A Semantic Account of Free Choice for Ability *can* in STIT Logic

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Abstract

The free choice effect appears in natural language when an interaction between a disjunction and a modal gives rise to a quasi-conjunctive interpretation. Problematic is that free choice inferences are not valid in classical modal logic. Free choice inferences are typically studied for the deontic modal *may*, but have also been observed for other modals, including ability *can*. STIT logic is a framework that can model agency, but it does not account for free choice inferences for ability modals. Fusco (2020) offers a semantic account for free choice ability with possible world semantics. In this thesis, I incorporate elements from from her account into STIT logic in order to create a modified version of STIT logic in which free choice inferences are valid for ability *can*. These elements include a two-dimensional account of disjunction that is sensitive to the 'actual world', and a view of ability as the historical possibility to externally realize something. This resulted in a STIT logic in which disjunctions are treated non-classically when in the scope of modal operators, allowing for free choice ability inferences.

Keywords: free choice, STIT logic, semantics, ability modals

Contents

1 Introduction

One of the main goals in AI is to develop agents that display intelligent behavior. The tasks of some of these agents might involve interpreting natural language produced by customers. The agents then might need to reason with the information they received and make inferences on which action they should take. Examples of such agents include chat bots, virtual assistants, or companion robots. These agents need to translate that natural language to some formal system with the appropriate semantics, so it can reason with the information provided by the customer and make inferences. A famous example of such a logical framework and semantics is modal logic with Kripke semantics. There are however cases where sentences in natural language are translated in such systems in such a way that incorrect inferences can be made or that correct inferences cannot be made. This thesis will consider one of these problems, called the *free choice* effect.

The free choice effect involves a certain interaction between a disjunction and a modal operator. The problem was introduced for deontic modals by Kamp (1974)[1], who named it *free choice permission.* Sentence (1) is an example of free choice permission. From sentence (1), we can infer both sentence (2) and sentence (3). The agent (you) is allowed a free choice between going to the beach or to the cinema.

(1) You may go to the beach or to the cinema.

(2) You may go to the beach.

(3) You may go to the cinema.

(4) $\Diamond(p \lor q) \to \Diamond p \land \Diamond q$

The inference from (1) to (2) and (3) can be translated into modal logic as (4), where the diamond represents the deontic *may.* This inference is however invalid in classical deontic logics.

The free choice effect is usually studied for deontic *may*, but can be observed for all sorts of modalities. One modality of particular interest for this thesis is ability. An example of this modality is the verb *can* as in the sentence "I can speak French". STIT logic is a well-known logic for modelling agency. In standard STIT logic however, free choice inferences are invalid. The research question of this thesis is whether STIT logic can be modified in such a way that free choice follows for ability modals.

In order to answer this question, I will first describe the free choice problem in a general fashion, then I will zoom in on free choice ability. I will then discuss one specific proposed solution to free choice ability, namely Fusco (2020), who uses a Kripke-style semantics to solve free choice. I will then try to apply ingredients from Fusco's account to STIT logic, to create a framework that handles free choice for ability modals correctly.

[1] Kamp was not the first to point this out, but he was the one who introduced the problem in linguistics, as it was treated as a logical problem before, see (Meyer, 2016).

2 Free Choice

2.1 The Problem of Free Choice

Disjunctions are often used to express uncertainty about something. When I am unsure whether John ate an apple or a pear, but know that he ate at least one of those, I could express this by saying (5). From that statement, a listener would not be able to infer (6), neither would they be able to infer (7). We can only conclude that John consumed at least one of the two[2].

(5) John ate an apple or a pear.

(6) John ate an apple.

(7) John ate a pear.

This pattern can also be seen for the disjunction operator of classical logic, $\vee$, which expresses that at least one of its arguments is true. From the logical statement that $a \vee p$, we cannot infer the truth of a, nor can we conclude that p must hold. We can however infer that at least one of them is true. Because of this similarity, the $\vee$ operator is typically used for translating the word *or* to logic.

However, something strange happens when we include the deontic modal *may*, as in (8). Suddenly, we can can conclude that both (9) and (10) are true. It seems like *may* does something to *or* that makes it possible to infer the truth of both disjuncts.

(8) John may eat an apple or a pear.

(9) John may eat an apple.

(10) John may eat a pear.

The standard way of modelling a modal like *may*, is with modal logic. Here, *may* is translated as the existential modal operator $\Diamond$. We can thus translate (8) to modal logic as $\Diamond(a \vee p)$[3]. Formulas in modal logic are interpreted using possible world semantics. We say $\Diamond\varphi$ is true if and only if there is some possible world (in case of deontic modality, we would say an *allowable* world) such that φ is true in that world (Garson, 2021).

(11) $\Diamond(a \vee p)$

(12) $\Diamond a$

(13) $\Diamond p$

Using this interpretation of $\Diamond$, could we infer from the truth of (11) the truth of (12) and the truth of (13)? The answer is clearly no. We could imagine some model with a possible world where only a is true. This would mean that both (11) and (12) are valid, but we don't have a world where p is true, so (13) is false. Just like $a \vee p$ does, $\Diamond(a \vee p)$ just guarantees that at least one disjunct is true. We cannot conclude for a specific disjunct its truth.

It appears that (8) has some sort of conjunctive interpretation. It is however clearly different from the corresponding conjunction as in (14):

[2] Typically, we would also conclude that John didn't eat both, something that is not accounted for by the classical interpretation of disjunction.

[3] Alternatively, we could translate it as $\Diamond a \vee \Diamond p$. This would have a different logical form than $\Diamond(a \vee p)$. $\Diamond(a \vee p)$ would have to the logical form [John may [[eat an apple] or [eat a pear]]], while $\Diamond a \vee \Diamond p$ would denote [[John may [eat an apple]] or [John may [eat a pear]]]. Both formulas are however equivalent in modal logic.

(14) John may eat an apple and a pear.

In (14), John is given permission to eat both fruits. In (8) on the other hand, John is not explicitly allowed to eat both. In fact, it seems to constrain him to eating only one. He gets to choose one of them.

Apparently, we can't solve this problem by simply interpreting *or* as *and*. There is a discrepancy between what we can infer in natural language and what we can infer in modal logic. We can thus attempt to somehow change the logic. We could try to add a 'free choice axiom' to our logic, like (15):

(15) Free choice axiom:
$\Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \wedge \Diamond\psi$

(16) $\Diamond\varphi \rightarrow \Diamond(\varphi \vee \psi)$

We would first of all need a new interpretation for modal logic, as possible world semantics predicts the validity of (16). Combining (16) with the free choice axiom, we could derive $\Diamond\psi$ from $\Diamond\varphi$. For deontic modality, this would mean that being allowed something implies being allowed everything, which is clearly not the case in natural language. Also, (16) is clearly not valid in natural language, demonstrated by the lack of entailment from (9) to (8), strengthening the need for a new interpretation of modal logic.
However, consider the following derivation:

$\Diamond(\neg\varphi \vee \neg\psi) \rightarrow \Diamond\neg\varphi$	Free choice axiom
$\neg\Diamond\neg\varphi \rightarrow \neg\Diamond(\neg\varphi \vee \neg\psi)$	Modus tollens
$\Box\varphi \rightarrow \Box\neg(\neg\varphi \vee \neg\psi)$	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
$\Box\varphi \rightarrow \Box(\varphi \wedge \psi)$	De Morgan's law

We derived (like Kamp (1974) did) the inference from $\Box\varphi$ to $\Box(\varphi \wedge \psi)$. The universal modal operator $\Box$ models obligation in deontic logic. Our free choice axiom allowed us to derive that from "John must eat an apple", we would be able to infer "John must eat an apple and a pear", which is clearly unacceptable. None of our inference steps used any theorems derived from possible world semantics (the justification of the third step is based on the logical definition of $\Diamond$ and $\Box$, not on their possible worlds interpretations). It thus seems impossible to create a logic that allows for free choice, while avoiding this inference.

Therefore, instead of trying to solve it logically, most accounts since Kamp approach it linguistically. Most of these accounts can be categorized as either a semantic or a pragmatic approach (Meyer, 2016). Semantic accounts change the semantic denotation of disjunction or the modal operator to make the free choice schema valid. Semantic analyses include Aloni (2007), Geurts (2005), Simons (2005), and Zimmermann (2000). Also Fusco (2020) proposes a semantic analysis, which I will discuss in section 4, and the account that I will propose is semantic as well.

