Hyperintensional modalities for many-valued logics

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Abstract

Work on hyperintensional modal logics, logics where modalities are invariant to substitution of logically equivalent formulas, has been mostly based on classical logic. We argue that many approaches to hyperintensionality can go hand-in-hand well with non-binary valuations, whether it is propositional variables, modal formulas or both that are many-valued. In particular, we generalize Sedlar[21]'s general framework for hyperintensional logics to be many-valued.

Contents

1 Introduction

Over the past few decades we have seen an increasing interest in non-classical logics. These logics do away with or change some of the assumptions of classical logic (**CL**), generally to create a weaker logic, in the sense that fewer theorems and inferences are valid.

One category of these are many-valued logics. They do away with the assumption that each formula is either true or false, and rather assign a value out of a larger set. Examples include Kleene 3-valued logic (**K₃**) in which a formula can at the same time be neither true nor false, Logic of Paradox (**LP**) where formulas can be both true and false at the same time and First-Degree Entailment logic (**FDE**) which combines these two. Continuum-valued logics also exist, such as fuzzy logic, with values ranging between 0, falsity, and 1, truth.

Another field of logic is that of modal logic, which studies the extension of an underlying logic, generally **CL**, with one or more modalities. These modalities have different interpretations, though they stem from a shared underlying framework. To name a few: in alethic logic modalities represent necessity and possibility, in deontic logic obligations and permissibility and in epistemic logic knowledge and possible knowledge.

Generally these modal logics are intensional, meaning that validity is preserved under substitution of equivalent (sub-)formulas. However, this is not always desired. For example, in knowledge representation and reasoning, a subfield of artificial intelligence, modalities are used to express knowledge of agents. Intensional modal logics would impose that agents must know all the equivalents of their knowledge. Conditions like these are generally considered to be too strong and are often dubbed as prescribing 'logical omniscience'.

As a result the field of hyperintensional logic emerged, in which weaker logics get around the logical omniscience problem. A number of approaches exist, but many have been shown to be special cases of a more general framework proposed by Sedlar [21]. It is mainly these approaches that we will consider.

Interestingly, so far these hyperintensional modal logics have been built on a framework of classical logic. In the present paper we will generalize Sedlar's framework from classical to many-valued logic in order to facilitate research into hyperintensional modalities regardless of background logic. We will also give a brief exploration of logics that may result from this framework.

In section 2 we will give an overview of classical (modal) logic and associated semantics. In section 3 we'll then give an overview of an abstract class of many-valued logic as well as some particular many-valued logics. The 4th section will explore different approaches to hyperintensionality as well as Sedlar's notion of a general hyperintensional logic. In section 5 we then show how these hyperintensional logics can be generalized to many-valued valuation. Section 6 highlights some particular types of logic resulting from this framework that may be interesting. Lastly, in section 7 we make some concluding remarks and turn our eye towards future work.

2 Classical logic

2.1 Non-modal classical logic

We will start by giving a summary of classical logic, and classical modal logic. Aside from forming a common base from which we depart, this will introduce the terminology used in this paper.

Classical logic deals with formulas built from propositional variables from a set $\mathsf{Prop} = p_1, p_2, \ldots$ using the connectives $\wedge, \vee, \rightarrow, \neg$ and $\leftrightarrow$. That is, all formulas are either just a propositional variable, or are finite constructions $\neg\varphi, \varphi \circ \psi$ where φ, ψ are formulas and $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. In classical logic a model $\mathcal{M}$ is just a valuation $\mathcal{V}$ assigning to each propositional variable either 0 or 1, respectively falsity and truth. Symbolically we will denote classical logic as $\mathbf{CL}$ and its set of truth values $\{0, 1\}$ as $\mathbf{2}$.

To find the validity of all formulas, we can extend a the valuation as follows:

$$\mathcal{V}(\varphi \vee \psi) = \max(\mathcal{V}(\varphi), \mathcal{V}(\psi))$$

$$\mathcal{V}(\varphi \wedge \psi) = \min(\mathcal{V}(\varphi), \mathcal{V}(\psi))$$

$$\mathcal{V}(\varphi \rightarrow \psi) = \max(\mathcal{V}(\psi), 1 - \mathcal{V}(\varphi))$$

$$\mathcal{V}(\neg\varphi) = 1 - \mathcal{V}(\varphi)$$

$$\mathcal{V}(\varphi \leftrightarrow \psi) = \mathcal{V}((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$$

We say that a formula φ follows from a set of formulas Γ if for every $\mathcal{V}$ where $\mathcal{V}(\gamma) = 1$ for all $\gamma \in \Gamma$, we also have $\mathcal{V}(\varphi) = 1$. We write $\mathcal{M}, \Gamma \models \varphi$, and say that in the model $\mathcal{M}$, φ is a valid inference from Γ. We abbreviate $\mathcal{M}, \emptyset \models \varphi$ as $\mathcal{M} \models \varphi$. If an inference hold for all models, we write $\Gamma \models \varphi$ or simply $\models \varphi$ in the case $\Gamma = \emptyset$. In the latter case, φ is said to be 'valid', or 'a theorem'.

Another important notion for many logics is that of a proof system. A proof system is a set of rules by which proofs can be created. For a given proof system, we write $\Gamma \vdash \varphi$ if a proof can be constructed from a finite subset of Γ which proves φ. A proper proof system should coincide with validity as defined on models. That is, $\Gamma \models \varphi$ should follow from $\Gamma \vdash \varphi$ and vice versa. In the former case we say that the proof system is sound, in the latter that it is complete. Unless otherwise specified, when we say proof system we will mean a sound and complete proof system.

One example of a proof system is a Hilbert-style calculus. Here a set of axioms is given as well as a set of inference rules. We say that $\Gamma \vdash \varphi$ if φ is an axiom, a formula of Γ, or if it can be obtained by a finite number of applications of the rules of the calculus to axioms and formulas of Γ. A list of axioms leading to a sound and complete proof system is called an axiomatization of a logic.

In the case of classical logic, such axioms might include, for every formula φ and ψ, $\varphi \rightarrow (\psi \rightarrow \varphi)$, stating that if a formula is true, then any implication with it as the consequent is too, and $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$, the axiom scheme of taking contrapositive. For classical

logic Hilbert-calculi, the only rule is generally modus ponens, stating that if you have a proof for φ and a proof for $\varphi \to \psi$, these can be combined to make a proof for ψ.

Many different axiomatizations of **CL** exist in the literature. Since their soundness and completeness with respect to the above valuation and validity is proven, they are all equivalent and we needn't concern ourselves with choosing any in particular. Different proof-systems exist too, such as Gentzen-style calculus and Tableaux. We will not cover them in particular, but note that proof systems of both types are well established for many-valued modal logics[1].

2.2   Kripke modal logics

Modal logics introduce additional symbols, next to the propositional variables and connectives. Generally, these are $\square$ and $\lozenge$. They can be put in front of any formula φ to obtain a modal formula, e.g. $\square\varphi$, whose interpretation depends on the application of the logic. We mentioned several of them in section 1. In this paper we will generally consider $\square$ to describe the knowledge or beliefs of an agent, unless mentioned otherwise. Normally further conditions are imposed on logics when modelling these two, such as that knowledge must always be true. We will only do so here when relevant. Note that our results are general, however, and can be applied to many different interpretations of the modalities.

In many modal logics, there is a nice duality between $\square$ and $\lozenge$, where $\lozenge$ is expressible in terms of $\square$, where $\lozenge$ has neat properties. This is why we generally consider two modalities. If one has no need for $\lozenge$, it can be omitted however. One might also easily add additional modalities and modal symbols, for example to model the knowledge of a number of agents.

To interpret modal formulas, we often make use of a set of worlds. These worlds are generally called possible worlds and assign valuations to propositional variables. Different worlds might give different valuations to variables. Between the worlds a relation exists, used for valuing modalities. In epistemic logic this is generally interpreted as a relation between worlds considered epistemically possible. Here worlds are ways the universe could be, with one world expressing the facts of the actual world. In a given world, the worlds seen by an agent are the worlds which for all he knows might be the actual world.

