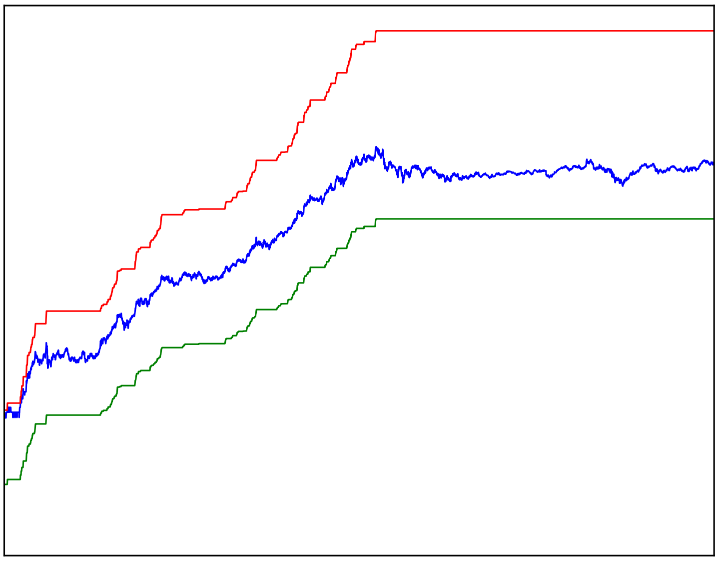
One-way trading with transaction costs



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Preface

This thesis was written as the final part of the Master's programme Stochastics and Financial Mathematics of Utrecht University (UU), VU University Amsterdam (VU) and University of Amsterdam (UvA).

I carried out my research at RiskCo B.V. in Utrecht. RiskCo's main activity is bridging the gap between financial product design and information technology. Although this thesis may be of limited importance to the business of RiskCo, it was the starting point for a research project on trading algorithms at RiskCo. This was initiated during the literature study for this thesis, where I encountered several trading algorithms that are now further developed at RiskCo.

Acknowledgments I'm very thankful to Bert de Bock for giving me the opportunity to do my research at RiskCo's, which has been a very nice place to work. Besides Bert, I would like to thank my friend, classmate and colleague Tomas Molenaars and colleague Marcus Hemminga for sharing ideas and thoughts.

I would especially like to thank Steven de Rooij who was my primary supervisor. He kept me being motivated and enthousiastic and came with new ideas and approaches to problems.

Nick Reinerink **Utrecht, October 2012**

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Chapter 1

Introduction

1.1 One-way trading

What is the best way for a trader to sell his shares on the stock market? This is the so-called *one-way trading problem*. Initially, the trader has his entire capital K_0 invested in shares that he wishes to sell. Each subsequent market day $t = 1, 2, \ldots$ after waking up and drinking a cup of coffee, the trader first decides how many of his shares he wishes to hang on to. Call this quantity p_t. The market subsequently reveals that day's share price X_t, and the trader's capital changes accordingly

$$K_t = K_{t-1} + p_t(X_t - X_{t-1}). \tag{1.1}$$

This protocol is summarized in Protocol 1.1.

Protocol 1.1 One-way trading

```
X_0 := 1, K_0 := 1, p_0 := 1
for t = 1, 2, ... do
    Trader announces p_t ≤ p_{t-1}
    Market announces X_t ∈ [0, ∞)
    K_t := K_{t-1} + p_t(X_t − X_{t-1})
end for
```

Note that in the protocol $p_t \leq p_{t-1}$, so the number of shares Trader holds can only decrease.

The one-way trading problem is a very universal problem that can be applied in various situations. Consider for example a trader who has a capital in some currency (e.g. euros) and wants to convert it to another currency (e.g. dollars). Each trading period the trader goes to the currency market, observes the dollar/euro rate and decides whether to exchange (a fraction of) his euros for dollars. Another example is a fund manager who has to change the position in his portfolio to reduce its portfolio risk, by selling one asset (e.g. a share) and buying another (e.g. gold). One-way trading can be applied by using the relative price of these two assets. The one-way trading problem can also be used for buying shares for cash, by considering the reciprocal prices.

Most investment strategies considered in the literature are analysed under heavy assumptions on the market dynamics, such as geometric Brownian motion. In Section 1.3 these assumptions will be further elaborated. However, the one-way trading problem is interesting in that nontrivial guarantees about the trader's final capital can be provided under no assumptions about market behaviour whatsoever. The problem was first studied without probabilistic assumptions in [4] by El Yaniv et al. However, they assume minimum and maximum share prices. Vovk et al. [3] were the first to completely characterize what capital guarantees can be achieved for this setting, and what strategies achieve this. This will be made more precise in Section 1.2. Their analysis considers the idealised case where there are no transaction costs. The optimal strategies they find generally trade whenever the share price attains a new maximum, which is clearly not realistic.

In this thesis we will extend the one-way trading problem with transaction costs and derive results with performance close to the results of Vovk et al. for the transaction cost-free model. In the rest of the introduction we will give an overview of the results of Vovk et al. and our results in the case of transaction costs. We will apply these results to some real-world examples. In Chapter 2 we will study the results of Vovk et al. for the idealized one-way trading problem more extensively. Then in Chapter 3 we extend the problem by including transaction costs. We will show it is possible to construct strategies which trade less often but have almost the same capital guarantee as the strategies of Vovk et al.

1.2 Idealised one-way trading

We now give a short overview of the main results for the one-way trading problem without transaction costs. Vovk et al. showed for which functions F there are trading strategies of which the capital K_t (the total value of cash and shares) for all times t satisfies under any possible sequence of prices,

$$K_t \geq F(M_t), \tag{1.2}$$

where M_t is the observed maximum price until time t. Such a function F is called a capital guarantee and it has a corresponding trading strategy. Vovk et al. showed that a capital guarantee F and its corresponding strategy are optimal in one-way trading *if and only if* the function F satisfies

$$\int_1^\infty \frac{F(y)}{y^2}\mathrm{d}y = 1. \tag{1.3}$$

As we will see later, a trader following such an optimal strategy should hold $p_t = \mathrm{P}(M_{t-1}, \infty)$ shares at time t, here P is a measure on $[1, \infty)$ such that the capital guarantee F satisfies

$$F(y) = \int_{[1,y]} u\mathrm{P}(\mathrm{d}u). \tag{1.4}$$

Because the maximum price M_t can only increase, the number of shares $p_t = \mathrm{P}(M_{t-1}, \infty)$ one should hold can only decrease, hence an optimal strategy only sells shares.

1.2.1 How Bill Gates could have sold his shares optimally

Let's apply this result to a concrete situation. Bill Gates is the co-founder of Microsoft Corporation and one of the world's richest people. In 1986 Microsoft went public on the NASDAQ stock exchange at an opening price of \$21, which is after adjustments for splits[1] \$0.07 in 2012. In Figure 1.1 the split-adjusted, normalized price of the Microsoft stock is displayed. On 9/7/2012 there were 8.4 billion tradeable Microsoft shares. Suppose Bill Gates owned the half of these shares in 1986 and he wanted to maximize his total cash by selling his shares.

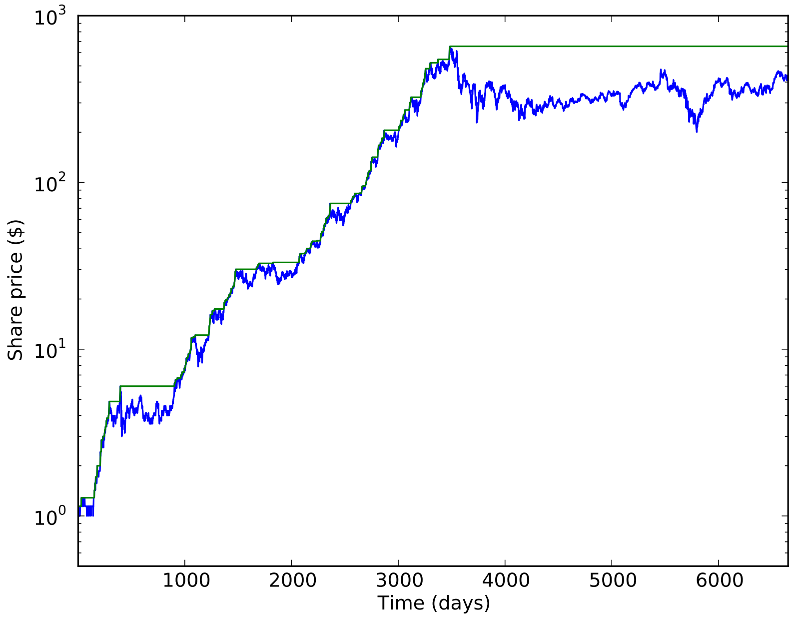

Figure 1.1: Split-adjusted and normalized price of Microsoft shares from 13/3/1986 until 9/7/2012 in \$ (blue) and its maximum (green).

We will show how Bill Gates would have sold his shares if he followed an optimal strategy as described above. Consider the function $F(y) = \alpha y^{1-\alpha}$ with $0 < \alpha < 1$. This function corresponds to an optimal strategy as it satisfies (1.3). One can show using equation (1.4) that the measure of this strategy is defined by $P\{1\} = 1-\alpha$ and $P(y, \infty) = (1-\alpha)y^{-\alpha}$ for $y \geq 1$. Let us take as an example $\alpha = 0.3$. According to the optimal strategy Gates should sell 30% of his shares at the introduction and subsequently hold a fraction $P(M_t, \infty) = 0.7(M_t)^{-0.3}$ of his initial shares at time t, where M_t is the maximum observed price at that time. Figure 1.2 displays the fraction of the shares Gates should hold through time. Observe that after the internet bubble of 2000 the share price never

[1]Since 1986 the number of Microsoft stocks outstanding was increased several times, consequently lowering the price. If for example every share is divided into two shares the price per share is also divided by two. The split-adjusted price is the corrected price for such changes in numbers of shares.

attained a new maximum, hence the fractions of shares remains fixed after this time at about 10%.

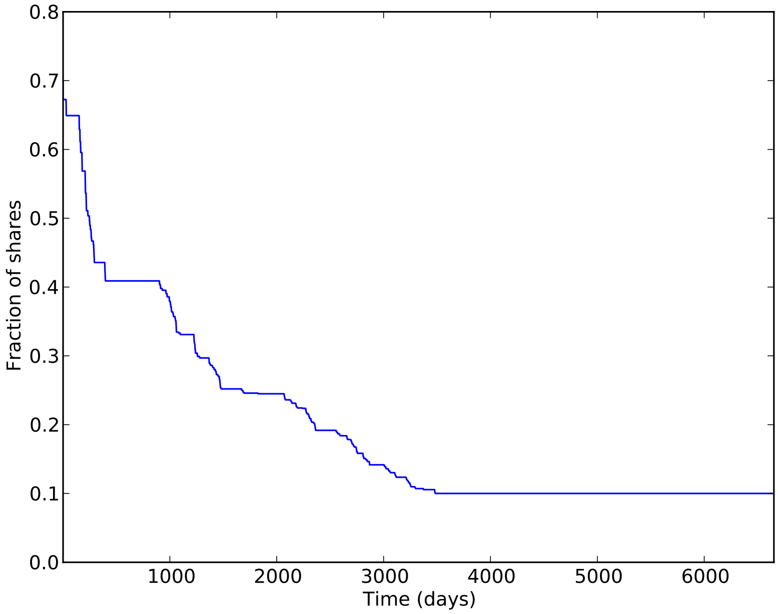

Figure 1.2: The fraction of his initial shares Bill Gates would hold if he followed the optimal strategy.

If Bill Gates followed this strategy his capital satifies inequality (1.2), i.e. $K_t \geq F(M_t) = 0.3(M_t)^{0.7}$ for all times t. In Figure 1.3 the capital and the lower bound are given as a function of time. Note that this lower bound $F(M_t)$ is valid for every possible price path. If the Microsoft stock collapsed to \$0 the capital K_t is still at least $F(M_t)$. However, in practice this will probably not happen and the capital of the optimal strategy is much higher than this worst-case guarantee $F(M_t)$.

The strategy would turn \$1 in 1986 into a capital of \$69.76 and has as guarantee of \$28.09 on 9/7/2012. Hence Bill Gates total capital at that day would be 293 billion dollar with a guarantee of 118 billion dollar, without considering things like transaction costs, taxes and reinvestments in other assets. Forbes Magazine estimates Bill Gates total capital in 2012 at approximately 66 billion dollar[2].

1.3 Transaction costs

The one-way trading problem is an *idealized model* of reality. In practice a trader must pay a fee to a broker for every transaction[3].

[2] http://www.forbes.com/profile/bill-gates/

[3] Brokers use various price structures for transaction costs. Often these transaction costs are a fixed amount per transaction plus an amount proportional to the transaction value. We

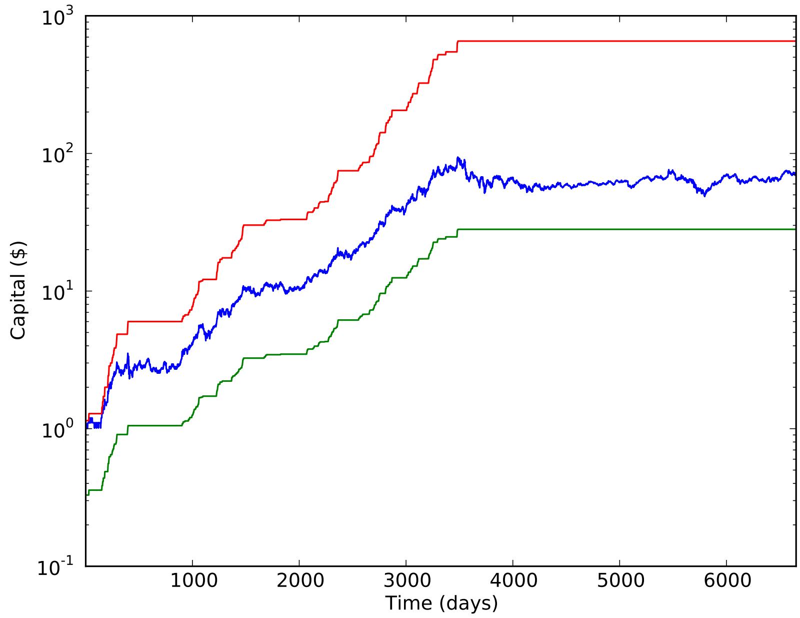

Figure 1.3: The capital of the optimal strategy K_t (blue) starting with 1$ is bounded from below by $F(M_t) = 0.3(M_t)^{0.7}$ (green). The maximum price M_t (red) gives the capital of the optimal strategy in hindsight starting with 1$ at time t.

The optimal strategies by Vovk maximize the cash in the idealized one-way trading problem. In the trading problem with transaction costs the obtained cash of the optimal strategies will be lower. Following these strategies may even result in very large losses, as we will see in the coming example.