Pragmatic analyses on the other hand maintain the standard denotation for the disjunction and the modal, but reach the desired effect by adding assumptions about the pragmatics involved. This may include assumptions about implicatures (something that is implied by a statement, even though it's is not stated explicitly) or reasoning about the speaker's intentions. Pragmatic analyses include Chemla (2009), Franke (2009), Kratzer and Shimoyama (2002), and Vainikka (1987). For a discussion of pragmatic approaches with regard to free choice for ability modals, see Nouwen (2018).

2.2 Free Choice Ability

The free choice effect is most famous with deontic modals, but it can also be observed for other modalities, consider for example (17) and (18):

(17) Mary might be in London or in Paris.

(18) I can meet on monday or on tuesday.

The free choice effect can be observed for (17) as well (be it that there is not actually anything to choose). This is a case of epistemic modality, where *might* can be modelled in modal logic using the existential modal operator $\Diamond$, just like *may*. Similarly, (18) is an example of circumstantial modality and again, free choice follows. The free choice effect is clearly not just a deontic phenomenon. In fact, it is thought to be valid for any (existential) modality (Meyer, 2016).

Before we go into the question whether free choice is available for ability, we need to briefly consider what ability is. Ability is usually associated with the word *can*, but it must be noted that *can* can also express other modalities. For example, (19) is a clear example of ability, but (20) up to (22) are examples of respectively deontic, epistemic, and circumstantial modalities.

(19) I can run a marathon.

(20) You can take a seat.

(21) There can be lots of fish here.

(22) I can come in the evening.

But can the *can* of ability be considered an existential modal like the other ones? Kenny (1976) argued that it cannot. He conceived of a situation with a deck of cards with the blind side up, where the speaker, who does not know which card is where in the stack, can rightfully utter (23), but not (24).

(23) I can bring it about that either I am picking a red card or I am picking a black card.

(24) Either I can bring it about that I am picking a red card or I can bring it about that I am picking a black card.

This suggests that ability *can* is not like deontic *may* or epistemic *might*, since $\Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$ is a theorem in modal logic. Moreover, the lack of entailment from (23) to (24) seems to deny free choice ability. On the other hand, Geurts (2010) notes the entailment from (25) to (26) and (27), clearly a free choice reading.

(25) Betty can balance a fishing rod on her nose or on her chin.

(26) Betty can balance a fishing rod on her nose.

(27) Betty can balance a fishing rod on her chin.

So apparently, the free choice effect is available for ability. This raises the question how we can explain the lack of entailment from (23) to (24). Here is what I think about it: imagine being in the situation sketched by Kenny, and the person in front of you utters (23). I would then expect that the speaker is about to perform a magic trick, so (24) does follow. If the person cannot guarantee that he will pick the color of his liking, then I don't think he is in a position to say (23)[4].

In conclusion, (25) is strong evidence for the availability of free choice for ability. The lack of entailment from (23) to (24) is doubtful. Hence, I will assume for the remainder of this thesis that free choice is a phenomenon that is generally available for ability. Returning to the question whether ability *can* can be considered an existential modal: there are analyses which treat ability that way (see for example Kratzer (1977) and Mandelkern, Schultheis, and Boylan (2017)). As we will see later in this thesis, I will instead treat it (like Fusco (2020) and Horty and Belnap (1995)) as a combination of an existential and a universal modal.

[4]More precisely, he can not make the claim that he has the ability to pick a red or a black card. He could rightfully say "I can either pick a red card or a black card", but this appears to have an epistemic reading of *can* instead of an ability one (and hence, free choice clearly follows).

3 STIT Logic

3.1 Description

STIT Logic is a logical framework that is suitable for modelling agents and their choices as well as time, in the sense of what is earlier and what is later. It adds the *stit* operator to the language of propositional logic, 'stit' being an acronym for 'sees to it that'. My description here is more or less a summary of chapter 2A from Belnap, Perloff, and Xu (2001). A more comprehensive description can be found there.

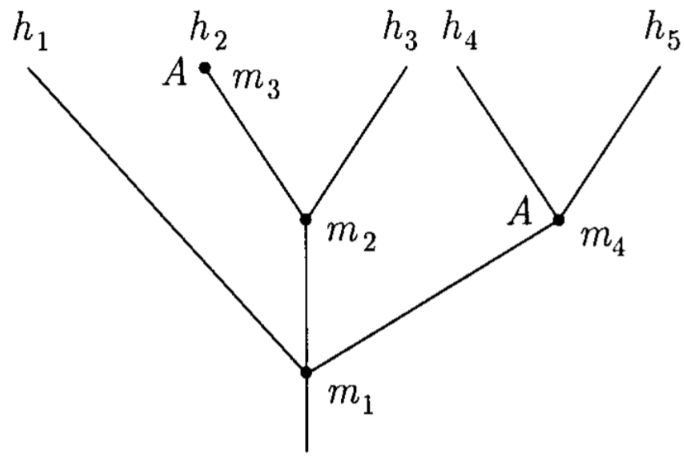

Figure 1: A depiction of a branching time tree, containing moments (labeled m_x) and histories (labeled h_x). Figure from Belnap et al. (2001).

STIT logic is founded on the theory of branching time, which was introduced by Prior (1967) and has been further developed by Thomason (1970, 1984). A frame consists of a set T, containing a set of nodes called *moments*, and a tree-ordering $<$. When we have two moments m and n and it is true that $m < n$, this can be thought of as that m happens in the past compared to n. The tree-like nature of $<$ allows us to follow a branch from a root moment to a leaf moment. We call such a maximal branch a *history*. We can consider a history as a possible course of events. We write H_m to denote the set of all histories through a moment m. In figure 1, an example tree has been drawn, where going upward represents going forward in time. We can see for example, that $m_1 < m_4$, or that $H_{m_2} = \{h_2, h_3\}$.

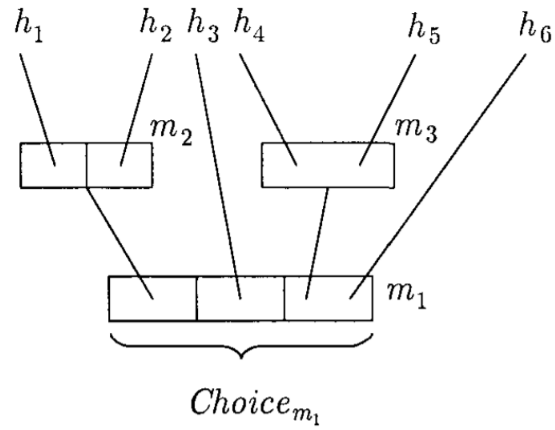

Figure 2: A depiction of choice cells at different moments. Figure from Belnap et al. (2001).

A STIT frame consists of such a tree, complemented with a set of agents and a choice function. For any moment m and agent α, this choice function provides a partitioning $Choice^m_\alpha$ of H_m, the set of all histories through m. The equivalence classes of this partition are called *choice cells*. As the name suggests, these choice cells can be thought of as the choices an agent has at a particular moment. A single choice cell may contain multiple histories (and therefore multiple possible futures), reflecting the intuition that the agent may not have the power to select any single history, but can narrow down what histories are possible by making certain choices. We write $Choice^m_\alpha(h)$ to denote the choice cell in m that h is part of. Figure 2 illustrates choice cells for some agent in different moments. We can see here that the agent has three options at m_1, but only one option (so effectively no choice) at m_3.

A STIT model adds a valuation function to a STIT frame. This function maps any atomic variable to a set of points of evaluation, where a point of evaluation is a moment-history pair. The truth conditions for atomic variables and Boolean operators like negation, conjunction, and disjunction are defined the regular way.

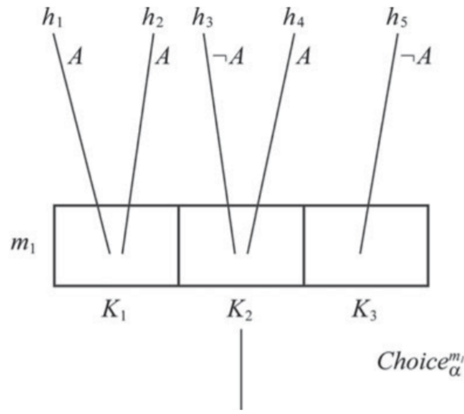

Figure 3: A model in which $m/h_1 \models [\alpha\ dstit:\ A]$ and $m/h_2 \models [\alpha\ dstit:\ A]$ are true, as A is true in both histories in their choice cell and there is a history (h_3 and h_5) in which A is false. In the other histories, $[\alpha\ dstit:\ A]$ is false, as they are in choice cells that contain histories where A is false. Figure from Horty and Pacuit (2017).

