' < k$ and the latter demands that $k' > k$. If such a node does not exists, every node forces $\varphi \to \psi$. Since for any model on this frame it is known if such a node exists, it is also known which one is the case. Therefore $\mathscr{K} \Vdash (\varphi \to \psi) \vee (\psi \to \varphi)$. ♡

Definition **Gödel-Dummett Logic** or LC is intuitionistic logic extended with the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$.

Definition A **height** n **tree** is a tree, for which the maximum distance between the root and a node is n.

Theorem 22 *A Kripke model on a height 1 tree, forces* $\varphi \vee (\varphi \rightarrow (\psi \vee \neg\psi))$.

Proof The root node will be called k. Every node that is not k is a maximal node, and therefore classical, thus forcing $\psi \vee \neg\psi$. If $k \Vdash \varphi$ then $k \Vdash \varphi \vee (\varphi \rightarrow (\psi \vee \neg\psi))$. If not then since all the child nodes force $\psi \vee \neg\psi$ and k does not force φ, all nodes force $\varphi \rightarrow (\psi \vee \neg\psi)$, and therefore $\varphi \vee (\varphi \rightarrow (\psi \vee \neg\psi))$. ♡

This theorem can be extended for any height n tree.

Theorem 23 *For a height n tree* $\mathscr{T}$*, a formula* φ *can be constructed for which* $\mathscr{T} \Vdash \varphi$*, but* $\nVdash_i \varphi$*, i.e. no height n tree is intuitionistically complete.*

Proof Given is a height n tree $\mathscr{T}$. To construct a formula φ with the desired properties take the atomic formulae $P_1, P_2, \ldots, P_n$. All nodes at height n are maximal nodes, therefore all nodes at height n force $P_n \vee \neg P_n$. Let's call this φ_n. Following theorem 22, it is clear that all nodes at height $n-1$ force $\varphi_{n-1} = P_{n-1} \vee (P_{n-1} \rightarrow \varphi_n)$. For all nodes at height $n-2$ it is the case that all their successors force φ_{n-1}, thus they themself force $\varphi_{n-2} = P_{n-2} \vee (P_{n-2} \rightarrow \varphi_{n-1})$. Inductively we can define $\varphi_1 = P_1 \vee (P_1 \rightarrow (P_2 \vee (P_2 \rightarrow (\ldots (P_n \vee \neg P_n) \ldots))))$. There is no Kripke model $\mathscr{K}$ on $\mathscr{T}$ that is a countermodel for φ_1, so $\mathscr{T} \Vdash \varphi_1$. But on a height $n+1$ tree this countermodel can easily be constructed, thus $\nVdash_i \varphi_1$. ♡

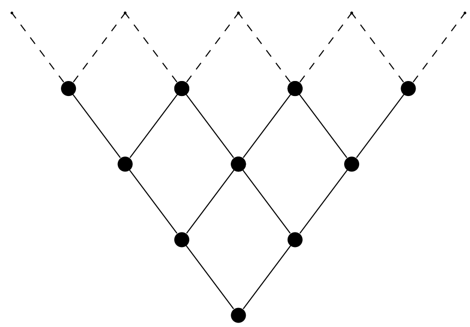
We can say that no finite Kripke frame is complete. However this is different from the claim that there are intuitionistically invalid formulae for which no Kripke countermodel with a finite frame can be constructed.

Definition The logic on trees of a maximum depth n is called a **Logic of Bounded Depth** n, or $\mathbf{BD_n}$

Definition A **beehive** is a frame in which:

- There is exactly one node with no predecessors,
- Every node has exactly two immediate succesors,
- If two nodes have a common immediate predecessor, they also have a common immediate successor.

A beehive looks as follows:



(5.11)

Observe, every two nodes have at least one common successor.

Theorem 24 *A Kripke model on a beehive forces* $\neg\varphi \vee \neg\neg\varphi$.