According to the optimal strategies a trader should sell shares every day the maximum observed price has increased. This results in a loss if the value of the trade is lower than the transaction costs. The loss will accumulate quickly if there are many of such loss-making trades.

In this thesis we will adjust the optimal strategies such that they trade less often. We will prove that these adjusted strategies guarantee almost the same capital in the transaction cost model as in the idealized transaction cost free model.

Transaction costs are in general difficult to handle in trading models. Especially fixed transaction costs are challenging because they are subtracted from the capital while a change in the share price works in a multiplicative sense. In the context of one-way trading there have been no articles published about the handling of transaction costs.

will consider this form of transaction costs in this thesis.

1.3.1 Trading Google every minute

As an example we will consider the one-way trading problem for shares of Google which are traded at the NASDAQ stock exchange. As described above, transaction costs can result in high losses if a strategy trades very often and the profit per trade is low. To illustrate this we will consider the share price of Google with intervals of a minute over a period of eight trading days, as given in Figure 1.4. Observe that in the period considered the share price of Google has many, relatively small increments.

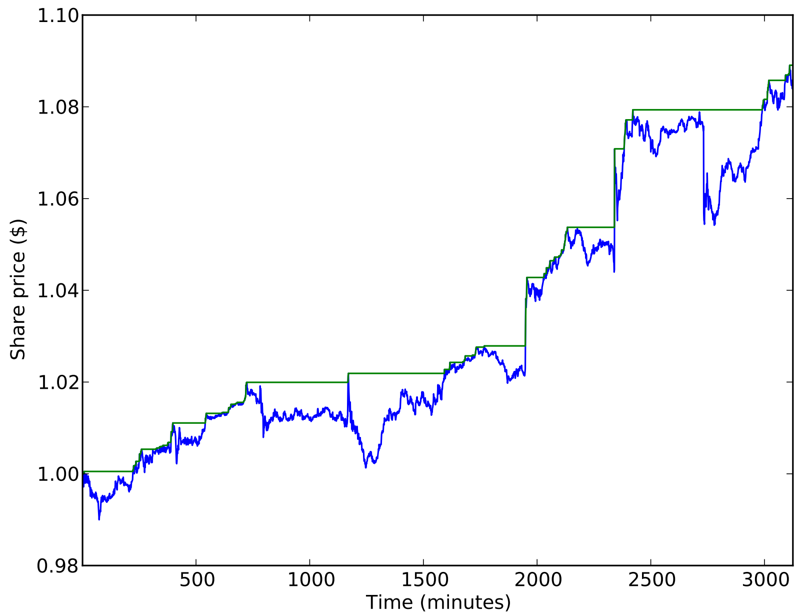

Figure 1.4: Share price of Google on the NASDAQ from 12/7/2012 9:30 until 23/7/2012 16:01 with intervals of a minute in \$ (blue) and its maximum (green).

Suppose we start with 1000\$ of Google shares and we have to pay a fixed amount of 5\$ per transaction. For simplicity we pretend it is possible to trade fractions of shares. We follow an optimal strategy (explained in the previous section) with capital guarantee[4] $F(M_t) = 0.5(M_t)^{0.5}$ and corresponding measure $P(M_t, \infty) = 0.5(M_t)^{-0.5}$. This measure gives the fractions of shares a trader should hold at time t. The blue line in Figure 1.5 shows this fraction of shares a trader holds if he follows this optimal strategy. Initially he sells 50% of his shares and subsequently sells 124 times a very small fraction until he finally is left with 48% of the initial shares.

Because of the large number of transactions the total transactions costs are large compared to the cash obtained from selling shares. In Figure 1.6 the blue line gives the capital through time if there were no transactions costs, while the

[4] This time we used $\alpha = 0.5$ for the capital guarantee $F(y) = \alpha y^{1-\alpha}$, this value was choosen for aesthetic reasons.

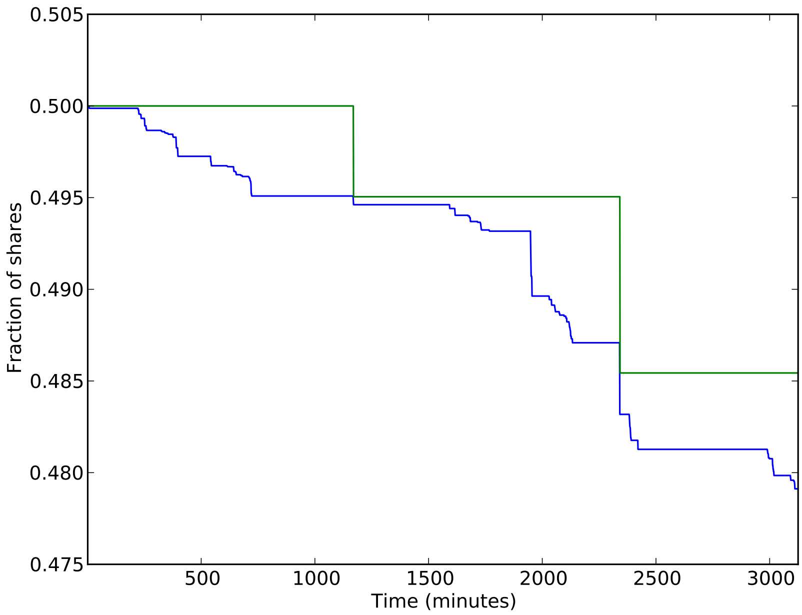

Figure 1.5: The fraction of shares a trader would hold if he followed an optimal strategy (blue line) or the adjusted strategy (green line).

light-blue line displays the capital with transaction costs subtracted. Because of the many transactions this last capital decreases very rapidly, even below the capital guarantee given by the green line (note that this guarantee only holds in case of zero transaction costs).

In this thesis we will construct a new strategy, which reduces the number of trades to reduce the total transactions costs. The green line in Figure 1.5 gives the fraction of shares of this strategy. This fraction decreases three times (note that initially it decreases from 1 to 0.5), so a trader should only sell his shares three times. The red line in Figure 1.6 gives the capital of this new strategy, which is as expected much closer to the capital of the strategy with zero transaction costs (blue line).

We could of course lower the number of transactions even further. But then there is the risk that the price rises to a certain maximum and will never exceed this maximum anymore. This maximum is not known beforehand, so if we trade not frequently enough we do not sell any shares close to the maximum price, hence we miss a potential profit. It turns out there is a trade-off between the frequency of trading (and therefore the total transaction costs) and the amount of missed profit. In this thesis we will show that there is a strategy which minimizes the sum of these two losses.

We will derive a lower bound for the capital of this new strategy, as we did for the optimal strategy without transaction costs in equation (1.2). Let F be the capital guarantee of the optimal strategy in the transaction cost free settting, then we show that the capital K_t of the new strategy satisfies for all

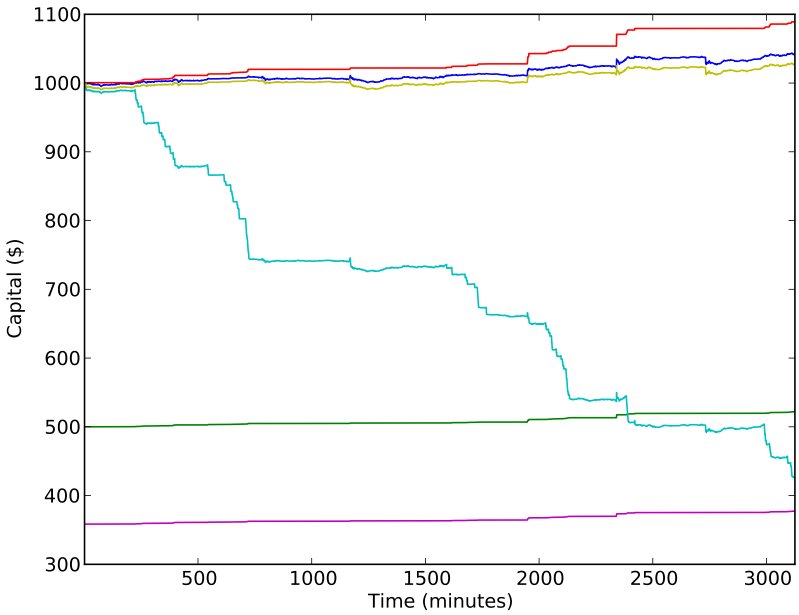

Figure 1.6: Capital through time of the optimal strategy without transaction costs (blue), with transaction costs (light-blue) and of the adjusted strategy with transaction costs (yellow) in $ starting with 1000$ with a cost per transaction of 5$. The capital guarantees of the optimal strategy (light blue) and the adjusted strategy (lila). The maximum price (red) gives the capital of the optimal strategy in hindsight.

times t

$$K_t \geq F(M_t) - \sqrt{8cF(M_t) + c^2}, \tag{1.5}$$

here c is the cost per transaction. For sufficiently large values of $F(M_t)$, the term $\sqrt{8cF(M_t) + c^2}$ is small compared to $F(M_t)$. Hence we can guarantee a capital for trading with transaction costs, which is close to the guarantee for trading optimally without transaction costs.

In our example the guarantee $F(M_t)$ is given by the green line in Figure 1.6 and the transaction cost guarantee $F(M_t) - \sqrt{8cF(M_t) + c^2}$ is the lila line. As in the Bill Gates example the actual capital processes (blue and red line) are much higher than these guarantees. These guarantees however hold under any market circumstance, even if the price of the share collapses to 0$.

1.4 Other practical considerations

Although we made the one-way trading more realistic by including transaction costs, there are other phenomena that we have not considered. In our trading model we suppose it is possible to trade exactly at the price of the market but in reality this is not possible, see for example [12]. However, this turns out not to be a problem for our strategies and their guarantees.

The strategies we consider can be implemented using *limit orders*, this are orders that are exercised if a certain price level is reached. Therefore we do not have to pay a *bid-ask spread*, this is the difference between the bid and ask price.

Placing an order can change the price before it is exercised, this phenomenom is called *market impact*. The guarantees we give are worst-case so the guarantees even hold for adverse markets.

The strategies we considered can be implemented using a sequence of limit orders for which we now the limits beforehand. Problems like a delay in data connection are therefore not an issue.

1.5 Probability-free models

In one-way trading there are no probabilistic assumptions about the price of the shares. Such a model is called a *probability-free model*. In most of the mathematical finance literature returns of financial securities are modeled using probabilistic models. There are however two major problems with probabilistic models.

Probabilistic models are based on so called *stylized facts*. Stylized facts are statistical properties obtained from empirical research on returns from financial markets. Examples of such stylized facts are absence of autocorrelations, heavy tails and volatility clustering of the returns series. A good probabilistic model must generate return series having these statistical properties. It turns out that most existing models are not able to reproduce all known stylized facts, as argued by Cont in [1].

Another even more fundamental problem with probabilistic models is that they are fitted on existing data. More advanced probabilistic models are able to model extreme events like market crashes by using probability distributions with heavy tails. But there is no guarantee such models will model future extreme events very well, as they cannot be validated using available data. Especially events wich are much more extreme than ever observed are a great risk, because these events will have a very large effect on portfolios of financial securities. Such events are called *Black Swans*[5] by Thaleb in [2].

Probability free models by definition do not suffer from these two problems. On the other hand, if one is truly confident that the market will behave according to certain assumptions, this can sometimes be used to provide more appealing capital guarantees.

1.6 Related work

A large part of this thesis is based on Vovk et al. in [3]. In fact they do not treat the one-way trading problem, but the more general two-way trading model. In the two-way trading model a trader can sell *and* buy shares every day. Their

[5] Black swans were discoverd in 1697 in Western Australia by the Dutch explorer Willem de Vlamingh. Rather ironically, in the centuries before its discovery the term 'black swan' was used as a metaphor for impossibilities.

results however also apply in the one-way trading problem as we will see in this thesis.

El Yaniv et al. [4] also consider the one-way trading model. However they concentrate on the model where the share price is assumed to have a fixed minimum and maximum, which is not realistic in real-world application.

Koolen and De Rooij [5] consider the two-way trading model. In the two-way trading model the optimal strategy in hindsight is selling at local maxima and buying at local minima. For this more difficult goal the optimal strategy is not known, but they derive a lower bound for the payoff of certain strategy in terms of the local maxima and minima. In contrast, we will show as Vovk et. al that there are strategies with a lower bound on the payoff in terms of the global maximum.

A problem related to two-way trading problem is the portfolio selection problem. In this problem a trader can buy and sell multiple assets. An approach to this problem is the theory of universal portfolios based on the work of Cover in [7] and [8], and summarized in [6] by Cesa-Bianchi and Lugosi.

Vovk et al. use the results for the two-way trading model to give an upper bound for so called lookback options in an arbitrage-free market. We will also study these results. In addition we will give an upper bound for these options in an arbitrage-free market with transaction costs. A similar approach for option pricing can be found in [9], here a lower and upper bound for the price of European call options is derived for the two-way trading model where is assumed the quadratic variation of the share price is bounded. There are several articles on the pricing of options using probabilistic models with transactions costs, see for example [10] and [11].

1.7 Conventions and notations

In this thesis we will make use of probability measures denoted by P which are defined on some interval $I \subset \mathbb{R}$ with a Borel sigma-algebra. In the rest of this thesis we will not mention such sigma-algebras of probability measures, for convenience. For simplicity we will denote the value of probability measure P of an interval $[a, b]$ by $\mathrm{P}[a, b]$. For a singleton $\{x\}$ we denote the value by $\mathrm{P}\{x\}$.

If not explicitly noted, the results in this thesis are own work.

Chapter 2

One-way trading

2.1 Introduction

In this chapter we will study the results of Vovk et al. [3] on one-way trading. They showed there are strategies for which it is possible to give a non-trivial guarantee for the capital. They also characterized the highest possible guarantee a strategy can achieve in the one-way trading problem.

We will first introduce the one-way trading problem. In the one-way trading problem a trader initially has a capital $K_0 = 1$ which is invested in shares. The shares have a price $X_0 = 1$, so the initial number of shares the trader holds is $p_0 = 1$. On day t the trader decides how many shares he wants to hold, denoted by p_t, subsequently the market determines the share price X_t. The capital of the trader becomes $K_t = K_{t-1} + p_t(X_t - X_{t-1})$. This protocol for one-way trading is given in Protocol 2.1. In one-way trading the trader may only *sell* shares, that is why we require $p_t \leq p_{t-1}$.