The best-known formalism of this is that of Kripke frames, from here on K-frames, which are a pair $\langle \mathcal{W}, \mathcal{R} \rangle$ with $\mathcal{R} : \mathcal{W} \times \mathcal{W} \to \mathbf{2}$ a function defining whether a world can see a given world, often called the accessibility relation.

A K-model is then a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ such that $\mathcal{W}$ and $\mathcal{R}$ form a K-frame and $\mathcal{V} :$ $\mathcal{W} \times \mathsf{Prop} \to \mathbf{2}$ a valuation determining whether a propositional variable is true at a given world.

Like in the case of non-modal **CL**, we can extend the valuation at each world. In effect, each

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[1]For tableaux, Priest gives an introduction for classical and many-valued logics in [18] and an extension to a simple modal logic in [19]. For a more general account, see [4], which also extensively covers many-valued logics and substructural logics, on which we base our account of general many-valued logics in section 3. For Gentzen-style, see [10].

world can be seen as a single non-modal **CL**-model. In formal terms, for each $w \in \mathcal{W}$:

$$\mathcal{V}(w, \varphi \wedge \psi) = \min(\mathcal{V}(w, \varphi), \mathcal{V}(w, \psi))$$

$$\mathcal{V}(w, \neg\varphi) = 1 - \mathcal{V}(w, \varphi)$$

and similarly for the other connectives. The truth of modal formulas is then defined as:

$$\mathcal{V}(w, \Box\varphi) = \min\{\mathcal{V}(v, \varphi) \mid \mathcal{R}(w, v) = 1\}$$
$$\mathcal{V}(w, \Diamond\varphi) = \max\{\mathcal{V}(v, \varphi) \mid \mathcal{R}(w, v) = 1\}$$

That is, $\Box\varphi$ is true precisely when all worlds w sees assign 1 to φ, and $\Diamond\varphi$ is true precisely when there is any world seen by w which assigns 1 to φ. Note that $\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$ and $\Box\varphi$ to $\neg\Diamond\neg\varphi$. These are called the duality axioms. Note that since in **CL** $\neg\neg\varphi = \varphi$, they follow from each other. In section 3.3 we will see that for many-valued logics, we don't always have this duality between modalities and need a second accessibility relation to express a second modality.

Validity is then expressed at worlds, where we write $\mathcal{M}, w, \Gamma \models \varphi$ when $\mathcal{V}(w, \varphi) = 1$ if $\mathcal{V}(w, \gamma) = 1$ for all $\gamma \in \Gamma$. If φ holds at all $w \in \mathcal{W}$, we write $\mathcal{M}, \Gamma \models \varphi$ and say it holds in the model. If it holds in all models, we write $\Gamma \models \varphi$. Again, if $\Gamma = \emptyset$, we omit it. If $\models \varphi$, we say that φ is valid, and a theorem.

An axiomatization of K(**CL**), the general modal logic of K-models, can be obtained by extending an axiomatization of **CL** with one axiom scheme, for any φ and ψ the axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, and one rule, if $\models \varphi$, then $\models \Box\varphi$. The former is called axiom K, and states that knowledge is closed under modus ponens, a result of the fact that modus ponens for non-modal formulas is valid at each world, and thus applies to the valuations seen from a given world. The latter is called necessitation, stating that all theorems are known, a result of them being true at all worlds and therefore at all worlds seen. Here too we see two conditions which might well be shared under logical omniscience: you wouldn't expect this quality of reasoning from an actual, bounded agent. We will see logics that do away with K and necessitation in the following sections.

One can make for even stronger rules of inference, by imposing conditions on the K-frame. For example, one could impose transitivity: for any $w, v, u \in \mathcal{W}$, $R(w, u) \geq \min(\mathcal{R}(w, v), \mathcal{R}(v, u))$. That is: if a world w sees a world v which sees another world u, then w must see u too. This results in the following implication being generally valid on the frame: $\Box\varphi \rightarrow \Box\Box\varphi$. When reasoning about knowledge this is called positive introspection: the condition that if you know something, you also know that you know it. Any proof system modelling logic on transitive frames would need to include this inference as an axiom scheme.

2.3   Montague-Scott modal logics

A more general case exists, these are modal logics on so-called Montague-Scott frames, hereafter MS-frames, independently developed by Montague [17] and Scott [20]. A MS-

frame is a pair $\langle \mathcal{W}, \mathcal{R} \rangle$ with $\mathcal{R} : \mathcal{W} \times 2^{\mathcal{W}} \to \mathbf{2}$ which defines whether a world can see a given subset of $\mathcal{W}$ or not.[2]

A MS-model is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ identical to the Kripke case except that $\mathcal{W}$ and $\mathcal{R}$ now form a MS-frame. The valuation of modalities is different however. We define $\mathcal{V}(w, \Box \varphi) = \mathcal{R}(w, \llbracket \varphi \rrbracket)$, where $\llbracket \varphi \rrbracket$ is called the proposition of φ: the set of worlds where φ is true.[3]A modal formula is true at a world, if the proposition of the modalized formula can be seen at that world.

$\mathcal{V}(w, \Diamond \varphi)$ is then defined as $\neg \mathcal{R}(w, \llbracket \neg \varphi \rrbracket)$: precisely as $\mathcal{V}(w, \neg \Box \neg \varphi)$. Observe that while $\Diamond$ still has a clear interpretation, it isn't as obviously meaningful a construct as in the Kripke case. We will see in section 3.4 that in the many-valued case it becomes somewhat superfluous. We include mainly so that MS is a more general logic than K, as we will show in section 2.3.1.

On MS-frames axiom K and necessitation no longer hold. There is one consequence of axiom K and necessitation, which still gives off a strong logical omniscience vibe, that does still hold: if $\models \varphi \leftrightarrow \psi$, then $\models \Box \varphi \leftrightarrow \Box \psi$. This is called the rule of equivalence, or RE. It holds because the propositions of φ and ψ must be equal. When discussing logics that get rid of this axiom too, we enter the territory of hyperintensional logic, which we will do in section 4.

In MS-frames too one can impose conditions on the neighbourhood function resulting in a stronger logic. For example, to obtain positive introspection we must have $\mathcal{R}$ such that for every $V \subset W$, the members of the set $U = \{ u \in \mathcal{W} \mid \mathcal{R}(u, V) \}$ of worlds which see V, must also see U. That is, $\mathcal{R}(w, V)$ implies $\mathcal{R}(w, U)$.

2.3.1   Augmented frames

One particular interesting subset of MS-frames is that of augmented frames[4]. These are MS-frames satisfying two conditions for every $w \in \mathcal{W}$. The first is that $\mathcal{R}(w)$, the set of sets of worlds seen by w, contains it's core: $\cap_{X \in \mathcal{R}(w)} X \in \mathcal{R}(w)$. The second is that $\mathcal{R}(w)$ is closed under supersets: if $X \in \mathcal{R}(w)$ and $X \subset Y$, then $Y \in \mathcal{R}(w)$.

The logic of augmented frames is equivalent to that of K-frames. To see this semantically, let $\mathcal{M}^{MS} = \langle \mathcal{W}, \mathcal{R}^{MS}, \mathcal{V} \rangle$ a MS-model. Define $\mathcal{M}^{K} = \langle \mathcal{W}, \mathcal{R}^{K}, \mathcal{V} \rangle$ such that $\mathcal{R}^{K}(w, v) = v \in \cap_{X \in \mathcal{R}(w)} X$ for all $w, v \in \mathcal{W}$, meaning that v is in the core of $\mathcal{R}^{MS}(w)$. Then $\mathcal{M}^{K}$ is a K-model in which at each world the exact same formulas are true as in the MS-model $\mathcal{M}^{MS}$, and as such the two are in a sense equivalent.