The language of STIT logic consists of the language of propositional logic extended with the *stit* operator. We write $[\alpha\ stit:\ \varphi]$ to denote that agent α sees to it that φ. There are multiple different *stit*-operators. One of them is *deliberative stit*, or *dstit*. According to Belnap et al., $[\alpha\ dstit:\ \varphi]$ should be interpreted as "that φ is guaranteed by the present choice of α". In a moment-history pair m/h, $[\alpha\ dstit:\ \varphi]$ holds if and only if two conditions are satisfied. The first condition, often called the positive condition, is that for each $h' \in Choice_{\alpha}^{m}(h)$, it is true that $m/h \models \varphi$. In words, φ needs to hold in every history in the choice cell that h is in. This condition seems obvious: clearly, α's choice would not guarantee φ if their choice might result in a future in which φ is false. More interesting is the second condition, or the negative condition. This condition states that there must be a history h'' in H_m in which φ is false. The intuition behind this is that α's choice must have actually mattered, that they could have made a choice that didn't guarantee φ, as it would not make sense for someone to deliberate something that is already settled. Figure 3 shows an example where $[\alpha\ dstit:\ A]$ is true. An alternative to *dstit* is *Chellas stit* or *cstit* (Segerberg, Meyer, & Kracht, 2020). It's truth condition is the same as *dstit*'s positive condition, lacking the negative condition.

Some additional operators include historical possibility and historical necessity. It is historically possible at a moment m that φ, denoted $\Diamond\varphi$, if and only if there is a history h in H_m such that $m/h \models \varphi$. Historical necessity, $\Box\varphi$, holds at m if and only if for all histories $h \in H_m$, $m/h \models \varphi$. The truth of historical possibility and necessity is thus not sensitive to the history in which they are evaluated, only to their moment.

3.2 Free Choice Ability in STIT Logic

To look at free choice ability in STIT logic, we first need a way to describe ability in it. Horty and Belnap (1995) propose to use a combination of historical possibility and *dstit*. The formula $\Diamond[\alpha\ dstit:\ \varphi]$ thus means that α is able to see to it that φ. Note that the evaluation of such a formula does not depend on the history of evaluation, only on the moment in which it is evaluated. For some agent to be able to φ, the following two conditions need to hold:

(i) there is a choice cell in which φ is valid in all histories and (ii) there is a history in which φ is false. An alternative definition of ability would a combination of historical possibility and *cstit* instead of *dstit*. In this case, only the first of the aforementioned conditions need to hold. Note that ability is thus not considered an existential modal in this analysis, but a combination of an existential and a universal modal.

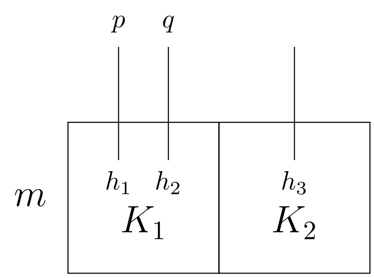

Figure 4: Counterexample demonstrating that free choice does not follow. For each history, only variables that are true are shown (so a variable that is not shown is false).

It is not difficult to see that free choice does not follow from these definitions. Figure 4 shows a countermodel. In this model, there is a choice cell (K_1) in which $p \vee q$ is true in all histories, and there is a history (h_3) in which it is false, satisfying the conditions for both the *dstit* as the *cstit* definition of saying the agent is capable of $p \vee q$. There is however no choice cell in which p is valid in all histories, so according to both definitions the agent is not able to p, and the same thing is the case for q. We can thus conclude that not only (using C for ability) $C(p \vee q) \not\rightarrow C(p) \wedge C(q)$, but even $C(p \vee q) \not\rightarrow C(p) \vee C(q)$. This is an interesting contrast to existential modals, as $\Diamond(p \vee q) \rightarrow \Diamond p \vee \Diamond q$ is a theorem in standard modal logic as well as STIT logic. Horty and Belnap (1995) as well as Horty and Pacuit (2017) see this as an advantage of this interpretation of ability, because it complies with Kenny's card case. However, as I discussed before, I consider the lack of entailment in that case doubtful and I assume that the free choice effect generally holds for ability.

Clearly, STIT logic doesn't correctly handle free choice ability cases. As we will see in the remainder of this thesis, the key to a solution for this lies in the definition of disjunction.

4 Fusco's Account of Free Choice Ability

Fusco (2020) proposes a framework of ability that handles free choice for ability. Her framework uses a historical modal base with a set of possible worlds. In this section, I will discuss Fusco's views on ability and disjunction and how these give rise to free choice.

4.1 Ability

In her discussion of when an agent α is able to φ, Fusco starts with the account of Mandelkern et al. (2017). They start from the idea that saying that "α can φ" means that "if α tried to φ, then α would φ". This might seem plausible, however, Mandelkern et al. come up with a case where this definition clearly fails. Imagine some situation where John is in an elevator with incorrectly wired buttons: the button for the first floor will take him to the basement and vice versa. John does not know about this defect, so according to the schema, (28) would be false: if John tried to take the elevator to the basement, he would actually bring it to the first floor.

(28) John can take the elevator to the basement.

(29) John can take the elevator to Budapest.

This contradicts our intuitions, which tell us that John can do (28), especially compared to a sentence like (29), which is clearly false (assuming the elevator isn't already in Budapest). Fusco argues this might be explained by distinguishing between intensional and extensional readings of *realizing* some outcome. Mandelkern et al.'s definition of ability is based on an intensional reading of realization, because the agents beliefs about their actions matter to whether they realize something. John believes that pressing the button that says 'basement' will bring him to the basement, and thus intensionally tries to realize going to the basement. Extensional realizition on the other hand is independent from the agent's inner states. Given that intensional realization causes this problem, Fusco proposes a reading of ability that depends on extensional realization.

According to Fusco, (28) is true because there is something John can do, namely pressing the button with a one it, that will cause him to extensionally realize the elevator going to the basement. On the other hand, there is nothing he can do to realize the elevator going to Budapest in any way, and therefore (29) is false. We say that it is historically possible for John to externally realize that the elevator goes to the basement, but it is not a historical possibility for him to externally realize that the elevator goes to Budapest. Fusco's definition of ability is thus that "α can φ" if and only if it is historically possible for α to externally realize φ. In the historical modal base, this means that there is some world w in the modal base, such that φ is valid in all worlds that can be reached from w.

4.2 Nonspecific De Re Disjunction

Fusco claims that disjunction can give rise to a *nonspecific de re* reading. I will start this subsection by describing nonspecific de re, and then I will discuss the connection with disjunctions.

4.2.1 Nonpecific De Re

Consider the following sentence:

(30) Mary wants a friend of mine to win.

Traditionally, there have been two reading of this ambiguous sentence: *de dicto* and *de re*. In the de dicto reading, Mary does not want a specific person to win, but she wants the winner to be a friend of mine. In the de re reading, there is one specific person she wants to win, who (possibly unbeknownst to her) happens to be a friend of mine. There is however a third possible reading called nonspecific de re. With this reading, there is no specific person she wants to win, yet there are some persons she want to win, who all happen to be my friends. von Fintel and Heim (2002) give as example a situation where Mary looks at ten contestants, points to the three at the right and says that she wants that one of them wins. She doesn't know that these three people are all my friends. So in this situation, there is not one specific person she wants to win, yet everyone in the set of persons she hopes contains the winner, is a friend of mine.

Nonspecific de re thus combines elements from de dicto and de re. The similarity between nonspecific de re and de dicto is the non-specificity. There is not one specific person Mary wants to win. What nonspecific de re and the traditional de re have in common is that their truth depends on the 'actual world'. If (30) is true, than it is true that the person or persons that Mary wants to win, are friends of mine in the actual world. Hence, we could evaluate a (nonspecific) de re reading based on two worlds: a 'world of evaluation' where we evaluate what Mary wants, and a 'world-as-actual' where we evaluate whether the person or persons she wants to win are actually friends of mine. A natural question to ask is why the world of evaluation and the world-as-actual would be different. As we will see later they are usually the same, but where the world of evaluation shifts when embedded under a modal operator, the world-as-actual stays the same, causing them to be different.

4.2.2 Disjunction and Nonspecific De Re

Fusco argues that a disjunction that triggers free choice, has nonspecific de re properties, so it is sensitive to the state of the actual world. For evidence for this, she turns to sentences combining 'whether ... or' with an intensional verb, as in (31).