Protocol 2.1 One-way trading

$X_0 := 1, K_0 := 1, p_0 := 1$
for $t = 1, 2, \dots$ **do**
  Trader announces $p_t \leq p_{t-1}$
  Market announces $X_t \in [0, \infty)$
  $K_t := K_{t-1} + p_t(X_t - X_{t-1})$
end for

To get a better understanding of the formula for the capital we can rewrite it as follows

$$
\begin{aligned}
K_t &= K_{t-1} + p_t(X_t - X_{t-1}) \\
&= K_0 + \sum_{s=1}^{t} p_s(X_s - X_{s-1}) \\
&= p_0 X_0 + \sum_{s=1}^{t} p_s X_s - \sum_{s=1}^{t} p_s X_{s-1} \\
&= p_0 X_0 + \sum_{s=1}^{t} p_s X_s - \sum_{s=0}^{t-1} p_{s+1} X_s \\
&= p_0 X_0 + \sum_{s=1}^{t-1} (p_s - p_{s+1}) X_s - p_1 X_0 + p_t X_t \\
&= \sum_{s=0}^{t-1} (p_s - p_{s+1}) X_s + p_t X_t. \qquad (2.1)
\end{aligned}
$$

At time s the number of shares sold by Trader is $p_s - p_{s+1}$ for the price X_s. Hence the sum in equation (2.1) represents Trader's total cash obtained by selling and borrowing shares until time t. The value of the shares Trader holds at time t is $p_t X_t$, this is the other part of equation (2.1).

At time t Trader can use information about the prices $X_0, X_1, \ldots, X_{t-1}$ to determine p_t, the number of shares he wants to hold, such that $p_t \leq p_{t-1}$. A *strategy* S for Trader is a sequence of functions $S = (S_t)_{t\geq 1}$ with $S_t : [0,\infty)^t \times \mathbb{R} \to \mathbb{R}$ that determine the number of shares p_t he holds if a certain price sequence is realized, $p_t = S_t(X_0, X_1, \ldots, X_{t-1}, p_{t-1})$. In the following we will, for readability, not use this formal description of a strategy. One has to keep in mind, however, that every strategy can be described as such a sequence of functions.

2.2 Capital guarantees

Consider the tragic case the share collapses at day t, which means the price of the share is $X_s = 0$ for all days $s \geq t$. In this scenario the value of the shares the trader holds is zero and the trader's capital K_t (sum of share value and cash) is equal to the obtained cash so far. Because the shares are worthless the capital K_t will be the same for all further times, hence $K_t = K_s$ for all $s \geq t$. Therefore we would like to find a strategy that gives a guarantee at any time.

To characterize such guarantees we define the following concepts.

Definition 2.1. *Let $F, G : [1,\infty) \to [0,\infty)$ be functions. We say F* dominates *the function G if $F(y) \geq G(y)$ for all $y \in [1,\infty)$. The function F* strictly dominates *the function G if F dominates G and $F(y) > G(y)$ for some $y \in [1,\infty)$.*

Definition 2.2. *An increasing function $F : [1,\infty) \to [0,\infty)$ is a* capital guarantee *if for some strategy Trader's capital satisfies*

$$K_t \geq F(M_t) \tag{2.2}$$

for all $t \geq 0$ *where* $M_t := \sup_{s \leq t} X_s$. *A capital guarantee is an* optimal capital guarantee *if it is not strictly dominated by any other capital guarantee.*

A capital guarantee F is a function which depends on the maximum price M_t. This is because the best strategy in hindsight at time t is selling all shares at the maximum observed price M_t. If Trader followed this strategy he would have a capital M_t in cash. Hence using F we can measure how close the guaranteed capital of a strategy is to the capital of the best strategy in hindsight.

Ideally we would like to find a strategy with a capital guarantee which dominates all other possible capital guarantees. But it is not possible to find such a guarantee, this can be seen as follows. A basic strategy for Trader is to sell all shares when the maximum M_t exceeded a certain price level u, this strategy has a capital guarantee $F_u(y) = u\mathbf{1}_{\{y \geq u\}}$. A strategy with a capital guarantee F which dominates all other capital guarantees, must dominate for all u the capital guarantees $F_u(y)$ of the basic strategies. This means the capital guarantee satisfies $F(y) \geq y$ for all y, which is as good as the best strategy in hindsight! Obviously it is not possible to find such a strategy without prior knowledge of the share prices. Because it is not possible to find a capital guarantee which dominates all other capital guarantees, we use a somewhat weaker definition for the optimal capital guarantee.

Rather surprisingly it turns out that capital guarantees and optimal capital guarantees can be fully characterized as in the following two theorems, due to Vovk et al. [3]. The proofs are a bit more extensive than the proofs of Vovk's, for example Vovk only provides an informal argument for implication (1) → (3).

Theorem 2.3. *Let* $F : [1, \infty) \to [0, \infty)$ *be an increasing function, then the following statements are equivalent*

1. *F is a capital guarantee,*
2. *For some probability measure* P *on* $[1, \infty]$ *for all* $y \in [1, \infty)$
$$F(y) = \int_{[1,y]} u \mathrm{P}(\mathrm{d}u),$$
3. *F is right-continuous and*
$$\int_1^\infty \frac{F(y)}{y^2} \mathrm{d}y \leq 1. \tag{2.3}$$

Proof. (2) → (1)
Let P be a probability measure on $[1, \infty]$ and define for all $y \in [1, \infty)$

$$F(y) = \int_{[1,y]} u \mathrm{P}(\mathrm{d}u).$$

For $u \geq 1$ consider as a strategy for Trader holding one share if $M_{t-1} < u$ and no shares if $M_{t-1} \geq u$ at time t, i.e. $p_t^{(u)} = 1_{\{M_{t-1}<u\}}$. Let $K_t^{(u)}$ be the capital at time t of this strategy. Now consider the mixture of these strategies with

$$p_t = \int_{[1,\infty]} p_t^{(u)} \mathrm{P}(\mathrm{d}u),$$

then one can show that

$$K_t = \int_{[1,\infty]} K_t^{(u)} \mathrm{P}(\mathrm{d}u). \tag{2.4}$$

This follows by induction; at $t = 0$ for all $u \geq 1$, $p_0^{(u)} = 1$ so $p_0 = \int_{[1,\infty]} \mathrm{P}(\mathrm{d}u) = 1$ and $K_0 = 1$. On the other hand $p_0^{(u)} = 1$ implies $K_0^{(u)} = 1$, so $\int_{[1,\infty]} K_0^{(u)} \mathrm{P}(\mathrm{d}u) = 1$, hence equation (2.4) holds for $t = 0$. And if equation (2.4) holds for $t-1$, it holds for t:

$$\begin{aligned}
K_t &= K_{t-1} + p_t(X_t - X_{t-1}) \\
&= \int_{[1,\infty]} K_{t-1}^{(u)} \mathrm{P}(\mathrm{d}u) + \int_{[1,\infty]} p_t^{(u)} \mathrm{P}(\mathrm{d}u)(X_t - X_{t-1}) \\
&= \int_{[1,\infty]} \left(K_{t-1}^{(u)} + p_t^{(u)}(X_t - X_{t-1})\right) \mathrm{P}(\mathrm{d}u) \\
&= \int_{[1,\infty]} K_t^{(u)} \mathrm{P}(\mathrm{d}u).
\end{aligned}$$

Then

$$\begin{aligned}
K_t &= \int_{[1,\infty]} K_t^{(u)} \mathrm{P}(\mathrm{d}u) \\
&\geq \int_{[1,M_t]} K_t^{(u)} \mathrm{P}(\mathrm{d}u) \\
&\geq \int_{[1,M_{t-1}]} K_t^{(u)} \mathrm{P}(\mathrm{d}u) + \int_{(M_{t-1},M_t]} K_t^{(u)} \mathrm{P}(\mathrm{d}u) \\
&\geq \int_{[1,M_t]} u \mathrm{P}(\mathrm{d}u) = F(M_t)
\end{aligned}$$

The last inequality follows because the left integral consists of the strategies that sold their shares, resulting in $K_t^{(u)} = u$. The right integral consists of strategies that still have all their shares, they have capital worth also $K_t^{(u)} = u$.

(1) → (3)
Let F be a capital guarantee for a strategy with capital sequence (K_t), then for all t we have $K_t \geq F(M_t)$. This holds for any price sequence, in the following we will consider a price sequence X_t which is a random walk starting at $X_0 = 1$ and stopped when it hits zero. Let $R = (R_t)_{t=0,1,2,\ldots}$ be a random walk with

$$\mathrm{P}(R_t - R_{t-1} = 1/N) = \mathrm{P}(R_t - R_{t-1} = -1/N) = 1/2,$$

$R_0 = 1$ and $N \in \mathbb{N}$. Define the stopping time $\tau = \inf\{t \geq 0 : X_t = 0\}$, the stopped process $X_t = R_{\min(\tau,t)}$ and its maximum $M_t = \max_{s \leq t} X_s$.

The capital sequence is a martingale

$$\begin{aligned}\mathrm{E}[K_t|X_0, X_1, \ldots X_{t-1}] &= \mathrm{E}[K_{t-1} + p_t(X_t - X_{t-1})|X_0, X_1, \ldots X_{t-1}] \\ &= K_{t-1} + p_t\mathrm{E}[X_t - X_{t-1}|X_0, X_1, \ldots X_{t-1}] \\ &= K_{t-1},\end{aligned}$$

therefore

$$\mathrm{E}F\left(\liminf_{t\to\infty} M_t\right) \le \mathrm{E}\liminf_{t\to\infty} K_t \le \liminf_{t\to\infty} \mathrm{E}K_t = \mathrm{E}K_0 = 1.$$

This holds for any maximum price sequence (M_t). In the following we will show that

$$\mathrm{E}F\left(\liminf_{t\to\infty} M_t\right) = \int_1^\infty \frac{F(y)}{y^2}\mathrm{d}y,$$

this then completes the proof. Define the (almost surely finite) stopping time

$$T = \inf\{t \ge 0 : R_t = 0 \text{ or } R_t = k/N \text{ for some } k \in \{N+1, N+2, \ldots, N+t\}\}$$

and the stopped process $Y = (Y_t)_{t=0,1,2,\ldots}$ defined by

$$Y_t = R_{\min(T,t)}.$$

Obviously R is a martingale so by Theorem 2.2 in [13] the stopped process Y is also a martingale. Define

$$Y_\infty = \lim_{t\to\infty} Y_t,$$

then by the martingale property of Y and dominated convergence (note that Y is bounded),

$$1 = \mathrm{E}[Y_0] = \mathrm{E}[Y_t] = \lim_{t\to\infty} \mathrm{E}[Y_t] = \mathrm{E}\left[\lim_{t\to\infty} Y_t\right] = \mathrm{E}[Y_\infty],$$

and

$$1 = \mathrm{E}[Y_\infty] = 0 \cdot \mathrm{P}\{Y_\infty = 0\} + k/N \cdot \mathrm{P}\{Y_\infty = k/N\},$$

hence

$$\mathrm{P}\{Y_\infty = k/N\} = \frac{N}{k}.$$

For the processes M and Y we have the relation

$$\mathrm{P}\{M_\infty \ge k/N\} = \mathrm{P}\{Y_\infty = k/N\} = \frac{N}{k},$$

which implies

$$\mathrm{P}\{M_\infty = k/N\} = \mathrm{P}\{M_\infty \ge k/N\} - \mathrm{P}\{M_\infty \ge (k+1)/N\} = \frac{N}{k} - \frac{N}{k+1} = \frac{N}{k(k+1)}.$$

We let the increments of the random walk become small by letting $N \to \infty$, then we have for F,

$$\begin{aligned}
\lim_{N\to\infty} \mathrm{E}F(M_\infty) &= \lim_{N\to\infty} \sum_{k=N+1}^{\infty} F\left(\frac{k}{N}\right) \mathrm{P}\{M_\infty = k/N\} \\
&= \lim_{N\to\infty} \sum_{k=N+1}^{\infty} F\left(\frac{k}{N}\right) \frac{N}{k(k+1)} \\
&= \lim_{N\to\infty} \sum_{k=N+1}^{\infty} F\left(\frac{k}{N}\right) \frac{1}{(k/N)^2} \frac{1}{N} \\
&= \int_1^\infty \frac{F(y)}{y^2} \mathrm{d}y.
\end{aligned}$$

(3) → (2) Let Q be the measure such that $\mathrm{Q}[1, y] = F(y)$ for all $y \in [1, \infty)$. Define the measure P on $[1, \infty]$ by

$$\mathrm{P}(\mathrm{d}u) = \frac{1}{u} \mathrm{Q}(\mathrm{d}u)$$

which means for a measurable set A

$$\mathrm{P}(A) = \int_A \mathrm{P}(\mathrm{d}u) = \int_A \frac{1}{u} \mathrm{Q}(\mathrm{d}u)$$

and

$$\mathrm{P}\{\infty\} = 1 - \int_{[1,\infty)} \frac{F(y)}{y^2} \mathrm{d}y.$$

Then

$$F(y) = Q[1, y] = \int_{[1,y]} u\mathrm{P}(\mathrm{d}u)$$

and

$$\begin{aligned}
\int_{[1,\infty)} \frac{F(y)}{y^2} \mathrm{d}y &= \int_{[1,\infty)} \int_{[1,y]} \frac{u}{y^2} \mathrm{P}(\mathrm{d}u)\mathrm{d}y \\
&= \int_{[1,\infty)} \int_{[u,\infty)} \frac{u}{y^2} \mathrm{d}y\mathrm{P}(\mathrm{d}u) = \int_{[1,\infty)} \mathrm{P}(\mathrm{d}u),
\end{aligned}$$

hence P is a probablity measure as $\mathrm{P}[1, \infty] = 1$. □

Theorem 2.3 is used to prove the following theorem.

Theorem 2.4. *Let $F : [1, \infty) \to [0, \infty)$ be an increasing function, then the following statements are equivalent*

1. *F is an optimal capital guarantee,*
2. *For some probability measure P on $[1, \infty)$ for all $y \in [1, \infty)$*

$$F(y) = \int_{[1,y]} u\mathrm{P}(\mathrm{d}u), \tag{2.5}$$

3. F is right-continuous and

$$\int_1^\infty \frac{F(y)}{y^2}\mathrm{d}y = 1. \tag{2.6}$$

Proof. (1) → (3)
Let F be an optimal capital guarantee and suppose $\int_1^\infty \frac{F(y)}{y^2}\mathrm{d}y < 1$. Then there exists a function $G : [1,\infty) \to [0,\infty)$ dominating F such that

$$\int_1^\infty \frac{F(y)}{y^2}\mathrm{d}y < \int_1^\infty \frac{G(y)}{y^2}\mathrm{d}y \le 1,$$

but then G is a capital guarantee by Theorem 2.3 that dominates F, this contradicts the fact that F is an optimal capital guarantee.

(3) → (1)
Let F be right-continous and $\int_1^\infty F(y)/y^2\mathrm{d}y = 1$, then by Theorem 2.3 F is a capital guarantee. Suppose F is strictly dominated by another capital guarantee G then

$$\int_1^\infty \frac{G(y)}{y^2}\mathrm{d}y > \int_1^\infty \frac{F(y)}{y^2}\mathrm{d}y = 1,$$

which cannot be true by Theorem 2.3. Hence there is no capital guarantee strictly dominating F.