Conversely from a K-model $\mathcal{M}^{K} = \langle \mathcal{W}, \mathcal{R}^{K}, \mathcal{V} \rangle$ we can construct an equivalent MS-model $\mathcal{M}^{MS} = \langle \mathcal{W}, \mathcal{R}^{MS}, \mathcal{V} \rangle$ where $\mathcal{R}^{MS}(w)$ consists of all supersets of $\mathcal{R}^{K}(w)$, the set of worlds

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[2]In the literature, $\mathcal{R}$ is sometimes called a neighborhood function/relation, where accessibility relation refers exclusively to the $\mathcal{R}$ of K-frames. In this paper we will use the name accessibility relation for both.

[3]Note that in the literature, a MS-model is often defined in terms of $\llbracket \cdot \rrbracket$, with $\mathcal{V}$ following straightforwardly from it at $\mathcal{V}(w, \varphi) = w \in \llbracket \varphi \rrbracket$. In the present paper we opt for this route to achieve greater symmetry between different types of models.

[4]For a more formal discussion and proof, see [1].

seen by w under $\mathcal{R}^K$.

Since every K-model has an equivalent MS-model, but not the other way around, we see that MS is indeed a more general logic than K, like we concluded when looking at the axioms of MS.

3   Many-valued logics

In this section we will look at logics that move away from the classic binary valuation. That is, instead of valuing formulas with ones and zero's, where a formula is either true or false, and frames as either fully seeing a world or proposition or not seeing it at all, we look at more nuanced logics. We will look at some logics in particular, but will make use of a general definition that should cover a broad class of many-valued logics.

3.1   Examples of many-valued logics

3.1.1   Truth-value gluts and gaps

The first logics we will look at still deal in binary notions of truth. Formulas are valued in terms of being true or not, and false and not. However, the presence of the one need not exclude the other, nor does the absence guarantee the other.

The former case leads to the Logic of Paradox (**LP**). Here, next to being true or being false, a formula can also be both true and false at the same time. Such a condition might be preferable in modelling the beliefs of an agent. After all, in practise it is hard to ensure that an agent has no conflicting beliefs. In **CL**, conflicting beliefs leads to a logical explosion: $\varphi \wedge \neg\varphi$ implies ψ for any φ and ψ. If you have any conflicting beliefs, you ought to infer everything. From believing two things in opposition with each other, and perhaps very obscurely, I'd have to conclude that pigs can fly. Clearly any situation with conflicting beliefs is to be avoided in **CL**.

Explicitly modelling the possibility of conflicting information helps combat this explosion. Next to 1 and 0 we add another value b (for both), often referred to as a truth-value glut. Then we define the connectives as one would expect: $\varphi \wedge \psi$ is true, though not also false, precisely when both φ and ψ are true but not both. We write $\mathcal{V}(\varphi \wedge \psi) = 1$. It is false and not both (valuation 0) in case $\mathcal{V}(\varphi) = 0$ or $\mathcal{V}(\psi) = 0$. Lastly, it is both true and false in the other cases: when φ and ψ are true (valuation 1 or b) and at least one of them is also false at the same time (valuation b). These relations can be captured in a truth-function-table such as in table 1

$p \wedge q$	0	b	1
0	0	0	0
b	0	b	b
1	0	b	1

Table 1: Truth-function of $\wedge$ for **LP**.

Another logic, Kleene 3-valued logic ($\mathbf{K_3}$) does the opposite. Here an extra value n is added, for neither true nor false, referred to as a truth-value gap. $\mathbf{K_3}$ is used to model situations where judgement about the valuation of a propositional variable is withheld for a later time, or even completely disregarded. One such instance might be considering propositional variables talking about facts of the future like 'it will rain tomorrow'. If we believe in a non-deterministic universe, such a proposition cannot be considered true or false until tomorrow actually arrives.

First-degree-entailment logic (**FDE**) is the combination of these two logics. It has four values: 0, 1, b and n. Connectives function as you would expect: $\wedge$ is true when both its inputs are, and (potentially: also) false when one of its inputs is. $\vee$ is false when both its inputs are and true when one of its inputs is. We consider a formula to be valid when it is true in all valuations. That is, if all valuations assign it either 1 or b. Note that in contrast to **CL**, validity is no longer defined in terms of just 1.

3.1.2   Łukasiewicz logic

Another interesting logic is the continuum-valued logic of Łukasiewicz ($Ł_{\aleph}$), where truth is expressed with real numbers in the interval $[0, 1]$. Here $\wedge$ takes the minimum of its inputs, and $\vee$ the maximum. $\mathcal{V}(\varphi \to \psi)$ is defined as 1 if $\mathcal{V}(\varphi) \leq \mathcal{V}(\psi)$ and $1 - (\mathcal{V}(\varphi) - \mathcal{V}(\psi))$ otherwise. $\mathcal{V}(\neg\varphi)$ can again be defined as $1 - \mathcal{V}(\varphi)$. Note the similarities to **CL**, and again, that **CL** and $Ł_{\aleph}$ coincide on the values 0 and 1.

3.2   General many-valued logic

In more general terms, we speak of a many valued logic $\mathfrak{A} = \langle \mathcal{A}, \mathcal{D}, \leq, \&, \to \rangle$ when[5]:

$\mathcal{A}$ is a set of values including 0 and 1;

$\langle \mathcal{A}, \leq \rangle$ form a bounded lattice. That is, $\leq$ orders the elements of $\mathcal{A}$ in such a way that 0 is the least element, 1 is the greatest element and every subset of $\mathcal{A}$ has an infimum, a largest element at least as small as all elements of the subset (also called greatest lower bound) and a supremum, a smallest element at least as large as all elements of the subset (also called least upper bound). $\wedge$ and $\vee$ then function as respectively the infimum and supremum;

$\mathcal{D} \subsetneq \mathcal{A}$ is a set of so-called designated values including 1 and not including 0. Informally, when a formula's valuation (see $\mathfrak{A}$-models below) is designated, we may consider it to be true;

$\&$ and $\to$ are connectives. $\&$ is associative and commutative and has 1 as an identity. $\to$ is an implication such that $\&$ and $\to$ form a residuated pair under $\leq$, i.e. $a\&b \leq c$ iff $a \leq b \to c$ for all $a, b, c \in \mathcal{A}$.

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[5]Here we follow the approach of [2]. Many-valued logics which fall outside of this framework exist, we leave broadening our framework to include them to future work, and mentioned a couple of them in section 7.

An $\mathfrak{A}$-model is a valuation $\mathcal{V}$ assigning to each propositional variable a value in $\mathcal{A}$. Again we can define an extensions of the valuations such that:

$$\mathcal{V}(\varphi \wedge \psi) = \inf(\mathcal{V}(\varphi), \mathcal{V}(\psi))$$

$$\mathcal{V}(\varphi \vee \psi) = \sup(\mathcal{V}(\varphi), \mathcal{V}(\psi))$$

$$\mathcal{V}(\varphi \& \psi) = \mathcal{V}(\varphi) \& \mathcal{V}(\psi)$$

$$\mathcal{V}(\varphi \rightarrow \psi) = \mathcal{V}(\varphi) \rightarrow \mathcal{V}(\psi)$$

$$\mathcal{V}(\varphi \leftrightarrow \psi) = \mathcal{V}((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$$

$$\mathcal{V}(\neg\varphi) = \mathcal{V}(\varphi) \rightarrow 0$$

Validity is defined in terms of $\mathcal{D}$. $\mathcal{M}, w, \Gamma \models \varphi$ holds when $\mathcal{V}(w, \gamma) \in \mathcal{D}$ for all $\gamma \in \Gamma$ implies that $\mathcal{V}(w, \varphi) \in \mathcal{D}$. As usual, if $\mathcal{M}, w\Gamma \models \varphi$ for all $w \in \mathcal{W}$ in a model, we may write $\mathcal{M}, \Gamma \models \varphi$. Similarly we omit $\mathcal{M}$ if it holds for all models and Γ if $\Gamma = \emptyset$. It is often desirable that certain closure rules apply to $\mathcal{D}$. For example, if it is closed under taking infinima, we have $\{\varphi, \psi\} \models \varphi \wedge \psi$.

