Implementing Active Management Risk in the Standard Model for the Required Funding Ratio

Hans Westrik

June 6, 2012

Summary

The funding ratio, calculated as the total assets divided by the total liabilities, is a frequently used indicator for the financial health of a pension fund. Using six risk elements the standard model calculates the required funding ratio, which is an additional buffer on top of a funding ratio of 100% such that there should exist only a 2.5% chance that within a year the funding ratio will be below 100%.

In this thesis we analyze this model and try to improve the calculation of the required funding ratio by adding an extra risk element, which we call the active management risk; the risk of (actively) deviating from a benchmark. We give measures and methods to calculate this risk element. After analyzing these measures and methods we try to implement this risk element in the standard model to see its effect on the required funding ratio.

Keywords: Required funding ratio, Standard model, Active management risk, Tracking error, Coherent risk measure, Monte Carlo simulation.

Preface

In the second year of my Master, early in 2011, I went looking for an internship to write my thesis. Consulting my internship coordinator Sasha Gnedin I came across a possibility at Syntrus Achmea. Although I was not completely sure if pension funds would be lively enough to find an interesting thesis subject, I took a shot at it, which turned out great. I would like to thank Sasha Gnedin for giving me this opportunity.

After some talks with Syntrus Achmea we decided I could get started in August. During this time I also approached Erik Balder, professor at Utrecht University, to ask him to supervise me. To this he agreed and I would like to thank him for keeping me sharp during my internship and helping me when needed. At the end of my intern Karma Dajani, senior researcher at Utrecht University, agreed to be my second reader, I would like to thank her for making time for this.

In the first stage of my internship I had a lot to learn about the company, about the Dutch pension system and pension funds and of course about subjects regarding my research area. I would like to thank Wim van Straten, my supervisor at Syntrus Achmea for guiding me through this process and giving me helpful feedback. Also I would like to thank Pepijn Carpay, my manager, who helped me a lot by discussing my progress, arranging meetings and other things and helping me to keep on schedule.

In the second stage my thesis began to take form. I often sat with Edward Bos, Abdel el Amrani, two other colleges at Syntrus Achmea, as well as Wim van Straten and Pepijn Carpay to discuss my progress in this thesis and to discuss what still needed to be done. Thanks for giving me helpful feedback and a clear goal for further improvements.

I also like to thank the whole Actuarial Control team I worked with during my internship. I experienced this team as a very close group and I am thankful to have been a part of it.

Finally I would like to thank my family, my friends and my girlfriend.

This is my thesis. I hope you enjoy it.

Hans Westrik De Meern, June 2012

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Chapter 1

Introduction

1.1 Pension risk

A pension provides income after a person retires, over his/her working years a participant makes a monthly contribution to a pension fund in the form of a percentage of his/her salary, in exchange for this contribution the participant wants security on his income upon retirement. The pension fund collects all these contribution and invests it. If a fund does nothing with the contributions and just saves the money to provide a fixed income after retirement, then this income will be very meager; the value of the money will have diminished over time due to inflation. In trying to keep up with the economic growth a pension fund will have to invest its money in stocks, bonds or other securities. But with these investments comes risk, one cannot be certain that these investments will be profitable. But how risky are these securities, how can a fund protect itself against these risks?

1.2 Required funding ratio

A foolproof protection against these risks is of course not feasible, but risk management theory provides ways to give some insight in these risks. A fund should then for itself decide how much risk it wants to take on, or, like in the Netherlands, the government provides rules and some guidelines to give pension funds a certain basis of protection against risk. These rules and guidelines are set up by the Dutch central bank, from now on called DNB[1], and focuses among other things on the funding ratios. The funding ratio is used as an indicator of the financial situation of a pension fund; it is calculated as the present value of the assets divided by the present value of the liabilities. For a pension fund the assets will consist of securities and the liabilities of provision, in the form of the promised payment for participants after retirement. A funding ratio of 100%

[1]DNB is short for De Nederlandsche Bank as it is called in Holland.

means that the value of 'everything owned' is equal to the value of 'everything owed'. The DNB wants all funds to have a funding ratio higher then 100% most of the time, all the time would be infeasible. Therefore DNB has provided a model, called the *standard model*, to calculate the *required funding ratio*. This required funding ratio is a funding ratio of 100% plus an additional buffer such that with this buffer there should exists only a 2.5% chance that within a year the funding ratio will be below 100%.

1.3 Problem statement

But how to make sure that this model calculates the required funding ratio precise enough to give such a 97.5% confidence level of not falling below a 100% funding ratio? Analyzing economics is a tricky business. Statistics only give estimations of probabilities. Furthermore, not all pension funds have similar investments. Does the standard model give a uniform way to calculate the required funding ratio? Or are there fund specific risks that should be accounted for? It's not realistic to cover all sides of these questions. That's why in this thesis we will focus on the risk of active management. And more precisely we'll focus on the question:
"How can we adjust the standard model, so that it takes into account the risk of active management?"

1.4 Thesis structure

The structure of this thesis is as follows. First we give an introduction of the standard model in Chapter 2. Then we introduce all the risks used in the standard model and the risk of active management in Chapter 3. To measure the risk of active management we'll focus on active returns and in Chapter 4 we give some basic notions and their properties needed to further examine these active returns. In Chapter 5 we introduce some measures for quantifying active risk; these measures are based on the assumption that we know the distribution of the active returns. To get this active return distribution there exist some methods which we'll discuss in Chapter 6. Now that we know the way to calculate active management risk we can try to adjust the standard model by implementing an active management risk component, for this there are already a few methods available. We will discuss these methods and try to work towards a 'best method' in Chapter 7.