(2) → (3)

$$\begin{aligned}\int_{[1,\infty)} \frac{F(y)}{y^2}\mathrm{d}y &= \int_{[1,\infty)}\int_{[1,y]} \frac{u}{y^2}\mathrm{P}(\mathrm{d}u)\mathrm{d}y = \int_{[1,\infty)}\int_{[u,\infty]} \frac{u}{y^2}\mathrm{d}y\mathrm{P}(\mathrm{d}u)\\ &= \int_{[1,\infty)} \mathrm{P}(\mathrm{d}u) = 1\end{aligned}$$

(3) → (2)
Let Q be the measure such that $\mathrm{Q}[1,y] = F(y)$ for all $y \in [1,\infty)$. Define the measure P on $[1,\infty)$ by

$$\mathrm{P}(\mathrm{d}u) = \frac{1}{u}\mathrm{Q}(\mathrm{d}u)$$

which means for a measurable set A

$$\mathrm{P}(A) = \int_A \mathrm{P}(\mathrm{d}u) = \int_A \frac{1}{u}\mathrm{Q}(\mathrm{d}u).$$

Then

$$F(y) = \mathrm{Q}[1,y] = \int_{[1,y]} u\mathrm{P}(\mathrm{d}u)$$

and

$$\begin{aligned}1 = \int_{[1,\infty)} \frac{F(y)}{y^2}\mathrm{d}y &= \int_{[1,\infty)}\int_{[1,y]} \frac{u}{y^2}\mathrm{P}(\mathrm{d}u)\mathrm{d}y\\ &= \int_{[1,\infty)}\int_{[u,\infty)} \frac{u}{y^2}\mathrm{d}y\mathrm{P}(\mathrm{d}u) = \int_{[1,\infty)} \mathrm{P}(\mathrm{d}u),\end{aligned}$$

hence P is a probablity measure. □

Note the small differences between both results. For a capital guarantee the measure P is defined on $[1, \infty]$, while for an optimal capital guarantee the measure is defined on $[1, \infty)$. Also note (2.3) is an inequality while (2.6) is an equality.

As one can see from the proofs, Theorem 2.4 follows from Theorem 2.3. In the following we will mainly use Theorem 2.4 because we are interested in optimal capital guarantees and their corresponding strategies.

The second statement of Theorem 2.4 can be used to define a strategy for Trader, this will be shown in Section 2.3. The third statement shows how fast an optimal capital guarantee F may increase. Equation (2.6) in this statement shows an optimal capital guarantee can increase almost as fast as the identity function on $[1, \infty)$. In Section 2.4 we will give examples of optimal capital guarantees and their corresponding strategies.

2.3 Optimal strategy

From Theorem 2.4 it is not clear which strategy corresponds to an optimal capital guarantee. We can interpret the characterization of F given in equation (2.5) as a trading strategy, as follows. Consider the basic strategy which initially holds one share and sells it when the maximum price y reaches the treshold u. This strategy has as capital guarantee $F_u(y) := u1_{\{y \geq u\}}$. Now consider a mixture of these basic strategies by using a probability measure P on u. Such a mixture is obtained by spreading the initial capital over the basic strategies according to P. The capital guarantee corresponding to this mixture strategy is

$$F(y) = \int_1^\infty F_u(y)\mathrm{P}(\mathrm{d}u) = \int_1^\infty u\mathrm{P}(\mathrm{d}u).$$

This capital guarantee is an optimal capital guarantee according to the second statement in Theorem 2.4.

This mixture of basic strategies requires the Trader to sell shares at every price level in $[1, M_t]$. In the protocol however Trader observes not a continuous stream of prices but the prices are given at discrete time instants. This means that when Trader observes M_t he has to execute all basic strategies with tresholds $u \in (M_{t-1}, M_t]$. Notice that the price at which the shares are sold is M_t, which is higher or equal than the treshold values $u \in (M_{t-1}, M_t]$, see also Figure 2.1.

In terms of the trading protocol this mixture strategy implies that at time t Trader announces $p_t = \mathrm{P}(M_{t-1}, \infty)$. This can be seen as follows. The maximum price observed until time t is M_{t-1} therefore $\mathrm{P}[1, M_{t-1}]$ shares should have been sold, according to the mixture strategy. After selling these shares Trader is left with $p_t = \mathrm{P}(M_{t-1}, \infty)$ shares. The following theorem shows that the strategy with $p_t = \mathrm{P}(M_{t-1}, \infty)$ at times t corresponds to an optimal capital guarantee.

Theorem 2.5. *Let F be an optimal capital guarantee and* P *be an probability measure on $[1, \infty)$ such that (2.5) holds. If Trader at times $s \leq t$ has $p_s =$*

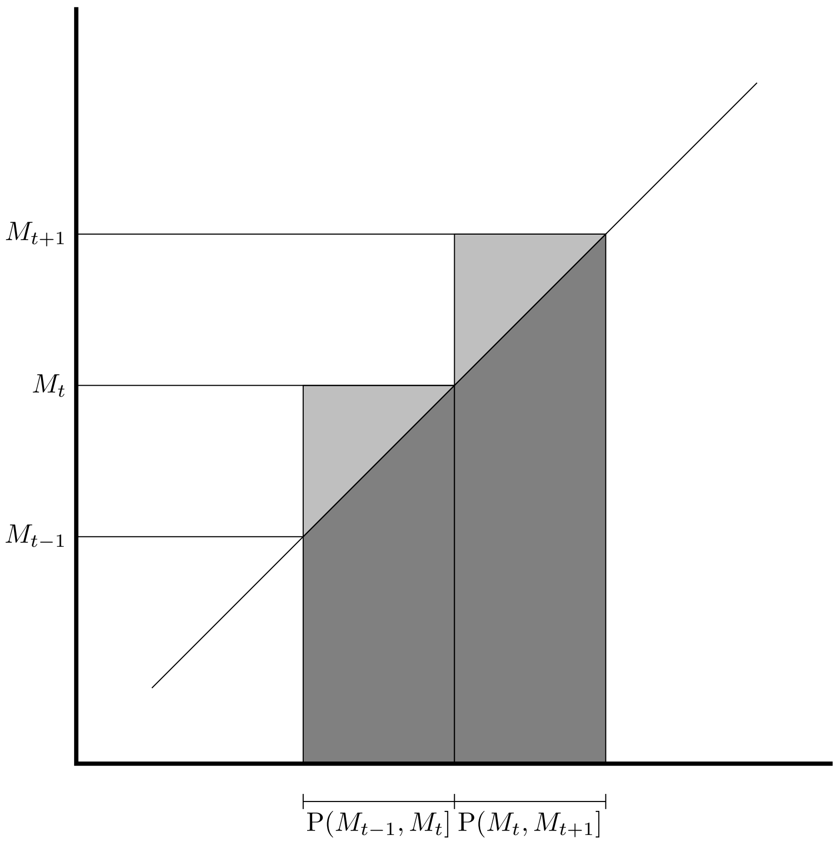

Figure 2.1: The dark grey area corresponds to optimal capital guarantee F, the light-grey area is the extra capital Trader obtains by selling at discrete price levels.

$\mathrm{P}(M_{s-1},\infty)$ *shares then*

$$K_t = \sum_{s=1}^{t-1} \mathrm{P}(M_{s-1}, M_s]M_s + \mathrm{P}(M_{t-1},\infty)M_t$$
$$\geq F(M_t) + \mathrm{P}(M_{t-1},\infty)M_t \tag{2.7}$$

Proof. For the capital K_t at time t we have

$$\begin{aligned}
K_t &= K_{t-1} + p_t(X_t - X_{t-1}) \\
&= \sum_{s=0}^{t-1}(p_s - p_{s+1})X_s + p_t X_t \\
&= \sum_{s=1}^{t-1} \mathrm{P}(M_{s-1}, M_s]M_s + \mathrm{P}(M_{t-1},\infty)M_t \\
&\geq \sum_{s=1}^{t} \mathrm{P}(M_{s-1}, M_s]M_s + \mathrm{P}(M_{t-1},\infty)M_t \\
&\geq \int_{[1,M_t]} u\mathrm{P}(\mathrm{d}u + \mathrm{P}(M_{t-1},\infty)M_t) \\
&= F(M_t) + \mathrm{P}(M_{t-1},\infty)M_t
\end{aligned}$$

The second equality follows from equation (2.1). Only when the maximum observed price has increased shares are sold, this implies the third equality. If the maximum price did not increase at time s then $\mathrm{P}(M_{s-1}, M_s] = 0$, if it did increase $X_s = M_s$. The fourth equality follows because $\mathrm{P}(M_{t-1},\infty) \geq \mathrm{P}(M_{t-1}, M_t]$. The last inequality holds because the left hand side is an upper Riemann sum of the integral on the right hand side, see also Figure 2.1. □

Equation (2.7) can be understood as follows. At time t Trader sold at times $s < t$ for price M_s an amount of $\mathrm{P}(M_{s-1}, M_s]$ shares. At time t he still holds $\mathrm{P}(M_{t-1},\infty)$ shares with a total worth of $\mathrm{P}(M_{t-1},\infty)M_t$.

2.4 Examples of optimal capital guarantees

We give some examples of optimal capital guarantees F in Table 2.1, they are plotted in Figure 2.2. In the Appendix we prove these capital guarantees are indeed optimal and we will derive the measures P corresponding to the guarantees.

In practice the maximum price M_t is unknown beforehand, so it is better if the capital guarantee is as high as possible for every possible maximum price. This means the capital guarantee has to increase as fast as possible for large maximum prices. Hence we are looking for the capital guarantee wich increases asymptotically the fastest, in Table 2.1 the capital guarantees are ordered according to their asymptotical increasingness.

It is possible to combine these examples of optimal capital guarantees, as the next corollary shows a convex combination of optimal capital guarantees as again an optimal capital guarantee.

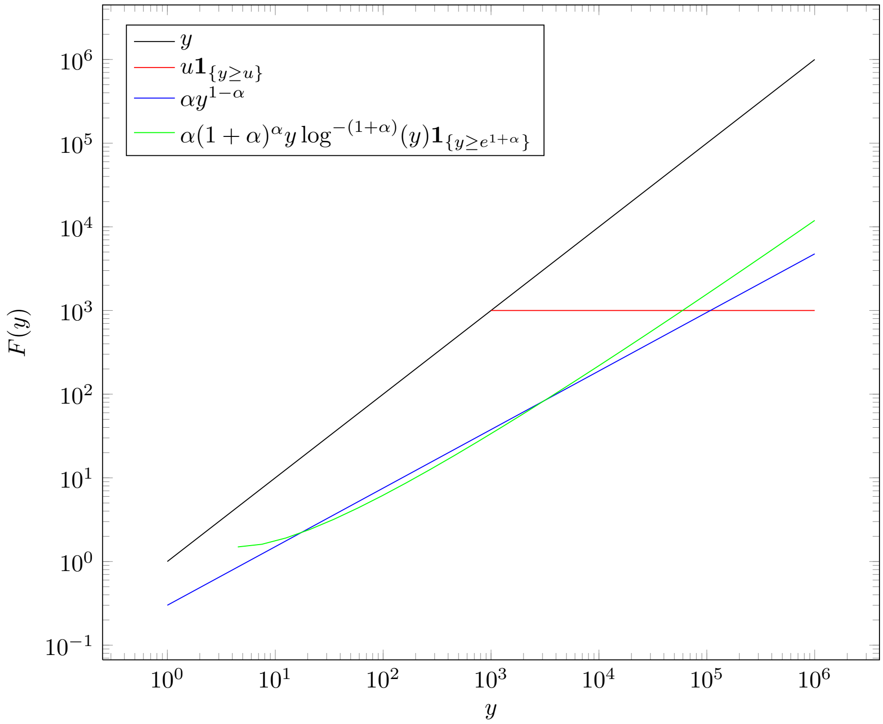

Figure 2.2: Some examples of optimal capital guarantees.

$F(y)$	Parameters	Asymptotically
$u\mathbf{1}_{\{y\geq u\}}$	$u \geq 1$	$\mathcal{O}(1)$
$\alpha y^{1-\alpha}$	$0 < \alpha < 1$	$\mathcal{O}\left(\frac{y}{y^{\alpha}}\right)$
$\alpha(1+\alpha)^{\alpha} y \log^{-(1+\alpha)}(y)\mathbf{1}_{\{y\geq e^{1+\alpha}\}}$	$\alpha > 0$	$\mathcal{O}\left(\frac{y}{\log^{\alpha} y}\right)$

Table 2.1: Examples of some optimal capital guarantees.

Corollary 2.6. *Let $p_i \in [0,1]$ for $i = 1,\dots,n$ such that $\sum_{i=1}^n p_i = 1$. For $i = 1,\dots,n$ let F_i be optimal capital guarantees with measures P_i then*

$$F(y) := \sum_{i=1}^n p_i F_i(y), \tag{2.8}$$

is also an optimal capital guarantees with measure defined by

$$\mathrm{P}(A) := \sum_{i=1}^n p_i \mathrm{P}_i(A), \tag{2.9}$$

for any measurable set A.

Proof. The functions F_i $(i = 1,\dots,n)$ are optimal capital guarantees, hence by Theorem 2.4 we have for every $y \in [1,\infty)$

$$F_i(y) = \int_{[1,y]} u\mathrm{P}_i(\mathrm{d}u).$$

Let $y \in [1,\infty)$ then

$$\int_{[1,y]} u\mathrm{P}(\mathrm{d}u) = \sum_{i=1}^n p_i \int_{[1,y]} u\mathrm{P}_i(\mathrm{d}u) = \sum_{i=1}^n p_i F_i(y) = F(y),$$

we conclude F is an optimal capital guarantee by Theorem 2.4. □

2.5 Two-way trading

The two-way trading problem can be formulated as in Protocol 2.2.

Protocol 2.2 Two-way trading

```
X_0 := 1, K_0 := 1, p_0 := 1
for t = 1, 2, ... do
    Trader announces p_t ∈ ℝ
    Market announces X_t ∈ [0, ∞)
    K_t := K_{t-1} + p_t(X_t − X_{t-1})
end for
```

This protocol is very similar to the one-way trading protocol. The only difference is that in this protocol there are no restrictions on the number of shares p_t Trader holds. This means Trader can buy and sell shares. Observe that p_t may also be negative, this means Trader can borrow shares[1]. Trader's capital K_t may also be negative, meaning he may borrow money.

In the proofs of Theorems 2.3 and 2.4 the restriction on the number of shares $p_t \leq p_{t-1}$ is actually not needed. So these results also hold for the more general two-way trading problem!

[1] In financial literature borrowing shares is also called going short, while buying shares is called going long.