Proof For each two nodes, there is a third node that is a successor of both. If there are no nodes that force φ we have $\neg\varphi$ and therefore $\neg\varphi \vee \neg\neg\varphi$. In any other case we have a node that forces φ. In that case every other node has a common successor with that node, which has to force φ. So no node can force $\neg\varphi$, which leads to every node forcing $\neg\neg\varphi$ and which implies $\neg\varphi \vee \neg\neg\varphi$. ♡

Definition De Morgan Logic or KC is Intuitionistic Logic extended with the axiom $\neg\varphi \vee \neg\neg\varphi$. This axioma is also known as the **Law of Weak Excluded Middle** or **WEM**.

5.3 Frame properties of Beth models

Theorem 25 *If a frame is finite, any Beth model on that frame is classical.*

Proof To prove that a Beth model is classical it suffices to prove that in every node $P \vee \neg P$ is true. It is easy to see that in a maximal node this is the case, for either P is the case in that node, or P will never be the case, being the maximal node. So all maximal nodes have $P \vee \neg P$. Since the model is finite, there can be no infinite paths. Thus every path will end in a maximal node. Therefore every node is barred by the set of maximal nodes and $P \vee \neg P$ is true in every node. ♡

Theorem 26 *If a node k has only one immediate successor* k', *then* $k \Vdash \varphi \Leftrightarrow k' \Vdash \varphi$.

Proof Trivial, $k \preccurlyeq k'$, so $k \Vdash \varphi \Rightarrow k' \Vdash \varphi$ and $\{k'\}$ bars k, so $k' \Vdash \varphi \Rightarrow k \Vdash \varphi$. ♡

Theorem 27 *A Beth model in which each node has at most one immediate successor is classical.*

Proof Again, to prove that a Beth model $\mathscr{B}$ is classical, it suffices to prove that in every node $P \vee \neg P$ is true. If $\mathscr{B}$ is finite, we have already proven that it is classical. We only need to prove that it is classical for infinite models. We will first presume that $\mathscr{B}$ only has one path (i.e. that it is a unary tree). By repeated application on theorem 26, it follows that all nodes are equal to eachother. So if $k \nVdash P$ then no $k' \in K$ forces P, and therefore, $k \Vdash \neg P$. Otherwise, $k \Vdash P$. In any case, $P \vee \neg P$ is the case. For more paths in $\mathscr{B}$ the argument goes the same for each path (since those paths don't cross, they can be treated as single linear Beth models). ♡

Theorem 28 *A Beth model with finitely many branches is classical*

Proof We will again prove that for every node in Beth model $\mathscr{B}$ the formula $P \vee \neg P$ is true. Suppose $\mathscr{B}$ has a finite number of branches. Then every path will eventually reach a node, after which there will be no more branching. By theorem 27 we can see that all successive nodes force the same and in that node $P \vee \neg P$ is true. Thus in every path we will come across a node where $P \vee \neg P$ is true. Therefore every node is barred by $P \vee \neg P$, and will force $P \vee \neg P$ itself. ♡

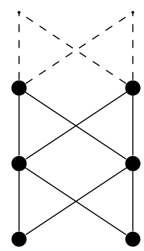
Definition A frame is a **spread** if it is a tree and every node has at least one successor.

Definition A frame is a **fan** if it is a spread and every node has at most finitely many immediate successors

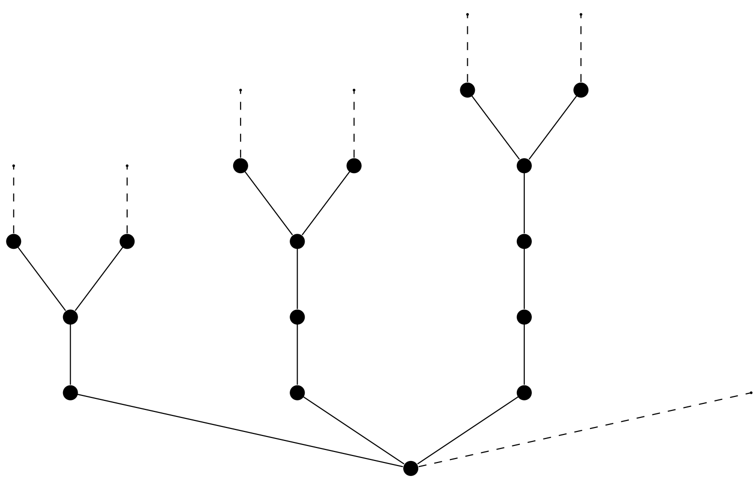
Theorem 29 *If there is a height in a tree after which every node has at most one successor, it is classical*

Proof The proof of this is not that hard. If there is a point after which the tree stops branching, every node at that height is classical because of theorem 27. These nodes bar all nodes below it, and therefore the frame is classical. ♡

All fans that are classical have a highest point of branching. There are however other frames that are classical, but do not have a highest point of branching. Here are two examples.



