University of Utrecht

Master Thesis

A no-go theorem for dyonic Lifshitz black branes

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Our imagination is stretched to the utmost, not, as in fiction, to imagine things which are not really there, but just to comprehend those things which are there.

Richard Feynmann

Acknowledgements

As a child I was taught to be humble and grateful. Writing down this note to thank all the people that have helped me during my master thesis is just a small gesture to show my appreciation.

When I was young, the great mystery of nature already attracted me. I wanted to know how people solved that great puzzle of nature. Especially how the world around me could be modelled in a mathematical way. When I grew older I realised that it was naive to think that I could study it all. At that time I learned about the existing of more exotic things. Black holes, quantum mechanics, relativity, etc suddenly became my main focus. For that reason I was very pleased that I was admitted to study theoretical physics at Utrecht university where all researchers are at the top of their domain. I will be eternal grateful to my parents for giving me this opportunity.

Secondly I would like to thank my supervisor professor Stefan Vandoren for guiding me through this thesis. Many times when I got stuck he helped me back on the right track. I would also like to thank him for all the fruitful discussions that we had. It always improved my understanding of the problem at hand.

Next I would like to thank all the other students that helped me. Especially Brecht Truijen. Many times we sat late at night discussing about how to interpret new results, how to solve the newly formed problems, etc. It was a pleasure to have someone as friend who really understood how it was to do research in theoretical physics and to motivate me when I was feeling down.

At last I would like to thank my girlfriend Michelle for all her love and moral support. She was always there to cheer me up. With her around I always felt relaxed and motivated which meant a lot to me.

Abstract

The original idea of this master thesis was to calculate a Hall conductivity from a field theory with anisotropic scaling in $2+1$ dimensions using a dyonic black brane in the $3+1$ dimensional dual Lifshitz spacetime with a $U(1)$ x $U(1)$ abelian gauge group. In order to calculate the Hall conductivity using the dual theory, one needs electric and magnetic charges originating from the same $U(1)$ gauge field in the bulk. However the dual theory didn't admit a solution for this system such that the Hall conductivity could not be calculated using holography. This led to a no-go theorem for dyonic Lifshitz black branes in $3+1$ dimensions.

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Chapter 1

Introduction

1.1 Thesis subject

Transport coefficients of physical systems provide you with a lot of important information. These coefficients can tell one how currents of matter will flow when certain potentials are turned on. For example, electrical conductivity states how much electrical current will flow upon application of an electrical potential

$$J = \sigma E \ . \tag{1.1}$$

For phases of matter at small couplings, these transport coefficient are well understood and can be analytically computed.

In the last decades strongly coupled phases were discovered such that known weakly coupled theories failed to describe the microscopic behaviour. A nice example of a physical system which exhibits those exotic phases are the high T_c superconductors.[1] Hence, one can not acquire results for the transport coefficients. This led to a large challenge in condensed matter theory in the beginning of this century. Luckily a solution was proposed in 2007 by Herzog, Kovtun, Sachdev and Son [1]. They proposed that the AdS/CFT correspondence can be used to calculate the transport coefficients of the strongly coupled phases.

In this thesis we will try to calculate a specific transport coefficient: the Hall conductivity. We want to find the Hall conductivity for a strongly coupled field theory, a possible condensed matter system. Furthermore we will assume that this field theory possesses an anisotropic scaling symmetry. The idea for this calculation is based on the papers [2] and [3]. In [2] a Hall conductivity

[1] This will be discussed in section 3.1.

was calculated for a field theory with isotropic scale invariance. The second paper [3] introduced an analytical solution for field theories at nonzero charge density with an anisotropic scale invariance using the dual theory. During the thesis project it became clear that our system couldn't be solved with the procedure of [1]. This led to a no-go theorem.

1.2 Outline

This thesis is divided over 2 parts. The first part explains all the concepts that are required to understand the physics of the thesis subject. We start with an introduction to the AdS/CFT correspondence. We give a motivation for the correspondence, we discuss how physical quantities of the two theories are related and extend the correspondence to theories with finite temperature, finite chemical potential and background magnetic fields. The next chapter handles on condensed matter theories. The correspondence will be generalised to condensed matter theories. Furthermore we derive the formula of the electrical conductivity and conclude that we are able to calculate the electrical conductivity with the correspondence.

The second part of this thesis covers all the calculations and interpretations of the thesis subject. We start with a basic model in chapter 4 to give the reader some comfort with the system. Afterwards we continue with an extended model in order to calculate the Hall conductivity. It shall become clear that this model admits no solution in the dual theory and leads to a no-go theorem. All these results are discussed in the last chapter to conclude this thesis.

Part I

Preliminaries

Chapter 2

AdS/CFT correspondence

We start with some preliminaries concerning the anti de Sitter/conformal field theory (AdS/CFT) correspondence. First we make a small journey back in time to motivate why the AdS/CFT correspondence received much attention in the physics community. At the same time, we will try to explain some interpretations of the correspondence. Then we will introduce the quantitative formalism of the AdS/CFT correspondence and show how to compute correlation functions. We conclude by discussing some generalisations and introduce concepts like non-zero temperature, electric charge and background magnetic fields in the conformal theory and their counterparts in the AdS-spacetime.

2.1 Motivation

The AdS/CFT correspondence was first proposed by Maldacena in 1998 [4].[1] The main idea in his paper was that supergravity or string theory on a $d+1$-dimensional AdS-spacetime together with a compact manifold corresponds with a special kind of conformal field theories in d-dimensions. To be precise, he stated that $\mathcal{N}=4$ super Yang-Mills theory in four dimensions with gauge group $SU(N)$ is equivalent to type IIB superstring theory on $AdS_5 \times S^5$, where it was assumed that N was large.

There was a lot of activity after the publication of the paper of Maldecena. Among them Gubser, Klebanov, Polyakov and Witten. They proposed a relation between observables of the conformal field theory and fields of the supergravity action [5],[6]. This made it possible to calculate correlation functions in a strongly coupled d-dimensional conformal field theory by performing much simpler perturbative calculations in a weakly coupled supergravity theory on a $d+1$-dimensional AdS spacetime. This is the true power of the correspondence: gathering information about a complicated strongly coupled theory by considering its much better understood and weakly coupled dual theory.

Another important feature of the AdS/CFT correspondence that was pointed out, was the holographic interpretation. One can understand the conformal field theories of the correspondence as living on a flat d-dimensional boundary M_d of the $d+1$-dimensional AdS-spacetime (AdS_{d+1}). See Figure 2.1 for a graphical interpretation. For example, it was shown in [4] that M_d is a copy of Minkowski space. In this interpretation, the AdS space of the correspondence is mostly referred to as the bulk and the the flat d-dimensional space-time on which the conformal field theories live as the boundary.

Likewise it was shown that the bulk spacetime and the boundary theory possess the same symmetry: $SO(2,d)$. On the boundary this symmetry group is the group of conformal symmetries of the CFT. In the bulk however, $SO(2,d)$ is just an ordinary symmetry of the AdS_{d+1} spacetime. One can conclude that both theories are different representations of the same symmetry.

The observation of the gauge/gravity duality was a major breakthrough in the nineties since the understanding of the strong force using QCD was still far from obvious. Now the duality made it possible to calculate specific properties of the strongly coupled theories using weakly coupled supergravity theories or string theories. In particular one hoped that the correspondence could be applied to supersymmetric versions of QCD. Ironically string theory

[1] The AdS/CFT correspondence is also called the gauge/gravity duality.

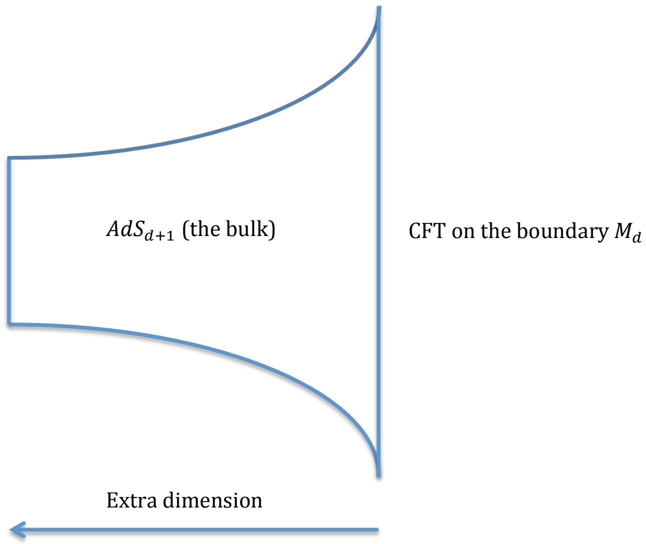

Figure 2.1: The AdS/CFT correspondence.

was invented in de sixties to describe the strong nuclear force but somewhere along the way it got lost of its original track. It was soon enough realised that string theory had more potential that just describing the strong nuclear force and that it could be used as a quantum gravity. The AdS/CFT correspondence links string theory back to strongly coupled field theories.

2.2 The AdS/CFT dictionary

In the previous section we introduced the AdS/CFT correspondence and its motivation. We still need to discuss the quantitative formalism of this theory. One can interpret it as a "dictionary". This dictionary consists of a list of statements how fields in the bulk should be translated in the boundary, how a global symmetry should be understood in the bulk, etc. It also enables one to calculate certain properties of the CFT in the bulk, for instance correlation functions.

We begin with discussing the formalism that was generated independent by Witten in [5] and Gubser, Klebanov and Polyakov in [6]. It is important to note that it was formulated for Euclidean signature. This means that AdS_{d+1} can be identified with the open unit ball B_{d+1}, such that the boundary of AdS_{d+1} is $\mathbf{S^d}$, the unit sphere in $d+1$-dimensions. Recall that $\mathbf{S^d}$ is just the union of $\mathbb{R}^d$ with a point at infinity. A recipe for Minkowski space is given in [7]. Secondly, gravity terms are not considered in the action of the bulk. The action is defined in a fixed background AdS_{d+1} spacetime. In other words, one looks at fluctuations of the fields around zero.

Consider an action $S[\phi]$, defined in the bulk, depending on a scalar field $\phi(x)$ with $x \in B_{d+1}$. The scalar field satisfies the boundary condition

$$\lim_{x\to\mathbf{x}} \phi(x) = \phi_0(\mathbf{x}) \ , \tag{2.1}$$

where $\mathbf{x} \in \mathbf{S^d}$ and $\phi_0(\mathbf{x}) < +\infty$. Since ϕ is a scalar field, one can always find a solution to the equations of motions in the bulk together with the boundary condition given in (2.1). See [5] for a broader discussion.

A first conjecture of the AdS/CFT correspondence states that ϕ_0 couples to a conformal operator $\mathcal{O}$ on the boundary such that the action of the CFT S_{CFT} acquires a new term: $\delta S_{CFT} = \int_{\mathbf{S^d}} \phi_0 \mathcal{O}$. The second and central conjecture of the correspondence links the expectation value of the exponential of δS_{CFT} to the partition function of the bulk in the limit $x \to \mathbf{x}$

$$\left\langle \exp \int_{\mathbf{S^d}} \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_S(\phi \to \phi_0) \ . \tag{2.2}$$

To put it differently: the perturbed partition function of the CFT is equal to the partition function of the bulk at the boundary.

Often we may approximate the supergravity partition function by

$$Z_S(\phi \to \phi_0) \approx \exp(-I_S(\phi \to \phi_0)) \ , \tag{2.3}$$

with $I_S(\phi \to \phi_0)$ the classical supergravity action. This is the saddle point approximation of the path integral. It is clear that this approximation can only be valid when quantum and string corrections are small. Let us now assume that this approximation is valid. We can state our AdS-CFT ansatz in the following way

$$\left\langle \exp \int_{\mathbf{S^d}} \phi_0 \mathcal{O} \right\rangle_{CFT} = \exp(-I_S(\phi \to \phi_0)) \ . \tag{2.4}$$

As before, $I_S(\phi \to \phi_0)$ is the value of the classical supergravity action at the boundary. However some subtleties can arise when calculating this term. It is possible that the action will diverge when we move towards the boundary. In that case we have to regularise the action by adding counter terms to render the action finite at the boundary. In this thesis we will only consider models that are renormalizable.

Next we turn our attention to gauge symmetries in the bulk. One is interested how these symmetries are translated on the boundary. Likewise one would like to know which operators couple to the gauge fields at the boundary. In [4] it was suggested that a gauge group G of dimension n with gauge fields A^i ($i = 1, \dots, n$) in the bulk becomes the global symmetry G of dimension n on the boundary which possesses n conserved currents J_i. These currents couple to the boundary values of the gauge fields via a coupling $\int_{\mathbf{S^d}} A_0^i J_i$, with A_0^i the value of the gauge field A^i at the boundary. For instance, a global $U(1)$ symmetry on the boundary is translated to an abelian $U(1)$ gauge group in the bulk.

The goal of this thesis is to calculate a Hall conductivity. As we will see the Hall conductivity will depend on correlation functions of the current operators. So we need to explain how to calculate those correlation functions using the dual theory. Before this can be done, we have to formulate the AdS/CFT ansatz similar to (2.2) for the gauge fields

$$\left\langle \exp \int_{\mathbf{S^d}} A_0^i J_i \right\rangle_{CFT} = Z_S(A \to A_0) \ , \tag{2.5}$$

with

$$\lim_{x \to \mathbf{x}} A^i(x) = A_0^i(\mathbf{x}) < +\infty \ . \tag{2.6}$$

Since the left hand side of this is equation is the expectation value of the generating functional, we can calculate the correlation functions of the currents by simply taking functional derivatives

$$\langle J_{i_1} \dots J_{i_k} \rangle_{CFT} = \frac{1}{Z_S} \frac{\delta}{\delta A_0^{i_1}} \dots \frac{\delta}{\delta A_0^{i_k}} Z_S(A \to A_0) \Big|_{A_0 = 0} \ . \tag{2.7}$$

In the low energy limit we can use the same approximation as (2.3)

$$Z_S(A \to A_0) \approx \exp(-I_S(A \to A_0)) \tag{2.8}$$

In this approximation (2.7) becomes

$$\langle J_{i_1} \dots J_{i_k} \rangle_{CFT} = (-1)^k \frac{\delta}{\delta A_0^{i_1}} \dots \frac{\delta}{\delta A_0^{i_k}} I_S(A \to A_0) \bigg|_{A_0=0} . \tag{2.9}$$

To conclude, the procedure for calculating correlation functions using the Witten formalism can be summarised as follows:

1. Define the action of the supergravity/string theory and derive the equation of motion for the desired bulk field.

2. Solve the equation of motion in accordance with the boundary conditions.

3. Put this solution back in the action. This will be the generator for the connected correlation functions of an operator that couples to the boundary value of the bulk field.