(31) Dr. Jones knows whether the test was positive or negative.
K_J (whether [N] OR [P])

The validity of this phrase depends on the result from the test in the actual world. If the test result in the actual world was negative, then (31) would be equivalent to (32).

(32) Dr. Jones knows the test was negative.
K_J(N)

Groenendijk and Stokhof (1982) and Lewis (1982) account for this by using a two-dimensional semantic entry for 'whether p or q'. This means that in order to evaluate the validity of such a formula, two worlds are needed: a world-as-actual and a world of evaluation. The evaluation of (31) then looks like (33), where w is the world-as-actual and v is the world of evaluation.

(33) $w, v \vDash K_J$ (whether [N] OR [P])

Note that a sentence like (31) shares some properties with nonspecific de re readings: it is actuality-sensitive and nonspecific (it is not specific about which disjunct is true). In addition, the disjuncts are mutually exclusive, as the test result can't be positive and negative at the same time. This mutual exclusivity seems to relate to free choice as well: when we say something like (34), we would assume he wouldn't be flying both at the same time. Meanwhile, a sentence like (35) seems like an odd thing to say[5]. Fusco thus argues that disjunctions in the scope of an ability modal have these nonspecific de re properties too.

(34) David can fly an airplane or a helicopter.

(35) David can fly a helicopter or recite a Shakespeare sonnet.

In line with this, Fusco presents a two-dimensional definition of disjunction. We thus have a world-as-actual w and a world of evaluation v. When the disjuncts are mutually exclusive and jointly exhaustive in w (exactly one of them is valid in w), the disjunction is valid if and only if the disjunct valid in w is valid in v as well. If both or none of the disjuncts are valid in w, the disjunction is equivalent to the classical disjunction with regard to v (so it is valid as long as any of the disjuncts is true in v).

4.3 Free Choice in Fusco

To see how this two-dimensional definition of disjunction can yield free choice readings, we need to understand how this two-dimensionality interacts with ability in a historical modal base. For one thing, we use diagonal consequence. This means that when interpreting formulas in the historical modal base h, we assume that the world-as-actual and the world of evaluation are the same. We thus say φ is true in h if and only if for every world w in h, we have $h, w, w \vDash \varphi$. Modal operators shift the world of evaluation, while the world-as-actual remains unchanged. Note that as a consequence of this diagonal interpretation, our new disjunction is equivalent to classical disjunction as long as it is not in the scope of a modal operator.

Remember that ability is defined as the historical possibility for an agent to externally realize something. Since we use a historical modal base, a possible world corresponds to a historical possibility. This means ability shifts the world of evaluation, without shifting the world-as-actual. Consider a world w where a propositional variable p is true and another propositional variable q is false. When it is claimed that in w, an agent is capable of $(p \text{ OR } q)$, this statement means that it is historically possible for the agent to make $(p \text{ OR } q)$ true. By the definition of disjunction, this last statement is equivalent to the statement that it is historically possible to realize p, as p is true in w while q is false. This statement in turn means that in w, the agent is capable of p. Since p is atomic, the truth of the statement that the agent can p is not sensitive to the world-as-actual (this in fact holds for every formula that does not contain a disjunction). Since an ability statement shifts the world of evaluation, it is not sensitive to the world of evaluation either. In other words, since there is a pair of worlds where it is true that an agent can p, it must be true for all possible pairs and thus for the entire modal base. We can make the same argument with a world where

[5] Although (35) seems like a strange way to phrase it, it does seem like free choice follows (though intuitions might differ, see for example Nouwen (2018)). As we will see in a moment, free choice does follow in Fusco's framework, as long as there are historically possible worlds where both disjuncts are mutually exclusive. That is: there is a world where the first disjunct is true and the second one is false, and a world where it is the other way around. The context of (35) is likely to assure that there is some possible world where David is flying a helicopter but not reciting Shakespeare and vice versa, so free choice would follow indeed.

q is true and p is false to show that an agent being capable of (p OR q) implies the agent being capable of q.

This means that from the truth of $C(\varphi \text{ OR } \psi)$ in a historical modal base h, it follows that $C(\varphi) \wedge C(\psi)$ is true in h, as long as φ and ψ do not contain a disjunction, and as long as there is a world in h where φ is true and ψ is false and vice versa.

5 Fusco and STIT Logic

Now that we have the necessary background in STIT logic and the framework proposed by Fusco, we can incorporate pieces from Fusco's solution to STIT logic to create a STIT logic that correctly handles free choice ability. I would like to start with a comparison of the two frames. A historical modal base cannot simply be equated to an entire branching time frame, as such a tree can express time throughout different moments. It can, relative to a single moment, express possible future moments and the past moment, as well as possible future moments to the possible future moments and the past moment to the past moment. A historical modal base on the other hand, expresses only one set of possible future worlds, which are all relative to the same moment. This leads me to conclude that a historical modal base corresponds to one moment in a STIT frame. This also means that we can equate the possible worlds from a historical modal base to the histories in a STIT moment. This can be seen using the definition of historical possibility in both frameworks. In a historical modal base, something is historically possible if there is a world in the modal base where that something is true. Meanwhile in a STIT moment, something is historically possible if there is a history in that moment where it is true.

Now that we know that a historical modal base corresponds to a moment in STIT logic and that a world in a historical modal base is similar to a history in a STIT moment, we can turn to two-dimensionality. A point of evaluation in a historical modal base consists of the base itself along with two worlds: the *world-as-actual* and the *world of evaluation*. In regular STIT logic, a point of evaluation consists of a moment and a history in that moment. Since I equated worlds to histories, we can simply say a point of evaluation in our 'new' STIT logic consists of a moment and two histories: a 'history-as-actual' and a 'history of evaluation'. We can now copy Fusco's definition of disjunction. Given a moment m and two histories $h_a, h_e \in H_m$, we say $m, h_a, h_e \vDash (\varphi \text{ OR } \psi)$ is true in the case that (i) any of the disjuncts true in h_a is also true in h_e or (ii) in case none of the disjuncts is true in h_a, any of the disjuncts is true in h_e. We also handle consequence and modal operators the same as Fusco. This means that φ is valid in a moment m if and only if $m, h, h \vDash \varphi$ for all $h \in H_m$, and that modal operators (such as historical possibility) shift only the history of evaluation, and thus not the history-as-actual.

The final definition that we need, is that of ability. Fusco defines ability as the historical possibility for an agent to externally realize something. This definition translates directly to $\Diamond[\alpha\ cstit:\ \varphi]$, which happens to be one of the definitions of ability already proposed. It is however tempting to use the *dstit* variant, because the intuition behind the negative condition that it adds, seems appealing: in order for an agent to be able to do something, that thing should not be a historical necessity. As an example, it is (probably) a historical necessity that the earth will make one rotation around its axis in the following 24 hours, yet it would be absurd for me to claim power over the earth's rotation.

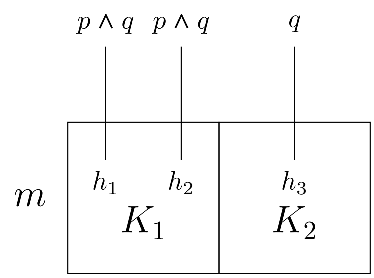

Figure 5: Counterexample in which $\Diamond[\alpha\ dstit:\ p \wedge q]$ is true but $\Diamond[\alpha\ dstit:\ q]$ is false, as the negative condition has not been satisfied. For each history, only variables that are true are shown (so a variable that is not shown is false).

The negative condition of *dstit* thus has an appealing intuition, however, it also creates a problem: as Belnap et al. (2001) note, if an agent deliberatively sees to it that $\varphi \wedge \psi$, this does not imply that they see to it that φ. This is still the case when in the scope of historical possibility, so the entailment $C(\varphi \wedge \psi) \rightarrow C\varphi \wedge C\psi$ does not hold when we define ability this way, as figure 5 shows. This would mean that the statement that someone can juggle three cones and sing a song does not imply that they can juggle three cones. This property sounds very undesirable, therefore I'll hold on to the *cstit* version of ability.