(5.12)



(5.13)

In model (5.12) if a formula φ is true in any node, then its immediate successors bar every node beneath them. So either φ is not true in any node, or φ is true in all nodes, making the model as expressive as a single node in the model, and clearly making it classical.

In model (5.13) the root node ω has infinitely many immediate successors, $k_1, k_2, \ldots$. Each immediate successor k_n has a 'branching node' s_n and $n-1$ fully ordered nodes between k_n and s_n. s_n has two immediate successors, l_n and r_n, both having infinitely many fully ordered successors. For each n it is easy to see that l_n and r_n are classical. Since they bar k_n, k_n must also be classical. And since this is true for all n's the set of k's bar ω, making it classical.

So not only frames that have a maximum branching height can be classical. However there is a property that every non-classical frame has to have.

Definition A path $\langle k_0, k_1, \ldots \rangle$ is a frame $\mathscr{F}$ is said to have the **offroad property** if there exists the set $L \equiv \langle l_0, l1, \ldots \rangle$ and

- $\vec{k}$ is infinite
- $\forall k \in \vec{k} \exists l \in L(l \geqslant k)$
- $\forall l \in L \forall k \in \vec{k}(k \ngeqslant l)$

i.e. each node in the path has a successor in L and no node in L has a successor in $\vec{k}$.

Theorem 30 *A frame $\mathscr{F}$ is non-classical iff there is a path with the offroad property.*

Proof To prove that a frame is non-classical if there is a path with the offroad property, we take a frame with a path with the meta-comb property and construct a model on it that is not classical.

Let $\mathscr{F}$ be a frame with a non-empty path $\vec{k}$ with the offroad property. Then for $\vec{k}$ there exists a set of nodes L as in the definition. We construct the model $\mathscr{B}$ on $\mathscr{F}$ in such a way that $\forall l \in L(l \Vdash p)$. All other $\Vdash$-relations are defined by the rules of Beth models.

Now we prove that no node in $\vec{k}$ forces $p \vee \neg p$. Take a node $k \in \vec{k}$. This node has by definition a successor $l \in L$. $l \Vdash p$ therefore $k \nVdash \neg p$. k has no predecessors in L and since the path $\vec{k}$ through k has no nodes forcing p, k is not barred by nodes forcing p. Therefore $k \nVdash p$ and thus $k \nVdash p \vee \neg p$. This means that $\mathscr{F}$ is non-classical.

To prove the reverse we take a non-classical frame and show that there is a path in that frame that has the offroad property.

Let $\mathscr{F}$ be a non-classical frame. Then there is a model on the frame not forcing $\varphi \vee \neg\varphi$ for a certain formula φ. This means that there is a node k not forcing $\varphi \vee \neg\varphi$. For k not to force $\varphi \vee \neg\varphi$, it has to be the case that k is not barred by formulae forcing $\varphi \vee \neg\varphi$. From this follows that there is an infinite path $\vec{k}$ through k that does not force $\varphi \vee \neg\varphi$. Since all nodes $k_n \in \vec{k}$ do not force $\varphi \vee \neg\varphi$, it is the case that:

- $k_n \nVdash \varphi$
- $k_n \nVdash \neg\varphi$

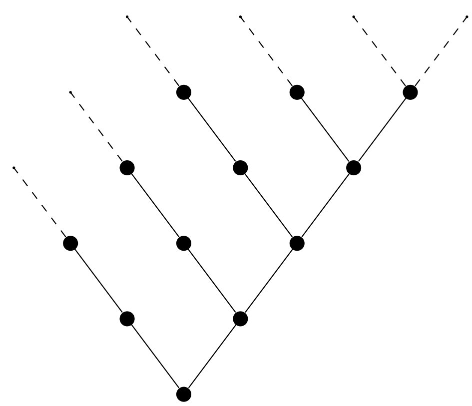
To make the latter true there is for every node k_n a successor l_n forcing φ. Let the set of all l's be called L. No node in L can have a succesor in $\vec{k}$ or the former condition will not be true. Therefore $\vec{k}$ satisfies the conditions for the offroad property. ♡

Definition A **comb** is a fan, for which it is the case that:

- The root node has two immediate successors,
- if a node has two immediate successors, then one of those successors has itself two immediate successors and the other has only one immediate successor,
- if a node has one immediate predecessor, it also has one immediate successor.

The **main branch** is that branch with all nodes with 2 immediate successors.

There is only one frame which is a comb, which is depicted in (5.14). We have already seen a comb in (3.2). All ways of depicting a comb can be folded to (5.14).



(5.14)

Theorem 31 *A Beth model on a comb forces* $(\varphi \vee \neg\varphi) \vee (\varphi \to (\psi \vee \neg\psi))$.

Proof Take a Beth model on a comb. Following the same reasoning as in theorem 27, it is clear that on each node, that is not in the main branch, it is the case that $\varphi \vee \neg\varphi$ and $\psi \vee \neg\psi$. If on the main branch there is a node that forces φ, then consequently $\varphi \vee \neg\varphi$ bars the root node. If however this is not the case, then every node that forces φ is not on the main branch, and for all those nodes we have already proven that $\psi \vee \neg\psi$. Therefore, given a Beth model $\mathscr{B}$ on a comb, we can prove that $\mathscr{B} \Vdash (\varphi \vee \neg\varphi) \vee (\varphi \to (\psi \vee \neg\psi))$. ♡

5.3.1 Beth models and KC

Lemma 32 *For each Beth model $\mathscr{B}$ on a beehive each formula φ for which $\mathscr{B} \nVdash \neg\varphi$ has one origin, that is* $\forall(k, k' \in K((k \Vdash \varphi \wedge k' \Vdash \varphi) \rightarrow \exists k'' \in K(k'' \Vdash \varphi \wedge k'' \preccurlyeq k \wedge k'' \preccurlyeq k')))$.

Proof Take two nodes that force φ, k and k'. If one of the two is a predecessor of the other, the property holds. If neither is a predecessor of the other, there is a third node k'', that is a predecessor of both nodes, and each path through k'' also goes through k or k' or any of their children. This can be proven as follows.

A beehive can be seen as a series of binary choices between 0 and 1 forming a sequence $\langle c_0, c_1, \dots, c_n\rangle$. Two sequences from a node reach the same node iff both sequences have the same amount of 0's and 1's, in whichever order. Now let the node k'' be that node for which one node (let this be k) can be reached by $\langle 0, 0, \dots, 0\rangle$ and the other (k') by $\langle 1, 1, \dots, 1\rangle$. All nodes that can be reached with sequences with as many or more zeroes as in the first sequence end up in k or a successor. For the sequences with as many or more ones as in the second sequence, the reached node is k' or a successor. So for each path through k'' there will eventually be a node that either is a successor of k or a successor of k', both forcing φ. If k'' still isn't the origin, then there is another node that is not a child of k'' and forces φ. We can preform the same trick as long as we have not found the origin. ♡

Theorem 33 *A Beth model on a beehive forces $\neg\varphi \vee \neg\neg\varphi$.*

Proof Same as the proof of theorem 24. ♡

As has been noted before the rule $\neg\varphi \vee \neg\neg\varphi$ is called the Weak Excluded Middle and Intuitionistic Logic extended with WEM is called KC.