Next we will discuss how the AdS/CFT correspondence is quantitatively realised when we start with a model with gravity included in the bulk action.[2] This prescription is reviewed in [8] by Hartnoll. In this second formalism, one doesn't make an assumption about the background of the bulk but one simultaneously generates the Einstein equations and the equations of motion of the fields from the action.[3] One can see this an extension of the formalism described above since Witten didn't take gravity terms into account in the action of the supergravity. This extended formalism will be used throughout the remainder of the thesis.

It should be clear that the central assumption of the duality that was formulated in (2.2) has to be modified. The fields of the bulk will now influence the AdS background spacetime. It is also no longer possible to look at fluctuations around zero, instead fluctuations around the equilibrium values of the fields will be considered. These fluctuations will now be the sources of the operators in the CFT. In the next paragraph we will make this statement mathematically precise.

[2] In this case one usually assumes a Lorentzian signature. We will follow this.

[3] When all these equations are solved, we will call the values of the fields equilibrium values.

Let ϕ be the equilibrium value of a field in the bulk, $\delta\phi$ a fluctuation around this equilibrium value which has a boundary value $\delta\phi_0$ and $\mathcal{O}$ an operator that couples to the source $\delta\phi_0$. The central conjecture of the AdS/CFT correspondence now becomes

$$\left\langle \exp\left(i\int_{\mathbf{S^d}} \delta\phi_0\mathcal{O}\right)\right\rangle_{CFT} = Z_S(\phi \to \phi_0 + \delta\phi_0)\ . \tag{2.10}$$

In this case the expectation value of the exponential of the perturbation of the field theory action $\delta S_{CFT} = \int_{\mathbf{S^d}} \delta\phi_0\mathcal{O}$ is equal to the partition function of the fluctuations around ϕ at the boundary. In the low energy limit this can again be rewritten

$$\left\langle \exp\left(i\int_{\mathbf{S^d}} \delta\phi_0\mathcal{O}\right)\right\rangle_{CFT} = \exp(iI_S(\phi \to \phi_0 + \delta\phi_0))\ , \tag{2.11}$$

with $I_S(\delta\phi \to \delta\phi_0)$ the renormalized classical action at the boundary.

As said before, we are interested in gauge fields. For those fields the assumption reads

$$\left\langle \exp\left(i\int_{\mathbf{S^d}} \delta A^i_{0,\mu} J^\mu_i\right)\right\rangle_{CFT} = \exp(iI_S(A \to A_0 + \delta A_0))\ . \tag{2.12}$$

Correlation functions of the currents can still calculated using (2.12)

$$\langle J_{i_1}\dots J_{i_k}\rangle_{CFT} = \frac{\delta}{\delta(\delta A_0^{i_1})}\cdots\frac{\delta}{\delta(\delta A_0^{i_k})} I_S(A \to A_0 + \delta A_0)\ . \tag{2.13}$$

The expectation value of a single operator can now be nonzero, since the equilibrium value of the field does not have to vanish. A similar analysis can made for fluctuations around the metric tensor. These fluctuations will source an energy-momentum tensor on the boundary.

The procedure for calculating correlation functions using the formalism of reviewed by Hartnoll is:

1. Define the action of the supergravity/string theory together with the gravity terms and derive the equations of motion for the desired bulk field and the metric.

2. Solve the equations of motion for the bulk fields and the metric

3. Introduce fluctuations in the equations of motion and solve these equations up to linear order in the fluctuations in accordance with the boundary conditions.

4. Put this solution back in the action. This will be the generator for the connected correlation functions of an operator that couples to the boundary value of the bulk field.

2.3 Finite nonzero temperature CFT and black holes/black branes

Suppose we want to extend the correspondence for CFT in an ensemble such that a finite nonzero temperature is introduced. In order to work with these theories in the AdS/CFT correspondence, one needs to know how a nonzero finite temperature can be established in the bulk.

AdS spacetimes are geometric realisations of scale invariance. Consider for instance AdS_{d+1} in the following coordinate system

$$ds^2 = L^2 \left(-\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + \frac{d\mathbf{x^2}}{r^2} \right) \ , \tag{2.14}$$

where L is a constant (the AdS radius), $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{d-1}})$ denotes the spatial coordinates of the boundary, t is the time coordinate and r is the extra dimension in the bulk. One easily verifies that the transformation $r \to \lambda r$, $t \to \lambda t$ and $\mathbf{x} \to \lambda \mathbf{x}$ doesn't change the metric. One can also check that a d-dimensional Minkowski space (the boundary) is recovered at $r = 0$.

In field theories one assigns temperatures to the period of the Euclidean time coordinate. See for instance [9]. Therefore, a natural solution is an deformation of the bulk AdS_{d+1} spacetime (2.14) such that the Euclidean time becomes periodic. Since the boundary may not be altered, the deformation can only depend on the r coordinate. In general it is possible that the deformation will break the scale invariance of the spacetime. We will restrict ourselves to deformations that are finite in extent, i.e. far away from the deformation we will recover the AdS_{d+1} spacetime. This is called an asymptotically AdS spacetime.

From these observations, it is natural to consider a modified AdS_{d+1} spacetime

$$ds^2 = L^2 \left(-\frac{f(r)dt^2}{r^2} + \frac{g(r)dr^2}{r^2} + \frac{h(r)d\mathbf{x^2}}{r^2} \right) \ . \tag{2.15}$$

Since we need an asymptotically AdS_{d+1} spacetime $f(r), g(r), h(r) \to 1$ when $r \to 0$.

Next we need to insert the proposed metric (2.15) in to the Einstein equations with a negative cosmological constant Λ

$$\mathcal{R}_{\mu\nu} = \frac{2\Lambda}{d-1} g_{\mu\nu} \ . \tag{2.16}$$

One can verify that this will lead to the following conditions on f, g and h

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^d \ , \tag{2.17a}$$

$$g(r) = \frac{1}{1 - \left(\frac{r}{r_h}\right)^d} \ , \tag{2.17b}$$

$$h(r) = 1 \ , \tag{2.17c}$$

with r_h the event horizon and $\Lambda = -\frac{d(d-1)}{L^2}$.

Insert this result now into (2.15)

$$ds^2 = \frac{L^2}{r^2}\left(-\left(1 - \left(\frac{r}{r_h}\right)^d\right)dt^2 + \frac{dr^2}{1 - \left(\frac{r}{r_h}\right)^d} + d\mathbf{x}^2\right) \ . \tag{2.18}$$

The obtained solution is called an AdS black brane. It has a close resemblance with a black hole. The only difference between black holes and black branes is the fixed-time topology of the event horizon. In a $d+1$-dimensional spacetime a black hole has a fixed-time topology of S^{d-1}, while a black brane has $\mathbb{R}^d$ as fixed-time topology. Since we are working in the coordinate system (2.15), we find a black brane. If we had considered the coordinate system

$$ds^2 = L^2\left(-\frac{f(r)dt^2}{r^2} + \frac{g(r)dr^2}{r^2} + \frac{h(r)d\Omega^2_{d-1}}{r^2}\right) \ , \tag{2.19}$$

we would have found a black hole.

From a first intuitive method we can understand that a suggested deformation of the AdS_{d+1} spacetime with specific boundary conditions leads to the formation of a black hole (BH) or a black brane (BB). We will further exploit the properties of these solutions in the next paragraphs. Since those properties are naturally discussed in Poincaré coordinates, we will start with transforming the metric (2.14) to the Poincaré coordinate system.

Consider AdS_{d+1} again in the following coordinates

$$ds^2 = L^2\left(-\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + \frac{dM^2_{k,d-1}}{r^2}\right) \ , \tag{2.20}$$

with[4]

$$dM^2_{k,d-1} = \begin{cases} d\phi^2 + \sin^2\phi \ d\Omega^2_{d-2} & \text{if } k = 1 \text{ and } \phi \in [0, \pi[\ , \\ d\chi^2 + \chi^2 \ d\Omega^2_{d-2} & \text{if } k = 0 \text{ and } \chi \in [0, +\infty[\ , \\ d\psi^2 + \sinh^2\psi \ d\Omega^2_{d-2} & \text{if } k = -1 \text{ and } \psi \in [0, +\infty[\ . \end{cases} \tag{2.21}$$

[4] $d\Omega^2_{d-2}$ is the spherical line element in $d-2$ dimensions.

This metric has an spherical spherical fixed-time topology when $k = 1$, a flat fixed-time topology when $k = 0$ and a hyperbolical fixed-time topology when $k = -1$. Since we are interested in black branes or black holes, we will only consider $k = 0$ and $k = 1$ in the remainder of this section. These coordinates are easily transformed into Poincaré coordinates by the following transformation:

$$r = \frac{L^2}{R} \ . \tag{2.22}$$

The other coordinates remain unchanged. In Poincaré coordinates (2.20) becomes

$$ds^2 = -\frac{R^2}{L^2}dt^2 + \frac{L^2}{R^2}dR^2 + \frac{R^2}{L^2}dM^2_{k,d-1} \ . \tag{2.23}$$

The boundary of the AdS_{d+1} space-time can now be found at $R \to +\infty$ and the radial coordinate R scales as $R \to \lambda^{-1}R$.

Next we plug a black brane/black hole into this space-time. This can be done by introducing a function $V_k(R)$ in the metric. Again $k = 0$ corresponds with a black brane and $k = 1$ with a black hole. The metric (2.23) becomes

$$ds^2 = -V_k(R)dt^2 + \frac{dR^2}{V_k(R)} + \frac{R^2}{L^2}dM^2_{k,d-1} \ . \tag{2.24}$$

An expression for $V_k(R)$ can be found by solving (2.16) with $\Lambda = -\frac{d(d-1)}{2L^2}$. This leads to

$$V_k(R) = k - \frac{m}{R^{d-2}} + \frac{R^2}{L^2} \ , \tag{2.25}$$

where m is an integration constant that is related to the mass of the black brane/black hole. Usually this solution is called the Schwarzschild-AdS solution in $d+1$ dimensions. One can check that (2.23) is recovered when we take $R \to +\infty$. This means that the asymptotical AdS requirement is satisfied. Likewise one obtains (2.23) when $m = 0$. This is nothing else then erasing the black brane/black hole from the AdS_{d+1} spacetime. The event horizon R_h of the BH/BB is the largest root of

$$V_k(R_h) = 0 \ . \tag{2.26}$$

Hence near this event horizon one can expand $V_k(R)$

$$V_k(R) = V_k'(R_h)\left(R - R_h\right) + \mathcal{O}\left(R - R_h\right)^2 \ . \tag{2.27}$$

Finally we can start discussing the temperature of a black brane/black hole. As previously mentioned, the temperatures of a field theory is defined as the

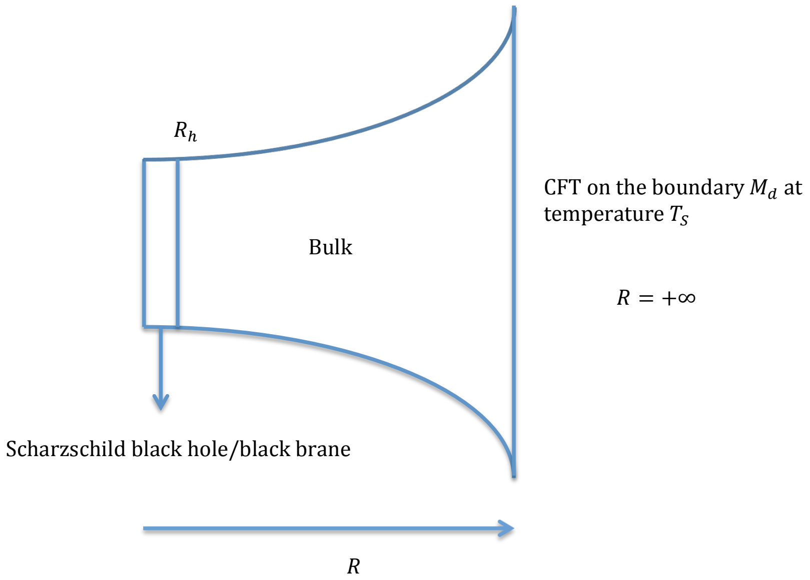

Figure 2.2: Scharzschild AdS_{d+1} black brane/black hole which leads to a temperature T_S on the boundary.

period of the Euclidean time. For this reason, we need to Wick rotate the Lorentzian time coordinate t to find the Euclidean time coordinate τ

$$\tau = it\ . \tag{2.28}$$

This implies that (2.24) transforms into

$$ds^2 = V_k(R)d\tau^2 + \frac{dR^2}{V_k(R)} + \frac{R^2}{L^2}dM^2_{k,d-1}\ . \tag{2.29}$$

Near the event horizon this becomes

$$ds^2 \approx V_k'(R_h)\,(R-R_h)\,d\tau^2 + \frac{dR^2}{V_k'(R_h)\,(R-R_h)} + \frac{R^2}{L^2}dM^2_{k,d-1}\ . \tag{2.30}$$

Consider now the following coordinate transformation

$$\rho = \frac{2}{\sqrt{|V_k'(R_h)|}}\sqrt{R-R_h}\ . \tag{2.31}$$

Combining this transformation with (2.30), we find

$$ds^2 \approx \frac{\left(V_k'(R_h)\right)^2}{4}\rho^2 d\tau^2 + d\rho^2 + \dots\ . \tag{2.32}$$

These are just polar coordinates of $\mathbb{R}^2$ where τ needs to satisfy the following condition to avoid a conical singularity

$$\tau = \tau + \frac{4\pi}{|V_k'(R_h)|} \ . \tag{2.33}$$

This defines the period P of τ

$$P = \frac{4\pi}{|V_k'(R_h)|} \ . \tag{2.34}$$

Since the period of the Euclidean time coordinate is inversely proportional to the temperature, we find the expression for the temperature of the black hole

$$T_S = \frac{|V_k'(R_h)|}{4\pi} \ . \tag{2.35}$$

If we now plug (2.25) into this equation, we find

$$T_S = \frac{1}{4\pi}\left(\frac{m(d-2)}{R_h^{d-1}} + \frac{2R_h}{L^2}\right) \ . \tag{2.36}$$

Using (2.26), we can rewrite m in function of R_h such that T_S becomes

$$T_S = \frac{1}{4\pi}\left(\frac{k(d-2)}{R_h} + \frac{R_h d}{L^2}\right) \ . \tag{2.37}$$

Introducing black branes/black holes into the AdS_{d+1} spacetime causes the Euclidean time coordinate to be periodic which gives rise to a nonzero temperature. This is dual to placing a CFT in an ensemble, see Figure 2.2. It also possible to derive the entropy of an AdS black brane/black hole, similar to the thermodynamic analysis of a black brane/black hole. We will not consider this. The interested reader can find a discussion on these topics in [10], [11].