Chapter 2

The standard model

In this chapter we introduce the standard model that is used to calculate the required funding ratio. We first cover some history in the following section. Then we'll discuss how the model was made and the assumptions made to get the parameters used in the standard model. And lastly we'll give some variations of the standard model which DNB has given as possible models to use.

2.1 The financial assessment framework

The financial assessment framework (FTK), which provides supervision over the Dutch pension funds, was implemented in the Dutch law in 2007. In this framework the Dutch government gives rules and some guidelines for pension funds, such that these funds take caution when making investment decisions and that the financial situation of a fund is transparent. Before the FTK there were the Actuarial Principles Pension funds (APP), implemented in Dutch law in 1997. In these principles the present value of the provision was calculated with a fixed interest rate of 4% and the rules for a buffer against risks were minimal as were the requirements for transparency. When the stock market crashed in the end of year 2001, the Dutch government began formulating stronger restrictions in keeping the funding ratios in control. And in 2004 a first version of the standard model was introduced by (de Geus, 2004)[1]. In this standard model a required funding ration was calculated taking into account the risks of an average pension fund. This form was further developed by (DNB, 2006)[2] and with this document the standard model was ready to be implemented in the law which happened on the first of January 2007. Thereafter the standard model was revised by (DNB, 2011)[3], this work proposes among other things to work with

[1]The title of this work "Hoofdlijnen voor een nieuwe Pensioenwet" translates to "Guidelines for regulating the financial supervision of pension funds".

[2]The title of this work "Advies inzake onderbouwing parameters FTK" translates to "Advice concerning the underpinning of the parameters in the FTK".

[3]The title of this work "Uitwerking herziening berekeningssystematiek vereist eigen vermogen" translates to "Revision of the calculation system for the required funding ratio".

new interest rate factors and to introduce a new risk for active management, see Section 3.9 for a list. Also the introduction of a partial internal model is discussed in this work.

2.2 The principles

In forming the standard model the main purpose was to guarantee the confidence level of 97.5% of not being underfunded within one year. Even if the statistic used to calculate risks would be very accurate, to make a model that works for every fund would still be extremely complex, because every fund has its own fund specific risks. Furthermore the standard model was meant to be fairly easy implementable. So in forming this model focus was also put for it being a general and easy to use model. To calculate the required funding ratio the total risk is divided into 9 risk elements denoted as $\mathbf{S_1},\ldots,\mathbf{S_9}$. These 9 elements are

$\mathbf{S_1}$: Interest rate risk

$\mathbf{S_2}$: Equity (and property) risk

$\mathbf{S_3}$: Currency risk

$\mathbf{S_4}$: Commodity risk

$\mathbf{S_5}$: Credit risk

$\mathbf{S_6}$: Underwriting risk

$\mathbf{S_7}$: Liquidity risk

$\mathbf{S_8}$: Concentration risk

$\mathbf{S_9}$: Operational risk

For each of these elements, parameters that should lead to a 97.5% confidence level have been estimated, the last 3 risks, however, were assumed to have a neglectable effect on the required funding ratio and were put to zero. For these 9 risk elements conditions were taken into account which (DNB, 2006, p. 7-8) nicely put in their work. The Dutch version of this text is included in Appendix A.4, freely translated it says

> The basis is to determine the required funding ratio in such a way that with a confidence level of 97.5% it is prevented that a pension funds has less assets then liabilities within a period of one year. Ideally the risk profile of a pension fund is exactly matched. This would however require an internal model, which is possible only for a few funds. This is the reason that a relative easy and tractable standard model is available. This standard model is less refined in comparison to an internal model and hence requires some prudence. With the standard model it is tested how sensitive a financial institution is to

different scenario's, like a decline in the stock market or a change in the interest rate structure. These scenario's are chosen in such a way that there exists a 1 in 40 chance it will happen (97.5%). The parameters in the standard model are therefore 'shock parameters' calibrated on a risk horizon of one year and a confidence level of 97.5%: they give the change in a risk factor[4] (for example a decline of 25% for developed market equities). The scenario's should match the mentioned confidence level of 97.5%. But here it also applies that an exact relation to the confidence level is not tractable, because this would suggest a precision that can never be reached. Determining a scenario that would happen once in 40 years is not easy. This is mainly due to lack of historic observations to make such an estimation; even for stock and interest rate markets where there is fairly much historic data available, this is not easy. On top of this, expected returns, volatility (standard deviation) and correlations are not stable variables over time.

Principles

With the considerations above a number of conditions are formulated. In choosing the parameters these conditions should be satisfied as much as possible.

1. A realistic estimation of the parameters should as much as possible be made with relevant and reliable historical data.
2. If insufficient relevant and reliable data exist then an assumption can be made on the probability distribution of the returns with a mean and a standard deviation.
3. Market valuation is an important principle in the pension law. If historical series are distorted by other valuation principles, then it could happen that the true risk is underestimated. A correction to this can be applied. Real property is an example of this.
4. Lack of transparency entails risk. For hedge funds, for example, this should lead to an application of a higher risk factor.
5. In changing parameters there is a conservative approach. If there is a lack of a significant advantage in changing the parameter, then the original parameter will hold.
6. To avoid the illusion of precision, the parameters are rounded.