Definition A path $\vec{k} \equiv \langle k_0, k_1, \dots \rangle$ in a frame $\mathscr{F}$ is said to have the **meta-comb property** if there exist the sets $L \equiv \{l_0, l_1, \dots\}$ and $M \equiv \{m_0, m_1, \dots\}$ and

- $\vec{k}$ is infinite
- $\forall k \in \vec{k} \exists l \in L(l \succcurlyeq k)$
- $\forall k \in \vec{k} \exists m \in M(m \succcurlyeq k)$
- $\forall l \in L \forall k \in K(k \nsucceq l)$

- $\forall m \in M \forall k \in K(k \not\geqslant m)$
- $\forall l \in L \forall m \in M \neg\exists o(o \geqslant l \wedge o \geqslant m)$

i.e. each node in the path has a successor in L and in M, no node in L has successors in common with a node in M and no nodes in either L or M have successors in the path. [2].

Theorem 34 *A frame $\mathscr{F}$ does not force Weak Excluded Middle, iff there is a path with the meta-comb property.*

Proof To prove that a frame does not force WEM if there is a path with the meta-comb property, we take a frame with a path with the meta-comb property and construct a model on it that does not force WEM.

Let $\mathscr{F}$ be a frame with a non-empty path $\vec{k}$ with the meta-comb property. Then for $\vec{k}$ there exist the sets of nodes L and M . We construct the model $\mathscr{B}$ on $\mathscr{F}$ in such a way that $\forall l \in L(l \Vdash p)$. All other $\Vdash$-relations are defined by the rules of Beth models.

Now we prove that no node in $\vec{k}$ forces WEM. Take a node $k \in \vec{k}$. This node has by definition a successor $l \in L$ and $m \in M$ as in the definition. $l \Vdash p$ therefore $k \nVdash \neg p$. m is not a successor of a node in L and no successor of m is in L or a successor of a node in L. Therefore no successor of m forces p, and thus $m \Vdash \neg p$. So we can see that $k \nVdash \neg\neg p$. Since we can show that this is the case for all nodes in $\vec{k}$ that are successors of k, we also know that k is not barred by a set of nodes that force WEM. Therefore $k \nVdash \neg p \vee \neg\neg p$, and thus $\mathscr{F}$ does not force WEM.

To prove the reverse we take a frame that does not force WEM and show that there is a path with the meta-comb property in that frame.

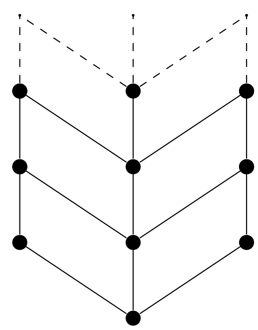
Let $\mathscr{F}$ be a frame that does not force WEM. Then there is a model $\mathscr{B}$ on $\mathscr{F}$ that does not force WEM and subsequently a node k in $\mathscr{B}$ that does not force $\neg\varphi \vee \neg\neg\varphi$ for a formula φ. Since we only deal with frames without infinite descending chains, there is a root node k_0 not forcing $\neg\varphi \vee \neg\neg\varphi$. For this to be the case, it must be so that k_0 is not barred by a set of nodes forcing $\neg\varphi \vee \neg\neg\varphi$. And since any maximal node forces this formula, there has to be an infinite path $\vec{k}$ of nodes not forcing $\neg\varphi \vee \neg\neg\varphi$, starting from k_0. For all those nodes $k_n \in \vec{k}$ not to force $\neg\varphi \vee \neg\neg\varphi$ it also has to be the case that:

[2]The meta-comb property is named this way, because a frame with such a path can be seen as a collection of subtrees, where each subtree can be seen as a node in a comb. Of course a meta-comb forces very different formulae than an actual comb.

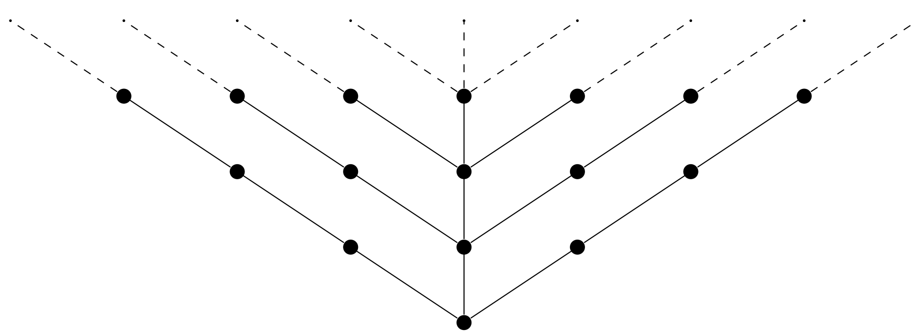
- $k_n \nVdash \neg\varphi$
- $k_n \nVdash \neg\neg\varphi$