This results in a version of free choice ability on the same conditions as in Fusco's framework, namely that there is for both disjuncts a history in which that disjunct is true and the other one is false, and that the disjuncts do not contain disjunctions themselves. An example model is shown in figure 6. When $C(p \text{ OR } q)$ is valid in a moment m, we have $m, h, h \vDash C(p \text{ OR } q)$ for all histories $h \in H_m$. This means that for each history, there is a choice cell where in all histories in that choice cell, $(p \text{ OR } q)$ is true in that history relative to h as history-as-actual. When relative to a history-as-actual in which p is true and q is false, such as h_1 in the example model, the formula $(p \text{ OR } q)$ is equivalent to p. This means that $m, h_1, h_1 \vDash C(p \text{ OR } q)$ entails $m, h_1, h_1 \vDash C(p)$. Since p is atomic, the history-as-actual is not relevant for the evaluation of $C(p)$, neither is the history of evaluation, since ability starts with historical possibility. This means $m, h, h \vDash C(p)$ holds for any h in H_m, therefore $m \vDash C(p)$. In a similar fashion, we can show $C(q)$.

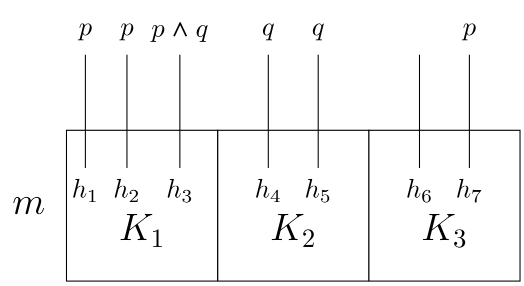

Figure 6: Example model illustrating free choice. For each history, only variables that are true are shown (so a variable that is not shown is false).

To prove this formally, we first need the formal definitions of disjunction and ability. Ability is evaluated as follows:

$m, h_a, h_e \vDash C_\alpha\varphi$ iff $\exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h_a, h' \vDash \varphi$
For disjunction, we first define an *answer* function relative to a moment m and a history-as-actual h:

$$Ans_{m,h}(\varphi,\psi) = \begin{cases} \{\varphi\} & \text{if } m,h,h \vDash \varphi \text{ and } m,h,h \nvDash \psi \\ \{\psi\} & \text{if } m,h,h \vDash \psi \text{ and } m,h,h \nvDash \varphi \\ \{\varphi,\psi\} & \text{otherwise} \end{cases}$$

Disjunction is then evaluated as follows:
$m, h_a, h_e \vDash (\varphi \text{ OR } \psi)$ iff $\exists \beta \in Ans_{m,h_a}(\varphi,\psi) : m, h_a, h_e \vDash \beta$
We can now proof free choice for our system:

Fact: $C_\alpha(p \text{ OR } q), \Diamond(p \wedge \neg q), \Diamond(q \wedge \neg p) \vDash C_\alpha p \wedge C_\alpha q$ (for any wffs p and q that do not contain a disjunction)

Proof. Consider a model in which the premises hold. By the first premise, it holds that $m, h, h \vDash C_\alpha(p \text{ OR } q)$ for all $h \in H_m$. Given the second premise, there is a history h_p such that $m, h_p, h_p \vDash p \wedge \neg q$. We instantiate h with h_p.

We get $m, h_p, h_p \vDash C_\alpha(p \text{ OR } q)$ iff
$\exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h_p, h' \vDash (p \text{ OR } q)$
Since $m, h_p, h_p \vDash p \wedge \neg q$, we have $Ans_{m,h_p}(p,q) = \{p\}$, so by the definition of OR, it follows that
$\exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h_p, h' \vDash p$
Since p does not contain a disjunction, the truth of this last statement does not depend on the history-as-actual, so it is equivalent to
$\forall h'' \in H_m : \exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h'', h' \vDash p$, which is the definition of
$\forall h'' \in H_m : m, h'', h'' \vDash C_\alpha p$
And therefore $m \vDash C_\alpha p$.

Using the third premise, we can make an analogous argument to show $m \vDash C_\alpha q$. □

In the appendix, I give the necessary definitions of the framework and prove some relevant properties. The above proof and example model can also be found there. In the final section, I will discuss the properties of the framework that are proven in the appendix.

6 Discussion

In this thesis, I proposed a modified semantics for STIT logic. Standard STIT logic as discussed by Belnap et al. (2001) does not account for the free choice effect for ability. With a few relatively minor changes, based on the account from Fusco (2020), the new framework correctly handles free choice. For the definition of ability *can*, I used a combination of historical possibility and the *cstit* operator, which is an alternative for the more commonly proposed combination of historical possibility and the *dstit* operator. The advantage of this combination of an existential and a universal modal compared to ability as just an existential modal is that the derivation to the problematic formula $\Box\varphi \to \Box(\varphi \wedge \psi)$ (see section 2.1) is denied. The change to the semantics itself was however the definition of disjunction. The new interpretation of disjunction is two-dimensional, meaning it uses two histories, a 'history-as-actual' and a 'history of evaluation', for its evaluation instead of one. When only one of the disjuncts is valid in the history-as-actual, the disjunction only holds if that disjunct is also valid in the history of evaluation. Typically, the two histories are the same (because of diagonal consequence), which makes the disjunction equivalent to the classical disjunction. However, when the disjunction is embedded under a modal operator (like ability *can*), the history of evaluation gets shifted, while the history-as-actual remains the same.

The resulting framework has certain properties that are desirable for free choice ability. Firstly, free choice ability itself. If a moment m validates $C(\varphi \text{ OR } \psi)$, then it also validates $C\varphi \wedge C\psi$, provided that φ and ψ do not contain any disjunctions themselves and that $\Diamond(\varphi \wedge \neg\psi)$ and $\Diamond(\psi \wedge \neg\varphi)$ are valid in m (appendix, fact 1). This second condition stems from the assumption that there is some form of mutual exclusivity between the disjuncts, that makes the nonspecific de re reading possible.

I think this is a reasonable assumption. As I mentioned in section 4.2.2 it is odd when a sentence with a disjunction in the scope of ability *can* has two disjuncts that are not mutually exclusive. Consider for example (36) again. This sentence is weird, because flying a helicopter and reciting Shakespeare seem independent. It would make more sense to phrase it with a conjunction already, like (37). This also matches the observation by Eckardt (2007) that free choice does not arise with mutually independent disjuncts (though admittedly, this framework only requires mutual exclusivity in *a* history for free choice, not in *all* histories, so mutual dependency is not guaranteed).

(36) David can fly a helicopter or recite a Shakespeare sonnet.

(37) David can fly a helicopter and recite a Shakespeare sonnet.

The first condition (that the disjuncts do not contain disjunctions themselves), suggests that free choice can only be derived for binary disjunctions. This is however not the case: a pleasant outcome is that free choice is valid for an any number of disjuncts (appendix, fact 6), as long as the two conditions hold. This is actually not the case for all analyses that attempt to explain free choice (Fox & Katzir, 2020). The constraint on the disjunct is in fact more specific: free choice might be blocked if the disjunct contains a modal that takes scope over a disjunction. I do nevertheless have a strong conjecture that free choice would then still be implied, because I could not construct a counterexample (with the mutual exclusivity assumption also valid for the second embedded disjunction). Unfortunately, I didn't get to proving it because of time constraints. It can however be questioned how useful that property would be: I cannot think of any sentence in English with ability *can* taking scope over another modal, let alone taking scope over a disjunction that takes scope over a modal that takes scope over a disjunction.

Another property is that $C\varphi$ does not entail $C(\varphi \text{ OR } \psi)$, not even when the conditions for free choice hold (appendix, fact 3). Clearly, this property is crucial, as the entailment from $C\varphi$ to $C(\varphi \text{ OR } \psi)$ in combination with the free choice property would allow us to derive $C\psi$ from $C\varphi$, which is clearly unacceptable (as already mentioned in section 2.1, this would mean that being able of something makes you able of everything). This schema is also supported empirically by the lack of entailment from (38) to (39).

(38) Mary can balance a fishing rod on her nose.

(39) Mary can balance a fishing rod on her nose or on her chin.

The next property is the lack of entailment (once again, even with the two conditions) from $(\varphi \text{ OR } \psi)$ to $C(\varphi \text{ OR } \psi)$ (appendix, fact 2). This means that when one of two things is going to happen, but we don't know which, we do not infer ability over this. This property is especially relevant with how Fusco views Kenny's card case. She argues that saying something like (23) in that situation is merely saying "I will pick a red card or a black card". This should not imply that the speaker is truly able to pick a red card or a black card in the sense that allows free choice. Fusco thus does not see Kenny's card case as a true case of ability, and with this property, the lack of entailment from (23) to (24) can be accounted for.