With these principles the scenarios for the risk elements were choosen. When reading "Advies inzake onderbouwing parameters FTK" by (DNB, 2006) and

[4] In this calibration the expected value of the risk factor is taken into account.

looking at the proposed adjustments[5] and their motivation by (DNB, 2011), there are, however, implicitly a few more conditions that the parameters can satisfy. These extra conditions or added principles we formulate as follows

Added principles

7. If further specifications in risk elements does not lead to noticeable changes, then the original model without these specifications should hold. In order to not make the model too complex.
8. The determination of a parameter should be consistent with the determination of the other parameters.
9. Since the parameters are chosen for extreme events (97.5% confidence), the correlation parameters should be adjusted for these extreme events and not solely on the average correlation.
10. Correlation parameters should also consider diversification advantages.

2.3 A standard pension fund

When these scenarios were calculated DNB took into account the average position of pension funds in the Netherlands. (de Geus, 2004) states that for a standard fund a position of 50% in bonds and 50% equities is assumed and a duration[6] gap of 11 years, that is, a duration of 5 for bonds and a duration of 16 for the liabilities of the pension fund. (DNB, 2006) specifies this position even further, in summary this gives an asset allocation of

- 50% in bonds
- 34% in developed market equities
- 3% in emerging market equities
- 3% in unlisted stocks
- 6% in indirect property
- 4% in direct property

If we look at historic data in the period from 2000 to 2006 taken from DNB, CBS[7] or OECD[8] we see that the 50/40/10 position in bonds, stocks and property respectively, holds roughly. Although for DNB the position seems to fluctuate

[5]These adjustments are discussed in Section 3.9.
[6]Duration is the weighted mean average of all cash flows, often used as a measure for interest rate sensitivity.
[7]Centraal Bureau voor de Statistiek, and in English: Statistic Netherlands.
[8]Organisation for Economic Co-operation and Development.

more around[9] 40/41/10/9, with the last 9 is for 'remaining' investments. For CBS the position seems to fluctuate around 45/50/5. And OECD the position seems to fluctuate around 43/48/5/4. The specification of equity in developed, emerging or unlisted equity, and of property in direct or indirect property is harder to check. This is because pension funds only have the obligation to report its investments in these specific elements since 2007; hence data about this before 2007 is not centrally documented.

This standard pension fund is used mainly to give an indication of what the required funding ratio will be for an average fund[10] and how it will change if parameters are changed. It was also mentioned by (de Geus, 2004) that the aim of the standard model was to keep the required funding ratio under 130% which is the required funding ratio in the simplified model discussed in Section 2.4. As we've seen in Section 2.2 the scenarios are worked out separately, so the asset allocation of the fund has little to do with the way the parameters are chosen. However, (DNB, 2006, p. 37-39) do make one adjustment that is based on the asset allocation of a fund. When analyzing funds with very large and very small positions in equities they found that an exchange between the correlation parameter between $\mathbf{S_1}$ and $\mathbf{S_2}$, discussed in Section 3.8, and the interest rate factors, discussed in Section 3.1 would better match these funds.

2.4 Other models

Alongside the standard model, DNB also gives the possibility for a simplified and an internal model. However, in practice these models are almost never applied by pension funds. (DNB, 2011) introduces the partial internal model as a possible alternative in the future.

Simplified model

The simplified model calculates the required funding ratio very straightforward as 130%, which is the boundary that was set for the required funding ratio calculated by the standard model. To qualify for use of this model however, a fund must have a simple pension scheme and its investment policy must be risk-averse. In practice this means that only small funds can qualify for these requirements and at the moment no pension funds use this model.

Internal model

With an internal model a pension fund creates its own model to calculate the required funding ratio. In contrast to the standard model which gives a fixed way for calculating the required funding ratio, the internal model can be very flexible and is meant to cover all the fund specific risks appropriately, which

[9]These positions are calculated as the mean position over the period 2000 to 2006, rounded to integers.
[10]Namely 125.64% as calculated by (DNB, 2006) with the then current interest rates.

makes this model complex. Furthermore, an internal model must satisfy some qualification requirements given by DNB and when in use, funds are required to inform DNB of its results relative to the standard model. Due to the complexity and the additional cost for regulatory reasons, this model is only realizable for large pension funds and even then the benefits are questionable.

Partial internal models

For a fund to go from the standard model to an internal model is a big step, that funds can not easily make. To offer an intermediate step (DNB, 2011) suggests introducing another model, or in fact, other models, namely the partial internal models. These models can be seen as an extension of the standard model in the following way. A fund uses a priori the standard model to calculate the required funding ratio, but if a fund deviates too much from the standard then DNB will advice and can oblige the fund to adjust its model to a partial internal model. Some possible adjustments, which we will also see further on in this thesis, are for example

- Adding an active management risk component.
- Using a fund specific currency basket.
- Using a fund specific benchmark and its risk.

With these partial internal models a fund can better match its fund specific risk profile.

Chapter 3

Risks from the standard model

To understand the standard model better, we here describe the risk elements that are used in the standard model and we briefly discuss how the parameters where chosen by (DNB, 2006), keeping in mind the principles stated in Section 2.2. We'll also mention the square root formula used to calculate the required funding ratio from the risk elements and we pay some attention to correlations between certain risks. In Section 3.9 we'll mention some adjustments that are suggested by (DNB, 2011). Lastly we'll discuss the risk of active management in Section 3.10.