Compared to the best strategy in hindsight an optimal capital guarantee is in general not very good guarantee. Let $z_0 := 1, z_1, \ldots, z_m$ be the local extrema of the price sequence $X_0, \ldots, X_t$. The best strategy in handsight is buying shares for all available cash at local minima and selling all shares at local maxima. To compare a strategy with the best strategy in hindsight one needs a guarantee in terms of the extrema $z_0, \ldots, z_m$. Koolen and De Rooij give in [5] a strategy which gives a bound in terms of the local extrema.

2.6 Lookback options

The results of the optimal capital guarantees for the two-way trading problem can also be used for the pricing of so called lookback options, as Vovk et al. showed in [3]. We will present these results here, in the next chapter we will use a similar analysis to price lookback options in a market with transaction costs.

A lookback option is a contract which gives Trader a payoff $G(M_t)$ when exercised at time t. Here $G : [0, \infty) \to [0, \infty)$ is assumed to be an increasing function and $M_0 > 0$. For the pricing of such options we need the notion of no-arbitrage, which says that it is impossible to make a risk-free profit at zero cost.

This can be formulated as follows, suppose we have two strategies P and Q starting with respectively K_0 and L_0 in cash. If the payoff of strategy P always dominates that of strategy Q, i.e. $K_t \geq L_t$ for all $t > 0$, then in a no-arbitrage market $K_0 \geq L_0$. Suppose this was not true then $K_0 < L_0$ and we could invest in P and short in Q giving at time zero $L_0 - K_0 > 0$ and a profit at all later times $t > 0$ given by $K_t - L_t \geq 0$.

Lemma 2.7. *If in a no-arbitrage market there exists a strategy P with a capital which satisfies for any time t the bound $K_t \geq G(M_t)$, then the price L of a lookback option is at most K_0.*

Proof. Consider the strategies

1. investing K_0 in strategy P,
2. buying a lookback option costing L.

The payoff of strategy 2 is $G(M_t)$ which dominates the payoff K_t of strategy 1, so by no-arbitrage $L \leq K_0$. □

So to get an upper bound for the price of a lookback option with payoff $G(M_t)$ when exercised at time t it suffices to find a strategy with capital sequence (K_t) such that $K_t \geq G(M_t)$ for all $t \geq 0$. The upper bound for the price is then given by the initial capital K_0 of that strategy. Vovk showed there exists such a strategy and gave the needed initial capital, which is given in the following theorem.

Theorem 2.8. *Let $G : [0, \infty) \to [0, \infty)$ be an increasing function and $X_0 > 0$. An upper bound for the price of an American lookback option that pays $G(M_t)$ when exercised at time t is*

$$X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x.$$

Proof. Consider the normalized price $Y_t = X_t/X_0$, its maximum price $N_t = \sup_{s\leq t} Y_t$ and

$$F(y) := \frac{G(X_0 y)}{X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x}.$$

The function F is an optimal capital guarantee by Theorem 2.4, because

$$\begin{aligned}\int_1^{\infty} \frac{F(y)}{y^2} \mathrm{d}y &= \left(X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x \right)^{-1} \int_1^{\infty} \frac{G(X_0 y)}{y^2} \mathrm{d}y \\ &= \left(X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x \right)^{-1} \int_{X_0}^{\infty} \frac{G(x)}{(x/X_0)^2} \frac{1}{X_0} \mathrm{d}x \\ &= 1.\end{aligned}$$

Hence there exists a strategy with a capital process (K_t) such that for any time t,

$$K_t \geq F(N_t) = F(M_t/X_0) = \frac{G(M_t)}{X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x}.$$

When Trader applies this strategy with initial capital $X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x$ he will have $K_t \geq G(M_t)$ for every t, so this initial capital is an upper bound for the price of the option. □

Note that the upper price of the lookback option may be infinite. For example this is the case if one considers the classical lookback option which pays M_t when exercised at time t. This lookback option has $G(x) = x$ and upper price

$$X_0 \int_{X_0}^{\infty} \frac{1}{x} \mathrm{d}x = \infty.$$

The hedging strategy that guarantees $K_t \geq G(M_t)$ at any time $t \geq 0$ is the strategy which starts with initial capital

$$K_0 = X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x,$$

and for all $s \leq t$ holds $p_s = \mathrm{P}(M_{s-1}, \infty]$ shares, where P is the measure on $[1, \infty)$ such that for all y,

$$F(y) := \frac{G(X_0 y)}{X_0 \int_{X_0}^{\infty} \frac{G(x)}{x^2} \mathrm{d}x} = \int_{[1,y]} u \mathrm{P}(\mathrm{d}u).$$

Chapter 3

Transaction costs

3.1 Introduction

In the previous chapter we considered the one- and two-way trading problems. These are idealized models of trading reality. In practice a trader has to pay a fee for a transaction. In this chapter we will consider the trading problem where a trader must pay transaction costs. As shown in the Google example of subsection 1.3.1 transaction costs can have dramatic effects on the capital of optimal strategies. In this chapter we construct strategies can guarantee a capital for the trader close to that of the optimal strategies of the transaction cost free settting.

The protocol for two-way trading is adjusted to include transaction costs to Protocol 3.1. Note that one-way trading is a special case of this protocol in which only selling is allowed. Because we will consider only strategies that sell shares all results also hold for the one-way trading problem.

Protocol 3.1 Two-way trading with transaction costs

```
X_0 := 1, K_0 := 1, p_0 := 1
for t = 1, 2, ... do
    Trader announces p_t ∈ ℝ
    Market announces X_t ∈ [0, ∞)
    K_t := K_{t−1} + p_t(X_t − X_{t−1})
    if p_t ≠ p_{t−1} then
        K_t = K_t − C(x)
    end if
end for
```

If Trader changes the number of shares ($p_t \neq p_{t-1}$) he wants to hold, he has to pay an amount $C(x)$ for the cost of the transaction where x is the value of the transaction. The value of the transaction is given by $x = |p_{t-1} - p_t|X_{t-1}$. In practice there are many different ways how the transaction costs are calculated. We will consider three different types of transactions costs:

- Proportional transaction costs, $C(x) = \gamma x$ where $\gamma \in (0, 1)$.
- Constant transaction costs, $C(x) = c > 0$.

- Fixed and proportional transaction costs, $C(x) = c + \gamma x$ where $c \geq 0$ and $\gamma \in (0, 1)$.

We will now give a short summary of the results from Chapter 2 that are relevant for this chapter. We derived the following strategy (see Theorem 2.5).

Strategy 1. *At time t hold $p_t = \mathrm{P}(M_{t-1}, \infty)$ shares, where* P *is a measure on* $[1, \infty)$ *of an optimal capital guarantee F.*

When Trader follows this strategy the capital K_t at all times t satisfies

$$K_t = \sum_{s=1}^{t-1} \mathrm{P}(M_{s-1}, M_s] M_s + \mathrm{P}(M_{t-1}, \infty) M_t \geq F(M_t) + \mathrm{P}(M_{t-1}, \infty) M_t,$$

where M_t is the observed maximum price. From Theorem 2.5 we know that F is an optimal capital guarantee if and only if for some measure P on $[1, \infty)$ we have

$$F(y) = \int_{[1,y]} u \mathrm{P}(\mathrm{d}u). \tag{3.1}$$

In two-way trading with transaction costs the capital of Strategy 1 will be obviously lower than in the transaction cost free model. We will see that for proportional transaction costs the above strategy is still optimal, while for constant transactions costs this strategy can result in big losses.

In this chapter we make the assumption that optimal capital guarantees $F : [1, \infty) \to [1, \infty)$ are continuous on the interval $[1, \infty)$. In the previous chapter we gave examples of optimal capital guarantees which are continuous (cf. Section 2.4).

3.2 Proportional transaction costs

If the transactions cost are proportional, the cost per transaction is given by $C(x) = \gamma x$ where $\gamma \in (0, 1)$ and $x = |p_{t-1} - p_t| X_{t-1}$ is the value of the transaction.

Theorem 3.1. *If F an optimal capital guarantee in the transaction cost free model, then $(1 - \gamma)F$ is an optimal capital guarantee in the model with proportional transaction costs given by $C(x) = \gamma x$.*

Proof. Let F be an optimal capital guarantee from the transaction cost free model. Suppose there exists a capital guarantee G in the model with proportional transaction costs that strictly dominates $(1 - \gamma)F$. Then also $G/(1 - \gamma)$ strictly dominates F. Let (p_t) be the sequence of the number of shares Trader holds according to the strategy which has as capital guarantee G. At any time t the share price can become zero and stay zero, hence the amount of cash at time t must be at least $G(M_t)$. The amount of cash satisfies

$$(1 - \gamma) \sum_{s=1}^{t-1} (p_s - p_{s+1}) X_s \geq G(M_t),$$

for any t. Suppose the transaction costs are zero, following the same strategy results in cash

$$\sum_{s=1}^{t-1}(p_s - p_{s+1})X_s \geq \frac{G(M_t)}{1-\gamma},$$

by the previous inequality. Hence in the transaction cost free model this same strategy has capital guarantee $G/(1-\gamma)$. This capital guarantee strictly dominates the optimal capital guarantee F, but this is in contradiction with the assumptions. So there is no capital guarantee dominating $(1-\gamma)F$.

Finally consider the strategy with optimal capital guarantee F (with zero transaction costs). Applying this strategy in the model with transaction costs results in cash reduced by a factor $1-\gamma$, while the share value stays the same. Hence the capital of this strategy is at least $(1-\gamma)F$, which shows $(1-\gamma)F$ is a capital guarantee for the model with transactions costs. We conclude $(1-\gamma)F$ is an optimal capital guarantee. □

3.3 Constant transaction costs

Suppose Trader has to pay a positive constant amount $C(x) = c \geq 0$ for every trade. Following Strategy 1 can result in a very high loss for Trader, as was shown in the Google example in the introduction. Suppose M_t increases very slowly, then at time t the value of the trade $\mathrm{P}(M_{t-1}, M_t]M_t$ may be lower than the cost c of the trade. To solve this Trader should use a strategy that trades more economically. In this section we will define such a strategy for which we show it guarantees a capital not much lower than the capital guarantee of Strategy 1.

3.3.1 Continuous price process

The protocol for trading is defined in *discrete time*. In the following we will consider trading in *continous time*, which allows us to consider *continuous* price processes. As it turns out continuous price processes are more convenient for deriving capital guarantees. Later on we will translate the results back to the discrete time setting. We will denote a continuous price process by Y and its maximum by $N_t = \sup_{s\leq t} Y_s$.

As noted before constant transaction costs can lower the capital guarantee significantly. This is a consequence of selling a too small amount of shares too often. To solve this we will adjust Strategy 1 such that there will be fewer transactions.

To this end we define a sequence of increasing price levels $(\nu_i)_{i\geq 0}$ with $\nu_i > \nu_{i-1} \geq 1$ for all $i \geq 1$ and $\nu_0 = 1$. The new strategy will only sell at these price levels (ν_i), see Figure 3.1. The resulting capital K_t at time t is

$$K_t = \sum_{i=1}^{n} \mathrm{P}(\nu_{i-1}, \nu_i]\nu_i + \mathrm{P}(\nu_n, \infty)\nu_n - nc$$

where n is the number of price levels below N_t excluding ν_0. Note that we assume the price function Y is continuous and therefore also its maximum N_t

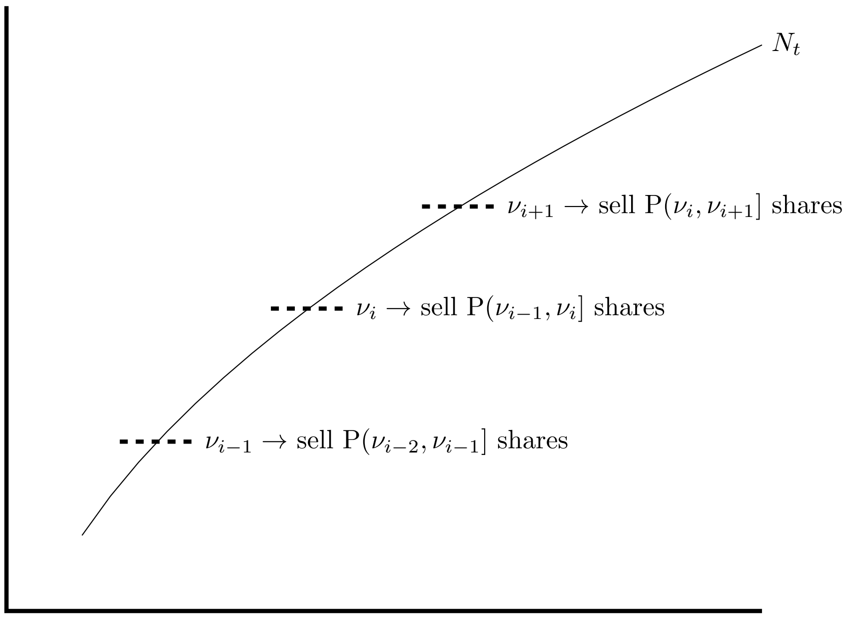

Figure 3.1: Sell shares at sequence of price levels (ν_i).

is continuous, hence every price level below N_t in the sequence (ν_i) is attained before time t .

By not selling at every price level but at a sequence of price levels (ν_i), the capital guarantee is reduced by another quantity in addition to the transaction costs. Suppose at time t the last selling price was ν_i with $\nu_i < N_t$, then Strategy 1 sold $\mathrm{P}(\nu_i, N_t]$ shares but the strategy with the sequence of price levels did not sell shares for those prices. Compared to Strategy 1 the capital is reduced by selling too late because $\nu_{i+1} > N_t$, this extra loss is called the slippage cost[1].

If the difference between selling prices is very small, the difference between the last trade and the current price level is small. Hence the slippage cost is small, but on the other hand there are a lot of transactions so the total transaction costs are large. Hence there is a trade-off between the total transaction costs and the slippage cost. This trade-off will be made more precise later.

To control the value of this slippage cost we define the following strategy. This strategy has as input a sequence of positive real numbers (p_i), where p_i is determines the value of the ith transaction. Suppose the selling prices $\nu_0, \ldots, \nu_i$ are determined, then we will define the next selling price ν_{i+1} such that the value of the $(i+1)$st transaction equals p_{i+1}:

$$\nu_{i+1}\mathrm{P}[\nu_i, \nu_{i+1}) = p_{i+1}.$$

To derive a capital guarantee it turns out is easier to approximate the left-hand

[1]In finance slippage cost is the difference between the expected price of a trade and the price of the actual trade. In our case we would like to trade at the current price N_t, but we trade only at the last price level ν_i

side by

$$\int_{(\nu_i,\nu_{i+1}]} u\mathrm{P}(\mathrm{d}u) \approx \nu_{i+1}\mathrm{P}[\nu_i,\nu_{i+1}),$$

this is a good approximation if the selling prices are close together. This approximation results in the following strategy.