2.4 Background magnetic fields and electric charges in CFT and charged black holes/black branes

The second extension of the AdS/CFT correspondence that we will discuss are electric charges and electromagnetic (EM) background fields on the boundary. In many theories, especially condensed matter theories, we can treat the electromagnetic $U(1)$ gauge (local) symmetry as a global symmetry. This means that we only consider an effective field theory. Charged matter is considered in electromagnetic background fields and the interactions with photons are neglected.

In subsection 2.2 we already argued that the dual of a global symmetry on the boundary is a gauge symmetry in the bulk. In the remainder of this section we will show that an electric charge density and a magnetic background field on the boundary correspond with the addition of an electromagnetic $U(1)$ gauge field in the bulk. We will first consider the purely electric case in the bulk where the $U(1)$ gauge field A only possesses one nonzero component A_t. Afterwards we will discuss the full electromagnetic model with other nonzero components.

2.4.1 Reissner-Nordström AdS_{d+1} black brane/black brane

Consider the $d+1$ dimensional Einstein-Maxwell action in the bulk

$$S_{EM} = \frac{-1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{-g}\left[\mathcal{R} - 2\Lambda - \frac{1}{4}F^2\right] \ , \tag{2.38}$$

with $F = dA$ and $\Lambda = \frac{-d(d-1)}{2L^2}$. The equations of motions corresponding to this Einstein-Maxwell action are

$$\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2}g_{\mu\nu} - \frac{d(d-1)}{2L^2}g_{\mu\nu} = \frac{1}{2}F_{\mu\rho}F^{\rho}_{\nu} - \frac{F^2}{8}g_{\mu\nu} \ , \tag{2.39a}$$

$$D_\mu F^{\mu\nu} = 0 \ . \tag{2.39b}$$

We expect the presence of a $U(1)$ gauge field A to change the AdS_{d+1} metric. For that reason we will search for deformations of (2.23). We will consider the ansatz for the metric as given in (2.24)

$$ds^2 = -W_k(R)dt^2 + \frac{dR^2}{W_k(R)} + \frac{R^2}{L^2}dM^2_{k,d-1} \ . \tag{2.40}$$

For the $d+1$ dimensional gauge field A we take

$$A = A_t(R)dt \ . \tag{2.41}$$

Assuming (2.40) and (2.41) we can solve the equations of motion given in (2.39) to find

$$W_k(R) = k - \frac{m}{R^{d-2}} + \frac{q^2}{2(d-1)(d-2)R^{2(d-2)}} + \frac{R^2}{L^2} \ , \tag{2.42a}$$

$$A_t(R) = K - \frac{q}{(d-2)R^{d-2}} \ , \tag{2.42b}$$

$$F_{Rt} = \frac{q}{R^{d-1}} \ , \tag{2.42c}$$

where q, m and K are integration constants. The obtained solution is a $d+1$ Reissner-Nordström (RN)-AdS black brane/black hole with a charge density ρ proportional to q (see below) and an event horizon $\tilde{R}_h$. A_t must vanish at $\tilde{R}_h$ in order to be well defined, see [12], [13]. This condition fixes the value of K

$$K = \frac{q}{(d-2)\tilde{R}_h^{d-2}} \ , \tag{2.43}$$

such that

$$W_k(R) = k - \frac{m}{R^{d-2}} + \frac{q^2}{2(d-1)(d-2)R^{2(d-2)}} + \frac{R^2}{L^2} \ , \tag{2.44a}$$

$$A_t(R) = \frac{q}{(d-2)\tilde{R}_h^{d-2}}\left(1-\left(\frac{\tilde{R}_h}{R}\right)^{d-2}\right) = \mu\left(1-\left(\frac{\tilde{R}_h}{R}\right)^{d-2}\right) \ , \tag{2.44b}$$

$$F_{Rt} = \frac{q}{R^{d-1}} \ , \tag{2.44c}$$

with $\mu = \frac{q}{(d-2)\tilde{R}_h^{d-2}}$. It will be shown that μ can be seen as a chemical potential on the boundary, hence its name. Again, we are only interested in the values $k = 0, 1$. Since the metric that we obtained has the same form as (2.24), we can associate a temperature to the black brane/black hole using formula (2.35). This gives

$$T_{RN} = \frac{1}{4\pi}\left(\frac{k(d-2)}{\tilde{R}_h} + \frac{\tilde{R}_h d}{L^2} - \frac{q^2}{2(d-1)\tilde{R}_h}\right) \ . \tag{2.45}$$

This RN-AdS_{d+1} solution has to be the consistent with the Scharzschild-AdS_{d+1} solution. In other words, we must obtain the solution found in section

2.3 when q vanishes in (2.44a). Indeed when we take $q \to 0$, we find

$$W_k(R) \to k - \frac{m}{R^{d-2}} + \frac{R^2}{L^2} = V_k(R) \ , \tag{2.46a}$$

$$A_t(R) \to 0 \ , \tag{2.46b}$$

$$F_{Rt} \to 0 \ , \tag{2.46c}$$

$$\tilde{R}_h \to R_h \ , \tag{2.46d}$$

$$T_{RN} \to T_S \ . \tag{2.46e}$$

Now we are ready to investigate what this RN-AdS_{d+1} solution introduces a charge density on the boundary. From the previous section we know that the temperature found in (2.45) corresponds with the temperature of the field theory in an ensemble on the boundary. However the main focus in this section is on the gauge field A and its field strength F_{rt}. We claim that a gauge field in the bulk with only A_t different from zero gives rise to a chemical potential and a charge densities on the boundary. To see this, recall (2.44b). When $R \to +\infty$, we have

$$A_t \to \mu \ . \tag{2.47}$$

When $d = 2$, this value diverges. In section 2.2 we argued that we only consider bulk fields which remain finite on the boundary. This means that $d > 2$. Since (2.47) represents an electrical potential at the boundary without an spacial dependence, it is often interpreted as a chemical potential. From (2.12) we know that fluctuations of the bulk field component A_t on the boundary $A_{0,t} = \mu$ couple to the charge density operator J^t such that a nonzero expectation value of the J^t will arise in the field theory on the boundary. We can calculate the charge density using (2.13)

$$\rho = \langle J_t \rangle = \frac{q}{16\pi G} \ , \tag{2.48}$$

which leads to a total charge

$$Q = \frac{qV_{d-1}}{16\pi G} \ , \tag{2.49}$$

with $V_{d-1} = \frac{1}{l^{d-1}} \int d^{d-1}x$ a dimensionless volume. This proves the claim that was made earlier. One finds a graphical illustration of this correspondence in Figure 2.3.

For the component of the field strength tensor, F_{rt}, we can see from (2.44c) that the magnitude of this component has a power law behaviour as we move towards the boundary and eventually vanishes on the boundary. From this observation, we can understand that an electrical field in the bulk doesn't lead to a background electrical field on the boundary. This is in contrast with a magnetic field as we will show in the next model.

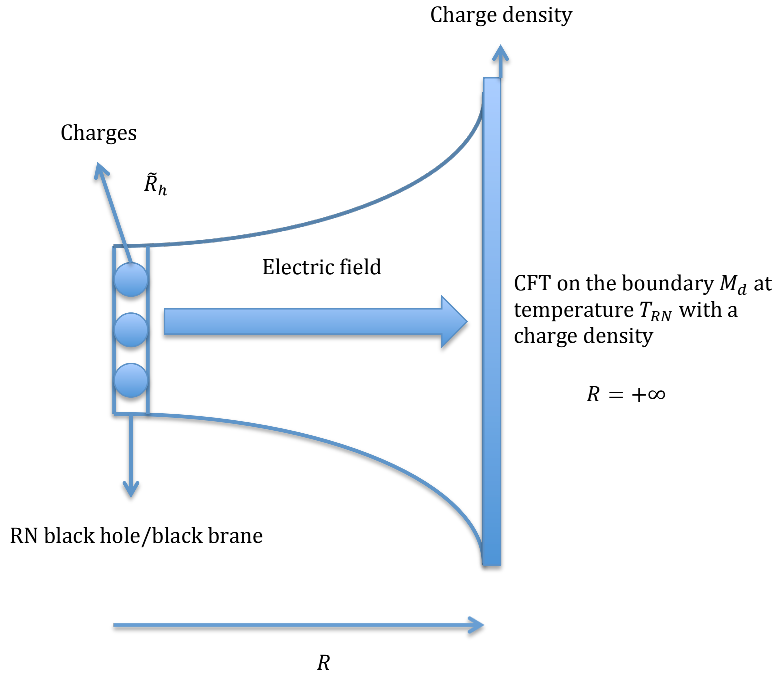

Figure 2.3: Reissner-Nordström AdS_{d+1} black brane/black hole which leads to a temperature T_{RH} and a net charge density on the boundary.

2.4.2 Dyonic AdS_4 black brane/black hole

Now we are going to extend the previous model to a dyonic black brane/black hole in $3+1$ dimensions. Dyonic refers to an electrical and a magnetically charge in the black brane/black hole. This requires that the gauge field also acquires a magnetic charge by considering a second nonzero spatial component of the gauge field

$$A = A_t(R)dt + Bydx \ , \tag{2.50}$$

with B a constant. This gauge field introduces, besides a charge density, a background magnetic field on the boundary as will become clear in this subsection. Since $d = 3$, there are only 2 spatial directions on the boundary. We will call them x and y in the remainder of this subsection.

We will consider the same action and metric as in the previous subsection. For $d = 3$, one finds

$$S_{EM} = \frac{-1}{16\pi G} \int \mathrm{d}^4x\sqrt{-g}\left[\mathcal{R} - 2\Lambda - \frac{1}{4}F^2\right] \ , \tag{2.51a}$$

$$ds^2 = -U_k(R)dt^2 + \frac{dR^2}{U_k(R)} + \frac{R^2}{L^2}dM^2_{k,d-1} \ . \tag{2.51b}$$

From (2.51a) we find the following equations of motion

$$\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} g_{\mu\nu} - \frac{3}{L^2} g_{\mu\nu} = \frac{1}{2} F_{\mu\rho} F^{\rho}_{\nu} - \frac{F^2}{8} g_{\mu\nu} \ , \tag{2.52a}$$

$$D_\mu F^{\mu\nu} = 0. \tag{2.52b}$$

Solving these equations gives

$$U_k(R) = k + \frac{R^2}{L^2} - \frac{m}{R} + \frac{B^2 L^4 + q^2}{4R^2} \ , \tag{2.53a}$$

$$A_t = \frac{q}{\hat{R}_h} \left(1 - \frac{\hat{R}_h}{R}\right) \ , \tag{2.53b}$$

$$F_{Rt} = \frac{q}{R^2} \ , \tag{2.53c}$$

$$F_{xy} = B \ . \tag{2.53d}$$

This solution is called a dyonic AdS_4 black brane/black hole. We obtain an extra magnetic charge in the black brane/black hole and a second nonzero component F_{xy} of the EM field strength tensor in comparison with subsection 2.4.1. The second nonzero component of F sources a magnetic field in the R-direction of the bulk which is perpendicular to the 2 spatial directions x and y. Rotational invariance is not broken on the boundary. The event horizon of this black brane/black hole is called $\hat{R}_h$ and is determined by the largest root of the following equation

$$U_k(\hat{R}_h) = 0 \ . \tag{2.54}$$

For future reference we explicitly state the dyonic AdS_4 black brane solution

$$U_k(R) = \frac{R^2}{L^2} \left(1 - \frac{mL^2}{R^3} + \frac{B^2 L^6 + q^2 L^2}{4R^4}\right) \ , \tag{2.55a}$$

$$A_t = \frac{q}{\hat{R}_h} \left(1 - \frac{\hat{R}_h}{R}\right) \ , \tag{2.55b}$$

$$F_{Rt} = \frac{q}{R^2} \ , \tag{2.55c}$$

$$F_{xy} = B \ . \tag{2.55d}$$

Again the temperature can be calculated using formula (2.35)

$$T_d = \frac{1}{4\pi} \left(\frac{k}{\hat{R}_h} + \frac{3\hat{R}_h}{L^2} - \frac{B^2 L^4 + q^2}{4\hat{R}_h^3}\right) \ . \tag{2.56}$$

It is readily checked that this result is consistent with the RN-AdS_4 black brane/black hole solution if $B \to 0$ and with the Scharzschild AdS_4 black brane/black hole solution if $q, B \to 0$.

As in the previous RN-AdS_{d+1} model, we investigate the boundary behaviour of the bulk fields. The analysis for A_t and F_{Rt} is the same as before. Here we will focus on the behaviour of F_{xy}. One can see from (2.55d) that the magnitude of F_{xy} does not depend on the bulk coordinate R. Hence as we move towards the boundary, F_{xy} doesn't decrease as F_{Rt} did. Since this magnetic field has a non-zero constant value at the boundary, one can interpret it as a background magnetic field for the field theory. It was already stated above that this background magnetic field is perpendicular to x and y. We can conclude that a dyonic BH/BB in the bulk gives rise to non-zero charge density and a background magnetic field for the field theory on the boundary, see Figure 2.4.