The final property I want to mention is the entailment from $\neg C(\varphi \text{ OR } \psi)$ to $\neg C\varphi \wedge \neg C\psi$, provided that φ and ψ do not contain disjunctions (appendix, fact 4). This schema, which Barker (2010) calls *double prohibition*, is demonstrated by the entailment from (40) to (41).

(40) Mary cannot balance a fishing rod on her nose or on her chin.

(41) Mary cannot balance a fishing rod on her nose and she cannot balance a fishing rod on her chin.

Contrary to the free choice property, mutual exclusivity is not required. That is because in cases where the history-as-actual contains either both or none of the disjuncts, the disjunction is equivalent to classical disjunction, for which the property already followed.

This new version of STIT logic, with relatively minor changes compared to traditional STIT logic, can now account for free choice. Unfortunately, there are still some problematic predictions concerning ability. With the definition for ability I used, the combination of historical possibility and *cstit*, historical necessity implies ability, which is does not seem right. As I already mentioned before, the fact that the earth will complete one rotation the next 24 hours does not make me capable of rotating the earth. Notice however, that free choice denies this property for disjunction, as $C(\varphi \text{ OR } \psi)$ implies $C\varphi \wedge C\psi$, while $(\varphi \text{ OR } \psi)$ does not imply $\varphi \wedge \psi$.

This property could be avoided by replacing *cstit* with *dstit* in the definition of ability, as *dstit*'s negative condition requires that the formula in its scope is not historically necessary. Unfortunately, the *dstit* variant has a problem that I think is worse, namely the lack of entailment from $C(\varphi \wedge \psi)$ to $C\varphi \wedge C\psi$. This property is not at all valid in the English language, where someone who can speak French and German can clearly speak French and can also speak German. Possible future research could focus on finding a way of defining ability in STIT logic such that these prediction are avoided, while maintaining the free choice schema valid.

Appendix

In this appendix, I give the definitions of the proposed framework and prove some of its properties.

Syntax

We define a language $\mathcal{L}$, which consists of the following well-formed formulas:

- If p is an atomic variable, then p is a wff.
- If φ and ψ are wffs, then $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \text{ OR } \psi)$ and $\Diamond(\varphi)$ are wffs.
- If φ is a wff and α is an agent, then $\mathsf{cstit}_\alpha(\varphi)$ is a wff.

Nothing else is a wff.

Semantics

Frame

A frame consists of a moment m and a set of histories through m. This moment can be part of a larger STIT frame which consists of a tree of moments. Though since we are not concerned with time, a single moment suffices for our purposes. We define H_m as the set of histories through moment m.

A frame also consists of a choice function *Choice* and a set of agents A. *Choice* maps an agent α and a moment m to a partitioning $Choice^m_\alpha$ of H_m. The equivalence classes of $Choice^m_\alpha$ are called 'choice cells'. If $h \in H_m$, we write $Choice^m_\alpha(h)$ to denote the choice cell that h is a member of. Choice cells can be thought of as the options available to an agent.

Model

A model consists of a frame and a valuation function V, which maps each atomic variable to a set of moment-history pairs.

Evaluation

A point of evaluation is a triple $\langle m, h_a, h_e \rangle$ consisting of a moment m and two histories $h_a, h_e \in H_m$, where h_a is the history-as-actual, and h_e is the history of evaluation. To evaluate disjunction, we need the *answer* function, which is defined as follows, given wffs φ and ψ, a moment m and a history $h \in H_m$:

$$Ans_{m,h}(\varphi, \psi) = \begin{cases} \{\varphi\} & \text{if } m,h,h \vDash \varphi \text{ and } m,h,h \nvDash \psi \\ \{\psi\} & \text{if } m,h,h \vDash \psi \text{ and } m,h,h \nvDash \varphi \\ \{\varphi, \psi\} & \text{otherwise} \end{cases}$$

For any point of evaluation $\langle m, h_a, h_e \rangle$, atomic p and wffs φ and ψ:

$$\begin{array}{lll} m,h_a,h_e \vDash p & \text{iff} & \langle m, h_e \rangle \in V(p) \\ m,h_a,h_e \vDash \neg\varphi & \text{iff} & m,h_a,h_e \nvDash \varphi \\ m,h_a,h_e \vDash (\varphi \wedge \psi) & \text{iff} & m,h_a,h_e \vDash \varphi \text{ and } m,h_a,h_e \vDash \psi \\ m,h_a,h_e \vDash (\varphi \text{ OR } \psi) & \text{iff} & \exists \beta \in Ans_{m,h_a}(\varphi,\psi):\ m,h_a,h_e \vDash \beta \\ m,h_a,h_e \vDash \Diamond\varphi & \text{iff} & \exists h \in H_m:\ m,h_a,h \vDash \varphi \\ m,h_a,h_e \vDash \mathsf{cstit}_\alpha\varphi & \text{iff} & \forall h \in Choice^m_\alpha(h_e):\ m,h_a,h \vDash \varphi \end{array}$$

Ability

We say an agent α is able to φ if and only if $\Diamond\mathsf{cstit}_\alpha\varphi$ is true. In the remainder of this appendix, I will abbreviate $\Diamond\mathsf{cstit}_\alpha\varphi$ to $C_\alpha\varphi$. This means ability is defined as:
$m, h_a, h_e \vDash C_\alpha\varphi$ iff $\exists h \in H_m : \forall h' \in Choice_\alpha^m(h) : m, h_a, h' \vDash \varphi$

Consequence

Just like Fusco, we use diagonal consequence, so $m \vDash \varphi$ iff $\forall h \in H_m : m, h, h \vDash \varphi$.

Example Model

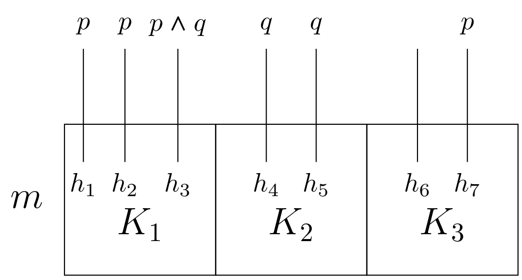

Figure 7: Example model. Shown is moment m. Each box represents a choice cell, labeled K_1, K_2 or K_3. Each line represents a history in that choice cell, with its label at the bottom of the line. On the top of each history, the propositional variables that are true in that history are shown.

Consider a model with a moment m, $H_m = \{h_1, ..., h_7\}$, $A = \{\alpha\}$, $Choice_\alpha^m = \{\{h_1, h_2, h_3\}, \{h_4, h_5\}, \{h_6, h_7\}\}$, $V(p) = \{h_1, h_2, h_3, h_7\}$ and $V(q) = \{h_3, h_4, h_5\}$. The model is drawn in figure 7.
In this model, $m \vDash C_\alpha p \wedge C_\alpha q$, because in choice cell K_1, p is true in all histories, while in choice cell K_2, q is true in all histories. $C_\alpha(p \text{ OR } q)$ is also true in m. For each history h, it is true that there is a choice cell in which the disjunct that is true in h (or any of the disjuncts, if p and q are both true in both false in h), is true in all histories of that choice cell (for h_1, h_2 and h_7, this is choice cell K_1, for h_4 and h_5, it is K_2, and for h_3 and h_6, it's both K_1 and K_2).

Properties

Fact 1: $C_\alpha(p \text{ OR } q), \Diamond(p \wedge \neg q), \Diamond(q \wedge \neg p) \vDash C_\alpha p \wedge C_\alpha q$ (for any wffs p and q that do not contain a disjunction)

Proof. Consider a model in which the premises hold. By the first premise, it holds that $m, h, h \vDash C_\alpha(p \text{ OR } q)$ for all $h \in H_m$. Given the second premise, there is a history h_p such that $m, h_p, h_p \vDash p \wedge \neg q$. We instantiate h with h_p.

We get $m, h_p, h_p \vDash C_\alpha(p \text{ OR } q)$ iff
$\exists h \in H_m : \forall h' \in Choice_\alpha^m(h) : m, h_p, h' \vDash (p \text{ OR } q)$
Since $m, h_p, h_p \vDash p \wedge \neg q$, we have $Ans_{m,h_p}(p, q) = \{p\}$, so by the definition of OR, it follows

that
$\exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h_p, h' \models p$
Since p does not contain a disjunction, the truth of this last statement does not depend on the history-as-actual, so it is equivalent to
$\forall h'' \in H_m : \exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h'', h' \models p$, which is the definition of
$\forall h'' \in H_m : m, h'', h'' \models C_\alpha p$
And therefore $m \models C_\alpha p$.