Strategy 2. *Let* $(p_i)_{i\geq 1}$ *be a sequence of positive real numbers. Suppose the selling prices* $\nu_0,\ldots,\nu_i$ *are determined, let the next selling price* ν_{i+1} *be determined by solving*

$$\int_{(\nu_i,\nu_{i+1}]} u\mathrm{P}(\mathrm{d}u) = p_{i+1}, \tag{3.2}$$

and sell a fraction $\mathrm{P}(\nu_i,\nu_{i+1}]$ *for price* ν_{i+1}.

We can rewrite the left side of equation (3.2) by using the expression for F in equation (3.1) into

$$\int_{(\nu_i,\nu_{i+1}]} u\mathrm{P}(\mathrm{d}u) = F(\nu_{i+1}) - F(\nu_i). \tag{3.3}$$

We assumed the capital guarantees F are continuous and the price function N is continuous, therefore there exists a price level ν_{i+1} which solves equation (3.2).

The following theorem gives a capital guarantee for Strategy 2.

Theorem 3.2. *Let* n *be the number of transactions done by Strategy 2 before maximum price* N_t*, then the resulting capital* K_t *of Strategy 2 at any time* t *satisfies*

$$K_t \geq F(N_t) - \underbrace{\int_{(\nu_n,N_t]} u\mathrm{P}(\mathrm{d}u)}_{\text{slippage cost}} - \overbrace{nc}^{\text{transaction costs}}. \tag{3.4}$$

Proof.

$$\begin{aligned}
K_t &= \sum_{i=1}^{n} \nu_i \mathrm{P}(\nu_{i-1},\nu_i] + \mathrm{P}(\nu_n,\infty)\nu_n - nc \\
&\geq \sum_{i=1}^{n} \nu_i \mathrm{P}(\nu_{i-1},\nu_i] - nc \\
&= \sum_{i=1}^{n} \nu_i \int_{(\nu_{i-1},\nu_i]} \mathrm{P}(\mathrm{d}u) - nc \\
&\geq \sum_{i=1}^{n} \int_{(\nu_{i-1},\nu_i]} u\mathrm{P}(\mathrm{d}u) - nc \\
&= \int_{[1,\nu_n]} u\mathrm{P}(\mathrm{d}u) - nc \\
&= \int_{[1,N_t]} u\mathrm{P}(\mathrm{d}u) - \int_{(\nu_n,N_t]} u\mathrm{P}(\mathrm{d}u) - nc \\
&= F(N_t) - \int_{(\nu_n,N_t]} u\mathrm{P}(\mathrm{d}u) - nc.
\end{aligned}$$

□

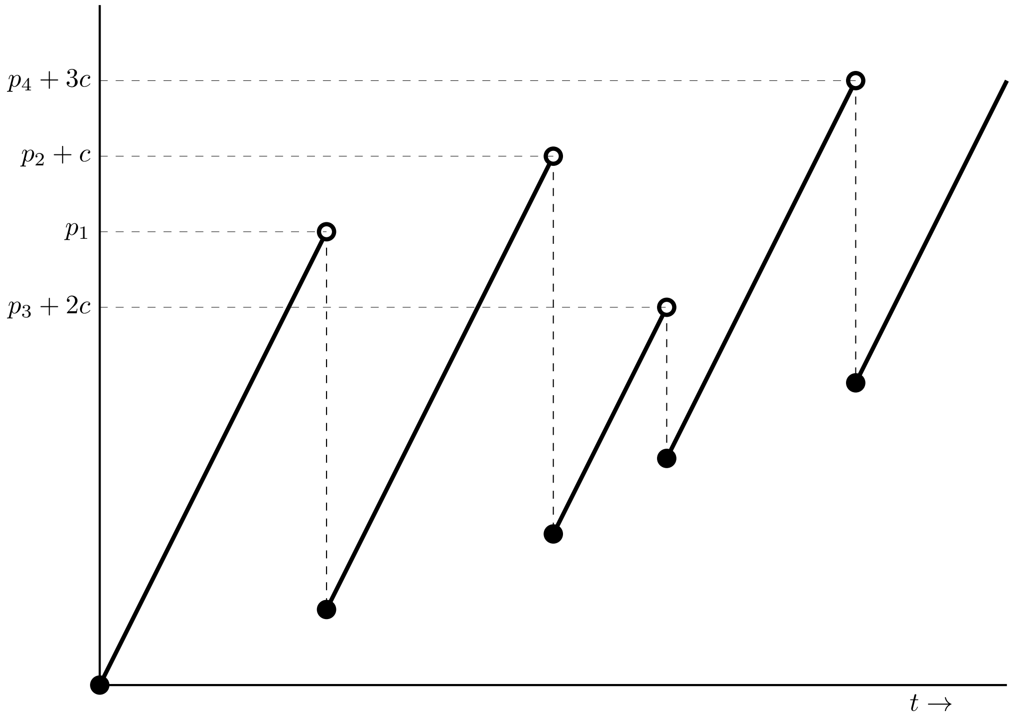

Figure 3.2: A typical loss function.

3.3.2 Optimal guarantee in hindsight

Theorem 3.2 gives a capital guarantee for Strategy 2, in this section we find the sequence (p_i) for which the guarantee would be maximized at time t. For this we define the loss function

$$L_t := \int_{(\nu_n, N_t]} u\mathrm{P}(\mathrm{d}u) + nc.$$

Because $F(N_t)$ is a constant at time t, maximizing the guarantee is the same as minimizing the loss function L. Figure 3.2 displays a typical loss function given a sequence (p_i). The loss function has a certain saw-tooth behaviour: as the maximum price N_t increases, the slippage cost $\int_{(\nu_n, N_t]} u\mathrm{P}(\mathrm{d}u)$ increases until N_t reaches the next selling price ν_{n+1}, then the slippage cost becomes zero while the transaction costs increase by c.

The following theorem shows which sequence of (p_i) results in the lowest loss, assuming the number of transactions n is fixed for the time being. In Theorem 3.4 we will optimize the capital guarantee with respect to n.

Theorem 3.3. *Using $p_i = p_1 - (i-1)c$ such that $\sum_{i=1}^n p_i = F(N_t)$ in Strategy 2 results in the lowest loss among all possible sequences (p_i) at time t.*

Proof. In the worst case for Trader after choosing (p_i), N_t equals the value for which the loss attains its maximum. Setting $p_i = p_1 - (i-1)c$ such that $\sum_{i=1}^n p_i = F(N_t)$ gives local maxima of the loss function which are all equal to p_1, hence the maximum is equal to p_1.

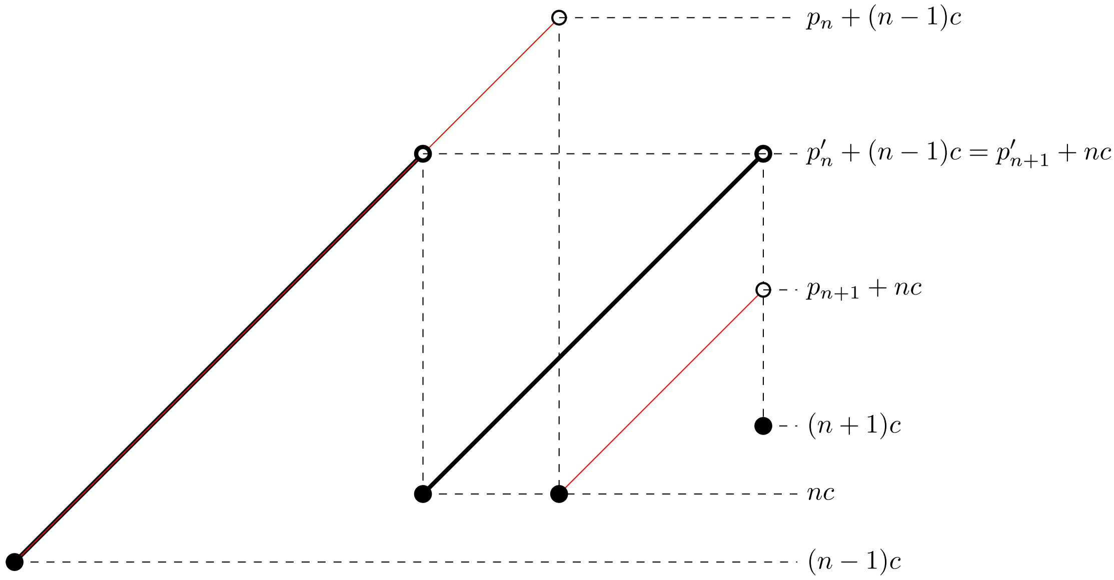

Figure 3.3: Selling later.

Suppose there exists a sequence (p_i) with a lower worst-case loss and the loss function has local maxima which are not all equal. Consider a maximum $p_n+(n-1)c$ and assume next to it is a local maximum $p_{n+1}+nc$ which is lower, so $p_n > p_{n+1} + c$. Such a local maximum can also lie on the left side, in that case the following analysis is similar. The situation is shown in Figure 3.3.

In this situation the strategy sold shares twice. Consider selling the first time such that we have equal maxima $p'_n + (n-1)c = p'_{n+1} + nc$, note that the second time we sell for the same price. The first maximum lies lower, hence $p'_n + (n-1)c = p'_{n+1} + nc < p_n + (n-1)c$. If $p'_n + (n-1)c$ was the only maximum of the loss function, the maximum can be lowered and hence a lower loss is possible. If there are more maxima the procedure can be applied for all such maxima, which will lead ultimately to a lower maximum and a lower worst-case loss. But this is in contradiction with the assumption, we conclude that all maxima must be equal. □

The number of transactions n depends on the value of p_1, the following theorem optimizes the guarantee with respect to p_1.

Theorem 3.4. *The capital guarantee of Strategy 2 with $p_i = p_1 - (i-1)c$ such that $\sum_{i=1}^n p_i = F(N_t)$ is maximized by $p_1 = \sqrt{2cF(N_t)} + \frac{1}{2}c$ and guarantees a capital*

$$K_t \geq F(N_t) - \sqrt{2cF(N_t)} + \frac{1}{2}c.$$

Proof. We have

$$F(N_t) = \sum_{i=1}^n p_i = \sum_{i=1}^n (p_1 - (i-1)c) = np_1 - \frac{1}{2}n(n-1)c,$$

or

$$p_1 = \frac{F(N_t)}{n} + \frac{1}{2}nc - \frac{1}{2}c.$$

By Theorem 3.2 the capital K_t of Strategy 2 satisfies

$$\begin{aligned} K_t &\geq F(N_t) - \int_{(\nu_n, N_t]} u\mathrm{P}(\mathrm{d}u) - nc \\ &\geq F(N_t) - \int_{(\nu_n, \nu_{n+1}]} u\mathrm{P}(\mathrm{d}u) - nc \\ &= F(N_t) - p_{n+1} - nc \\ &= F(N_t) - p_1 \\ &= F(N_t) - \frac{F(N_t)}{n} - \frac{1}{2}nc + \frac{1}{2}c. \end{aligned} \tag{3.5}$$

The guarantee is maximized by $n = \sqrt{2F(N_t)/c}$ with maximum

$$F(N_t) - \sqrt{2cF(N_t)} + \frac{1}{2}c.$$

□

We showed the lowest worst-case loss is obtained for Strategy 2 by $p_i = p_1 - (i-1)c$ such that $\sum_{i=1}^n p_i = F(N_t)$. The lowest possible loss for Strategy 2 is $\sqrt{2cF(N_t)} - c/2$. To achieve this loss $F(N_t)$ must be known beforehand, hence this is not a practical strategy. However, in the following section we will show that it is possible to come very close to this ideal loss.

3.3.3 A practical strategy

In Theorem 3.2 we gave a capital guarantee which can be further bounded like in the proof of Theorem 3.4 in (3.5) by

$$K_t = F(N_t) - p_{n+1} - nc.$$

As discussed before there is a trade-off between the slippage cost and the transaction costs, here we bounded the slippage costs by p_{n+1}. If we want a small number of transactions, to reduce the total transaction costs nc, the transaction value sequence (p_i) must be increasing quickly. But this leads to a high value of p_{n+1}. If we on the other hand let (p_i) be slowly increasing such that p_{n+1} is small, the number of transactions is high and therefore also nc.

In the proof of Theorem 3.4 the optimal number of transactions is $n = \sqrt{2F(N_t)/c}$, by using $p_i = ci$ we get the same number of transactions without prior knowledge of $F(N_t)$. The following theorem shows to which capital guarantee this leads.

Theorem 3.5. *Let (p_i) be given by $p_i = ci$ in Strategy 2. The resulting capital K_t at any time t satisfies*

$$K_t \geq F(N_t) - \sqrt{8cF(N_t) + c^2}. \tag{3.6}$$

Proof. Let n be the number of transactions done by Strategy 2 before N_t, then

$$\begin{aligned} F(N_t) &= \int_{[1,N_t]} u\mathrm{P}(\mathrm{d}u) \\ &\geq \int_{[1,\nu_n]} u\mathrm{P}(\mathrm{d}u) \\ &= \sum_{i=1}^{n} \int_{(\nu_{i-1},\nu_i]} u\mathrm{P}(\mathrm{d}u) \\ &= \sum_{i=1}^{n} p_i = c\sum_{i=1}^{n} i = \frac{1}{2}cn(n+1), \end{aligned}$$

from which we get a upper bound on the number of transactions

$$n \leq \sqrt{\frac{2F(N_t)}{c} + \frac{1}{4}} - \frac{1}{2}.$$

Then by Theorem 3.2

$$\begin{aligned} K_t &\geq F(N_t) - p_{n+1} - nc \\ &= F(N_t) - c(n+1) - nc \\ &= F(N_t) - 2nc - c \\ &\geq F(N_t) - 2c\left(\sqrt{\frac{2F(N_t)}{c} + \frac{1}{4}} - \frac{1}{2}\right) - c \\ &= F(N_t) - \sqrt{8cF(N_t) + c^2}. \end{aligned}$$

□

By using $p_i = ci$ in Strategy 2 we get a loss of

$$\sqrt{8cF(N_t) + c^2} \approx 2\sqrt{2cF(N_t)}.$$

The best possible loss in hindsight for Strategy 2 (cf. Theorem 3.4) is

$$\sqrt{2cF(N_t)} - c/2 \approx \sqrt{2cF(N_t)}.$$

These two approximations hold for large $F(N_t)$. We conclude that $p_i = ci$ leads only to an extra factor of approximately two, compared to the best choice of (p_i) in hindsight.