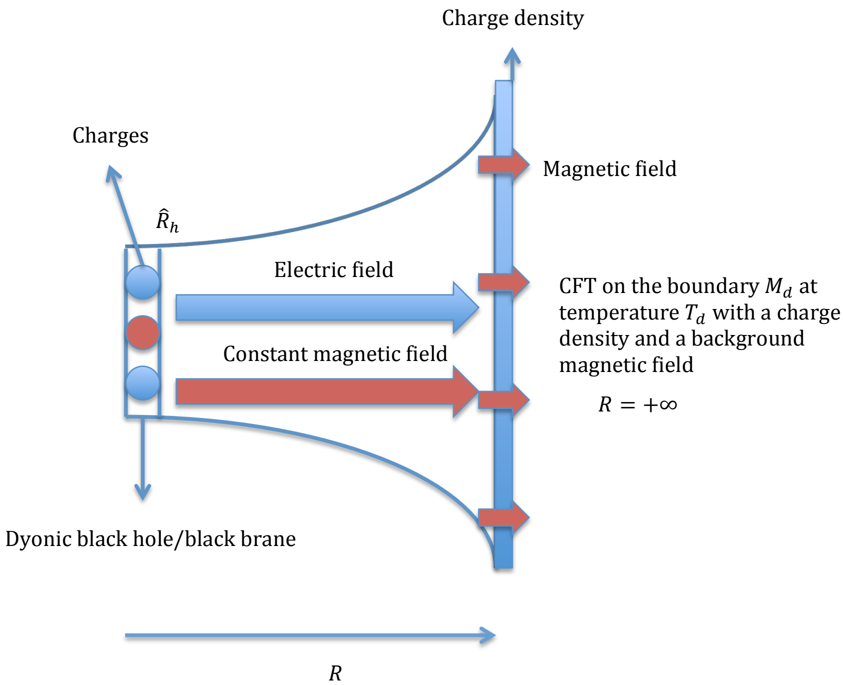

Figure 2.4: Dyonic AdS_4 black brane/black hole solution which leads to a temperature T_d, a net charge density and a constant magnetic background field on the boundary.

Chapter 3

Holography and condensed matter

In this chapter we will discuss condensed matter theories (CMT) and their applications in the AdS/CFT correspondence. When the correspondence is applied to CMT, we call it the AdS/CMT correspondence. Since the area of condensed matter is quite broad, we will focus on some specific topics that will be used in this thesis. One of these topics is the (Hall) conductivity.

We start with motivating the AdS/CMT correspondence. In the second section we will introduce the general concepts of linear response theory. Afterwards we will apply linear response theory to derive the Kubo formula, which is used to calculate (Hall) conductivities.

3.1 From AdS/CFT to AdS/CMT

Condensed matter physicists have discovered new materials and new phases of materials in the last decades that behave like strongly coupled systems. One can think of several examples: high T_c superconductors, cold atoms, graphene etc. The microscopic behaviour of these materials and phases can not be described by standard weakly coupled quasiparticle theories. The examples we just have listed are part of a bigger class of systems, namely quantum critical systems.

Quantum critical systems undergo quantum phase transitions. These phase transitions, mostly continuous, occur at zero temperature and are driven by quantum fluctuations, hence the name. The quantum fluctuations even dominate the thermal fluctuations at nonzero temperatures where $\hbar\omega > k_B T$, with ω the frequency of the quantum fluctuation. This region is called the quantum critical region. In Figure 3.1 one can find a graphical example of a quantum phase transition.

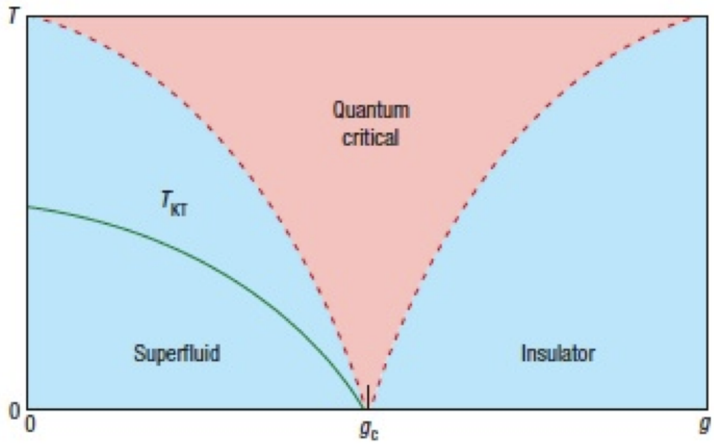

Figure 3.1: Phase diagram of a superfluid-insulator quantum phase transition. The coupling constant is on the horizontal axis and the temperature on the vertical axis. The pink region is the quantum critical region and the green line represents the classical Kosterlitz-Thouless phase transition. Illustration from [14].

The point in the phase diagram that indicates the zero temperature quantum phase transition is called the quantum critical point. It is at the critical point and in the quantum critical region that the known theories fail to describe the microscopic behaviour of the system. The mass of the quasiparticles diverges such that the system becomes strongly coupled. The system becomes also scale invariant. See [14], [15] for more information on these topics.

From these observations, one can naturally suggest the application of the AdS/CFT correspondence in this regime to calculate correlation functions. The duality has already been applied to calculate transport coefficients of certain CMT. An example is found in Figure 3.2. In this thesis we will try to calculate a Hall conductivity for a strongly coupled field theory at finite temperature with anisotropic scale invariance using the duality described in the previous chapter.

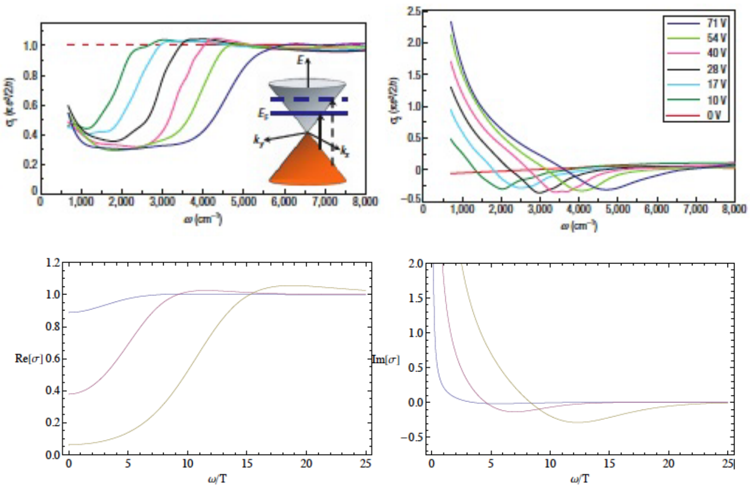

Figure 3.2: Electrical conductivity of graphene in function of the frequency of the applied AC current. The first row represents the experimental data and the second row represents results computed from the AdS/CFT correspondence. Figures from [8].

3.2 Linear response and transport coefficients

We will proceed with computing the electrical conductivity tensor and its place in the AdS/CFT correspondence in this section. The electrical conductivity tensor, $\boldsymbol{\sigma}$, is defined by the equation

$$J_i(\mathbf{r},t) = \sum_{j=1}^{2} \int d^3\mathbf{r} \int_{-\infty}^{+\infty} dt' \sigma_{ij}(\mathbf{r},\mathbf{r}',t,t') E_l^p(\mathbf{r}',t') \ . \tag{3.1}$$

We will start with a general introduction to linear response theory as given in [16]. Afterwards apply this formalism to derive a formula for $\boldsymbol{\sigma}$.

3.2.1 Linear response theory

Consider a physical system described by an unperturbed time independent Hamiltonian H_0 and a small perturbation $\tilde{H}(t)$ at a time t. We work in the Schrödinger picture. The Hamiltonian of the total system is

$$H(t) = H_0 + \tilde{H}(t) \ . \tag{3.2}$$

Furthermore we use grand canonical ensemble. The density operator of the unperturbed system is

$$\rho_0 = \frac{1}{\mathcal{Z}_0} \exp(-\beta \mathcal{H}_0) \ , \tag{3.3}$$

with $\mathcal{H}_0 = H_0 - \mu N$ and $\mathcal{Z}_0 = \mathrm{Tr}(\exp(-\beta\mathcal{H}_0))$. For the total system the density operator is given by

$$\rho(t) = \frac{1}{\mathcal{Z}} \exp(-\beta \mathcal{H}(t)) \ , \tag{3.4}$$

where $\mathcal{H}(t) = H(t) - \mu N$ and $\mathcal{Z} = \mathrm{Tr}(\exp(-\beta\mathcal{H}(t)))$. One can approximate this expression by

$$\rho(t) \approx \rho_0 + \tilde{\rho}(t) \tag{3.5}$$

where $\tilde{\rho}(t)$ only contains terms of first order in $\tilde{H}(t)$ such that $\rho(t) = \rho_0$ when the perturbation is absent.

Recall from a course on statistical physics that the expectation value of a general operator A is completely fixed by the the density operator

$$\langle A(t) \rangle = \mathrm{Tr}\left(\rho(t) A\right) \ . \tag{3.6}$$

This expectation value is the physical and measurable quantity associated with the operator A at time t. From equation (3.6) one can understand why it is crucial to have an expression for the density operator $\rho(t)$. This expression can be calculated using the equation of motion[1]

$$i\hbar\frac{d\rho}{dt} = [\mathcal{H}(t), \rho(t)] \approx [\mathcal{H}_0 + \tilde{H}(t), \rho_0 + \tilde{\rho}(t)] \ . \tag{3.7}$$

Since $\tilde{H}(t)$ is a small perturbation, it suffices to consider up to linear order in $\tilde{H}(t)$[2]

$$i\hbar\frac{d\tilde{\rho}}{dt} \approx [\mathcal{H}_0, \tilde{\rho}(t)] + [\tilde{H}(t), \rho_0] \ . \tag{3.8}$$

Now we switch to the Dirac picture

$$i\hbar \exp(\frac{i}{\hbar}\mathcal{H}_0 t)\frac{d\tilde{\rho}}{dt}\exp(-\frac{i}{\hbar}\mathcal{H}_0 t) = [\mathcal{H}_0, \tilde{\rho}_H(t)] + [\tilde{H}_H(t), \rho_0] \ , \tag{3.9}$$

with $\tilde{\rho}_D(t) = \exp(\frac{i}{\hbar}\mathcal{H}_0 t)\tilde{\rho}(t)\exp(-\frac{i}{\hbar}\mathcal{H}_0 t)$ and $\tilde{H}_D(t) = \exp(\frac{i}{\hbar}\mathcal{H}_0 t)\tilde{H}(t)$ $\exp(-\frac{i}{\hbar}\mathcal{H}_0 t)$. This can be rearranged to

$$i\hbar\frac{\tilde{\rho}_D(t)}{dt} = [\tilde{H}_D, \rho_0] \ . \tag{3.10}$$

This differential equation together with the boundary condition (no fluctuations at the beginning)

$$\lim_{t\to-\infty}\tilde{\rho}_D(t) = 0 \ , \tag{3.11}$$

is solved by

$$\tilde{\rho}_D(t) = -\frac{i}{\hbar}\int_{-\infty}^{t} dt'[\tilde{H}_D(t'), \rho_0] \ . \tag{3.12}$$

Eventually (3.5) becomes

$$\rho(t) = \rho_0 - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \exp(-\frac{i}{\hbar}\mathcal{H}_0 t)[\tilde{H}_D(t'), \rho_0]\exp(\frac{i}{\hbar}\mathcal{H}_0 t) \ . \tag{3.13}$$

Now we are finally ready to calculate the expectation value of operator A in (3.6)

$$\begin{aligned}\langle A(t)\rangle &= \langle A\rangle_0 - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \ \mathrm{Tr}\left(\exp(-\frac{i}{\hbar}\mathcal{H}_0 t)[\tilde{H}_D(t'), \rho_0]\exp(\frac{i}{\hbar}\mathcal{H}_0 t).A\right) \\ &= \langle A\rangle_0 - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \ \mathrm{Tr}\left([\tilde{H}_D(t'), \rho_0]A_D(t)\right)\end{aligned}$$

[1] It might be confusing due to the resemblance with the Heisenberg EOM but it is important to stress that this equation is in the Schrödinger picture. This equation is often referred to as the "von Neumann-Liouville equation".

[2] Since $0 = i\hbar\frac{d\rho_0}{dt} = [\mathcal{H}_0, \rho_0]$, we can drop $[\mathcal{H}_0, \rho_0]$ and $\frac{d\rho_0}{dt}$.

$$\begin{aligned} &= \langle A \rangle_0 - \frac{i}{\hbar} \int_{-\infty}^{t} dt' \; \mathrm{Tr} \left(\rho_0 [A_D(t), \tilde{H}_D(t')] \right) \\ &= \langle A \rangle_0 \underbrace{- \frac{i}{\hbar} \int_{-\infty}^{t} dt' \langle [A_D(t), \tilde{H}_D(t')] \rangle_0}_{\tilde{A}(t)} , \end{aligned} \tag{3.14}$$

with $\langle \ldots \rangle_0 = \mathrm{Tr}(\rho_0 \ldots)$ and $A_H(t) = \exp\left(\frac{i}{\hbar}\mathcal{H}_0 t\right) A \exp\left(-\frac{i}{\hbar}\mathcal{H}_0 t\right)$. It is clear that $\tilde{A}(t)$ can be interpreted as the expectation value of the response of the operator A at a time t due to the perturbation of the system. This is a linear response because we used a first order approximation.

3.2.2 The Hall conductivity tensor

This thesis focusses on condensed matter systems with anisotropic scale invariance in 2+1-dimensions, with a charge density ρ_i and background magnetic field B_i, all originating from the fields A^i or the associated conserved current J_i, with $i = 1, \ldots, n$. For such a system we would like to calculate the response of the current J_i to a time dependent electrical field E^i. This response gives us all the information about the electrical conductivity, see (3.1). Since the magnetic fields B_i will influence the currents J_i, the electrical conductivity will require off diagonal terms. These terms are mainly called the Hall conductivity. Let us derive the formula for a conductivity tensor using linear response theory introduced in the previous subsection.

Consider now the condensed matter system as described above. We perturb this system using a time dependent vector potential $\mathbf{A}^{p,i}(\mathbf{r}, t)$. This induces a time dependent electric field $\mathbf{E}^{p,i}(\mathbf{r}, t)$ which in turn induces an AC electric current $\mathbf{J}_i^p(\mathbf{r}, t)$.[3] Mathematically this becomes

$$\tilde{H}(t) = -\sum_{i=1}^{2}\sum_{j=1}^{2} \int d^3\mathbf{r} J_j^{p,i}(\mathbf{r}) A_j^{p,i}(\mathbf{r}, t) , \tag{3.15}$$

where 1 is the x-direction and 2 the y-direction.