To make the former true, k_n has to have a successor l forcing φ, and for the latter to have a successor m forcing $\neg\varphi$. Let the set of all l's be L and the set of all m's be M. Since no node in $\vec{k}$ may force either $\neg\varphi$ or $\neg\neg\varphi$, L and M may not have any successors in $\vec{k}$, and since all nodes in L force $\neg\varphi$ and all nodes in M force $\neg\neg\varphi$ they can not have any common successors. Therefore $\vec{k}$ fulfills the definition of a meta-comb. ♡

Following are two frames that do not force WEM.



(5.15)



(5.16)

5.3.2 Beth models and LC

Theorem 35 *A beehive forces the formula* $(\varphi \to \psi) \vee (\psi \to \varphi)$.

Proof Naturally this follows from the fact that KC is stronger than LC – of which $(\varphi \to \psi) \vee (\psi \to \varphi)$ is the defining formula – and that the beehive forces KC. But for instructive purposes we give a semantic proof.

Let frame $\mathscr{F}$ be a beehive and φ and ψ be two formulae. If there is no node in $\mathscr{F}$ that forces φ then the frame forces $\psi \to \varphi$ and therefore $(\varphi \to \psi) \vee (\psi \to \varphi)$. The same reasoning goes for a frame without nodes that force ψ. By case distinction we are left with one case, the case that there are both nodes that force φ and nodes that ψ. As proven in lemma (32) there are nodes k_φ and k_ψ that are predecessors of all other nodes forcing respectively φ and ψ. These nodes – as noted in (32) – are characterised by the amount of 0's (times left) and 1's (times right) are needed to reach it. Let the coordinate of a node be $\langle n, m\rangle$ where n is the amount of 0's and m the amount of 1's. Then let $\langle n_\varphi, m_\varphi\rangle$ be the coordinate of k_φ and $\langle n_\psi, m_\psi\rangle$ be the coordinate of k_ψ. Then $\langle \max(n_\varphi, n_\psi), \max(m_\varphi, m_\psi)\rangle$ is the coordinate of the lowest node $k_{\varphi\wedge\psi}$ that forces $\varphi \wedge \psi$. It is of course the case that $k_{\varphi\wedge\psi} \Vdash (\varphi \to \psi) \vee (\psi \to \varphi)$. Any node k on the same height as $k_{\varphi\wedge\psi}$ (note that the height can be defined by the sum of m and n) either:

- $k \Vdash \varphi$, then $k \Vdash (\varphi \to \psi) \vee (\psi \to \varphi)$,
- $k \Vdash \psi$, then $k \Vdash (\varphi \to \psi) \vee (\psi \to \varphi)$,
- $k \nVdash \varphi$ and $k \nVdash \psi$, then the coordinate $\langle n_k, m_k\rangle$ has either a lower n_k than $\min(n_\varphi, n_\psi)$, or a lower m_k than $\min(m_\varphi, m_\psi)$. Note that they cannot be both lower, because $n_k + m_k$ is fixed. Suppose $n_k < n_\varphi \leq n_\psi$ and $m_k > m_\varphi \geq m_\psi$. Therefore any node $k' \geqslant k$ for which $k' \Vdash \psi$ then it also must force φ because the coordinate $\langle n_{k'}, m_{k'}\rangle$ has an $n_{k'}$ equal or greater than n_ψ and which is equal or greater to n_φ, and since it is a successor of k which has an m_k greater than m_φ it also has an $m_{k'}$ greater than m_φ. Therefore it is a successor of k_φ and forces φ. So $k \Vdash \psi \to \varphi$ and thus $(\varphi \to \psi) \vee (\psi \to \varphi)$. The same reasoning holds if m_k it less than $\min(m_\varphi, m_\psi)$ and if $n_\varphi \geq n_\psi$ or $m_\varphi \leq m_\psi$.