In some of the following logics $\& = \wedge$. Note however, that sometimes this is not possible because no proper $\rightarrow$ could be defined such that $\wedge$ and $\rightarrow$ would form a residuated pair.[6]

CL is, as one might expect, a special case of this framework. Take $\mathcal{A} = \mathbf{2}, \mathcal{D} = \{1\}$ and $\leq$ as normal on $\{0, 1\}$. With $\& = \wedge$, $\&$ and $\rightarrow$ indeed form a residuated pair, as is easily checked.

Of note should be that axiomatizations of many-valued logics can differ wildly from that of **CL**. Some axioms do hold in all many-valued logics as described above[7], such as $\varphi \rightarrow \varphi$ and $\varphi \wedge \psi \rightarrow \varphi$[8]. Rules differ too. For example, modus ponens holds only if there's no $x \notin \mathcal{D}$ such that for some $y \in \mathcal{D}$ we have $\mathcal{V}(\varphi) = y, \mathcal{V}(\psi) = x$ and $\mathcal{V}(\varphi \rightarrow \psi) \notin \mathcal{D}$, with an obvious counterexample arising when there does exist such an x.

The lattice for a logic can often be well represented in a diagram. The lattice for **FDE** is shown in figure 1. This diagram also hints at the possibility of removing n or b as a value: we still have a bounded lattice after the removal, respectively that of **LP** and $\mathbf{K_3}$. Note that both have the same lattice. The difference between the two logics is in the designated values, where $b \in \mathcal{D}$ whereas $n \notin \mathcal{D}$.

$Ł_\aleph$ has infinite values and can therefore not been shown in the same way. However, one might envision a diagram as the number line from 0 and 1, with the usual ordering. Observe also that with $\& = \wedge$, $\rightarrow$ uniquely follows from creating a residuated pair with $\&$, and is indeed equal to how we defined $\rightarrow$ above.

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[6]Consider for example a 5-valued logic with three elements $0 < a, b, c < 1$ where a, b and c are not comparable.

[8]Both follow since $x \rightarrow y = 1$ iff $x \leq y$, see lemma 2 of [2]

[8]Some many-valued logics are said to have no axioms. These logics, however, do not fit in the framework we use in this paper. One such example is **FDE** without $\rightarrow$, or with $\rightarrow$ defined differently than here.
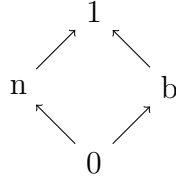
Figure 1: Lattice for **FDE**

3.3   Many-valued Kripke modal logics

Given a many-valued logic $\mathfrak{A}$, we can give the definition of an $\mathcal{A}$-valued Kripke frame, or K($\mathcal{A}$)-frame, as a pair $\langle \mathcal{W}, \mathcal{R} \rangle$ such that $\mathcal{R} : \mathcal{W} \times \mathcal{W} \to \mathcal{A}$ determines to which degree a world sees another world.

A model is then a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ such that $\langle \mathcal{W}, \mathcal{R} \rangle$ is a K($\mathcal{A}$)-frame and $\mathcal{V} : \mathcal{W} \times$ Prop $\to \mathcal{A}$. Like in section 2.2, we can extend the valuation so that each world functions just like $\mathfrak{A}$ for non-modal formula's.

Modalities are also defined analogously, but this will require more elaboration. For this we extend the subsethood function $\subseteq$ to be $\mathcal{A}$-valued. Traditionally $\subseteq$ deals with binary sets where an element is either part of the set or not, and then a set is either a subset of the other or not. Here we introduce the notion of $\mathcal{A}$-valued sets of worlds, or $\mathcal{A}$-sets[9]. Such a set can be seen as a function $\mathcal{W} \to \mathcal{A}$ which determines to which degree a world is a part of the set, with 1 and 0 representing full membership and no membership at all respectively. We will write application of this function analogously to the binary case with $\in$, such that $s \in S$ is the degree to which s is a member of S.

For $\mathcal{A}$-sets A and B then, we define $A \subseteq B = \inf_{w \in \mathcal{W}}(w \in A \to w \in B)$. This can be thought of as the degree to which A is a subset of B. If each world is 'more strongly' a member of B than of A, we have $w \in A \to w \in B = 1$ for all w, and therefore $A \subseteq B = 1$: A is 'fully' as subset of B. If some world is less-so an element of B than of A, the value of $A \subseteq B$ lowers, reaching 0 when there's a world in A to degree 1 and in B to degree 0, just like in the binary case.

Similarly, we can define a binary function of degree of overlap $\odot$ as $A \odot B = \sup_{w \in \mathcal{W}}(w \in X \& w \in Y)$, which is equal to 1 if there's an element in both A and B to degree 1, and gets lower as A and B coincide less on their 'strongest' members.

We can then define:

$$\mathcal{V}_w(\Box \varphi) = \mathcal{R}(w) \subseteq [\![\varphi]\!]^{10}$$

$$\mathcal{V}_w(\Diamond \varphi) = \mathcal{R}(w) \odot \mathcal{V}(\varphi)$$

The interpretation of the modalities is still basically the same. $\mathcal{V}_w(\Box \varphi)$ expresses how strongly φ is represented as true at the worlds seen by w, and $\mathcal{V}_w(\Diamond \varphi)$ expresses how strongly w can see a truth of φ somewhere.

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[9]First introduced by [23].
[10]Note that $\mathcal{R}(w)$ is now an $\mathcal{A}$-valued set.

Note that in the case $\mathcal{A} = \mathbf{2}$ we find the traditional binary definition of subsets, and the same validity as described in section 2.2. We can therefore consider K as the logic K($\mathbf{CL}$) on K($\mathbf{2}$)-frames.

The valid formulas and inferences, and therefore the axioms and rules of a sound and complete proof system, of K($\mathfrak{A}$) depend on $\mathfrak{A}$. The axioms of $\mathfrak{A}$ are always included and so is RE[11]. Necessitation, duality and K might hold. Most notably, they do in the case $\mathfrak{A} = \mathbf{CL}$. Note that if duality between $\square$ and $\lozenge$ does not hold, then RE is needed as a rule for both modalities, since $\lozenge\varphi \leftrightarrow \lozenge\psi$ no longer follows from $\square\varphi \leftrightarrow \square\psi$.

3.4 Many-valued Montague-Scott modal logics

A more general case is again a form of Montague-Scott semantics, this time for many valued logics. A change here is that we move to a pair of accessibility relations, where $\lozenge$ makes use of a relation different from that of $\square$. Note that we could still use a single relation and define $\lozenge$ as the dual of $\square$, using $\neg\varphi = \varphi \to 0$. However, for some $\to$ this might not have the properties we want. In particular, the resulting logic might not be a proper generalization of K($\mathfrak{A}$), a notion we will discuss in section 3.4.2.

We define a MS($\mathcal{A}$)-frame as a triple $\langle \mathcal{W}, \mathcal{R}^\square, \mathcal{R}^\lozenge \rangle$ with $\mathcal{R} : \mathcal{W} \times \mathcal{A}^\mathcal{W} \to \mathcal{A}$, which describes to which degree a world sees as proposition. Note that the nature of propositions has changed. Because we no longer binarily value formulas at worlds we move away from the notion of propositions as a set, and instead see them as $\mathcal{A}$-valued sets. Validity for modal formulas follows as being $\mathcal{V}_w(\square\varphi) = \mathcal{R}^\square(w, \llbracket\varphi\rrbracket)$ and $\mathcal{V}_w(\lozenge\varphi) = \mathcal{R}^\lozenge(w, \llbracket\varphi\rrbracket)$. Note that $\lozenge$ functions identical to $\square$. It may be thought of as just a second modality, and omitted if one is not interested in it. It is only when we impose conditions on the relation between $\square$ and $\lozenge$ that they become related again.