We will assume that the time dependence of $\mathbf{A}^p(\mathbf{r}, t)$ will be governed by $\exp(-i\omega t)$. This gives

$$\tilde{H}(t) = -\sum_{i=1}^{2}\sum_{j=1}^{2} \int d^3\mathbf{r} J_j^{p,i}(\mathbf{r}) A_j^{p,i}(\mathbf{r}) \exp(-i\omega t) . \tag{3.16}$$

[3]The superscript p isn't an index but indicates that the quantity at hand represents a perturbation.

Using the identity $-\partial_t A_j^{p,i}(\mathbf{r},t) = E_j^{p,i}(\mathbf{r},t)$, we can transform the previous equation to[4]

$$\begin{aligned}\tilde{H}(t) &= -\sum_{i=1}^{2}\sum_{i=j}^{2}\int d^3\mathbf{r} J_j^{p,i}(\mathbf{r})\frac{E_j^{p,i}(\mathbf{r})}{i\omega}\exp\left(-i\omega t\right)\\ &= \frac{i}{\omega}\sum_{i=1}^{2}\sum_{i=j}^{2}\int d^3\mathbf{r} J_j^{p,i}(\mathbf{r})E_j^{p,i}(\mathbf{r},\mathbf{t}) \ . \qquad (3.17)\end{aligned}$$

In the Dirac picture, the perturbation can be written as

$$\begin{aligned}\tilde{H}_D(t) &= \frac{i}{\omega}\sum_{i=1}^{2}\sum_{j=1}^{2}\int d^3\mathbf{r}\exp\left(\frac{i}{\hbar}\mathcal{H}_0 t\right)J_j^{p,i}(\mathbf{r})\exp\left(-\frac{i}{\hbar}\mathcal{H}_0 t\right)E_j^{p,i}(\mathbf{r},t)\\ &= \frac{i}{\omega}\sum_{i=1}^{2}\sum_{j=1}^{2}\int d^3\mathbf{r} J_{D,j}^{p,i}(\mathbf{r},t)E_j^{p,i}(\mathbf{r},t) \ . \qquad (3.18)\end{aligned}$$

Now we can calculate the expectation value of the current with the formula given in (3.14)

$$\begin{aligned}\langle J_k^{p,j}(\mathbf{r},t)\rangle &= \langle J_k^{p,j}(\mathbf{r})\rangle_0 + \frac{1}{\hbar\omega}\sum_{i=1}^{2}\sum_{l=1}^{2}\int d^3\mathbf{r}'\int_{-\infty}^{+\infty}dt'\langle[J_{D,k}^{p,j}(\mathbf{r},t),J_{D,l}^{p,i}(\mathbf{r}',t')]\rangle_0\\ &\quad E_l^{p,i}(\mathbf{r}',t') \ . \qquad (3.19)\end{aligned}$$

This can be rewritten using the cyclic property of the trace and that currents from different gauge fields commute

$$\begin{aligned}\langle J_k^{p,j}(\mathbf{r},t)\rangle &= \langle J_k^{p,i}(\mathbf{r})\rangle_0 \ + \frac{1}{\hbar\omega}\sum_{i=1}^{2}\sum_{l=1}^{2}\int d^3\mathbf{r}'\int_{-\infty}^{+\infty}dt'\theta(t-t')\delta^{ji}\\ &\quad \langle[J_{D,k}^{p,j}(\mathbf{r},t-t'),J_{D,l}^{p,i}(\mathbf{r}',0)]\rangle_0 E_l^{p,i}(\mathbf{r}',t')\\ &= \langle J_k^{p,j}(\mathbf{r})\rangle_0 +\\ &\quad \underbrace{\sum_{i=1}^{2}\sum_{l=1}^{2}\int d^3\mathbf{r}'\int_{-\infty}^{+\infty}dt'\sigma_{kl}^{ji}(\mathbf{r},\mathbf{r}',t-t')E_l^{p,i}(\mathbf{r}',t')}_{\delta J_k^{p,i}(\mathbf{r},t)} \ . \qquad (3.20)\end{aligned}$$

We have found an expression for the conductivity tensor $\boldsymbol{\sigma}$ using linear response theory as we intended. One is often interested in the frequency dependence of the conductivity tensor. This becomes more transparent after a

[4] $E_j^{p,i}(\mathbf{r},t) = i\omega A_j^{p,i}(\mathbf{r})\exp\left(-i\omega t\right) = E_j^{p,i}(\mathbf{r})\exp\left(-i\omega t\right)$

Fourier transformation of $\delta J_k^p(\mathbf{r}, t)$

$$
\begin{aligned}
\delta J_k^{p,j}(\mathbf{r}, \omega') &= \int_{-\infty}^{+\infty} dt \frac{1}{\hbar\omega} \sum_{i=1}^{2} \sum_{l=1}^{2} \int d^3\mathbf{r}' \int_{-\infty}^{+\infty} dt' \theta(t-t') \delta^{ji} \\
&\quad \langle [J_{D,k}^{p,j}(\mathbf{r}, t-t'), J_{D,l}^{p,i}(\mathbf{r}', 0)] \rangle_0 E_l^{p,i}(\mathbf{r}', t') \exp{(i\omega' t)} \\
&= \frac{1}{\hbar\omega} \sum_{i=1}^{2} \sum_{l=1}^{2} \int d^3\mathbf{r}' \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} \frac{d\omega''}{2\pi} E_l^{p,i}(\mathbf{r}', \omega'') \\
&\quad \exp{(-i\omega'' t')} \theta(t-t') \delta^{ji} \langle [J_{D,k}^{p,j}(\mathbf{r}, t-t'), J_{D,l}^{p,i}(\mathbf{r}', 0)] \rangle_0 \exp{(i\omega' t)} \\
&= \frac{1}{\hbar\omega} \sum_{i=1}^{2} \sum_{l=1}^{2} \int d^3\mathbf{r}' \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} \frac{d\omega''}{2\pi} E_l^{p,i}(\mathbf{r}', \omega'') \\
&\quad \exp{(-i(\omega''-\omega') t')} \theta(t-t') \delta^{ji} \langle [J_{D,k}^{p,j}(\mathbf{r}, t-t'), J_{D,l}^{p,i}(\mathbf{r}', 0)] \rangle_0 \\
&\quad \exp{(i\omega'(t-t'))} \ . \qquad (3.21)
\end{aligned}
$$

Now perform a coordinate transformation

$$
u = t - t' \ , \qquad (3.22a)
$$
$$
v = t' \ . \qquad (3.22b)
$$

to find

$$
\begin{aligned}
\delta J_k^{p,j}(\mathbf{r}, \omega') &= \frac{1}{\hbar\omega} \sum_{i=1}^{2} \sum_{l=1}^{2} \int d^3\mathbf{r}' \int_{0}^{+\infty} du \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} \frac{d\omega''}{2\pi} E_l^{p,i}(\mathbf{r}', \omega'') \\
&\quad \exp{(-i(\omega''-\omega') v)} \delta^{ji} \langle [J_{D,k}^{p,j}(\mathbf{r}, u), J_{D,l}^{p,i}(\mathbf{r}', 0)] \rangle_0 \exp{(i\omega' u)} \\
&= \frac{1}{\hbar\omega} \sum_{i=1}^{2} \sum_{l=1}^{2} \int d^3\mathbf{r}' \int_{0}^{+\infty} du \ \delta^{ji} \langle [J_{D,k}^{p,j}(\mathbf{r}, u), J_{D,l}^{p,i}(\mathbf{r}', 0)] \rangle_0 \\
&\quad \exp{(i\omega' u)} E_l^{p,i}(\mathbf{r}', \omega') \\
&= \sum_{i=1}^{2} \sum_{l=1}^{2} \int d^3\mathbf{r} \ \sigma_{kl}^{ji}(\mathbf{r}, \mathbf{r}', \omega') E_l^{p,i}(\mathbf{r}', \omega') \ . \qquad (3.23)
\end{aligned}
$$

Since $E_l^{p,i}(\mathbf{r}, \omega') = 2\pi E_l^{p,i}(\mathbf{r}) \delta(\omega - \omega')$, there can only be a non-zero contribution when $\omega' = \omega$ and we will only consider the conductivity tensor $\boldsymbol{\sigma}$ at this particular frequency. We obtain the following result for the conductivity tensor

$$
\sigma_{kl}^{ji}(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{\hbar\omega} \sum_{i=1}^{2} \sum_{l=1}^{2} \int d^3\mathbf{r}' \int_{0}^{+\infty} dt \ \delta^{ji} \langle [J_{D,k}^{p,j}(\mathbf{r}, t), J_{D,l}^{p,i}(\mathbf{r}', 0)] \rangle_0 \exp{(i\omega t)} \ . \qquad (3.24)
$$

At the point it should be clear why the duality is interesting to use. Mostly it will take a lot of work to compute the expectation value of the commutator

(3.24). If a condensed matter system can be described by a scale invariant theory, we can calculate this expectation value easily using the AdS/CFT formalism that was developed in subsection 2.2 and 2.4.2. A nice example of a Hall conductivity with one global $U(1)$ field and an associated conserved current on the boundary can be found in [2].

We need to stress that this conductivity tensor only makes sense when the currents, i.e. the charge densities, and the magnetic fields have the same index i. This is obvious for the electric charges since the expression for the conductivity tensor is zero when the charge densities of the currents have different values. For the magnetic fields this is more subtle. However the Hall conductivity will be zero when we consider one magnetic background field and one charge density originating from a field and a conserved current with different index, due to the Maxwell equations.

Part II

Thesis project

Chapter 4

The magnetic Lifshitz black brane

In this second part of the thesis we will look at the dual theory of a model with anisotropic scale invariance. The specific anisotropic scale behaviour that we are going to investigate is given by

$$t \to \lambda^z t \ , \qquad \mathbf{x} \to \lambda \mathbf{x} \ , \tag{4.1}$$

where z is called the dynamical exponent. This scaling is also known as Lifshitz scaling. One should notice that in the limit $z \to 1$, the isotropic scaling behaviour of the AdS/CFT correspondence is recovered.

As was shown in [17], these theories exhibit gravity duals, called Lifshitz spacetimes. For a d-dimensional field theory on the boundary with Lifshitz scaling, the $d+1$ dimensional dual spacetime has a Lifshitz metric that can be written as

$$ds^2 = -\frac{R^{2z}}{L^{2z}}dt^2 + \frac{L^2}{R^2}dR^2 + \frac{R^2}{L^2}dM^2_{k,d-1} \ . \tag{4.2}$$

The constant L is the radius of the corresponding AdS_{d+1} spacetime, z the dynamical exponent and the line element $dM^2_{k,d-1}$ was previously defined in (2.21). This is an extension of the AdS_{d+1} spacetime that we considered earlier. Furthermore we let R scale as $R \to \lambda^{-1}R$, such that this metric will be invariant under Lifshitz scalings.

In this chapter we will have a look at a magnetic Lifshitz black brane. The first part of this chapter further explains this model. The equations of motion are also derived. Afterwards we will explicitly solve the equations of motion and finally the obtained solutions will be discussed.

4.1 Introducing the magnetic Lifshitz model

As was mentioned before, we will consider a magnetic Lifshitz black brane. We will work in a 3+1 dimensional bulk. A natural choice of the metric in the bulk is

$$ds^2 = \frac{L^2}{R^2}\frac{dR^2}{b_0(R)} - b_0(R)\frac{R^{2z}}{L^{2z}}dt^2 + \frac{R^2}{L^2}(dx^2 + dy^2) \ , \tag{4.3}$$

where $b_0(R)$ is a scalar function depending only on R.[1] This metric is just a deformation of (4.2). We also constrain $b_0(R)$, such that we recover a Lifshitz spacetime when $R \to \infty$

$$b_0(R) \to 1 \ , \tag{4.4}$$

This will guarantee that Lifshitz scale invariance is recovered close to the boundary.

In [3] and [18], it was shown that one $U(1)$ gauge field does not suffice to obtain an electric degree of freedom for Lifshitz spacetimes. This gauge field is fixed by the equations of motions to support the asymptotic Lifshitz spacetime. For that reason, it was proposed to consider a $U(1)$ x $U(1)$ abelian gauge symmetry with 2 gauge fields to obtain an electric field with a free parameter. We will follow the same approach for the magnetic Lifshitz black brane.

Next we need to define the action of our model

$$S = -\frac{1}{16\pi G_4}\int \mathrm{d}^4x\ \sqrt{-g}\left[\mathcal{R} - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] \ , \tag{4.5}$$

where ϕ is a scalar field that only depends on R, the λ_i are coupling constants and $\Lambda < 0$ is the cosmological constant. Notice that the coupling between ϕ and the gauge fields A_i is exponential. Every power of ϕ couples to the gauge fields. This model was solved for an electric Lifshitz black brane in $d+1$ dimensions, see [3].

The gauge fields are given by

$$A_i = B_i x dy \ , \tag{4.6}$$

where $i = 1, 2$ and x, y are the two spatial components on the boundary. This is the simplest model that can be constructed with a nonzero magnetic field and no electric field.

[1] The subscript zero in $b_0(R)$ indicates that we consider the spatial line element $dM_{0,2}^2$.

In the remainder of this chapter we will mainly be working with the electromagnetic field tensor instead of the gauge field itself. For that reason let us calculate the components of this tensor using (4.6). The only nonzero components of the electromagnetic field strength tensor are

$$(F_i)_{xy} = B_i \ , \tag{4.7a}$$

$$(F_i)^{xy} = B_i \frac{L^4}{R^4} \ . \tag{4.7b}$$

From these equations it is obvious that the physical magnetic field can be thought of as a constant field along the R direction, perpendicular to the two spatial directions.

We have all the ingredients to calculate the equations of motion from this model. These are derived by varying the action (4.5) to the various fields

$$\mathcal{R}_{\mu\nu} - \Lambda g_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\sum_{i=1}^{2} e^{\lambda_i\phi}\left[(F_i)_{\mu\sigma}(F_i)^{\sigma}_{\nu} - \frac{1}{4}F_i^2 g_{\mu\nu}\right] = 0 \ , \tag{4.8a}$$

$$D_\mu(e^{\lambda_i\phi}F_i^{\mu\nu}) = 0 \ , \tag{4.8b}$$

$$\Box\phi - \frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2 = 0 \ . \tag{4.8c}$$

Here we used the notation $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$ and D_μ is the covariant derivative. In the next section we will solve these equations using (4.3) and (4.7).