3.1 Interest rate risk

The funding ratio is calculated by dividing the present value of the assets by the present value of the liabilities. For fixed income securities and liabilities these present values are calculated using a yield curve. DNB calculates the yield curve using the euro swap curve based on interest rates swaps, this swap curve is in Dutch called "rentetermijnstructuur". For each maturity the level of the interest rate is given in this swap curve. With these interest rates the fixed income securities and liabilities are discounted to give the present values.

The risk involved in this risk element is the risk that the interest rate will change and hence the present value of these securities and provision will change. This risk is unique in the sense that it gives a change on the asset side and on the liability side of the balance sheet. Hence the risk taken into account when calculating the required funding ratio can come from an increase or a decrease interest rate. Usual for a pension fund is that a decrease in interest rate will give the most negative effect on the funding ratio since the duration of the liabilities (16 years on average) is greater then the duration of fixed income securities (5 years on average) and the amount of liabilities is usually bigger than the amount of fixed income securities.

In order to calculate this risk element, the standard model uses interest rate factors. DNB calculated these factors in 2006 using the model

$$\ln(r_{t+1}^n) - \ln(r_t^n) = \varepsilon_t^n \quad \text{with } \varepsilon_t^n \sim N(0, \sigma_n^2).$$

Here n stands for the maturity of the risk-free interest rate. So these factors represent the logarithmic interest changes and a normal distribution around zero on these logarithmic changes was assumed. In this way a boundary for the 97.5% confidence level follows for every maturity n by $1.96\sigma_n^2$. The data used for maturities up to $n = 10$ were the German Zinsstrukturkurve on the time interval 1973 to 2003. For the other maturities 11to 25 the German Zinsstrukturkurve was extrapolated and the Euribor curve was interpolated between the maturities 1, 5, 10, 15, 20, 25 and 30 using data of the Euribor curve on the time interval 1997 to 2005. These curves were considered to be the most robust and representative for calculating the interest rate factors.

Lastly it should be noted that the eventual interest rate (increase) factors presented in the next table have been artificially raised by multiplying with a factor 1.13. This raise in interest rate factors goes together with a reduction of the correlation factor between interest rate risk and equity risk. This exchange, that we mentioned before in Section 2.3, between the correlation and interest rate factors was done to better match pension funds with relative large or relative small equity positions. The factors used are

Maturity	Increase (factor)	Decrease (factor)
1	1,60	0,63
2	1,51	0,66
3	1,45	0,69
4	1,41	0,71
5	1,37	0,73
6	1,35	0,74
7	1,34	0,75
8	1,33	0,75
9	1,33	0,75
10	1,32	0,76
15	1,29	0,77
20	1,28	0,78
25	1,27	0,79

To calculate the risk element $\mathbf{S_1}$ the swap curve is shifted with these factors. Then the present value of the fixed income securities and the liabilities is calculated with the shifted swap curve and the actual swap curve and from this the risk element $\mathbf{S_1}$ is calculated as the difference.

3.2 Equity (and property) risk

This risk element covers the risk that investments in shares and property will loose value cause of market dynamics. DNB prescribes that a fund should account for a loss in value of

$\mathbf{S_{2A}}$: 25% for developed market equities (including indirect property).

$\mathbf{S_{2B}}$: 35% for emerging market equities.

$\mathbf{S_{2C}}$: 30% for unlisted stocks (private equity).

$\mathbf{S_{2D}}$: 15% for direct property.

These scenarios represent an overall decline in the market value of these shares or properties. That is, if a fund has 20% of their capital in developed market equities, they have to account for a $20\% \cdot 25\% = 5\%$ loss of their capital and therefore should hold a buffer of 5% over a funding ratio of 100%.

These percentages, used to calculate $\mathbf{S_{2A}}$ to $\mathbf{S_{2D}}$, have been determined by (DNB, 2006) by analyzations of data from important indices such as the MSCI world, etc. These indices are (weighted) combinations of certain developed market equities which are intended to represent changes in the entire stock market. Pension funds often use them as a standard also called a benchmark: it is an index to which a fund compares its performance. A fund follows a benchmark if it uses this index as a standard. It can outperform or underperform a benchmark meaning that its return is higher or lower than the return of the benchmark.

Ideally there should be a scenario percentage for every benchmark that can be followed, representing the risk in this benchmark. But this would make the standard model too complex. Therefore, a number of benchmarks are taken together to form one parameter. Here special attention should be placed on the criterion of relevant and reliable historical data. The data should be representative; it should reflect the stocks in which Dutch pension funds invest on average, and there should be enough available history. Also survivorship bias must be taken into account; stocks that perform badly are often taken out of the index or not included, this could lead to an optimistic view of equity risk.

With these consideration in mind 3 datasets were put next to each other to estimate the scenario for developed market equities, namely the datasets

- the MSCI world index 1970-2002, which due to a relative short history gives the most optimistic view
- the Dimson dataset 1900-2000, which contains a long history of 16 countries (one of them the Netherlands) put together to make a world benchmark
- the US dataset by Shiller 1871-2002.

For each the volatility is estimated and assuming a normal distribution around an expected return of 8% the downside scenario is estimated (and rounded) to 25% for developed market equities.

The downside scenarios for emerging market equities, unlisted equities and direct and indirect property have been determined in a similar way. Here however there exists fewer historical observations and survivorship bias plays a more important role. Furthermore due to autocorrelation the volatilities could be underestimated. The market value of stocks, driven by supply and demand, are (almost) constantly available, but for private equity and real property the returns are based on book values or other less frequent valuation methods, reported on monthly, quarterly or a yearly basis. For periods in between the value of such an investment is estimated or set equal to the book value. Because of this, the returns on these investments over time are dependent on each other, that is, they have a positive autocorrelation. With all these considerations in mind DNB came to the scenarios mentioned above.