Of note should be that simply taking $\mathfrak{A} = \mathbf{CL}$ in the above formulation does not directly yield the MS logic of section 2.3, since $\lozenge$ and $\square$ are no longer dual. Disregarding our second accessibility relation and considering $\lozenge$ as an abbreviation for $\neg\square\neg$ is an option. Equivalently, one could require that $\mathcal{R}^\lozenge(w, \varphi) = \neg\mathcal{R}^\square(w, \neg\varphi)$.

An axiomatization of MS($\mathfrak{A}$) is that of $\mathfrak{A}$ together with RE for both modalities[12]. Note that since we lost duality, they no longer follow from each other. Note that every many-valued logic has some equivalent statements, such as φ and $\varphi \wedge \varphi$.

3.4.1 Binary versus many-valued frames

Of note should be that the formulations we give above are the most general, but they are not the only generalizations of modal logic to many-valued logics that exist in the literature. Priest [19], for example, gives an account of many-valued modal logic based on binary K-frames.

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[11]See [2]. The result for K($\mathfrak{A}$) is the result of it holding for MS($\mathfrak{A}$), which is a generalization of K($\mathfrak{A}$), as we will discuss in section 3.4.2.

[12]See [2].

He then defines $\mathcal{V}(w, \Box\varphi) = \inf\{\mathcal{V}(v, \varphi) \mid \mathcal{R}(w, v)\}$ and the same for $\Diamond$ substituting sup for inf[13]. This follows from our definition above if one sees the K-frame as an K($\mathfrak{A}$)-frame using only the values 0 and 1, since as a result from our definition of $\to$, we have $0 \to a = 1$ which is the identity for $\wedge$, and $1 \to a = a$.

Note that no straightforward generalization to MS-semantics exists. The difficulty in this results from the fact that under many-valued logics, propositions are no longer clearly delineated sets. In the Kripke case, we can define a set of worlds as seen and define the valuation of modalities as a function of the values of the modalized formula at the seen worlds. In the Montague-Scott case, the valuation of modalities is, however, not defined in terms of truth at seen worlds, but rather on whether a certain truth at worlds is seen.

One attempt may involve creating a dichotomy between designated and non-designated values. Validity can then be defined as in the MS(**CL**) case, where the former are treated as 1 and the latter as 0: a function $\mathcal{R}^* : 2^{\mathcal{W}} \to \mathbf{2}$ is given, and a modal formula $\Box\varphi$ is valuated at world w as $\mathcal{R}(w, [\![\varphi]\!]) = \mathcal{R}^*(w, [\![\varphi]\!]^*)$ where $[\![\varphi]\!]^* = \{w \in \mathcal{W} \mid \mathcal{V}(w, \varphi) \in \mathcal{D}\}$. Note that as a result modal formulas are only valued 0 or 1, which may or may not be desirable. We will discuss that further in section 3.4.3.

3.4.2   Augmented frames

For MS($\mathcal{A}$)-frames too a subclass of augmented frames exist[14]. We call a MS($\mathfrak{A}$)-frame augmented when for each $w \in \mathcal{W}$ there exists a unique $C_w \in \mathcal{A}^{\mathcal{W}}$ such that for any $X \in \mathcal{A}^{\mathcal{W}}$ we have

$$C_w \subseteq X = \mathcal{R}^{\Box}(w, X)$$
$$C_w \odot X = \mathcal{R}^{\Diamond}(w, X)$$

Here is mostly why we define $\Diamond$ using a separate relation, if we defined it as a dual to $\Box$, the second condition might not hold. Note that if $\mathcal{A} = \mathbf{2}$, the first condition is the same condition placed on $\Box$ in section 2.3.1. The second condition already holds there, which means that in the case $\mathcal{A} = \mathbf{2}$, imposing the second condition is equal to setting duality.

The logics on augmented MS($\mathcal{A}$)-frames are equivalent to those on K($\mathcal{A}$)-frames. To see this semantically, let $\mathcal{M}^{MS} = \langle \mathcal{W}, \mathcal{R}^{\Box MS}, \mathcal{R}^{\Diamond MS}, \mathcal{V} \rangle$ a MS-model. Define $\mathcal{M}^K = \langle \mathcal{W}, \mathcal{R}^K, \mathcal{V} \rangle$ such that $\mathcal{R}^K(w, v) = \inf_{X \in \mathcal{A}^{\mathcal{W}}}\{X \in \mathcal{R}^{\Box MS}(w) \to v \in X\}$ for all $w, v \in \mathcal{W}$. As a result, the degree to which v is seen from w, is the value of v in C_w[15]. Then $\mathcal{M}^K$ is a K-model in which at each world the exact same formulas are true as in the MS-model $\mathcal{M}^{MS}$, and as such the two are in a sense equivalent.

Conversely from a K-model $\mathcal{M}^K = \langle \mathcal{W}, \mathcal{R}^K, \mathcal{V} \rangle$ we can construct an equivalent MS-model $\mathcal{M}^{MS} = \langle \mathcal{W}, \mathcal{R}^{\Box MS}, \mathcal{R}^{\Diamond MS}, \mathcal{V} \rangle$ where $\mathcal{R}^{\Box MS}(w, X) = \mathcal{R}^K(w) \subseteq X$ and $\mathcal{R}^{\Diamond MS}(w, X) = \mathcal{R}^K(w) \odot X$.

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[13]Priest uses Glb (greatest lower bound) and Lub (least upper bound) instead of supremum and infinimum. For clarity I have changed his notation to the one we use in the present paper.

[14]For a more formal discussion and proofs of claims in this section, see [2].

[15]Equivalently, we could define $\mathcal{R}^K(w, v) = \inf_{X \in \mathcal{A}^{\mathcal{W}}}\{v \in X \to X \in \mathcal{R}^{\Diamond MS}(w)\}$.

Since every K($\mathfrak{A}$)-model has an equivalent MS($\mathfrak{A}$)-model, but not the other way around, we conclude that MS($\mathfrak{A}$) is a more general logic than K($\mathfrak{A}$), like we saw when looking at the axioms of MS.

3.4.3   Different valuations for modal and non-modal formulas

The desired set of valuations for modal and non-modal formulas need not coincide. One might argue that valuation of formulas containing no modalities ought to be classical, whereas the modalities might be valued differently. For example, propositional variables might be thought to express statements about the physical world which are either true or false, whereas modal formulas express the knowledge of an agent which can vary in degrees, or perhaps the agent can be in a position of both knowing and not knowing something.

This cannot be expressed within our definitions above in the Kripke case. After all, if ϕ is binary at all worlds, then so must $\Box\phi$ and $\Diamond\phi$. In the MS case we do however have the option for binary-valued propositions to be seen to some non-binary degree. The latter can be modelled in MS, and therefore on K, by requiring that any ($\mathcal{A}$-valued) proposition is either seen or not. We saw one such way at the end of section 3.4.1, where a proposition was considered seen if all worlds assigned designated values made a set of worlds seen by a binary accessibility relation.

Alternatively, truth of propositional variables at worlds might be considered many-valued while truth of modal formulas is not. We might see this in the case where propositional formulas express facts about the future, which are yet to be determined true or false, hence they are neither. For example, one may claim that if $\Box$ represents necessity, it ought to be binary-valued, as some fact is either necessary or not. This case, modelled on a set of $\mathbf{K_3}$ worlds, is discussed in section 7 of [19].

Note that requiring either propositional variables or modal formulas to be binary isn't the only way to dichotomize truth values. Both could be assigned values in only partially overlapping sets larger than $\mathbf{2}$. Depending on $\mathfrak{A}$, formulas containing parts with modalities and parts with no modalities might be valued in their union, or with values present in neither of the two.

4   Hyperintensionality

4.1   The logical omniscience problem

All modal logics discussed so far have the property that if $\varphi \leftrightarrow \psi$, then $\Box\varphi \leftrightarrow \Box\psi$: the rule of equivalence RE. Some logics also have other properties such as axiom K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and necessitation: from $\models \varphi$ follows $\models \Box\varphi$.