3.3.4 Selling prices

Strategy 2 determines the selling prices by solving recursively equation (3.2). From this equation it is possible to explicitly determine the selling prices. Equation (3.2) can be written as

$$\int_{(\nu_i,\nu_{i+1}]} u\mathrm{P}(\mathrm{d}u) = F(\nu_{i+1}) - F(\nu_i) = p_{i+1},$$

by rearranging

$$\begin{aligned} F(\nu_{i+1}) &= F(\nu_i) + p_{i+1} \\ &= F(\nu_{i-1}) + p_{i+1} + p_i \\ &= \cdots = F(\nu_0) + \sum_{j=1}^{i+1} p_j. \end{aligned}$$

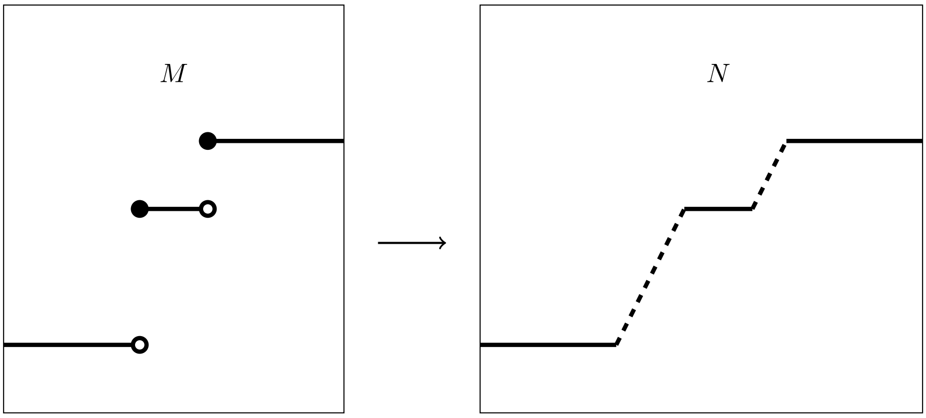

Figure 3.4: From the discontinuous price process M we construct a continous price process N by pasting linear interpolations at the discontinuities.

Because $F : [1, \infty) \to [0, \infty)$ is assumed to be a continuous and increasing function its inverse F^{-1} exists on $[0, \infty)$, so the selling prices are given by

$$\nu_{i+1} = F^{-1}\left(F(\nu_0) + \sum_{j=1}^{i+1} p_j\right).$$

For the optimal capital guarantee $F(y) = \beta y^{1-\beta}$ with $\beta \in (0, 1)$ and $p_i = \alpha i$ one obtains

$$\nu_i = \left[\frac{1}{\beta}\left(F(1) + \sum_{j=1}^{i} p_i\right)\right]^{\frac{1}{1-\beta}} = \left[1 + \frac{\alpha}{\beta}\sum_{j=1}^{i} j\right]^{\frac{1}{1-\beta}} = \left[1 + \frac{\alpha}{2\beta} i(i+1)\right]^{\frac{1}{1-\beta}}.$$

3.3.5 Discontinuous price process

In the previous section we derived capital guarantees for the continuous price process Y with maximum price N, which is also continuous. In this section we consider a price process $X : [0, \infty) \to [0, \infty)$ which may contain a finite number of discontinuities, as we will later embed this price process in discrete time. The maximum price process M is defined by $M_t = \sup_{s \leq t} X_t$ and can also contain a finite number of discontinuities.

From the price process M we construct a continuous price process N by pasting a linearly interpolated part between discontinuities, see Figure 3.4. By pasting parts between discontinuities the time scale of N changes but this does not change the analysis. In the following we will refer to the time scale of the original price process M and will assume for every time t of M that $N_t = M_t$.

We can apply Strategy 2 to the price process N. The resulting sequence of selling prices of this strategy is used in the following strategy for price process M. The strategy will sell at a sequence of prices denoted by $(\mu_i)_{i \geq 0}$ with $\mu_{i+1} > \mu_i$ for all $i \geq 0$ and $\mu_0 = 1$.

Strategy 3. *Let $(p_i)_{i \geq 1}$ be a sequence of real numbers. Let (ν_n) be the continuous selling prices determined using Strategy 2 using price process N and*

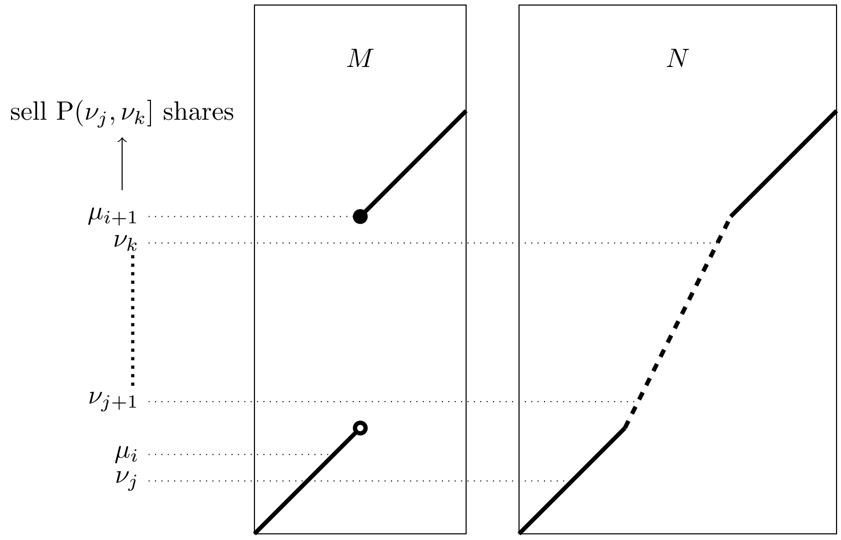

Figure 3.5: The next selling level μ_{i+1} is defined as the lowest price after the discontinuity of N. At this price $\mathrm{P}(\nu_j, \nu_k]$ shares are sold.

sequence (p_i)*. Suppose* $\mu_0, \ldots, \mu_i$ *are determined. Let* ν_j *be the highest continuous selling price such that* $\nu_j \leq \mu_i$*. Let* $\mu_{i+1} = \inf\{M_t : M_t \geq \nu_{j+1}\}$ *and let* ν_k *be the highest continuous selling price such that* $\nu_k \leq \mu_{i+1}$*. Sell* $\mathrm{P}(\nu_j, \nu_k]$ *shares at price* μ_{i+1}*.*

Figure 3.5 shows how the next selling price μ_{i+1} is determined in Strategy 3. We have the following result for Strategy 3.

Theorem 3.6. *Let* (L_t) *be the capital sequence of Strategy 2 on the continuous maximum price process* N *(constructed using* M*) and let* (K_t) *be the capital sequence of Strategy 3 on the piecewise-constant maximum price process* M*. Then at any time* $t \geq 0$*,*

$$K_t \geq L_t \geq F(M_t) - \sqrt{8cF(M_t) + c^2}. \tag{3.7}$$

Proof. We will proof this by induction on the selling levels (μ_i) constructed by Strategy 3 on M. Let $\mu_0, \ldots, \mu_i$ be the selling levels determined and let K_s be the capital at the time μ_i was determined. Let L_s be the capital at the same time of Strategy 2, for the inductive step assume $K_s \geq L_s$. Let (ν_i) be the sequence of selling prices determined by Strategy 2. Let μ_{i+1} be the next selling level determined by Strategy 3 and let ν_j be Strategy 2's highest selling price such that $\nu_j \leq \mu_i$ and let ν_k be Strategy 2's highest selling price such that $\nu_k \leq \mu_{i+1}$. Then at the time when $M_t = N_t = \mu_{i+1}$ the capital L_t of Strategy

2 satisfies

$$\begin{aligned}
L_t &= L_s + \sum_{l=j+1}^{k} \nu_l \mathrm{P}(\nu_{l-1}, \nu_l] \\
&\leq L_s + \sum_{l=j+1}^{k} \nu_k \mathrm{P}(\nu_{l-1}, \nu_l] \\
&= L_s + \nu_k \mathrm{P}(\nu_j, \nu_k] \\
&\leq K_s + \mu_{i+1} \mathrm{P}(\nu_j, \nu_k] \\
&= K_t.
\end{aligned}$$

Hence the K_t capital of Strategy 3 is higher than the capital L_t of Strategy 2 at the next selling price μ_{i+1}. By choosing the sequence (p_i) like in Theorem 3.5, i.e. $p_i = ci$, the right hand side of equation (3.7) is obtained. Note that $F(M_t) = F(N_t)$ for every time t. □

Remark 3.7. Strategy 3 applied to a continuous price function results in precisely the same trades as if Strategy 2 is applied. If this continuous price function had some discontinuities, by Theorem 3.6 a higher capital guarantee could be given than for the continuous price function. Hence Theorem 3.6 states that for Trader it is better that the price functions contains jumps than that the price function is continuous.

3.3.6 Discrete time

The trading protocol is defined in discrete time, but it can be embedded in continuous time by a piecewise constant price process. Let $X : \mathbb{N} \to [0, \infty]$ be the price process in discrete time, then it can be embedded by a price function in continuous time $X' : [0, \infty) \to [0, \infty)$ by defining $X'_t = X_{\lfloor t \rfloor}$. Note that at time t the price process X' has a finite number of discontinuities. Strategy 3 can be applied to the price process X'. Since this strategy only trades when the price process changes it will only trade at times $t = 1, 2, \ldots$, hence the strategy can also be applied in discrete time. Therefore we can apply Theorem 3.6 to the trading protocol (which is defined in discrete time), which shows Strategy 3 has a capital sequence (K_t) such that

$$K_t \geq F(M_t) - \sqrt{8cF(M_t) + c^2}. \tag{3.8}$$

3.3.7 Optimal capital guarantee

We did not show that Strategy 3 gives an optimal capital guarantee for the trading protocol with transaction costs. We can however give bounds for such an optimal capital guarantee. Let F^* be an optimal capital guarantee for the trading protocol with transaction costs, then there exists an optimal capital guarantee F in the transaction cost free protocol such that for all t

$$F(M_t) \geq F^*(M_t) \geq F(M_t) - \sqrt{8cF(M_t) + c^2}.$$

This can be reasoned as follows. Obviously there exists such an F which dominates F^*, as the presence of transaction costs can only lower the guarantee. By (3.8) we know that $F'(M_t) := F(M_t) - \sqrt{8cF(M_t) + c^2}$ is a capital guarantee. Because F^* is an optimal capital guarantee, it dominates F'.

3.4 Mixed transaction costs

The results for proportional and fixed transaction costs can be combined to get a capital guarantee for transaction costs of the form $C(x) = c + \gamma x$ where $c > 0$, $\gamma \in (0, 1)$ and x is the value of the transaction.

Theorem 3.8. *If Trader follows Strategy 3 with $p_n = \frac{c}{1-\gamma} n$ then at any time $t \geq 0$ his capital K_t satisfies*

$$K_t \geq (1-\gamma)F(M_t) - \sqrt{8F(M_t)(1-\gamma)c + c^2}.$$

The precise proof of this theorem can be found in the Appendix. We can also obtain this result in an informal way by reasoning as follows. Suppose first $c = 0$, then we have by Theorem 3.1 for every t

$$K_t \geq (1-\gamma)F(M_t).$$

Define $F'(M_t) = (1-\gamma)F(M_t)$, suppose $c > 0$, and use Strategy 2 then we have similarly to Theorem 3.2 after n transactions

$$K_t \geq F'(M_t) - (1-\gamma)p_{n+1} - nc.$$

Because we can guarantee at most $F'(M_t)$ by the proportional costs, we have to pay nc constant costs and in the worst case lose $(1-\gamma)p_{n+1}$. To minimize the loss we equate the last two terms, so $p_n = \frac{c}{1-\gamma} n$. Analoguously to the case without proportional costs this leads to

$$\begin{aligned} K_t &\geq F'(M_t) - \sqrt{8F'(M_t)c + c^2} \\ &= (1-\gamma)F(M_t) - \sqrt{8F(M_t)(1-\gamma)c + c^2}. \end{aligned}$$

3.5 Lookback options with transaction costs

In the previous chapter we reasoned that an upper price for the lookback option with return $G(M_t)$ when exercised at time t, is the initial capital K_0 needed for a strategy such that the capital sequence (K_t) satisfies $K_t \geq G(M_t)$ for all $t \geq 0$. This of course still holds for the two-way trading problem with transaction costs.

The following theorem gives an upper bound for the price of a lookback option in the two-way trading problem with mixed transaction costs given by $C(x) = c + \gamma x$ where $c > 0$ and $\gamma \in (0, 1)$.

Theorem 3.9. *Let $G : [0, \infty) \to [0, \infty)$ be an increasing function and $X_0 > 0$. An upper bound for the price of an American lookback option that pays $G(M_t)$ when exercised at time t is the value of α which solves*

$$\frac{X_0}{1-\gamma} \int_{X_0}^{\infty} \left[\frac{G(x)}{\alpha} + 4c + \sqrt{17c^2 + \frac{8cG(x)}{\alpha}} \right] \frac{1}{x^2} \mathrm{d}x = 1 \tag{3.9}$$

Proof. Consider the normalized price $Y_t = X_t/X_0$, its maximum price $N_t = \sup_{s \leq t} Y_t$ and

$$F(y) := \frac{1}{1-\gamma} \left[\frac{G(X_0 y)}{\alpha} + 4c + \sqrt{17c^2 + \frac{8cG(X_0 y)}{\alpha}} \right].$$

The function F is an optimal capital guarantee, because $\int_1^\infty F(y)/y^2 \mathrm{d}y = 1$ by Equation 3.9. Consider the equation

$$(1-\gamma)f - \sqrt{8c(1-\gamma)f + c^2} = \frac{g}{\alpha}, \tag{3.10}$$

we will solve f in this equation. Substitute $h := \sqrt{8c(1-\gamma)f + c^2}$ for which

$$f = \frac{h^2 - c^2}{8c(1-\gamma)},$$

holds. Then we get

$$(1-\gamma)f - \sqrt{8c(1-\gamma)f + c^2} = (1-\gamma)\frac{h^2 - c^2}{8c(1-\gamma)} - h = \frac{g}{\alpha},$$

or

$$\frac{h^2}{8c} - h - \left(\frac{c}{8} + \frac{g}{\alpha}\right) = 0.$$

This quadratic equation has as only positive solution

$$h = 4c + \sqrt{17c^2 + \frac{8gc}{\alpha}}.$$

Substituting this back into f gives

$$\begin{aligned} f &= \frac{\left(4c + \sqrt{17c^2 + \frac{8gc}{\alpha}}\right)^2 - c^2}{8c(1-\gamma)} \\ &= \frac{16c^2 + 8c\sqrt{17c^2 + \frac{8gc}{\alpha}} + 17c^2 + \frac{8gc}{\alpha} - c^2}{8c(1-\gamma)} \\ &= \frac{1}{1-\gamma}\left[\frac{g}{\alpha} + 4c + \sqrt{17c^2 + \frac{8cg}{\alpha}}\right]. \end{aligned}$$

By Theorem 3.8 there exists a strategy with a capital process (K_t) such that for any time t,

$$K_t \geq (1-\gamma)F(N_t) - \sqrt{8c(1-\gamma)F(N_t) + c^2} = \frac{G(M_t)}{\alpha}$$

The right-hand side follows by taking $f = F(M_t)$ in Equation 3.10. When Trader applies this strategy with initial capital α he will have $K_t \geq G(M_t)$ for every t, so this is an upper bound for the price of the option. □

By multiplying equation (3.9) with α and some rearranging we obtain

$$\alpha = \underbrace{\frac{1}{1-\gamma}}_{\geq 1} \underbrace{X_0 \int_{X_0}^\infty \frac{G(x)}{x^2} \mathrm{d}x}_{\text{cost-free upper price}} + \underbrace{\frac{X_0}{1-\gamma} \int_{X_0}^\infty \left[4c + \sqrt{17c^2 + \frac{8cG(x)}{\alpha}}\right] \frac{1}{x^2} \mathrm{d}x}_{>0}.$$

Hence the upper price of the the lookback option it at least the price of the option in the transaction cost free trading model, which was to be expected.