To get the risk element $\mathbf{S_2}$ from the elements $\mathbf{S_{2A}}$ to $\mathbf{S_{2D}}$ (DNB, 2006) uses a "a strong, but not perfect correlation" between these elements of $\rho' = 0.75$. This represents the view that, especially in extreme events, the equity market tends to 'move together', but that diversification advantages exist between the different types. With this correlation the equity risk element can now be calculated as

$$\mathbf{S_2} = \sqrt{\mathbf{S_{2A}}^2 + \mathbf{S_{2B}}^2 + \mathbf{S_{2C}}^2 + \mathbf{S_{2D}}^2 + 2\rho'(\text{cross terms})}$$

with

$$\text{cross terms} = \mathbf{S_{2A}}\mathbf{S_{2B}} + \mathbf{S_{2A}}\mathbf{S_{2C}} + \mathbf{S_{2A}}\mathbf{S_{2D}} + \mathbf{S_{2B}}\mathbf{S_{2C}} + \mathbf{S_{2B}}\mathbf{S_{2D}} + \mathbf{S_{2C}}\mathbf{S_{2D}}.$$

3.3 Currency risk

Currency risk is the risk that the value of investments in foreign currency will diminish due to movements in foreign exchange rates. The 20% parameter for this risk element is based on the exchange rates of a basket of foreign currencies against the euro. This basket is based on the average position of pension funds in foreign currencies and the exchange rate is the weighted average of these foreign currency exchange rates. By (DNB, 2006) the parameter was estimated (and rounded) to 20% using the basket

Currency	**Weight**
US Dollar	35%
British Pound	24%
Argentine Pesos (with a correction)	13%
Japanese Yen	8%
Swedish Crown	7%
Swiss Franc	7%
Australian Dollar	6%

The correction takes into account investments in emerging markets; the volatility of these investments is similar to the volatility of the peso. To calculate $\mathbf{S_3}$ for a fund, this 20% is used for all its investment in foreign currencies, making the assumption that the funds investments in foreign currencies is diversified according to the basket above.

3.4 Commodity risk

Commodity risk is the risk that the value of investments in commodities will decrease. Here the 30% parameter was determined using the Goldman Sachs Commodity Index (GSCI) which is a basket of 24 commodities. And for a fund this 30% fall is used for all its investment in commodities, making the assumption that the funds investments in commodities is diversified according to the GSCI. With this 30% parameter the element $\mathbf{S_4}$ can be calculated.

3.5 Credit risk

Credit risk plays a role when investing in bonds, it is the risk that the creditworthiness of counterparties will diminish and that the credit spread will increase. (DNB, 2012) defines credit spread by

> **Credit spread:**
>
> “Credit risk is reflected in the interest margin on credits (i.e. the credit spread). This is the difference between the redemption yields on a collection of cash flows whose payment depends upon the creditworthiness of counterparties and the redemption yields on the same collection of cash flows where they are certain to be paid. The risk-free redemption yield is a consequence of the prescribed term structure of interest rates.”

So the credit spread is a measure for the creditworthiness of the counterparty, i.e. the confidence that the counterparty will make the payments as promised. In the model the credit risk element is then determined by assuming that the average credit spread will increase by 40% which will affect the value of investments. This 40% is estimated using the credit spread on investment grade corporate bonds[1] in the period 1999-2004. The risk element $\mathbf{S_5}$ is calculated as the difference between the present value of the bonds using the actual credit spread and the present value of the bonds using the 40% increased credit spread.

[1] Rating BBB or higher

3.6 Underwriting risk

For pension funds the underwriting risk consists of mortality related risks, i.e. the risks that the life expectations of the participants don't match with reality. Deviations from the expected mortality rate are considered and deviations from the expected mortality trend (longevity risk). For more on mortality rates and mortality trends see (Schaeffers, 2010).

When there is such a deviation from the expected mortality the calculated provisions will not match with the actual provisions, the risk that the calculated provisions are too low is called the underwriting risk. With the characteristics of a fund the risk element $\mathbf{S_6}$ can be calculated using parameters determined by DNB.

3.7 Other risks

Beside these 6 risk elements, DNB also mentions 3 other risks, namely liquidity risk, concentration risk and operational risk. These risks are assumed to have a neglectable effect and (DNB, 2006) puts these parameters to 0%. A fund is, however, required to monitor these risks. In this section we'll shortly explain these risks.

3.7.1 Liquidity risk

An asset is called liquid if it can easily and quickly be converted into cash. Liquidity risk is then the risk that a fund does not have sufficient liquid assets to meet their liabilities at some point. If this happens, the fund can be obliged to sell an illiquid asset for less than its market price. For most pension funds the pension contribution to be received in a certain period is higher then the pension liabilities to be paid in the same period. Therefore this risk is set to zero.

3.7.2 Concentration risk

Concentration risk is the risk that the investments of a fund are not diverse enough. If, for example, a fund invests mainly in Dutch companies, then it will be highly dependent on the performance of the Dutch market. If some external event affects only these Dutch companies, for example a change in government rules and regulations, then this will have a great effect for this fund and only a small effect on a diversified fund. Concentration risk can also be present if a fund is (relatively) concentrated in one company or one sector, but also on the liability side, if the average age of the participants of a pension fund is high or the age dispersion is low, then this can be seen as a form of concentration risk. However, if a fund uses the standard model it is assumed that its investments, and liabilities, are well diversified, so the concentration risk can be set to 0%. If a fund is not well diversified then DNB advises the use of an internal model.