These properties can be problematic. If we use our modality to model knowledge of an agent, then an agent knowing φ must also know ψ for all ψ equivalent to φ. In **CL** each formula has an infinitude of equivalent formulas. A formula φ is equivalent to $\neg\neg\varphi$, and $\neg\neg\neg\neg\varphi$ for example, but also to formula's that seemingly have less to do with it, such as $\varphi \wedge (\psi \vee \neg\psi)$. Even worse, all formulas which are always true are equivalent to each other. Suppose then

that an agent knows that $1 + 1 = 2$, should we expect him to also know other mathematical truths like the Poincaré conjecture, whose proof alone runs hundreds of pages long? And even the mathematical truths which are yet to be discovered?

Max Cresswell articulates it well: "there is no reason why someone should not take a different propositional attitude (belief, say) to two propositions that are logically equivalent. And when a mathematician discovers the truth of a mathematical principle he does not thereby discover the truth of all mathematical principles." [3]

Clearly if propositional variables express truths at worlds and modalities are to express properties of actual, non-ideal agents, then we must find a logic where RE does not hold. Such logics are called hyperintensional logics. The name stems from the fine-grained nature of intensions in such logic. MS modal logics consider intensions of formulas as propositions on possible worlds. This is not specific enough to individuate equivalent formulas to non-identical content.

Hyperintensional logics consider content to be more fine-grained, such that formulas that are logically equivalent can still have different intensions. Several such logics exist in the literature, most of them based on **CL**. We will summarize a number of them in the following sections[16].

4.2   Sedlar

A general framework for classical hyperintensional modal logics, which covers all logics we will consider in this paper, is given by Sedlar [21]. To bring notation more in line with the rest of this paper, I will give an analogous definition here:

A hyperintensional model, or H-model, is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{C}, O, \mathcal{R}, I \rangle$. Here $\mathcal{C}$ is a set which is said to express the content of a formula. These contents are assigned to formulas by $O : \mathsf{Form} \to \mathcal{C}$. The exact forms of content vary among various frameworks, we will encounter a number of them in the coming sections. $I : \mathcal{W} \times \mathcal{C} \to \mathbf{2}$ then decides to which degree certain content is true at each world. One condition is that $\mathcal{V}(w, \varphi) = I(w, O(\varphi))$ behaves as a valuation function as defined in section 2.3 in regards to non-modal formulas. That is, it satisfies the rules for the connectives for all non-modal formula. E.g. $\mathcal{V}(w, \varphi \wedge \psi) = \inf(\mathcal{V}(w, \varphi), \mathcal{V}(w, \psi))$.

$\mathcal{R} : \mathcal{W} \times \mathcal{C} \to \mathbf{2}$ behaves a bit differently than before. Most notably, it is not necessarily a relation between worlds. As such we no longer speak of frames we considering H-models. Rather, $\mathcal{R}$ assigns to each world which content it can see. The valuation of modal formulas is then defined as $\mathcal{V}(w, \Box\varphi) = \mathcal{R}(w, O(\varphi))$.

It is straightforward to see that H-models are a generalization of MS-models. Given an MS-model $\langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ we define a H-model $\langle \mathcal{W}, C, O, \mathcal{R}, I \rangle$ where $C = 2^{\mathcal{W}}$, I such that $I(c, w) = c \in w$ and O such that $(O(\varphi) = [\![\varphi]\!]$. Now the content is just the proposition of the formula and $\mathcal{R}$ designates whether a world sees that proposition. Note that $\langle \mathcal{W}, \mathcal{R} \rangle$ now forms a MS-frame. However, it should be noted that in some of the approaches described

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[16]For a further introduction into hyperintensional logics, see [13].

in the sections to come modalities aren't valued in terms of propositions of worlds. As such the only thing necessarily taken from Kripke and MS semantics is the notion of possible worlds. One shouldn't think of H-models as being built on top of MS-models: one could go straight from learning about K-models to many of the hyperintensional logics of the sections to come.

An axiomatization of the logic H is an axiomatization of **CL** extended with the duality axiom[17]. Most importantly: RE does not hold. All of the approaches below have in their base form the same axiomatization as H. The main difference between them is therefore only in the semantic interpretation of the logics. Of course, conditions can be added to make the logics stronger.

4.3   Syntactic approaches

The first approach we discuss is often called the syntactic approach: rather than being valued based on a relation to the content of a formula, as in the non-hyperintensional cases as well as in the approaches to follow below, modalities are (also) valued based on relations to the syntactic form of the formula itself. In epistemic logic for example, this might mean that the agent in question needs to have consciously considered the formula, contrasting our earlier interpretations of the modality which often considered knowledge as more implicit.

Perhaps the conceptually easiest approach is to set $\mathcal{C} = \mathsf{Form}, O = ID$, the identity function, and I a normal valuation function. We can then set $\mathcal{R}(w, \varphi)$ to whatever we want the valuation of $\Box\varphi$ to be at a given world.

It allows us to define knowledge precisely how we want it, without imposing any unwanted conditions on our agents knowledge. Its main critique[18] is that it is only a way of representing knowledge rather than modelling knowledge: it doesn't yield any insight about properties of knowledge. There is no semantics, like logically or epistemically possible worlds, to give an interpretation of our syntactical valuation of modal formulas. It is for this reason, that other approaches are generally preferred.

A more intricate example is the logic of general awareness by Fagin and Halpern [7]. They model belief instead of knowledge, but their approach might easily be applied to knowledge. Their logic considers an awareness modality in addition to an implicit belief modality functioning based on relations to possible worlds like the modalities in section 2.3. Explicit belief then occurs at the intersection of implicit belief and awareness. The awareness is modelled as a set of formulas at each world of which the agent is aware. This be modelled as the modality in Sedlar's framework setting $\mathcal{R}(w, \varphi) = 1$ whenever φ is seen at w. Implicit belief could then be modelled with a MS-modality as in section 2.3, and explicit belief can then be said to occur when both modalities are true. Alternatively, one can consider the awareness and epistemic possibility relations implicit and directly model the resulting explicit belief operator.

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[17]See [21].

[18]See e.g. [11, 8].

4.4   State-based approaches

A common approach is to expand the set of possible worlds to a set of states, which includes states which function differently from possible worlds. These states are not metaphysically possible, but the agent might consider them epistemically possible. These are worlds where $1 + 1 = 2$ might hold whereas the Poincaré conjecture does not. Logically this is impossible, but unless our agent is well-versed in mathematics he is not aware of that. As such he might consider such worlds in forming his knowledge, and hence they affect the valuation of modalities in such logics.

In Sedlar's framework this is achieved by defining a set of states $\mathcal{S} \supset \mathcal{W}$ and setting $\mathcal{C} = 2^{\mathcal{S}}$. O then assigns each formula a proposition of classical and non-classical worlds. This proposition is subject to valuation at worlds and states, we will give a number of accounts of such valuations below. As before $\mathcal{R} : \mathcal{W} \times \mathcal{W} \to \mathbf{2}$ designates which propositions are seen. Lastly, I designates whether a world is part of a proposition.

A simple account of non-classical worlds is given by Jago [11]. Here non-classical worlds are worlds where connectives still function as usual, but contradictions can occur by allowing formulas to be both true and false, or neither of the two. That is, impossible worlds have **FDE**-valued valuations. He then defines $\Box\varphi$ to be two-valued, following Kripke-semantics: true if φ is true at all worlds seen, and false otherwise. In a way, the resulting model is just modal **FDE** with binary modalities as discussed in section 3.4.3. The only difference is that validity is considered only at worlds with classical valuations.

The resulting logic still 'suffers' from RE however. In [11], Jago makes use of a different implication, defined as $\varphi \to \psi = \neg\varphi \vee \psi$. As he points out, however, this logic too is free from logical omniscience problems. The resulting logic has condition $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box(\psi \vee (\varphi \wedge \neg\varphi)))$, which is known as closure under known implication-or-contradiction, stating that for all inferences from his knowledge, an agent must know they are either true, or that he believes a contradiction. This does not help with the logical omniscience problem: the agent is still to be in a cognitive relation to the logical conclusions of his knowledge. Only now he ought to know that they might not be true in the case his knowledge is contradictory. Even worse, we might want the agent to consider his beliefs non-contradictory, for who believes that his own knowledge is? As such in this case we'd expect him to infer ψ anyhow, and we're back at axiom K.