Thus all nodes at the height of $k_{\varphi\wedge\psi}$ force $(\varphi \to \psi) \vee (\psi \to \varphi)$ This bars all nodes below it, thus a beehive forces $(\varphi \to \psi) \vee (\psi \to \varphi)$. ♡

Remember that LC is Intuitionistic Logic + $(\varphi \to \psi) \vee (\psi \to \varphi)$.

Definition A path $\vec{k} \equiv \langle k_0, k_1 \ldots\rangle$ in a frame $\mathscr{F}$ is said to have the **ghostbuster property** if there exist the sets $L \equiv \{l_0, l_1, \ldots\}$ and $M \equiv \{m_0, m_1, \ldots\}$ and

- $\vec{k}$ is infinite
- $\forall k \in \vec{k} \exists l \in L (l \geqslant k)$

- $\forall k \in \vec{k} \exists m \in M(m \geqslant k)$
- $\forall l \in L \forall k \in \vec{k}(k \ngeqslant l)$
- $\forall m \in M \forall k \in \vec{k}(k \ngeqslant m)$
- $\forall l \in L \exists \vec{l}(\vec{l}$ is a path through l and $\forall m \in M \neg \exists o \in \vec{l}(o \geqslant m))$
- $\forall m \in M \exists \vec{m}(\vec{m}$ is a path through m and $\forall l \in L \neg \exists o \in \vec{m}(o \geqslant l))$

i.e. each node in the path has a successor in L and M, through each node in L there is a path that does not contain successors of nodes in M and the other way around, and the nodes in L and M have no successors in $\vec{k}$. [3].

Theorem 36 *A frame $\mathscr{F}$ does not force* LC*, iff there is path with the ghostbuster property.*

Proof To prove that a frame does not force LC if there is a path with the ghostbuster property, let us assume we have a frame with a path that has the ghostbuster property and construct a model on this frame that is a countermodel to LC.

Let $\mathscr{F}$ be a frame with a path $\vec{k}$ with the ghostbuster property. Then for that path $\vec{k}$ there exist the sets L and M as in the definition. Let $\mathscr{B}$ be a model on $\mathscr{F}$ where all nodes in L force p and all nodes in M force q. All other $\Vdash$-relations are defined by the rules of Beth models.

Now we prove that no node in $\vec{k}$ forces $(p \to q) \vee (q \to p)$. Take a node $k \in \vec{k}$. For that k there are an $l \in L$ and $m \in M$, where $l \Vdash p$ and $m \Vdash q$. There is a path $\vec{l}$ through l for which no nodes are successors of nodes in M, therefore those nodes do not force q. Thus $l \nVdash q$ and $l \nVdash p \to q$. The opposite is true for m, thus $m \nVdash q \to p$. Therefore $k \nVdash p \to q$ and $k \nVdash q \to p$, and since none of the nodes in $\vec{k}$ force this, k is also not barred by $(p \to q) \vee (q \to p)$, so $k \nVdash (p \to q) \vee (q \to p)$, implying that $\mathscr{B}$ and in extention $\mathscr{F}$ do not force LC.

To prove the reverse we take a frame that does not force LC and show that it has a path with the ghostbuster property.

Let $\mathscr{F}$ be a frame that does not force LC. Then there is a model $\mathscr{B}$ that has a node k, for which $k \nVdash (\varphi \to \psi) \vee (\psi \to \varphi)$ for formulae φ and ψ. Because we do not allow infinite descending chains, there is a root node k_0 that does not force

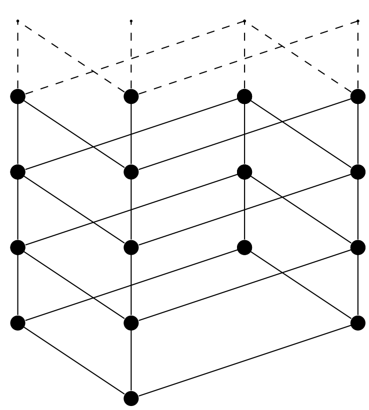
[3]The ghostbuster property is named this way as a reference to a famous quote from the movie The Ghostbusters: "There is something very important I forgot to tell you. Don't cross the streams".