4.2 Solving the equations of motion

Lets us solve the equations of motion (4.8) by starting with equation (4.8b)

$$D_\mu(e^{\lambda_i\phi}F_i^{\mu\nu}) = \partial_\mu(e^{\lambda_i\phi}F_i^{\mu\nu}) + \Gamma^\mu_{\mu\rho}e^{\lambda_i\phi}F_i^{\rho\nu} + \Gamma^\nu_{\mu\rho}e^{\lambda_i\phi}F_i^{\mu\rho} = 0 \ . \tag{4.9}$$

Since ϕ and F_i only depend on R and $(F_i)^{\mu\nu}$ is zero for $\mu = R$ or $\nu = R$, we can drop the term $\partial_\mu(e^{\lambda_i\phi}F_i^{\mu\nu})$. Before we can proceed, we need to calculate the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \ . \tag{4.10}$$

Using (4.3) we can see that the only non-vanishing components are

$$\Gamma^R_{RR} = -\frac{\partial_R b_0}{2b_0} - \frac{1}{r} \ , \tag{4.11a}$$

$$\Gamma^R_{tt} = \frac{1}{L^{2z+2}}\left(R^{2z+2}b_0\partial_R b_0 + 2zR^{2z+1}b_0^2\right) \ , \tag{4.11b}$$

$$\Gamma^R_{ij} = -\delta_{ij}\frac{R^3 b_0}{L^4} \ , \tag{4.11c}$$

$$\Gamma^t_{Rt} = \Gamma^t_{tR} = \frac{\partial_R b_0}{2b_0} + \frac{z}{R} \ , \tag{4.11d}$$

$$\Gamma^i_{Rj} = \Gamma^i_{jR} = \frac{\delta^i_j}{R} \ . \tag{4.11e}$$

In these formulas i, j are used to denote the spatial coordinates on the boundary, x and y. We have to be careful not to confuse these spatial Lorentz indices with the indices of the gauge fields. I will explicitly state which indices correspond with the former and which ones to the latter if confusion is possible. With the Christoffel symbols (4.11), the reader can straightforwardly verify that equation (4.9) is trivially satisfied for each value of ν.

Next consider equation (4.8a). We start by raising the index ν

$$E^\rho_\mu = \mathcal{R}_{\mu\nu}g^{\nu\rho} - \Lambda\delta^\rho_\mu - \frac{1}{2}\partial_\mu\phi\partial^\rho\phi - \frac{1}{2}\sum_{i=1}^{2}e^{\lambda_i\phi}\left[(F_i)_{\mu\sigma}(F_i)^{\rho\sigma} - \frac{F_i^2}{4}\delta^\rho_\mu\right] = 0 \ . \tag{4.12}$$

We begin by solving the linear combination $E^t_t - E^R_R = 0$

$$E^t_t - E^R_R = \mathcal{R}_{tt}g^{tt} - \mathcal{R}_{RR}g^{RR} + \frac{1}{2}(\partial_R\phi)^2 g^{RR} = 0 \ . \tag{4.13}$$

The Ricci tensor is defined by

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu} \ . \tag{4.14}$$

With the expressions given in (4.11) and (4.14) we can calculate $\mathcal{R}_{tt}$ and $\mathcal{R}_{RR}$

$$\mathcal{R}_{tt} = \frac{b_0 R^{2z}}{2L^{2z+2}} \left(R^2 \partial_R^2 b_0 + (3z+3) R \partial_R b_0 + 2z(z+2) b_0 \right) \ , \tag{4.15a}$$

$$\mathcal{R}_{RR} = -\frac{1}{2R^2 b_0} \left(R^2 \partial_R^2 b_0 + (3z+3) R \partial_R b_0 + 2(z^2+2) b_0 \right) \ . \tag{4.15b}$$

Using (4.15) we can rewrite (4.13) to find

$$(\partial_R \phi)^2 = \frac{4(z-1)}{R^2} \ . \tag{4.16}$$

Solving this differential equation gives an expression for ϕ

$$\phi = \ln \left(\frac{R}{R_0} \right)^{\pm 2\sqrt{z-1}} \ , \tag{4.17}$$

with R_0 an integration constant. Equivalently one can write

$$e^{\phi} = \mu R^{\pm 2\sqrt{z-1}} \ , \tag{4.18}$$

with $\mu = R_0^{\mp 2\sqrt{z-1}}$. Note that μ is not related to any chemical potential.

Secondly we consider (4.12) with $\rho = i$ and $\mu = j$

$$E^i_j = \mathcal{R}_{ji} g^{ii} - \Lambda \delta^i_j - \frac{1}{2} \partial_j \phi \partial^i \phi - \frac{1}{2} \sum_{k=1}^{2} e^{\lambda_k \phi} \left[(F_k)_{j\sigma} (F_k)^{i\sigma} - \frac{F_k^2}{4} \delta^i_j \right] = 0 \ . \tag{4.19}$$

Recall that i and j denote the spatial coordinates on the boundary. k is the gauge field index. Again we have to calculate a component of the Ricci tensor namely R_{ji}. Using (4.11) the following result is obtained

$$\mathcal{R}_{ji} = \frac{\delta_{ji} R^2}{L^4} \left[-R \partial_R b_0 - b_0 (z+2) \right] \ . \tag{4.20}$$

Inserting this in (4.19) gives

$$\frac{\delta^i_j}{L^2} \left[-R \partial_R b_0 - b_0(z+2) \right] - \Lambda \delta^i_j - \sum_{k=1}^{2} \frac{e^{\lambda_k \phi}}{2} \left[(F_k)_{j\sigma} (F_k)^{i\sigma} - \frac{(F_k)_{xy} (F_k)^{xy}}{2} \delta^i_j \right] = 0 \ . \tag{4.21}$$

It is obvious that this equation is satisfied when $i \neq j$. When $i = j$, we have

$$-R \partial_R b_0 - b_0(z+2) - \Lambda L^2 - \frac{L^2}{2} \sum_{k=1}^{2} e^{\lambda_k \phi} \frac{(F_k)_{xy} (F_k)^{xy}}{2} = 0 \ . \tag{4.22}$$

If we now use the results (4.7) and (4.18), we obtain the following differential equation

$$-R\partial_R b_0 - b_0(z+2) - \Lambda L^2 - \frac{L^6}{4}\sum_{k=1}^{2}\mu^{\lambda_k} R^{-4\pm 2\lambda_k\sqrt{z-1}} B_k^2 = 0 \ . \tag{4.23}$$

This equation uniquely determines b_0 up to an integration constant. We find the following expression for b_0 using variation of constants

$$b_0(R) = -\frac{\Lambda L^2}{(z+2)} - mR^{-(z+2)} - \frac{L^6}{4}\sum_{k=1}^{2}\frac{\mu^{\lambda_k} R^{-4\pm 2\lambda_k\sqrt{z-1}} B_k^2}{z-2\pm 2\lambda_k\sqrt{z-1}} \ . \tag{4.24}$$

The integration constant m can be related to the mass of the black brane.

Let us continue by solving (4.8c). We start by expanding the equation

$$\Box\phi = \frac{1}{2}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}(F_i)_{xy}(F_i)^{xy} \ . \tag{4.25}$$

Substituting (4.18) and (4.7), we find the following expression

$$\Box\phi = \frac{L^4}{2}\sum_{i=1}^{2}\lambda_i\mu^{\lambda_i}B_i^2 R^{-4\pm 2\lambda_i\sqrt{z-1}} \ . \tag{4.26}$$

The $\Box$ operator was defined to be

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) \ . \tag{4.27}$$

Using (4.3) and (4.18) we can rewrite this expression

$$\Box\phi = \frac{\pm 2\sqrt{z-1}}{L^2}\left[(z+2)b_0 + R\partial_R b_0\right] \ . \tag{4.28}$$

Combining this result with (4.26) leads to

$$\frac{\pm 2\sqrt{z-1}}{L^2}\left[(z+2)b_0 + R\partial_R b_0\right] = \frac{L^4}{2}\sum_{i=1}^{2}\lambda_i\mu^{\lambda_i}B_i^2 R^{-4\pm 2\lambda_i\sqrt{z-1}} \ . \tag{4.29}$$

Insert the result from (4.24)

$$\pm 4\Lambda\sqrt{z-1} = \mp L^4\sum_{i=1}^{2}\mu^{\lambda_i}B_i^2 R^{-4\pm 2\lambda_i\sqrt{z-1}}(\pm\lambda_i + \sqrt{z-1}) \ . \tag{4.30}$$

In order to solve this equation, we have to fix the values of λ_i and B_i^2 ($i = 1, 2$). We are going to keep B_2 as a free parameter such that it can be seen as a physical magnetic field like we have discussed above. From equation (4.30), we can deduce that the condition on λ_2 will be

$$\pm\lambda_2 + \sqrt{z-1} = 0 \ . \tag{4.31}$$

Since λ_2 has to satisfy this equation, (4.30) reduces to the following expression

$$\pm 4\Lambda\sqrt{z-1} = \mp L^4 \mu^{\lambda_1} B_1^2 R^{-4\pm 2\lambda_1\sqrt{z-1}}(\pm\lambda_1 + \sqrt{z-1}) \ . \tag{4.32}$$

The left hand side is independent of R such that λ_1 must satisfy

$$-4 \pm 2\lambda_1\sqrt{z-1} = 0 \ . \tag{4.33}$$

If we use this result together with equation (4.32), we obtain an expression for B_1^2

$$\pm 4\Lambda\sqrt{z-1} = \mp L^4 \mu^{\pm\frac{2}{\sqrt{z-1}}} B_1^2 \frac{z+1}{\sqrt{z-1}} \ . \tag{4.34}$$

We can conclude from (4.33), (4.31) and (4.34) that λ_1, λ_2 and B_1^2 are restricted by the following identities in order for B_2 to remain a genuine degree of freedom[2]

$$\lambda_1 = \pm\frac{2}{\sqrt{z-1}} \ , \tag{4.35a}$$

$$\lambda_2 = \mp\sqrt{z-1} \ , \tag{4.35b}$$

$$B_1^2 = -\frac{4\Lambda(z-1)}{L^4(z+1)}\mu^{\mp\frac{2}{\sqrt{z-1}}} \ . \tag{4.35c}$$

With these expressions at hand, we can rewrite (4.24), (4.18) and (4.6)

$$b_0(R) = \frac{-2L^2\Lambda}{(z+2)(z+1)} - mR^{-(z+2)} + \frac{L^6\mu^{\mp\sqrt{z-1}}}{4z}B_2^2 R^{-2(z+1)} \ , \tag{4.36a}$$

$$A_1 = \frac{2}{L^2}\sqrt{-\frac{\Lambda(z-1)}{z+1}}\mu^{\mp\frac{1}{\sqrt{z-1}}} \ xdy \ , \tag{4.36b}$$

$$A_2 = B_2 \ xdy \ , \tag{4.36c}$$

$$e^{\phi} = \mu R^{\pm 2\sqrt{z-1}} \ . \tag{4.36d}$$

We now choose Λ in the following way ($b_0 \to 1$ when $R \to +\infty$)[3]

$$\Lambda = -\frac{(z+1)(z+2)}{2L^2} \ . \tag{4.37}$$

[2]Note that expression (4.35c) makes sense because $\Lambda < 0$.
[3]Since $z \geq 1$ as will be shown in the next section, we have that $\Lambda < 0$.

With this choice (4.36) becomes

$$
\begin{aligned}
b_0(R) &= 1 - mR^{-(z+2)} + \frac{L^6 \mu^{\mp\sqrt{z-1}}}{4z} B_2^2 R^{-2(z+1)} \ , && (4.38a)\\
A_1 &= \frac{1}{L^3}\sqrt{2(z-1)(z+2)}\mu^{\mp\frac{1}{\sqrt{z-1}}} \ xdy \ , && (4.38b)\\
A_2 &= B_2 \ xdy \ , && (4.38c)\\
e^{\phi} &= \mu R^{\pm 2\sqrt{z-1}} \ . && (4.38d)
\end{aligned}
$$

The only remaining equation of motion that we did not yet consider is the linear combination

$$E_t^t + E_R^R = 0 \ . \tag{4.39}$$

From equation (4.13) we already knew that $E_t^t = E_R^R$, such that the only thing left to verify is

$$E_t^t = 0 \ . \tag{4.40}$$

If we now use the definition given in (4.12) and use the results (4.15a), (4.7),(4.18) and (4.24) we find the following algebraic equation

$$
\begin{aligned}
\Lambda(1-z) \quad = \quad & \frac{L^4}{8}\sum_{i=1}^{2} B_i^2 \mu^{\lambda_i} R^{-4\pm 2\lambda_i\sqrt{z-1}} \left[\frac{(\pm 2\lambda_i\sqrt{z-1}-4)(\pm 2\lambda_i\sqrt{z-1}-5)}{z-2\pm 2\lambda_i\sqrt{z-1}} \right.\\
& \left. + \frac{(3z+3)(\pm 2\lambda_i\sqrt{z-1}-4)+2z(z+2)}{z-2\pm 2\lambda_i\sqrt{z-1}} + 2 \right] \ . \qquad (4.41)
\end{aligned}
$$

The reader can verify that, upon substituting the values of λ_1, λ_2 and B_1^2, this equation is satisfied.

For the temperature we find

$$T = \frac{R_h^z}{4\pi L^{z+1}}\left(z+2-\frac{L^6 B_2^2 \mu^{\mp\sqrt{z-1}}}{4}\right) \tag{4.42}$$

We have solved the model given in the previous section for a Lifshitz magnetic black brane. The solution is given in (4.38) and (4.35). This is a new solution and possesses some interesting features. We will discuss the solution in the next section.

4.3 A discussion on the solution

The expressions that we have found in (4.38) together with the restrictions on the parameters (4.35) solve all the equations of motion given in (4.8). Now we will discuss the physics behind the obtained solutions. Lets us begin with equation (4.38a)

$$b_0(R) = 1 - mR^{-(z+2)} + \frac{L^6\mu^{\mp\sqrt{z-1}}}{4z} B_2^2 R^{-2(z+1)} \ , \tag{4.43}$$

The form of this solution clearly suggests that we are dealing with a Reissner-Nordström Lifshitz black brane with a magnetic charge proportional to B_2 in $3+1$ dimensions. The event horizon R_h can be found by searching for the largest root of $b_0(R_h) = 0$.