As such Jago calls for a different approach, which is relevant to our logics in much the same way. He argues for so-called 'open worlds': worlds not closed under any rule of inference. That is, connectives are no longer truth-functional and a valuation must be created that assigns a value to every single formula. This removes any logical order from these worlds, and as such from modalities. We are now safe from any logical omniscience expectations on our agents. If we want any condition, we can explicitly model it. For example, if we want RE to hold only for φ, ψ in some set $\Omega \subset \mathsf{Form}$, we can require that all worlds see the propositions of formulas $\omega \in \Omega$[19].

The downside is that our open-world approach means that our world-based semantics has

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[19]See also Sedlar's discussion of Rantala models in section 5.1 of [21].

lost all meaning. What is the semantic meaning of a world seeing certain states? It shows the propositions the agents sees at the world, but these are completely arbitrary. In a sense, it is no different from a sentential approach.

This can be remedied by adding certain conditions instilling some meaning in this framework. One type of these is frame conditions, such as transitivity which leads to positive introspection. Another would be to require connectives to behave as normal in formulas which fit some measure of low complexity[20]. The result of this is that agents are considered to be able to spot easy contradictions and implications of their knowledge. It is only more complex conclusions that they're not able to draw.

4.5   Structuralist approaches

Structuralist approaches take it that modalities express properties of semantic contents of formulas which are neither the syntax of the formula itself, nor sets of worlds, but rather some other structure. Here we can take $\mathcal{C}$ to be the set of those structures, O designating the specific structure expressing the contents of a given formula, and $\mathcal{R}$ having certain conditions defining the relation between the structures in question.

One such case takes structure to be Russellian propositions: expressions relating formulas to statements about individuals and relations between them. An example by King [15]: consider the differences between the sentences 'Sarah believes that every equation on page ten characterizes a circle' and 'Sarah believes that every equation on page ten characterizes a set of all points equidistant from a given point'. Logically, these express the same situation. Structurally they are very similar too, expressing Sarah's beliefs about equations appearing on page ten. 'a circle' and 'a set of all points equidistant from a given point' are different individuals however, even if they are logically the same. As a result, the two sentences have different intensions. As a result, we may consider the first sentence to be true whilst the latter is false. The difference with a syntactic approach here is that we are not valuing modalities based on an attitude of an agent towards the syntax of the sentence itself. Rather, the valuations represent an attitude towards the structured content expressed by the sentence.[21]

Another approach is Transparent Intensional Logic, which considers structures as constructions, procedures which construct possible-world propositions. 'Procedure' here is to be read in the sense of 'recipe', not as 'process'. It is how one could construct the proposition. It is this procedure that one becomes in relation to when one performs it and attains knowledge of the sentence of which the procedure is the intension. These constructions are adequately fine-grained since different constructions might give rise to the same proposition without being identical. An example from [6]: 'Bill walks' and 'Bill walks and whales are mammals' are true in the same worlds by virtue of 'whales are mammals' being true in all of them. However, the procedures by which one could come to know the truth or falsity of both sentences are different. As such, someone who has never come to learn that whales are mammals

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[20]See e.g. [12].
[21]See also [14] and [22].

might believe the former sentence while not believing the latter.[22]

5 Many-valued hyperintensional logic

5.1 General statement

Now, we can finally formulate our general framework of many-valued hyperintensional logic. For a many-valued logic $\mathfrak{A}$, we define a hyperintensional $\mathfrak{A}$-valued model, or H($\mathfrak{A}$)-model, as a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{C}, O, \mathcal{R}, I \rangle$ such that $\mathcal{W}$ is a set of worlds, $\mathcal{C}$ is a set of semantic contents of formulas, $O : \mathsf{Form} \to \mathcal{C}$ a function that assigns each formula its content, $\mathcal{R} : \mathcal{W} \times \mathcal{C} \to \mathcal{A}$ a function assigning to which degree a world sees specific content and $I : \mathcal{W} \times \mathcal{C} \to \mathcal{A}$ determining to which degree content is true in each world.

The following condition applies, that the valuation constructable from O and I as $\mathcal{V} = I(w, O(\varphi))$, satisfies the rules for non-modal formulas. That is, at any given world, the valuation of formulas of the form $\varphi \circ \psi$ with $\circ \in \{\wedge, \vee, \to\}$ follows from the valuations of φ and ψ as it would in section 3.4. Valuation of modal formulas is defined as $\mathcal{V}(w, \Box \varphi) = \mathcal{R}(w, O(\varphi))$.

Here we omitted mention of $\Diamond$. Like in section 3.4 for MS($\mathfrak{A}$), if we are interested in $\Diamond$ we add a second accessibility relation[23]$\mathcal{R}^{\Diamond}$ and label the first one as $\mathcal{R}^{\Box}$. In this paper we will only do this when discussing state-based approaches. In other cases, this second modality is not particularly meaningful.

Note that for H($\mathfrak{A}$)-models there is no clear notion of a frame. Previously we considered pairs of worlds and an accessibility relation between them. With the propositional formulas considered fixed, all that was necessary to go from a frame to a model is a valuation of those propositional formulas at each world.

In the case of hyperintensional models we also deal with content. The set of content cannot be considered fixed irrespective of the worlds and valuations considered. After all, $\mathcal{C}$ might be considered as a set of impossible worlds, which ties in closely with the worlds which are possible. Considering content as directly related to worlds isn't always congenial either, for example, when we consider $\mathcal{C} = \mathsf{Form}$ it hardly seems like content is in any way specific to a given set of worlds and therefore to a given frame, if we were to entertain such a notion.

The same holds for $\mathcal{R}$, which is not necessarily a relation between worlds anymore: which content is seen at a given world does not have to relate in any metaphysical sense to other possible worlds the world is in some kind of relation with. However, it can, as in the case of state-based approaches. Perhaps it is good to not define a notion of a frame, as to leave it open to specific approaches to define an appropriate notion.

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[22]For more, see [6, 5].

[23]Note that the term accessibility relation does not always make perfect sense for H-models. When talking about approaches other than state-based, it's hard to say what it is that's being considered accessible. Regardless, for terminological ease we will stick with it.

5.2   Axioms of many-valued hyperintensional modal logic

An axiomatization of the our logic is just the axiomatization of $\mathfrak{A}$. Intuitively this is easy to see: compared to H($\mathbf{2}$) we lose the duality axiom by moving to many-valued logic, and compared to MS($\mathfrak{A}$) we lose RE by virtue of our hyperintensional definition of the modality.

For brevity, we will only sketch a proof of this. The interested reader is referred to look at axiomatization proofs for MS($\mathfrak{A}$) in [16] and [9] for a more detailed analogous proof and explication of concepts as 'consistency' and 'maximally consistent sets'. A more general, though more complex, proof is given in [2]. The changes to this proof for H($\mathfrak{A}$) follow straightforwardly from the definition of the canonical model below, which is itself analogous to the proof sketched for axiomatization of H($\mathbf{2}$) in [21].

Soundness holds since all axioms and rules of $\mathfrak{A}$ hold at all worlds, each of which functions like non-modal $\mathfrak{A}$ with $\mathcal{V} = \mathcal{V}(w)$, and are therefore generally valid in H($\mathfrak{A}$).