The hedging strategy which guarantees $K_t \geq G(M_t)$ at any time $t \geq 0$ is the strategy which starts with initial capital $K_0 = \alpha$ which solves equation (3.9) and for all $s \leq t$ holds $p_s = \mathrm{P}(M_{s-1}, \infty]$ shares, where P is the measure on $[1, \infty)$ such that for all y,

$$F(y) := \frac{1}{1-\gamma}\left[\frac{G(X_0 y)}{\alpha} + 4c + \sqrt{17c^2 + \frac{8cG(X_0 y)}{\alpha}}\right] = \int_{[1,y]} u\mathrm{P}(\mathrm{d}u).$$

Chapter 4

Conclusion

4.1 Summary

For the one-way trading problem Vovk et al. showed in [3] it is possible to give non-trivial guarantees for the capital a trader holds. They showed there are strategies for which the capital K_t at any time t satisfies

$$K_t \geq F(M_t),$$

where F is an increasing function of the maximum price $M_t = \sup_{s \leq t} X_t$. They found that such capital guarantees F are optimal if and only if they satisfy

$$\int_1^\infty \frac{F(y)}{y^2} \mathrm{d}y.$$

In this thesis we considered the trading problem with transaction costs, which is more realistic. First we considered proportional transaction costs $C(x) = \gamma x$, for which we showed the optimal strategies still give optimal capital guarantees, but they are scaled down by a factor $1 - \gamma$.

For constant transaction costs the optimal strategies cannot guarantee a similar capital anymore. We modified the strategies in such a way that they trade more economically. The new strategies trade when the transaction value reached a certain value: trade i is exercised if the trade value equals p_i. We showed a good choice was $p_i = ci$ as it results in only a factor two in loss (the diffence in the capital guarantee with or without transactionc costs) compared to the best (p_i) in hindsight. If a strategy had as capital guarantee F in the transaction cost free model, then this modified strategy has a capital K_t which satisfies

$$K_t \geq F(M_t) - \sqrt{8cF(M_t) + c^2}.$$

4.2 Future work

Vovk et al. characterized the optimal capital guarantees for the transaction free trading problem. The question is whether it is possible to find optimal capital guarantees in the model *with transaction costs*. And furthermore whether it is possible to characterize these optimal capital guarantees.

We did not find such optimal capital guarantees, but we found bounds for them. Suppose F^* is an optimal capital guarantee in the model with transaction costs, we showed that there exists an optimal capital guarantee F in the transaction cost free model such that for all $y \geq 1$,

$$F(y) \geq F^*(y) \geq F(y) - \sqrt{8cF(y) + c^2}.$$

It would also be interesting to see if a similar approach for handling transaction costs is possible for the strategies of Koolen and De Rooij [5] for the two-way trading problem.

We considered proportional and fixed transaction costs and a mixed form of both. Other types of transaction costs could also be studied. For example transaction costs which are proportional to the number of shares.

Appendix A

Some proofs

A.1 Examples of optimal capital guarantees

The following functions F are optimal capital guarantees:

1. *For $u \geq 1$*

$$F(y) = u\mathbf{1}_{\{y \geq u\}}.$$

2. *For $\alpha \in (0, 1)$*

$$F(y) = \alpha y^{1-\alpha}.$$

3. *For $\alpha > 0$*

$$F(y) = \alpha(1+\alpha)^\alpha \frac{y}{\log^{1+\alpha} y} \mathbf{1}_{\{y \geq e^{1+\alpha}\}}.$$

With measures P defined by respectively:

1. $\mathrm{P}\{u\} = 1.$
2. $\mathrm{P}\{1\} = \alpha$ *and for $y \geq 1$*

$$\mathrm{P}(y, \infty) = (1-\alpha)y^{-\alpha}.$$

3. $\mathrm{P}\{e^{1+\alpha}\} = \alpha/(1+\alpha)$ *and for $y \geq e^{1+\alpha}$*

$$\mathrm{P}(y, \infty) = (1+\alpha)^\alpha \log^{-\alpha} y - \alpha(1+\alpha)^\alpha \log^{-(1+\alpha)} y.$$

Proof. Using the third statement of Theorem 2.4 we will check these are indeed optimal capital guarantees. For this we have to check the condition

$$\int_1^\infty \frac{F(y)}{y^2} \mathrm{d}y = 1.$$

For $u \geq 1$ we have

$$\int_1^\infty \frac{u\mathbf{1}_{\{y \geq u\}}}{y^2} \mathrm{d}y = u \int_u^\infty \frac{1}{y^2} \mathrm{d}y = u\frac{1}{u} = 1.$$

for $\alpha \in (0,1)$

$$\int_1^\infty \frac{\alpha y^{1-\alpha}}{y^2}\mathrm{d}y = \alpha \int_1^\infty y^{-1-\alpha}\mathrm{d}y = \alpha \left[-\frac{y^{-\alpha}}{\alpha} \right]_1^\infty = 1.$$

Finally for $\alpha > 0$ consider

$$\begin{aligned}
&\int_1^\infty \frac{\alpha(1+\alpha)^\alpha y \log^{-(1+\alpha)}(y)\mathbf{1}_{\{y \geq e^{1+\alpha}\}}}{y^2}\mathrm{d}y \\
&= \alpha(1+\alpha)^\alpha \int_{e^{1+\alpha}}^\infty \left(\frac{1}{y \log^{1+\alpha} y} \right) \mathrm{d}y \\
&= \alpha(1+\alpha)^\alpha \left[-\frac{1}{\alpha \log^\alpha y} \right]_{e^{1+\alpha}}^\infty \\
&= \alpha(1+\alpha)^\alpha \frac{1}{\alpha(1+\alpha)^\alpha} = 1.
\end{aligned}$$

It is clear the first two examples ($F(y) = u\mathbf{1}_{\{y \geq u\}}$ and $F(y) = \alpha y^{1-\alpha}$) are right-continous and increasing, from which we conclude they are optimal capital guarantees. The last example is also right-continuous, to show it is increasing on $[e^{1+\alpha}, \infty)$ it is sufficient to show that its derivative is nonnegative on $[e^{1+\alpha}, \infty)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}y} \left[\alpha(1+\alpha)^\alpha y \log^{-(1+\alpha)} y \right] \\
&= \alpha(1+\alpha)^\alpha \left[\log^{-(1+\alpha)} y - (1+\alpha) \log^{-(2+\alpha)} y \right] \\
&= \alpha(1+\alpha)^\alpha \left(1 - \frac{1+\alpha}{\log y} \right) \log^{-(1+\alpha)} y \geq 0,
\end{aligned}$$

for all $y \in [e^{1+\alpha}, \infty)$. Hence $F(y) = \alpha(1+\alpha)^\alpha y \log^{-(1+\alpha)}(y)\mathbf{1}_{\{y \geq e^{1+\alpha}\}}$ is also an optimal capital guarantee.

How to derive a strategy from a given optimal capital guarantee was explained in the previous section. At time t Trader should sell a fraction $\mathrm{P}[M_{t-1}, M_t)$ of his shares, this measure P is uniquely defined by the optimal capital guarantee F. For the examples given in Table 2.1 we will derive the corresponding measure P. From the proof of Theorem 2.4 we have the following relations, from which it is possible to obtain the measure P. Define the measure Q by $\mathrm{Q}[1, y] := F(y)$ for all $y \in [1, \infty)$. The measure P is then defined for every measurable set A by

$$\mathrm{P}(A) = \int_A \frac{1}{u} \mathrm{Q}(\mathrm{d}u).$$

For the guarantee $F(y) = u\mathbf{1}_{\{y \geq u\}}$ we have the point measure $\mathrm{P}\{u\} = 1$.

The guarantee $F(y) = \alpha y^{1-\alpha}$ has as corresponding measure for all $y \in [1, \infty)$

$$\begin{aligned}
\mathrm{P}[1, y) &= \int_{[1,y)} \frac{1}{u} \mathrm{Q}(\mathrm{d}u) \\
&= \int_1^y \frac{1}{u} \frac{\mathrm{d}}{\mathrm{d}u} F(u) \mathrm{d}u \\
&= \int_1^y \frac{1}{u} \alpha(1-\alpha) u^{-\alpha} \mathrm{d}u \\
&= \alpha(1-\alpha) \int_1^y u^{-(1+\alpha)} \mathrm{d}u \\
&= \alpha(1-\alpha) \left[-\frac{u^{-\alpha}}{\alpha} \right]_1^y \\
&= (1-\alpha)(1-y^{-\alpha}).
\end{aligned}$$

The guarantee $F(y) = \alpha(1+\alpha)^\alpha y \log^{-(1+\alpha)}(y) \mathbf{1}_{\{y \geq e^{1+\alpha}\}}$ has a point measure at $e^{1+\alpha}$

$$\mathrm{P}\{e^{1+\alpha}\} = \int_{\{e^{1+\alpha}\}} \frac{1}{u} \mathrm{Q}(\mathrm{d}u) = \frac{\mathrm{Q}\{e^{1+\alpha}\}}{e^{1+\alpha}} = \frac{\alpha}{1+\alpha}.$$

For $y \in (e^{1+\alpha}, \infty)$ we have

$$\begin{aligned}
\mathrm{P}[1, y) &= \int_{[e^{1+\alpha},y)} \frac{1}{u} \mathrm{Q}(\mathrm{d}u) \\
&= \int_{e^{1+\alpha}}^y \frac{1}{u} \frac{\mathrm{d}}{\mathrm{d}u} F(u) \mathrm{d}u \\
&= \int_{e^{1+\alpha}}^y \frac{1}{u} \alpha(1+\alpha)^\alpha \left(\log^{-(1+\alpha)} u - (1+\alpha) \log^{-(2+\alpha)} u \right) \mathrm{d}u \\
&= \alpha(1+\alpha)^\alpha \int_{e^{1+\alpha}}^y \frac{1}{u \log^{1+\alpha} u} \mathrm{d}u - \alpha(1+\alpha)^{1+\alpha} \int_{e^{1+\alpha}}^y \frac{1}{u \log^{2+\alpha} u} \mathrm{d}u \\
&= \alpha(1+\alpha)^\alpha \left[-\frac{1}{\alpha \log^\alpha u} \right]_{e^{1+\alpha}}^y - \alpha(1+\alpha)^{1+\alpha} \left[-\frac{1}{(1+\alpha) \log^{1+\alpha} u} \right]_{e^{1+\alpha}}^y \\
&= (1+\alpha)^\alpha \left[\frac{1}{(1+\alpha)^\alpha} - \frac{1}{\log^\alpha y} \right] - \alpha(1+\alpha)^\alpha \left[\frac{1}{(1+\alpha)^{1+\alpha}} - \frac{1}{\log^{1+\alpha} y} \right] \\
&= \frac{1}{1+\alpha} + \alpha(1+\alpha)^\alpha \frac{1}{\log^{1+\alpha} y} - (1+\alpha)^\alpha \frac{1}{\log^\alpha y}.
\end{aligned}$$

□

A.2 Proof of Theorem 3.8

If Trader follows Strategy 3 with $p_i = \frac{c}{1-\gamma} i$ then at any time $t \geq 0$ his capital K_t satisfies

$$K_t \geq (1-\gamma) F(M_t) - \sqrt{8 F(M_t)(1-\gamma)c + c^2}.$$

Proof. Let (L_t) be the capital of Strategy 2 on the continuous maximum price N constructed from M by pasting linear interpolations at the discontinuities. Let (ν_i) be the sequence of sellin prices constructed by Strategy 2, then analoguously to Theorem 3.2 we obtain for the capital L_t after n transactions

$$\begin{aligned}
L_t &= \sum_{i=1}^{n}(1-\gamma)\nu_i \mathrm{P}(\nu_{i-1},\nu_i] + \mathrm{P}(\nu_n,\infty)\nu_n - nc \\
&\geq (1-\gamma)\sum_{i=1}^{n}\nu_i \mathrm{P}(\nu_{i-1},\nu_i] - nc \\
&= (1-\gamma)\sum_{i=1}^{n}\nu_i \int_{(\nu_{i-1},\nu_i]} \mathrm{P}(\mathrm{d}u) - nc \\
&\geq (1-\gamma)\sum_{i=1}^{n}\int_{(\nu_{i-1},\nu_i]} u\mathrm{P}(\mathrm{d}u) - nc \\
&= (1-\gamma)\int_{[1,\nu_n]} u\mathrm{P}(\mathrm{d}u) - nc \\
&= (1-\gamma)\int_{[1,\nu_{n+1}]} u\mathrm{P}(\mathrm{d}u) - (1-\gamma)\int_{(\nu_n,\nu_{n+1}]} u\mathrm{P}(\mathrm{d}u) - nc \\
&\geq (1-\gamma)\int_{[1,N_t]} u\mathrm{P}(\mathrm{d}u) - (1-\gamma)p_{n+1} - nc \\
&= (1-\gamma)F(N_t) - (1-\gamma)p_{n+1} - nc.
\end{aligned}$$

Then like in Theorem 3.5 we choose $p_i = \frac{c}{1-\gamma}i$ to get

$$n \leq \sqrt{\frac{2(1-\gamma)F(N_t)}{c} + \frac{1}{4}} - \frac{1}{2}$$

and

$$\begin{aligned}
L_t &\geq (1-\gamma)F(N_t) - 2c\sqrt{\frac{2F(N_t)(1-\gamma)}{c} + \frac{1}{4}} \\
&= (1-\gamma)F(N_t) - \sqrt{8F(N_t)(1-\gamma)c + c^2}.
\end{aligned}$$

Now consider Strategy 3 applied to the price process M and let (K_t) be the capital sequence then by Theorem 3.6

$$\begin{aligned}
K_t \geq L_t &\geq (1-\gamma)F(N_t) - \sqrt{8F(N_t)(1-\gamma)c + c^2} \\
&= (1-\gamma)F(M_t) - \sqrt{8F(M_t)(1-\gamma)c + c^2}.
\end{aligned}$$

□

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