Next, consider the solutions of the gauge fields

$$A_1 = \frac{1}{L^3}\sqrt{2(z-1)(z+2)}\mu^{\mp\frac{1}{\sqrt{z-1}}} \ xdy \ , \tag{4.44a}$$

$$A_2 = B_2 \ xdy \ . \tag{4.44b}$$

The second gauge, A_2, has a free parameter and leads to the EM field strength tensor given in (4.7). This can be interpreted as a constant magnetic field along the R-direction with magnitude B_2. As was explained in subsection 2.4.2, this leads to a constant magnetic background field at the boundary perpendicular to the 2 spatial directions x and y. The first gauge field, A_1, is fixed in terms of the constant μ. However it still corresponds with a magnetic field. This differs from the observations in [3]. They found for an electric Lifshitz black brane in the same model that one of the two gauge field could not be interpreted as a electrical potential.

Another interesting feature of this solution is the limit $z \to 1$. In this limit we should recover a Reissner-Nordström AdS_4 black brane. From (4.38), (4.35) and (4.42) we can see that

$$b_0(R) \to 1 - mR^{-3} + \frac{L^6 B_2^2 R^{-4}}{4} \ , \tag{4.45a}$$

$$A_1 \propto \sqrt{z-1}\mu^{\mp\frac{1}{\sqrt{z-1}}} \ xdy \ , \tag{4.45b}$$

$$A_2 \to B_2 \ xdy \ , \tag{4.45c}$$

$$\lambda_1 \to \pm\infty \ , \tag{4.45d}$$

$$\lambda_2 \to 0 \ , \tag{4.45e}$$

$$e^{\phi} \to \mu \ , \tag{4.45f}$$

$$T \to \frac{1}{4\pi}\left(\frac{3R_h}{L^2} - \frac{L^4 B_2^2}{4R_h^3}\right) \tag{4.45g}$$

From these results, we can derive whether the plus or the minus sign has to be used because we do not want A_1 to diverge in the limit $z \to 1$. This gives

$$\text{if } \mu < 1 \to \text{"+"-sign} \ , \tag{4.46a}$$

$$\text{if } \mu > 1 \to \text{"-"-sign} \ , \tag{4.46b}$$

This choice ensures $A_1 \to 0$ when $z \to 1$. Next we have a second look at the action given in (4.5). The first gauge field will decouple from the theory in the limit $z \to 1$

$$-\frac{1}{4}e^{\lambda_1 \phi}F_1^2 = -\frac{1}{2L^2}(z-1)(z+2) \to 0 \ . \tag{4.47}$$

For the second gauge field we find that the coupling with the scalar field disappears if $z \to 1$

$$-\frac{1}{4}e^{\lambda_2 \phi}F_2^2 = -\frac{1}{4}\mu^{\mp(z-1)}R^{-(z-1)}\frac{L^4}{R^4}B_2^2 \to \frac{L^4}{R^4}B_2^2 = F_2^2 \ . \tag{4.48}$$

Since the scalar field becomes a constant for $z \to 1$, we can also drop the term $\frac{1}{2}(\partial\phi)^2$. We conclude that the action in the limit $z \to 1$ takes the following form

$$S = -\frac{1}{16\pi G_4}\int \mathrm{d}^4x \ \sqrt{-g}\left[\mathcal{R} - 2\Lambda - \frac{1}{4}F_2^2\right] \ . \tag{4.49}$$

This action clearly corresponds to the action considered in section 2.4.2. Now one can check that the results (4.45), are consistent with the ones given in (2.55) with only one nonzero magnetic charge.

Furthermore we can consider the limit $B_2 \to 0$ of (4.38) and (4.35)

$$b_0(R) = 1 - mR^{-(z+2)} \tag{4.50a}$$

$$A_1 = \frac{1}{L^3}\sqrt{2(z-1)(z+2)}\mu^{\mp\frac{1}{\sqrt{z-1}}} \ xdy \tag{4.50b}$$

$$A_2 = 0 \tag{4.50c}$$

$$\lambda_1 = \pm\frac{2}{\sqrt{z-1}} \tag{4.50d}$$

$$\lambda_2 = \mp\sqrt{z-1} \tag{4.50e}$$

$$e^{\phi} = \mu R^{\pm 2\sqrt{z-1}} \tag{4.50f}$$

We find a Scharzschild Lifshitz solution but we still have a magnetic field originating from A_1. This means that A_1 has to vanish. This is not possible if $z \neq 1$. This means that we can not compare this result with the uncharged

Lifshitz solution given in [18]. If we also consider the limit $z \to 1$, we recover the Schwarzschild black brane in AdS_4 from section 2.3

$$b_0(r) = 1 - mR^{-3} \tag{4.51a}$$
$$A_1 = 0 \tag{4.51b}$$
$$A_2 = 0 \tag{4.51c}$$
$$\lambda_1 = \pm\infty \tag{4.51d}$$
$$\lambda_2 = 0 \tag{4.51e}$$
$$e^{\phi} = \mu \tag{4.51f}$$

To conclude this chapter we remark that we have interpreted z as a dynamical variable. However, not all values of z are permitted. It turns out that causality of the strongly coupled field theory at the boundary is incompatible when $z < 1$. From now on we will always assume that $z \geq 1$. The interested reader can find more information on this topic in [19].

Chapter 5

The dyonic Lifshitz black brane

The goal of this thesis is trying to calculate a Hall conductivity for an anisotropic scale invariant field theory using its dual theory. In order to obtain such a conductivity, the field theory must have a nonzero charge density and a background magnetic field. Obviously, the dual theory that has to be used to calculate this Hall conductivity is a dyonic Lifshitz black brane/black hole. We will only consider the black brane here.

We will continue with the action defined in (4.5), but both gauge fields will have an extra nonzero t component, A_t. It turns out that a dyonic Lifshitz black brane does not admit a solution with an electric and magnetic degree of freedom originating from the same gauge field. This results in a no-go theorem which we will prove.

5.1 A first no-go theorem for the dyonic Lifshitz black brane

As was said in the introduction of this chapter, we consider a dyonic Lifshitz black brane. We will work with the same model as the previous chapter

$$ds^2 = \frac{L^2}{R^2}\frac{dR^2}{b_0(R)} - b_0(R)\frac{R^{2z}}{L^{2z}}dt^2 + \frac{R^2}{L^2}(dx^2 + dy^2) \ , \tag{5.1a}$$

$$S = -\frac{1}{16\pi G_4}\int \mathrm{d}^4x \ \sqrt{-g}\left[\mathcal{R} - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] \ . \tag{5.1b}$$

Only the ansatz for the 2 gauge fields will be different from the previous chapter

$$A_1 = V_1(R)dt + B_1 x dy \ , \tag{5.2a}$$

$$A_2 = V_2(R)dt + B_2 x dy \ . \tag{5.2b}$$

Again some of the gauge field components will be determined in terms of other quantities due to the equations of motion. The idea is to search for a solution of this model with a free magnetic and electric charge originating from the same gauge field as was explained in subsection 3.2.2.

The equations of motion that we found in section 4.1

$$E_\mu^\rho = 0 \ , \tag{5.3a}$$

$$D_\mu(e^{\lambda_i\phi}F_i^{\mu\nu}) = 0 \ , \tag{5.3b}$$

$$\Box\phi - \frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2 = 0 \ . \tag{5.3c}$$

We solve the equations of motion using the same techniques one outlined in section 4.2. We start with the equation of motion of the EM field strength tensor (5.3b). Expand this equation by using the definition of the covariant derivative

$$\partial_\mu(e^{\lambda_i\phi}F_i^{\mu\nu}) + \Gamma^\mu_{\mu\rho}e^{\lambda_i\phi}F_i^{\rho\nu} + \Gamma^\nu_{\mu\rho}e^{\lambda_i\phi}F_i^{\mu\rho} = 0 \ . \tag{5.4}$$

Since we are considering the same metric as in the previous chapter, the Christoffel symbols have already been given in (4.11). The only equation that is not trivially satisfied arises for $v = R$. We find the following first order differential equation

$$\partial_R(e^{\lambda_i\phi}F_i^{Rt}) + \frac{z+1}{R}e^{\lambda_i\phi}F_i^{Rt} = 0 \ . \tag{5.5}$$

which is solved by

$$(F_i)_{Rt} = \rho_i e^{-\lambda_i \phi} R^{z-3} , \tag{5.6a}$$
$$(F_i)^{Rt} = -L^{2z-2} \rho_i e^{-\lambda_i \phi} R^{-(z+1)} . \tag{5.6b}$$

where $\rho_1, \rho_2 \in \mathbb{R}$. There are two more nonzero components of the EM field strength tensor. They are given by

$$(F_i)_{xy} = B_i , \tag{5.7a}$$
$$(F_i)^{xy} = B_i \frac{L^4}{R^4} . \tag{5.7b}$$

The next step is solving the linear combination $E_t^t - E_R^R = 0$. This gives the same differential equation as (4.13) such that we find the same result for ϕ

$$e^{\phi} = \mu R^{\pm 2\sqrt{z-1}} . \tag{5.8}$$

Let us proceed by solving $E_j^i = 0$. This equation reads

$$E_j^i = \mathcal{R}_{ji} g^{ii} - \Lambda \delta_j^i - \frac{1}{2} \partial_j \phi \partial^i \phi - \frac{1}{2} \sum_{k=1}^{2} e^{\lambda_k \phi} \left[(F_k)_{j\sigma} (F_k)^{i\sigma} - \frac{F_k^2}{4} \delta_j^i \right] = 0 , \tag{5.9}$$

where i and j denote the spatial coordinates of the boundary and k is the gauge field index. Use now result (4.20) to obtain

$$\begin{aligned} \frac{\delta_j^i}{L^2} \left[-R \partial_R b_0 - b_0 (z+2) \right] &= \frac{1}{2} \sum_{k=1}^{2} e^{\lambda_k \phi} \Big[(F_k)_{j\sigma} (F_k)^{i\sigma} \\ &\quad - \frac{(F_k)_{xy} (F_k)^{xy} + (F_k)_{Rt} (F_k)^{Rt}}{2} \delta_j^i \Big] \\ &\quad + \Lambda \delta_j^i . \end{aligned} \tag{5.10}$$

Again, the above equation is trivially satisfied when $i \neq j$. When we take $i = j$ and insert the results (4.7) and (5.6), we find

$$\begin{aligned} -R \partial_R b_0 - b_0 (z+2) &= \Lambda L^2 + \frac{1}{4} \sum_{k=1}^{2} \Big(L^6 \mu^{\lambda_k} B_k^2 R^{-4 \pm 2\lambda_k \sqrt{z-1}} \\ &\quad + L^{2z} \mu^{-\lambda_k} \rho_k^2 R^{-4 \mp 2\lambda_k \sqrt{z-1}} \Big) . \end{aligned} \tag{5.11}$$

This leads to a solution for $b_0(R)$

$$\begin{aligned} b_0(R) &= -\frac{1}{4} \sum_{k=1}^{2} \left(\frac{L^6 \mu^{\lambda_k} R^{-4 \pm 2\lambda_k \sqrt{z-1}} B_k^2}{z - 2 \pm 2\lambda_k \sqrt{z-1}} + \frac{L^{2z} \mu^{-\lambda_k} R^{-4 \mp 2\lambda_k \sqrt{z-1}} \rho_k^2}{z - 2 \mp 2\lambda_k \sqrt{z-1}} \right) \\ &\quad - \frac{\Lambda L^2}{(z+2)} - m R^{-(z+2)} . \end{aligned} \tag{5.12}$$

The next equation that we need to consider is (5.3c). Using (4.28), (5.12), (4.7) and (5.6), this equation can be rewritten in the following way

$$\begin{aligned}\mp 4\Lambda\sqrt{z-1} \;\; &= \;\; \sum_{i=1}^{2}(L^4\mu^{\lambda_i}B_i^2R^{-4\pm 2\lambda_i\sqrt{z-1}}(\lambda_i \pm \sqrt{z-1}) \\ &\qquad +L^{2z-2}\mu^{-\lambda_i}\rho_i^2R^{-4\mp 2\lambda_i\sqrt{z-1}}(-\lambda_i \pm \sqrt{z-1}))\ . \end{aligned} \tag{5.13}$$

In the next section we will prove that this equation does not admit a solution with an electric and a magnetic degree of freedom originating from the same gauge field. This is called a no-go theorem since it specifies that a specific situation is not physically possible.

No-go theorem 1

The model defined in (5.1) *with an abelian gauge group* $U(1)$ *x* $U(1)$ *does not admit a dyonic Lifshitz black brane solution with an electric and a magnetic degree of freedom originating from the same gauge field for* $z > 1$*.*

5.2 Proof of the no-go theorem

Let us prove No-go theorem 1. The central idea of this proof is to fix the values of λ_1 and λ_2 such that R-dependent terms can cancel or immediately vanish. Before we start, we can see that the R-dependence of some terms in (5.13) drops out when

$$-4 \pm 2\lambda_i \sqrt{z-1} = 0 \ , \tag{5.14}$$

with $i = 1, 2$. This gives the following requirements on λ_i

$$\lambda_i = \frac{\pm 2}{\sqrt{z-1}} \ , \tag{5.15}$$

This observation will be very useful when we are proving No-go theorem 1.

Let us rewrite (5.13)

$$\begin{aligned}
\mp 4\Lambda\sqrt{z-1} \ = \ & \overbrace{L^4 \mu^{\lambda_1} B_1^2 R^{-4\pm 2\lambda_1\sqrt{z-1}} (\lambda_1 \pm \sqrt{z-1})}^{A_1} \\
& + \overbrace{L^{2z-2} \mu^{-\lambda_1} \rho_1^2 R^{-4\mp 2\lambda_1\sqrt{z-1}} (-\lambda_1 \pm \sqrt{z-1})}^{A_2} \\
& + \overbrace{L^4 \mu^{\lambda_2} B_2^2 R^{-4\pm 2\lambda_2\sqrt{z-1}} (\lambda_2 \pm \sqrt{z-1})}^{C_1} \\
& + \overbrace{L^{2z-2} \mu^{-\lambda_2} \rho_2^2 R^{-4\mp 2\lambda_2\sqrt{z-1}} (-\lambda_2 \pm \sqrt{z-1})}^{C_2} \ .
\end{aligned} \tag{5.16}$$

Furthermore we have that $\mu > 0$. This follows from the definition of μ given in section 4.2.