Completeness can be proven by a canonical model construction, constructing a model $\mathcal{M} = \langle \mathcal{W}, \mathcal{C}, O, \mathcal{R}, I \rangle$ where $\mathcal{W}$ is the set of maximal consistent theories of $\mathfrak{A}$, $\mathcal{C} = \mathsf{Form}$ with O the identity, $\mathcal{R}(w, \varphi)$ and $I(w, \varphi)$ respectively equal to the value of $\Box\varphi$ and φ at the theory at w. For any maximally consistent theory $\Gamma \in \mathcal{W}$, we have that $\varphi \in \Gamma$ iff $\mathcal{V}(w, \varphi) \in \mathcal{D}$. Suppose then that X where a theorem of H($\mathfrak{A}$) not derivable by the axioms of $\mathfrak{A}$. Then there exists a maximally consistent set Γ such that X is not derivable from Γ. But since Γ is maximally consistent, X like all formulas, must have a valuation. Therefore we find a countermodel.

The main conclusion to be taken here is that H($\mathfrak{A}$)-models allow us to introduce modalities to many-valued logics without them being subjected to strong conditions like necessitation, axiom K and, in particular, RE.

6   A brief look at some cases of the framework

Discussing the axiomatization of H($\mathbf{2}$), Sedlar [21] notes that "unsurprisingly, the logic of all hyperintensional models is not very interesting". Losing the duality axiom, many-valued hyperintensional logic appears even less interesting.

It is therefore only in imposing certain conditions on the nature of $\mathcal{C}$ that our logic starts to shine. In the following sections we will discuss a number of these.

6.1   State-based approaches

6.1.1   Augmented frames and Kripke semantics on impossible worlds

First we consider $\mathcal{C} = 2^{\mathcal{S}}$, a set of propositions on an extension of $\mathcal{W}$, like in section 4.4. Together with $\mathcal{R} : \mathcal{W} \times \mathcal{A}^{\mathcal{S}} \to \mathcal{A}$, which as per usual designates to which degree propositions are seen at a world, we now have a notion of a H($\mathfrak{A}$)-frame: the pair $\langle \mathcal{W}, \mathcal{R} \rangle$.

As is to be expected, we have a notion of augmented frames leading to a special case also

interpretable with Kripke semantics. We call a H($\mathfrak{A}$)-frame augmented when for each $w \in \mathcal{W}$ there exists a unique $C_w \in \mathcal{A}^{\mathcal{S}}$ such that for any $X \in \mathcal{A}^{\mathcal{S}}$ we have

$$C_w \subseteq X = \mathcal{R}^{\square}(w, X)$$

$$C_w \odot X = \mathcal{R}^{\diamond}(w, X)$$

Note that we now use propositions on $\mathcal{S}$ rather than on $\mathcal{W}$. However, we still only consider $w \in \mathcal{W}$ since modal formulas, like all formulas, are considered to be arbitrarily valuated at states $s \in \mathcal{S} \setminus \mathcal{W}$.

One might well argue that modalities themselves should at non-classical worlds still function truth-functionally like at classical worlds. In the epistemic case, for example, an agent might be in a situation where he considers worlds where statements about the world are, unbeknownst to him, logically contradictory as epistemically possible, but this does not mean that he expects his reasoning to be different at such worlds. After all: he is not aware that these worlds are contradictory, or he would not consider them possible.

We could then extend $\mathcal{R}$ to a function from all of $\mathcal{S}$, rather than just from $\mathcal{W}$, which gives the valuation of modalities at non-classical worlds. In the Kripke case, we'd also extend the above conditions to all $s \in \mathcal{S}$.

6.1.2   Binary relations

As discussed in section 3.4.1, we might consider binary-valued accessibility relations, as the fact that propositions and modalities are $\mathcal{A}$-valued does not need to mean that the way we consider worlds to be epistemically possible needs to be $\mathcal{A}$-valued as well.

In the Kripke case we can consider an augmented H($\mathcal{A}$)-frame whose core labels states only with values in $\mathbf{2}$. In the MS case we can give a behind the scenes binary accessibility relation $\mathcal{R}^* : \mathcal{W} \times \mathbf{2}^{\mathcal{S}} \to \mathbf{2}$ and define $\mathcal{R}(w, X) = \mathcal{R}^*(w, \{s | (s \in X) \in \mathcal{D}\})$. Like discussed, this results in modal formulas being valued only in $\mathbf{2}$.

6.2   Syntactic approaches

In section 4.3 we considered syntactic approaches to hyperintensionality in the case of classical logic. An analog for many-valued logic is easy to find: again, set $\mathcal{C} = \mathsf{Form}, O = ID, I$ assigning truth in worlds like $\mathcal{V}$ in MS($\mathfrak{A}$) and let $\mathcal{R}$ designate to which degree the agent is at a given world aware of a given formula.

Again, notice how easily when can vary the values of propositional variables at worlds separately from those of modal formulas. If we consider an agent to be either aware or unaware of a formula, $\mathcal{R}$ may be chosen to be binary, where only facts about non-modal formulas are many-valued. Alternatively, we may use many-valued valuations to model degrees of awareness, for example by real numbers in $[0, 1]$ in Ł$_{\aleph}$, where 0 is total unawareness and 1 total awareness.

In case of the double modality of [7], described in section 4.3, there are some interesting interactions if the modalities are many-valued. If facts at worlds are to be $\mathbf{K_3}$-valued, with

n expressing that a propositional valuable is yet to be instantiated because it expresses a fact about the future, awareness indicates whether the agent would be able to know the fact when it gets instantiated. Similarly, degrees of awareness might vary among worlds and non-classical values could be used to express whether the agent might come to be aware or will forever be unaware.

6.3   Structuralist approaches

Lastly we will consider many-valued structuralist approaches, similar to section 4.5. Again we can take $\mathcal{C}$ as our set of preferred structures, with O designating the structural content of formulas, I giving the propositions expressed by these structures and $\mathcal{R}$ giving the degree of relation towards a structure. This degree may still be binary, with only possible-world propositions being valued, or it may be many-valued.

These values could for example express levels of confidence. In the circle on page ten example of section 4.5 Sarah may vaguely remember that circles are the same as sets of points equidistant from a given point, without having absolute confidence in it. Similarly, the effects of performing a belief-producing procedure may wear over time, with the agent being to some degree aware of the produced knowledge but no longer having a foundation for it at full strength as at the time of inference.

7   Conclusions and future work

In this paper we set out to give an account of many-valued logics with hyperintensional modalities. We discussed a couple of many-valued modal logics and a number of approaches to hyperintensional logics based on classical logic. We showed that these hyperintensional logics could be well generalized to many-valued logics, and are all special cases of a many-valued generalization of the hyperintensionality-framework by Sedlar [21].

Further work might consist of looking into special cases of our framework. In section 6 we gave some ideas as to how state-based, syntactic and structuralist approaches to hyperintensionality might benefit from non-binary valuations. All of these could be explored further, and there are many many-valued logics and approaches to hyperintensionality we did not discuss, which might benefit from being combined.

More generally, many logics exist which fall outside the range of many-valued logics covered here. These include logics with different, more free interpretations of $\rightarrow$, intuitionistic logic, logics with more or fewer connectives and logics with quantifiers. It may prove fruitful to look into how our framework can be extended to include these logics.

Lastly, work is to be done on the interpretation of many-valued modal logics and whether non-binary valuation is relevant to express facts about propositional variables at possible worlds and about modalities, and how these interact with each other. Additionally, it may be worth inquiring more into the interpretation of many-valued modalities. If truth of non-modal formulas is to be adequately expressed in many values, modalities maybe should reflect awareness of that. Rather, under current interpretation, the many-valuedness of non-binary

modalities expresses something about the relation to the modalized formula, rather than the formulas valuation. In epistemic modal Ł$_\aleph$-logic, for example, $\Box\varphi$ being valued 0.7 might be interpreted as an agent being 70% aware of φ. This is not the same, however, as the agent (fully) believing that φ is 70% true, which is not modelled by our accounts of many-valued modal logic. Similarly, lacking belief of φ's truth or falsity, isn't the same as believing that φ is neither true nor false.

All this should expand our understanding of how facts about the world and about modalities, such as knowledge and belief, are to be well expressed and how they relate to each other. As a result, we might create more effective agents, whether they are human or artificial, and understand them better.

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