3.7.3 Operational risk

Operational risk is the risk that arises from failures or defects in the operational process. It should be taken very broad, so fraud and miscalculations are included, internal events like the breakdown of an IT system is included and also external events like a natural disaster is included. This risk is hard to model and therefore cannot be easily included in the standard model. Although this risk is inevitable, its effect is often minimal, therefore this risk is set to 0%.

3.8 Square root formula and correlation

When each risk element $\mathbf{S_1}$ to $\mathbf{S_6}$ has been calculated, then the total buffer needed for all risks $\mathbf{S_t}$ is calculated by the square root formula

$$\mathbf{S_t} = \sqrt{\mathbf{S_1}^2 + \mathbf{S_2}^2 + 2\rho\mathbf{S_1}\mathbf{S_2} + \mathbf{S_3}^2 + \mathbf{S_4}^2 + \mathbf{S_5}^2 + \mathbf{S_6}^2}.$$

Here we see that most risk elements are assumed to happen independently, without correlation. The only correlation here is between interest rate risk and equity risk, with $\rho = 0.5$. This correlation represents the expectation that these risks can happen at the same time, a decline in the swap curve indeed often goes together with a decline in the stock market when analyzed. This correlation is based on the adjustment mentioned in Section 2.3, before this exchange between correlation and interest rate factors the correlation was assumed to be 0.65. Another reason to change this correlation from 0.65 to 0.5 was to avoid the illusion of precision, see principle 6. However, the historical correlation varies over time, (Andersson et al., 2004) shows evidence and reasons for this phenomenon. The mean of the historical correlation can be shown to be lower then 0.65. The reason why a higher (than the mean) correlation is assumed, is because the 97.5% scenario is based on an extreme event and in extreme events the correlations often rises.

3.9 Proposed adjustments

After reviewing the risk scenarios (DNB, 2011, p. 3) proposes a list of adjustments. These are not yet included in the standard model, but for further discussion it's good to keep them in mind. Here's the list

scenario	risk factor	sub factor	current	new
$\mathbf{S_1}$	interest rate risk	15 year decrease factor	0.77	0.75
$\mathbf{S_2}$	equity risk	developed market equities	25%	30%
		emerging market equities	35%	40%
		unlisted stocks	30%	40%
		property	15%	15%
$\mathbf{S_3}$	currency risk		20%	15%
$\mathbf{S_4}$	commodity risk		30%	35%
$\mathbf{S_5}$	credit risk	AAA	40%	60 bps
		AA	40%	80 bps
		A	40%	130 bps
		BBB	40%	180 bps
		$\leq$BB	40%	530 bps

The bps in this list stands for basis point, 1 basis point equals 0.01%. For correlations the adjustments are

scenario	scenario	ρ_{current}	ρ_{new}	
$\mathbf{S_1}$ (interest rate)	$\mathbf{S_2}$ (equity)	0.5	0.4	if $\mathbf{S_1}$ is based on a decrease
$\mathbf{S_1}$ (interest rate)	$\mathbf{S_5}$ (credit)	0	0.4	if $\mathbf{S_1}$ is based on a decrease
$\mathbf{S_1}$ (equity)	$\mathbf{S_5}$ (credit)	0	0.5	

After evaluating the model, it came to light that some risk elements were underestimated. These changes should, therefore, better fit the 97.5% confidence criteria. Further proposals made in this document are the introduction of a partial internal model, discussed in Section 2.4 and the introduction of a new risk element, namely active management risk, which will be discussed in the following section.

3.10 Active management risk

Active management, in general, is any investment strategy where a fund actively buys and sells securities. Usually this is done in comparison to a benchmark; a fund tries to deviate from a benchmark, using analytic research, forecasts, and their judgment and experience, in the hope of outperforming the benchmark. Active management can, however, also exist for other reasons then outperformance for example a fund can invest actively when it wants to

- *Lower the volatility of the benchmark and hence reducing the risk.*
  This can be done by buying securities in the benchmark with low volatility, which is called low volatility investing.

- *Lower the volatility of the total portfolio.*
  Securities that have a low correlation with the benchmark can be bought to give a greater diversification effect which lowers the volatility of the total portfolio.

- *Invest ethically also called socially responsible investing.*
  Here a fund seeks out companies that are engaged in environmental sustainability, human rights, consumer protection, etc. and it avoids investments in companies that are involved with alcohol, tobacco, gambling, weapons, etc.

As a counterpart of active management there is passive management. This management style often takes the form of index tracking where a fund tries to mirror a benchmark. In this way a fund can profit from market growth with minimal costs. In theory it will get the exact same return as the benchmark, however, the composition of the benchmark is not kept constant; sometimes weights are changed or securities are left out and new securities are put in. If a fund wants to mimic these changes there will be transaction costs and, furthermore, these changes can't be done immediately. Hence the return of the index tracking fund will never be exactly the same as the return of the benchmark and there will be an error between these returns. This error is called the tracking error[2].