Proof: Case 1: $B_1 \neq 0, B_2 \neq 0, \rho_1 \neq 0$ and $\rho_2 \neq 0$.
Suppose we fix λ_1 different from (5.15) and different from zero.[1] It is clear from (5.16) that A_1 and A_2 can not cancel each other since they have different powers of R. It is also impossible for the two terms to be simultaneously zero. First we assume that none of these two terms are zero. In that case we have to fix λ_2 in C_1 and C_2 to cancel the R-dependent terms A_1 and A_2. This is only possible when

$$\begin{cases} \lambda_2 = \lambda_1 & \text{(I)} \ , \\ \lambda_2 = -\lambda_1 & \text{(II)} \ . \end{cases} \tag{5.17}$$

[1] An analogous argument can be made when you fix λ_2 instead of λ_1.

Case (I) leads to

$$\begin{cases} \rho_1^2 = -\rho_2^2 \ , \\ B_1^2 = -B_2^2 \ , \\ -4\Lambda\sqrt{z-1} = 0 \ . \end{cases} \tag{5.18}$$

This set of equations does not admit a solution in $\mathbb{R}$. Case (II) also leads to a set of equations without solution

$$\begin{cases} \rho_1^2 L^{2z-2} = -B_2^2 L^4 \ , \\ B_1^2 L^4 = -\rho_2^2 L^{2z-2} \ , \\ -4\Lambda\sqrt{z-1} = 0 \ . \end{cases} \tag{5.19}$$

The reader can verify that nothing changes when A_1 or A_2 is zero. You still have to consider case (I) or (II) for further cancellation. So this will neither generate a solution.

When we take $\lambda_1 = 0$ there are 2 possibilities. The first is that A_2 cancels A_1, but this leads to $B_1^2 L^4 = -\rho_1^2 L^{2z-2}$ which has no solution in $\mathbb{R}$. Otherwise A_1 and A_2 can cancel with C_1 and C_2. This can only happen when $\lambda_2 = 0$ and results in the following set of equations

$$\begin{cases} (\rho_1^2 + \rho_2^2) L^{2z-2} + (B_1^2 + B_2^2) L^4 = 0 \ , \\ -4\Lambda\sqrt{z-1} = 0 \ . \end{cases} \tag{5.20}$$

Again, this does not admit a real-valued solution.

Suppose that we now take $\lambda_1 = \frac{\pm 2}{\sqrt{z-1}}$. A_1 becomes R-independent but A_2 will have an R-dependence. Again we have to choose case (I) or (II) to cancel the R-dependent term A_2. Case (I) gives

$$\begin{cases} \rho_1^2 = -\rho_2^2 \ , \\ -4\Lambda\frac{z-1}{z+1}\mu^{\frac{\mp 2}{\sqrt{z-1}}} L^{-4} = B_1^2 + B_2^2 \ . \end{cases} \tag{5.21}$$

Case (II) gives

$$\begin{cases} \rho_1^2 L^{2z-2} = -B_2^2 L^4 \ , \\ -4\Lambda\frac{z+1}{z-1}\mu^{\frac{\mp 2}{\sqrt{z-1}}} = B_1^2 L^4 + \rho_2^2 L^{2z-2} \ . \end{cases} \tag{5.22}$$

Both sets of equations do not generate a physical solution in $\mathbb{R}$.

<u>Case 2</u>: $B_1 \neq 0$, $B_2 \neq 0$, $\rho_1 = 0$ and $\rho_2 \neq 0$.[2]
In this case A_2 obviously vanishes. Suppose we fix λ_2 different from (5.15)

[2]An analogous proof can be made if one of the parameters other then ρ_1 is zero.

and different from zero. It is clear from (5.16) that C_1 and C_2 can not cancel each other since they have different powers of R. It is also impossible for the two terms to simultaneously equal zero and we will assume that both terms are nonzero. This means that we have to fix λ_1 to cancel the R-dependent terms C_1 and C_2. Since A_1 is the only remaining term with a tuneable R-dependence, we are only able to let either C_1 or C_2 vanish, but not both. So there will always remain a nonzero term on the right hand side of (5.16) that depends on R. The resulting equation does not have real-valued solutions

If we take $\lambda_2 = 0$, we are able to have C_1 and C_2 cancel against each other. This gives $B_2^2 L^4 = -\rho_2^2 L^{2z-2}$, which is an equation with no solution in $\mathbb{R}$. If λ_1 also vanishes, we get $(B_1^2 + B_2^2)L^4 + \rho_2^2 L^{2z-2} = 0$, which again does not admit a solution in $\mathbb{R}$.

In the case where one of the C- terms vanishes because λ_2 is given by (5.15), it is impossible to find a solution. If C_1 is zero, we need

$$\begin{cases} \lambda_1 = -\lambda_2 \ , \\ \rho_1^2 L^{2z-2} = -B_2^2 L^4 \ , \\ -4\Lambda\sqrt{z-1} = 0 \ . \end{cases} \tag{5.23}$$

If C_2 is zero, one finds

$$\begin{cases} \lambda_1 = \lambda_2 \ , \\ B_1^2 = -B_2^2 \ , \\ -4\Lambda\sqrt{z-1} = 0 \ . \end{cases} \tag{5.24}$$

Both (5.23) and (5.24) do not admit a real-valued solution.

Now take $\lambda_2 = \frac{\pm 2}{\sqrt{z-1}}$. C_1 becomes R-independent. Again we need to cancel C_2 with A_1. This gives

$$\begin{cases} \lambda_1 = -\lambda_2 \ , \\ B_1^2 L^4 = -\rho_2^2 L^{2z-2} \ , \\ -4\Lambda\frac{z-1}{z+1}\mu^{\frac{\mp 2}{\sqrt{z-1}}} L^{-4} = B_2^2 \ . \end{cases} \tag{5.25}$$

Again a set of equations with no solution in $\mathbb{R}$. An analogous argument can be made when $\lambda_2 = \frac{\mp 2}{\sqrt{z-1}}$.

On the other hand, we can also start by fixing λ_1. Suppose $\lambda_1 = \mp\sqrt{z-1}$ then A_1 vanishes and we are left with C_1 and C_2. Since these two terms can not simultaneously vanish and can only be R-independent when $\lambda_2 = 0$. This leads to $\rho_2^2 L^{2z-2} = -B_2^2 L^4$, which has no real-valued solution. The same argument holds when we take $\lambda_1 = \frac{\pm 2}{\sqrt{z-1}}$. If λ_1 is different from these 2 values,

A_1 is R-dependent. This means that we need C_1 or C_2 to cancel A_1. This can be done by choosing

$$\begin{cases} \lambda_1 = \lambda_2 \ , \\ \text{or} \\ \lambda_1 = -\lambda_2 \ . \end{cases} \tag{5.26}$$

These 2 options both lead to sets of equations without a physical solution.

Case 3: $B_1 = 0$, $B_2 \neq 0$, $\rho_1 = 0$ and $\rho_2 \neq 0$.[3]
Only A_1 and A_2 are different from zero. Since they can not simultaneously be R-independent or zero, we have to fix $\lambda_1 = 0$ such that A_1 can cancel A_2. This results in

$$\begin{cases} \lambda_1 = 0 \\ \rho_1^2 L^{2z-2} = -B_1^2 L^4 \\ -4\Lambda\sqrt{z-1} = 0 \end{cases} \tag{5.27}$$

which does not admit a solution in $\mathbb{R}$.

Case 4: $B_1 = 0$, $B_2 \neq 0$, $\rho_1 \neq 0$ and $\rho_2 = 0$.
We fix $\lambda_1 = \frac{\mp 2}{\sqrt{z-1}}$ and $\lambda_2 = \mp\sqrt{z-1}$. This leads to

$$\begin{cases} \lambda_1 = \frac{\mp 2}{\sqrt{z-1}} \\ \lambda_2 = \mp\sqrt{z-1} \\ -4\Lambda\frac{z-1}{z+1}\mu^{\frac{\mp 2}{\sqrt{z-1}}} L^{2-2z} = \rho_1^2 \end{cases} \tag{5.28}$$

This system of equations admits a solution but it has only one free charge, namely B_2. The first gauge field doesn't lead to a free electric charge on the boundary since $(F_1)_{Rt}$ has no R-dependence like an electric field. It will look like

$$(F_1)_{Rt} = 2L^{-z}\sqrt{d-2}\mu^{\frac{\pm 2}{\sqrt{z-1}}} R^{z+1} \ . \tag{5.29}$$

This will blow up a large values of R.

Case 5: $B_1 \neq 0$, $B_2 = 0$, $\rho_1 = 0$ and $\rho_2 \neq 0$.
We fix $\lambda_1 = \frac{\pm 2}{\sqrt{z-1}}$ and $\lambda_2 = \pm\sqrt{z-1}$. This leads to

$$\begin{cases} \lambda_1 = \frac{\mp 2}{\sqrt{z-1}} \\ \lambda_2 = \mp\sqrt{z-1} \\ -4\Lambda\frac{z-1}{z+1}\mu^{\frac{\mp 2}{\sqrt{z-1}}} L^{-4} = B_1^2 \end{cases} \tag{5.30}$$

This system of equations admits a solution and has 1 free charge, ρ_2 and one fixed charge, B_1. The first gauge field leads to a magnetic background field at

[3] The same argument will hold when $B_1 \neq 0$, $B_2 = 0$, $\rho_1 \neq 0$ and $\rho_2 = 0$.

the boundary at the second gauge field gives rise to a nonzero charge density at the boundary. However they do not originate from the same gauge field. This means that we can not compute an electrical conductivity.

The situations with $\rho_1 = \rho_2 = 0$ or $B_1 = B_2 = 0$ are not considered here since they obviously won't generate a dyonic Lifschitz BB. The same can be said when we choose 3 or more parameters of the set $\{\rho_1, \rho_2, B_1, B_2\}$ to be equal to zero.

$\square$

5.3 Extending the no-go theorem for dyonic Lifshitz black branes

In the previous section we proved No-go theorem 1. We shall argue that this theorem can be generalised to the same model but with an $U(1)^N$ abelian gauge group. The statement reads

No-go theorem 2
The model defined in (5.1) *with an abelian gauge group* $U(1)^N$ *does not admit a dyonic Lifshitz black brane solution with an electric and a magnetic degree of freedom originating from the same gauge field for* $z > 1$.

An explicit proof of this theorem shall not be given here. We provide an intuitive argument:

Suppose that we add a gauge field to the system defined in section 5.1. After a similar analysis as done in section 5.1, one finds the following algebraic equation

$$\begin{aligned}
\mp 4\Lambda\sqrt{z-1} \;=\; & \overbrace{L^4 \mu^{\lambda_1} B_1^2 R^{-4\pm 2\lambda_1\sqrt{z-1}}(\lambda_1 \pm \sqrt{z-1})}^{A_1} \\
& + \overbrace{L^{2z-2} \mu^{-\lambda_1} \rho_1^2 R^{-4\mp 2\lambda_1\sqrt{z-1}}(-\lambda_1 \pm \sqrt{z-1})}^{A_2} \\
& + \overbrace{L^4 \mu^{\lambda_2} B_2^2 R^{-4\pm 2\lambda_2\sqrt{z-1}}(\lambda_2 \pm \sqrt{z-1})}^{C_1} \\
& + \overbrace{L^{2z-2} \mu^{-\lambda_2} \rho_2^2 R^{-4\mp 2\lambda_2\sqrt{z-1}}(-\lambda_2 \pm \sqrt{z-1})}^{C_2} \\
& + \overbrace{L^4 \mu^{\lambda_2} B_3^2 R^{-4\pm 2\lambda_3\sqrt{z-1}}(\lambda_3 \pm \sqrt{z-1})}^{D_1} \\
& + \overbrace{L^{2z-2} \mu^{-\lambda_3} \rho_3^2 R^{-4\mp 2\lambda_3\sqrt{z-1}}(-\lambda_3 \pm \sqrt{z-1})}^{D_2} \; . \qquad (5.31)
\end{aligned}$$

The extra 2 terms D_1 and D_2 have the same form as the terms A_i and C_i ($i = 1, 2$). Suppose that they are both different from zero. In order to get rid of the R-dependence in D_1 or D_2 we need λ_3 or $-\lambda_3$ to equal λ_2 and/or λ_1. Again this leads to equations where sums of quadratic terms have to vanish similar to the proof in section 5.2. This does not lead to an solution with a magnetic and an electric degree of freedom originating from the same gauge field.

When D_1 or D_2 is zero, we can have an R-independent or an R-dependent term. R-dependent terms have to be cancelled like was mentioned in the previous paragraph. This does not lead to a solution with 2 EM degrees of freedom originating from the same gauge field. When the term is R-independent we can absorb this term on the left hand side of (5.31) such that No-go theorem 1 holds.

This argument can be repeated up to N $U(1)$ gauge fields such that No-go theorem 2 makes sense.

Chapter 6

Conclusion

In this thesis we considered a $3+1$ dimensional system with a $U(1)$ x $U(1)$ abelian gauge group defined by

$$ds^2 = \frac{L^2}{R^2}\frac{dR^2}{b_0(R)} - b_0(R)\frac{R^{2z}}{L^{2z}}dt^2 + \frac{R^2}{L^2}(dx^2+dy^2) \ , \tag{6.1a}$$

$$S = -\frac{1}{16\pi G_4}\int \mathrm{d}^4x\ \sqrt{-g}\left[\mathcal{R} - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] \ . \tag{6.1b}$$

In particular we searched for 2 types of solutions:

1. Magnetic Lifshitz black branes
2. Dyonic Lifshitz black branes

In chapter 4 we looked at the magnetic case. We found a Lifshitz black brane solution. Contrary to what was expected, the gauge fields both led to a magnetic field. One of them was completely fixed by the equations of motion and the other had a free parameter. This solution turned out to have an *AdS* limit that was consistent with the magnetic *AdS* black brane. Since we were interested in calculating a Hall conductivity, this model needed to be extended.

For that reason we looked for dyonic Lifshitz BB solutions. We found that this system did not admit such a solution which led to a first no-go theorem. This theorem was proved in section 5.2. It turned out that the no-go theorem for a dyonic Lifshitz BB could be extended to a system with a $U(1)^N$ abelian gauge group. This was briefly discussed in section 5.3. The underlying physics which causes the dyonic Lifshitz BB solution to fail for this particular model remains unclear.

Since we have focussed on black branes in this thesis, one might wonder if black holes would be better suited to solve the model (6.1). This remains a possibility since the algebraic equation (5.13) that will be generated for black holes will be different. This raises an interesting question for future research.

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