Using the third premise, we can make an analogous argument to show $m \models C_\alpha q$. □

Fact 2: $(\varphi \text{ OR } \psi), \Diamond(\varphi \wedge \neg\psi), \Diamond(\psi \wedge \neg\varphi) \nvDash C_\alpha(\varphi \text{ OR } \psi)$

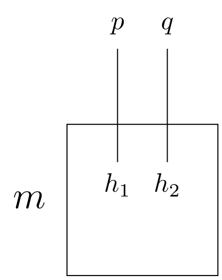

Figure 8: For this model, $m \models (p \text{ OR } q), \Diamond(p \wedge \neg q), \Diamond(q \wedge \neg p)$, but $m \nvDash C_\alpha(p \text{ OR } q)$

Proof. Consider the model shown in figure 8, consisting of a moment m with only one choice cell consisting of histories h_1, with only p true, and h_2, with only q true. We instantiate φ with p and ψ with q.

We have $m \models (p \text{ OR } q)$ iff $m, h_1, h_1 \models (p \text{ OR } q)$ and $m, h_2, h_2 \models (p \text{ OR } q)$. Both these conditions hold, so $(p \text{ OR } q)$ is true in this model.

We have $m \models \Diamond(p \wedge \neg q)$, because $m, h_1, h_1 \models p \wedge \neg q$. We also have $m \models \Diamond(q \wedge \neg p)$, because $m, h_2, h_2 \models q \wedge \neg p$.

We have $m \models C_\alpha(p \text{ OR } q)$ iff $\forall h \in H_m : m, h, h \models C_\alpha(p \text{ OR } q)$. We instantiate h with h_1. We get:
$m, h_1, h_1 \models C_\alpha(p \text{ OR } q)$ iff
$\exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h_1, h' \models (p \text{ OR } q)$
Since there is only one choice cell, we can instantiate h' with h_2. We get: $m, h_1, h_2 \models (p \text{ OR } q)$
But this last statement is false, therefore $m \nvDash C_\alpha(p \text{ OR } q)$ □

Fact 3: $C_\alpha\varphi, \Diamond(\varphi \wedge \neg\psi), \Diamond(\psi \wedge \neg\varphi) \nvDash C_\alpha(\varphi \text{ OR } \psi)$

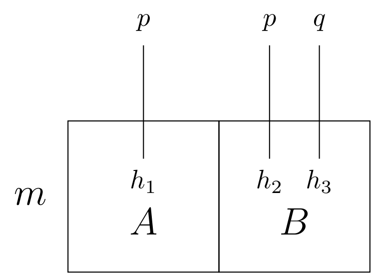

Figure 9: In this model, $m \models C_\alpha p, \Diamond(p \wedge \neg q), \Diamond(q \wedge \neg p)$, but $m \not\models C_\alpha(p \text{ OR } q)$

Proof. Consider the model displayed in figure 9. We instantiate φ with p and ψ with q.

We have $m \models C_\alpha p$ iff
$\forall h \in H_m: \; m, h, h \models C_\alpha p$ iff
$\forall h \in H_m: \; \exists h' \in H_m: \; \forall h'' \in Choice^m_\alpha(h'): m, h, h'' \models p$
Which is true if we instantiate h' with h_1, so $m \models C_\alpha p$

We have $m \models \Diamond(p \wedge \neg q)$, because $m, h_1, h_1 \models p \wedge \neg q$. We also have $m \models \Diamond(q \wedge \neg p)$, because $m, h_3, h_3 \models q \wedge \neg p$.

We have $m \models C_\alpha(p \text{ OR } q)$ iff $\forall h \in H_m: m, h, h \models C_\alpha(p \text{ OR } q)$. We instantiate h with $h3$. We get:
$m, h_3, h_3 \models C_\alpha(p \text{ OR } q)$ iff
$\exists h \in H_m: \; \forall h' \in Choice^m_\alpha(h): \; m, h_3, h' \models (p \text{ OR } q)$
Since $Ans_{m,h_3}(p, q) = \{q\}$, this is equivalent to
$\exists h \in H_m: \; \forall h' \in Choice^m_\alpha(h): \; m, h_3, h' \models q$
But no such h exists, therefore $m \not\models C_\alpha(p \text{ OR } q)$. □

Fact 4: $\neg C_\alpha(p \text{ OR } q) \models \neg C_\alpha p \wedge \neg C_\alpha q$ (for any wffs p and q that do not contain a disjunction)

Proof. Consider some model in which the premise holds. This means that $m, h, h \models \neg C_\alpha(p \text{ OR } q)$ for all $h \in H_m$. This is true if it is not the case that
$\exists h' \in H_m: \; \forall h'' \in Choice^m_\alpha(h'): \; m, h, h'' \models (p \text{ OR } q)$, which is false iff
$\forall h' \in H_m: \; \exists h'' \in Choice^m_\alpha(h'): \; m, h, h'' \models \neg(p \text{ OR } q)$ is true.
We can distinguish four cases for h:

Case 1: $m, h, h \models p \wedge \neg q$.
In this case, we have $Ans_{m,h}(p, q) = \{p\}$, so this means that
$\forall h' \in H_m: \; \exists h'' \in Choice^m_\alpha(h'): \; m, h, h'' \models \neg(p \text{ OR } q)$ is true iff
$\forall h' \in H_m: \; \exists h'' \in Choice^m_\alpha(h'): \; m, h, h'' \models \neg p$, which means it is not the case that
$\exists h' \in H_m: \; \forall h'' \in Choice^m_\alpha(h'): \; m, h, h'' \models p$, so this means that
$m, h, h \models \neg C_\alpha p$
Since p does not contain a disjunction, this final statement is insensitive to both the history-as-actual and the history of evaluation, so we have $m, h*, h* \models \neg C_\alpha p$ for all $h* \in H_m$, so $m \models \neg C_\alpha p$.
In one of the steps, we proved $m, h, h'' \models \neg p$ for some $h'' \in H_m$. This is insensitive to the history-as-actual, so it must be true that $m, h'', h'' \models \neg p$. This means that there is a history that triggers either case 2 or case 3. These cases both prove $m \models \neg C_\alpha q$.

Case 2: $m, h, h \models q \wedge \neg p$.

This case is similar to case one. With an analogous argument, we can show that $m \models \neg C_\alpha q$ and that there is a history which is either in case 1 or case 3, which prove that $m \models \neg C_\alpha p$.

Case 3: $m, h, h \models \neg p \land \neg q$.
In this case, we have $Ans_{m,h}(p, q) = \{p, q\}$, so this means that
$\forall h' \in H_m : \exists h'' \in Choice^m_\alpha(h') : m, h, h'' \models \neg(p \text{ OR } q)$ is true iff
$\forall h' \in H_m : \exists h'' \in Choice^m_\alpha(h') : m, h, h'' \models \neg p \land \neg q$
This case is thus analogous to case 1 and case 2 combined, and using the same argument for both individual disjuncts, it is shown that $m \models \neg C_\alpha p \land \neg C_\alpha q$.

Case 4: $m, h, h \models p \land q$.
In this case, we have $Ans_{m,h}(p, q) = \{p, q\}$. This case is thus the same as case 3.

In all cases, we have $m \models \neg C_\alpha p \land \neg C_\alpha q$. □

Fact 5: If $m, h, h \models \neg a_1 \land ... \land \neg a_{i-1} \land a_i \land \neg a_{i+1} \land ... \land \neg a_n$, then $m, h, h' \models (a_1 \text{ OR } ... \text{ OR } a_n)$ iff $m, h, h' \models a_i$ (for any wffs $a_1, ..., a_n$ that do not contain a disjunction).

Proof. By induction on the length of n.

Base Case. $n = 2$, so $(a_1 \text{ OR } a_2)$. There are two cases: either $a_i = a_1$ or $a_i = a_2$.

Case 1: $a_i = a_1$.
To show: $m, h, h' \models (a_1 \text{ OR } a_2)$ iff $m, h, h' \models a_1$.
This follows straightforwardly from the definition of OR.

Case 2: $a_i = a_2$.
This case is symmetric to case 1.

Inductive Case. $n > 2$, so $((a_1 \text{ OR } ... \text{ OR } a_{n-1}) \text{ OR } a_n)$. There are two cases: either $a_i \in \{a_1, ..., a_{n-1}\}$ or $a_i = a_n$.
Induction Hypothesis (IH): if $a_i \in \{a_1, ..., a_{n-1}\}$, then $m, h, h' \models (a_1 \text{ OR } ... \text{ OR } a_{n-1})$ iff $m, h, h' \models a_i$.