For any benchmark there exists the risk that the value of this benchmark drops. Active management risk is the additional risk that comes from active investing in this benchmark. If no benchmark is followed then, because we are investigating the standard model for the required funding ratio, the additional active management risk could be calculated relative to the benchmark(s) on which the scenario for the risk in question is based. But we must keep in mind that these scenarios are rough estimations of the risk in the market, therefore it is better to measure active management risk relative to the benchmark in which actively is invested, then relative to a benchmark on which the standard model is based, provided of course that the former benchmark is representable enough for the market. However, not for all scenarios in the standard model there exist good benchmarks. That's why in herziening uitwerking (2011) it is suggested that active management risk is applied at first only to

- **Investments in developed market equities.**
  The majority of the pension funds use the MSCI or S&P500 as benchmark, these benchmarks fit the way the equity risk scenario is calculated for developed market equities.

Active management risk could later on also be applied for other categories as

- **Investments in emerging market equities.**
  For this the MSCI Emerging Markets is fairly representable as a benchmark. Therefore we can quite easily include an active management risk component.

- **Investments in private equity or unlisted stock.**
  This part of a pension fund is often already actively managed. Hence the

[2]Note that the tracking error is essentially a notion for funds that track a benchmark, but we'll see that it can also be used for active investing.

30% scenario in the standard model in some sense already includes an active management risk component.

- **Investments in direct property.**
  Like unlisted stock, this is also often already actively managed. And hence an active management risk part is in some sense already included in the 15% scenario.

- **Active currency management.**
  A fund can actively manage its currency risk and in doing so it will deviate from the basket given in Section 3.3.

- **Investments in commodity.**
  This is often already done actively. Hence the commodity risk scenario of 30% in some sense already includes an active management risk component.

- **Investments in bonds.**
  Since bonds are fixed income securities, there is a great part of interest rate risk. But here active management risk doesn't play a role because 2 comparable bonds have by construction the same interest rate risk. There is only additional active management risk on top of the credit risk of the bond. (DNB, 2006) based the risk scenario for credit risk on bonds with credit rating BBB or higher. (DNB, 2011) suggests to further specify this credit risk using all the ratings, see Section 3.9. This specification already covers most of the active management risk on top of credit risk.

- **Asset allocation.**
  Even if the strategic asset allocation is fixed, the floating asset allocation will not be constant. For suppose a fund invest 50% in bonds and 50% in stocks then after some time, if one of them has a higher return, the allocation will become for example 49%/51% if no transactions are made. There are several ways to deal with this phenomenon; there is the buy-and-hold strategy and the constant-weight (rebalancing) strategy, which are passive management strategies. But there are also active management strategies like a dynamic asset allocation strategy or a tactical asset allocation strategy, where managers are given some freedom to deviate from the strategic asset allocation in order to respond to market movements.

Since the risk in these types of active management is harder to evaluate we will keep our focus, like (DNB, 2011) suggests, mainly on the active management risk in developed market equities.

Total Expense Ratio

An important part of active management is the TER (Total Expense Ratio) of a fund. It is calculated as the total costs (management fees, administration costs, etc.) divided by the total assets. Or it can be more specified to, for example developed market equities: the total costs in developed markets divided by the total developed market equities. The TER gives the percentage of the investments which is lost to the costs. If for example the return in a year on the developed market equities is 2%, but the TER (costs) is 1% then these investments will only have a net return of 1%. It should be clear that for an active strategy the TER will be often higher than for a passive strategy.

Chapter 4

Returns

To analyze the performance of a fund portfolio compared to a benchmark over time we define 2 notions of returns: linear returns and logarithmic or log returns. We also give some properties of these returns and some basic notions used for evaluating these returns.

4.1 Linear and log returns

Let $P(t)$ be the value of a portfolio or a security at time t, suppose that $P(t) \geq 0$ for all t, i.e. we only analyze long positions, and let us assume that the value is adjusted for dividend. Then the linear return is defined as

$$R(t) = \frac{P(t) - P(t-1)}{P(t-1)} = \frac{P(t)}{P(t-1)} - 1,$$

such that

$$P(t) = (1 + R(t))P(t-1).$$

The log return is defined as

$$r(t) = \ln(1 + R(t)) = \ln \frac{P(t)}{P(t-1)} = \ln P(t) - \ln P(t-1) = p(t) - p(t-1)$$

where $p(t) := \ln P(t)$ is the natural logarithm of the portfolio value. For log returns we have

$$P(t) = \exp(r(t))P(t-1)$$

or

$$p(t) = p(t-1) + r(t).$$

4.2 Properties

The linear return has a very handy property when analyzing portfolios and the log return has a very handy property. Unfortunately one return does not have both properties. In this section we'll give these properties and we'll see that approximately these returns are equivalent, which makes it possible to use these notions interchangeably.

4.2.1 Aggregation over securities

An important property of the linear returns is that they aggregate over securities. Let $S_i(t)$ $(i = 1, 2, \ldots, m)$ be m securities and $R_i(t)$ there returns. Let portfolio $P(t)$ equal $\sum_{i=1}^m n_i S_i(t)$ with holdings n_i, we define weights w_i by

$$w_i = \frac{n_i S_i(t)}{P(t)}.$$

Then if weights are kept constant over time

$$\begin{aligned}
1 + R(t) = \frac{P(t)}{P(t-1)} &= \sum_{i=1}^m \frac{n_i S_i(t)}{P(t-1)} \\
&= \sum_{i=1}^m \frac{n_i S_i(t-1)}{P(t-1)} \frac{S_i(t)}{S_i(t-1)} \\
&= \sum_{i=1}^m w_i (1 + R_i(t)) \\
&= \sum_{i=1}^m w_i + \sum_{i=1}^m w_i R_i(t) \\
&= 1 + \sum_{i=1}^m w_i R_i(t).
\end{aligned}$$

So it follows that

$$R(t) = \sum_{i=1}^m w_i R_i(t).$$

If we work this out for the log return with $r_i(t)$ the return for security i, we get

$$r(t) = \ln\left[\frac{P(t)}{P(t-1)}\right] = \ln\left[\sum_{i=1}^m \frac{n_i S_i(t-1)}{P(t-1)} \frac{S_i(t)}{S(t-1)}\right] = \ln\left[\sum_{i=1}^m w_i \exp r_i(t)\right].$$

This we can't simplify further.