$(\varphi \to \psi) \vee (\psi \to \varphi)$. For k_0 not to force $(\varphi \to \psi) \vee (\psi \to \varphi)$ it at least has to be the case that k_0 is not barred by nodes forcing $(\varphi \to \psi) \vee (\psi \to \varphi)$. Any maximal node forces $(\varphi \to \psi) \vee (\psi \to \varphi)$ therefore there has to be an infinite path $\vec{k}$ from k_0 of nodes not forcing $(\varphi \to \psi) \vee (\psi \to \varphi)$. For each node $k \in \vec{k}$ not to force $(\varphi \to \psi) \vee (\psi \to \varphi)$ it also has to be the case that $k \nVdash \varphi \to \psi$ and $k \nVdash \psi \to \varphi$. This means for one that $k \nVdash \varphi$ and $k \nVdash \psi$. It also means that k has to have successors l and m for which:

- $l \Vdash \varphi$
- $l \nVdash \psi$
- $m \Vdash \psi$
- $m \nVdash \varphi$

Let the set of the l's corresponding to each of the $k \in \vec{k}$ be L and the set of m's corresponding to each of the $k \in \vec{k}$ be M. For any node $l \in L$, if it has a successor $k \in \vec{k}$ then, $k \Vdash \varphi$ which is not the case. Therefore $\forall l \in L \forall k \in \vec{k}(k \ngeq l)$. The same is true for M. Now if a node l has no path $\vec{l}$ such that no node in $\vec{l}$ is a successor of a node in M, then this means that l is barred by successors of m, and therefore $l \Vdash \psi$, which we have excluded. Therefore there must be a path $\vec{l}$ through l for which no nodes in $\vec{l}$ are successors of nodes in M. The same is true for all nodes $m \in M$. Therefore the path $\vec{k}$ satisfies the demands for the ghostbuster property. ♡

The frames shown as examples for frames that do not force KC – frame (5.15) and (5.16) – can also be used as examples for frames not forcing LC. The following frame does not force LCbut does force KC.



(5.17)

Chapter 6

Conclusions

In chapter 4 we have compared Kripke to Beth models and we have seen that Kripke and Beth models are easily defined in each others definition. We have seen two transformations from Kripke to Beth and we have seen that transformations from Beth to Kripke can only happen in very specific cases – i.e. when the Beth model is strong – and in that case the transformation is immediate.

In chapter 5 we have looked at frame properties. First of all we looked at how to rewrite frames into trees in such a way that there is an injection from the frame to the tree. It became clear that for frames without infinite descending chains it is easy to rewrite by only using two simple methods, named splitting and rerooting. For frames that do contain infinite descending chains the method of grafting can be used, which is more brute force and has some possibly unwanted side-effects, like infinite branching for infinite models.

After that we looked at frames for Kripke models and looked at a couple of frames to see which formulae it forced. We have seen frames forcing different formulae, most notably frames forcing KC and LC. These properties have been studied more thoroughly in Fiorentini [3].

In section 5.3 we have looked at frame properties for Beth models. These properties have not been studied before. We find that the defining features for frame properties for certain logics are often the non-existence of paths with certain properties. We define those paths for classical logic, KC and LC.

6.1 Future research

For two intermediate logics – KC and LC – we have given properties defining frames that force those logics. It would be very interesting to see if there are defining properties for other logics and what they are, e.g. for the logic of bounded depth/branching, the Kreisel-Putnam Logic and Medvedev Logic.

Also it could be of interest to see if there are frames that are complete in Beth semantics for a given intermediate logic.

6.2 Relevance for artificial intelligence

Intuitionistic logic is a topic that is relevant for the field of artificial intelligence. It produces more intuitive proofs than classical logic and is therefore often more closely related to human reasoning. Also since constructive proofs are stronger, they are more desirable.

Furthermore the constructive proofs are used in a couple of programming languages – most notably COQ. In these programming languages countermodels are very important. Research into Beth semantics might come up with relevant information. However this thesis has come out inconclusive.

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