Case 1: $a_i \in \{a_1, ..., a_{n-1}\}$.
To show: $m, h, h' \models ((a_1 \text{ OR } ... \text{ OR } a_{n-1}) \text{ OR } a_n)$ iff $m, h, h' \models a_i$.
We have $m, h, h \models a_i$, so by the IH, $m, h, h \models (a_1 \text{ OR } ... \text{ OR } a_{n-1})$. Also, $m, h, h \not\models a_n$. Therefore $Ans_{m,h}((a_1 \text{ OR } ... \text{ OR } a_{n-1}), a_n) = \{(a_1 \text{ OR } ... \text{ OR } a_{n-1})\}$. We thus get $m, h, h' \models ((a_1 \text{ OR } ... \text{ OR } a_{n-1}) \text{ OR } a_n)$ iff $m, h, h' \models (a_1 \text{ OR } ... \text{ OR } a_{n-1})$, which by the IH is true iff $m, h, h' \models a_i$.

Case 2: $a_i = a_n$.
To show: $m, h, h' \models ((a_1 \text{ OR } ... \text{ OR } a_{n-1}) \text{ OR } a_n)$ iff $m, h, h' \models a_n$.
Clearly $m, h, h \not\models (a_1 \text{ OR } ... \text{ OR } a_{n-1})$ since all of the disjuncts are false in h. Also, $m, h, h \models a_n$. Therefore $Ans_{m,h}((a_1 \text{ OR } ... \text{ OR } a_{n-1}), a_n) = \{a_n\}$. By the definition of OR, it thus follows that $m, h, h' \models ((a_1 \text{ OR } ... \text{ OR } a_{n-1}) \text{ OR } a_n)$ iff $m, h, h' \models a_n$. □

Fact 6: $C_\alpha(a_1 \text{ OR } ... \text{ OR } a_n), \Diamond(\neg a_1 \land ... \land \neg a_{i-1} \land a_i \land \neg a_{i+1} \land ... \land a_n) \models C_\alpha a_i$ (for any wffs $a_1, ..., a_n$ that do not contain a disjunction)

Proof. Consider a model in which the premises hold. By the second premise, there is a history $h_i \in H_m$ in which a_i is true and all other disjuncts are false. By the first premise, it is true that $m, h_i, h_i \models C_\alpha(a_1 \text{ OR } ... \text{ OR } a_n)$, which is true iff
$\exists h \in H_m : \forall h' \in Choice^m_\alpha(h) : m, h_i, h' \models (a_1 \text{ OR } ... \text{ OR } a_n)$
By fact 5, the previous statement is true iff

$\exists h \in H_m : \ \forall h' \in Choice^m_\alpha(h) : \ m, h_i, h' \vDash a_i$, so this means that $m, h_i, h_i \vDash C_\alpha a_i$ Since a_i does not contain a disjunction, the previous statement is not sensitive to the history-as-actual or the history of evaluation, so for all $h \in H_m$ it is true that $m, h, h \vDash C_\alpha a_i$, so $m \vDash C_\alpha a_i$. □

References

Aloni, M. (2007). Free choice, modals, and imperatives. *Natural Language Semantics*, *15*(1), 65–94. doi: https://doi.org/10.1007/s11050-007-9010-2

Barker, C. (2010). Free choice permission as resource-sensitive reasoning. *Semantics and Pragmatics*, *3*(10), 1–38. doi: https://doi.org/10.3765/sp.3.10

Belnap, N., Perloff, M., & Xu, M. (2001). *Facing the future*. Oxford University Press.

Chemla, E. (2009). Similarity: Towards a unified account of scalar implicatures, free choice permission and presupposition projection. *Under revision for Semantics and Pragmatics*.

Eckardt, R. (2007). Licensing *or*. In U. Sauerland & P. Stateva (Eds.), *Presupposition and implicature in compositional semantics*. Palgrave Macmillan, London. doi: https://doi.org/10.1057/9780230210752_3

Fox, D., & Katzir, R. (2020). *Notes on iterated rationality models of scalar implicatures*.

Franke, M. (2009). *Signal to act: Game theory in pragmatics* (PhD thesis). Institute for Logic, Language and Computation, Universiteit van Amsterdam.

Fusco, M. (2020). Agential free choice. *Journal of Philosophical Logic*, *50*(1), 57–87. doi: https://doi.org/10.1007/s10992-020-09561-w

Garson, J. (2021). Modal logic. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Summer 2021 ed.). Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/entries/logic-modal/.

Geurts, B. (2005). Entertaining alternatives: Disjunctions as modals. *Natural Language Semantics*, *13*(4), 383–410. doi: https://doi.org/10.1007/s11050-005-2052-4

Geurts, B. (2010). *Quantity implicatures*. Cambridge University Press.

Groenendijk, J., & Stokhof, M. (1982). Semantic analysis of wh-complements. *Linguistics and Philosophy*, *5*(2), 175–233. doi: https://doi.org/10.1007/bf00351052

Horty, J. F., & Belnap, N. (1995). The deliberative stit: A study of action, omission, ability, and obligation. *Journal of Philosophical Logic*, *24*(6), 583–644. doi: https://doi.org/10.1007/bf01306968

Horty, J. F., & Pacuit, E. (2017). Action types in stit semantics. *The Review of Symbolic Logic*, *10*(4), 617–637. doi: https://doi.org/10.1017/s1755020317000016

Kamp, H. (1974). Free choice permission. *Proceedings of the Aristotelian Society*, *74*(1), 57–74. doi: https://doi.org/10.1093/aristotelian/74.1.57

Kenny, A. (1976). Human abilities and dynamic modalities. In J. Manninen & R. Tuomela (Eds.), *Essays on explanation and understanding* (pp. 209–232). Springer, Dordrecht. doi: https://doi.org/10.1007/978-94-010-1823-4_11

Kratzer, A. (1977). What 'must' and 'can' must and can mean. *Linguistics and philosophy*, *1*(3), 337–355. doi: https://doi.org/10.1007/bf00353453

Kratzer, A., & Shimoyama, J. (2002). Indeterminate pronouns: The view from japanese. In Y. Otsu (Ed.), *Third tokyo conference on psycholinguistics* (pp. 1–25). Tokyo: Hituzi Syobo.

Lewis, D. (1982). 'Whether' report. In T. Pauli (Ed.), *Philosophical essays dedicated to Lennart Åqvist on his fiftieth birthday* (pp. 194–206). University of Uppsala Press.

Mandelkern, M., Schultheis, G., & Boylan, D. (2017). Agentive modals. *Philosophical Review*, *126*(3), 301–343. doi: https://doi.org/10.1215/00318108-3878483

Meyer, M. C. (2016). *An Apple or a Pear.* free choice disjunction. *The Wiley Blackwell Companion to Semantics*. doi: https://doi.org/10.1002/9781118788516.sem070

Nouwen, R. (2018). Free choice and distribution over disjunction: the case of free choice ability. *Semantics and Pragmatics*, *11*(4). doi: https://doi.org/10.3765/sp.11.4

Prior, A. N. (1967). *Past, present, and future.* Oxford University Press.

Segerberg, K., Meyer, J.-J., & Kracht, M. (2020). The logic of action. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Summer 2020 ed.). Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/entries/logic-action/.

Simons, M. (2005). Dividing things up: The semantics of *or* and the modal/*or* interaction. *Natural Language Semantics*, *13*(3), 271–316. doi: 10.1007/s11050-004-2900-7

Thomason, R. H. (1970). Indeterminist time and truth-value gaps. *Theoria*, *36*(3). doi: https://doi.org/10.1111/j.1755-2567.1970.tb00427.x

Thomason, R. H. (1984). Combinations of tense and modality. In D. Gabbay & F. Guenthner (Eds.), *Handbook of philosophical logic* (Vol. II, pp. 135–165). Springer, Dordrecht.

Vainikka, A. (1987). Why can or mean and or or? *University of Massachusetts Occasional Papers in Linguistics*, *13*(2), 148–186.

von Fintel, K., & Heim, I. (2002). *Lecture notes on intensional semantics.* Massachusetts Institute of Technology.

Zimmermann, T. E. (2000). Free choice disjunction and epistemic possibility. *Natural Language Semantics*, *8*(4), 255–290. doi: https://doi.org/10.1023/A:1011255819284