4.2.2 Aggregation over time

The linear returns do not aggregate over time, to see this let $R^k(t) = \frac{P(t)}{P(t-k)} - 1$ be the return over k time intervals t, so that $R^1(t) = R(t)$. Then we have

$$\begin{aligned}
1 + R^k(t) &= \frac{P(t)}{P(t-k)} \\
&= \frac{P(t)}{P(t-1)} \frac{P(t-1)}{P(t-2)} \cdots \frac{P(t-(k-1))}{P(t-k)} \\
&= (1 + R(t))(1 + R(t-1)) \cdots (1 + R(t-(k-1))),
\end{aligned}$$

which we can't simplify further. For the k-interval log return $r^k(t) = \ln(1 + R^k(t))$ we get from the definition of the logarithm that

$$\begin{aligned}
r^k(t) &= \ln\left[1 + R^k(t)\right] \\
&= \ln\left[(1 + R(t))(1 + R(t-1)) \cdots (1 + R(t-(k-1)))\right] \\
&= \ln\left[1 + R(t)\right] + \ln\left[1 + R(t-1)\right] + \ldots + \ln\left[1 + R(t-(k-1))\right] \\
&= r(t) + r(t-1) + \ldots + r(t-(k-1)) \\
&= \sum_{j=0}^{k-1} r(t-j).
\end{aligned}$$

So log returns do aggregate over time.

4.2.3 Taylor approximation

We'll now show that the linear return and the log return are approximately the same. Looking at the Taylor series for the log return

$$\begin{aligned}
r(t) = \ln(1 + R(t)) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} R(t)^n \\
&= R(t) - \frac{(R(t))^2}{2} + \frac{(R(t))^3}{3} - \ldots \\
&= R(t) + O(R(t)^2)
\end{aligned}$$

we see that approximately we have

$$r(t) \approx R(t)$$

if $R(t)$ is small, which will be the case for short time horizons. Since linear returns and log returns are nearly equivalent and both have useful properties they are often used interchangeably. But great care should be taken when these returns are used in formulas or are quoted in research. For a further discussion see (Meucci, 2010).

> **Note:** *In the following chapters we will use the log return in our definitions. But keep in mind that linear returns could also be used and that then the results would be somewhat different.*

4.3 Basic notions and assumptions

Basic notions

To analyze the return distribution we summarize some basic notions from statistics. Let X be a random variable and let $x_1,\ x_2,\ \ldots,\ x_n$ be any sample[1] of this random variable. We have the following notions when dealing with random variables

Notion	Notation	Random variable X (and Y)
Mean	$\mu(X) = \mu_x = \mu$	$= \mathbb{E}(X)$
Variance / $(\text{St.dev.})^2$	$\sigma^2(X) = \sigma_x^2 = \sigma^2$	$= \mathbb{E}\left[(X-\mu)^2\right]$
Skewness	$\mathrm{S} = \gamma_1(X) = \gamma_1$	$= \mathbb{E}\left[(\frac{X-\mu}{\sigma})^3\right]$
Kurtosis[2]	$\mathrm{K} = \gamma_2(X) = \gamma_2$	$= \mathbb{E}\left[(\frac{X-\mu}{\sigma})^4\right] - 3$
Covariance	$\Sigma_{xy} = \mathrm{Cov}(X,Y)$	$= \mathbb{E}\left[(X-\mu_x)(Y-\mu_y)\right]$
Correlation	$\rho_{xy} = \mathrm{Corr}(X,Y)$	$= \frac{\Sigma_{xy}}{\sigma_x\sigma_y}$

In samples we have for the estimators of the above notions, using the definitions that have been adjusted for bias,

[1] When calculating with returns the index will usually be related to the time giving the notation $x(t)$ for t over some interval.

Notion	Notation	Sample $x_1,\ \ldots,\ x_n$ (and $y_1,\ \ldots,\ y_n$)
Mean	$\bar{x} = \hat{\mu}$	$= \frac{1}{n}\sum_{i=1}^{n} x_i$
Variance / (St.dev.)^2	$s^2 = \hat{\sigma}^2$	$= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
Skewness	$g_1 = \hat{\gamma}_1$	$= \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}(\frac{(x_i-\bar{x})}{s})^3$
Kurtosis	$g_2 = \hat{\gamma}_2$	$= \frac{n(n+1)}{(n-1)(n-2)(n-3)}\sum_{i=1}^{n}(\frac{(x_i-\bar{x})}{s})^4 - 3\frac{(n-1)^2}{(n-2)(n-3)}$
Covariance	$c_{xy} = \hat{\Sigma}_{xy}$	$= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$
Correlation	$r_{xy} = \hat{\rho}_{xy}$	$= \frac{c_{xy}}{s_x s_y}$

Quantiles

We define quantiles as follows. Let $F(x) = P(X \leq x)$ denote the cumulative distribution function of a random variable X. For $\alpha \in (0,1)$ the lower α-quantile is defined as

$$x_{(\alpha)} = q_\alpha(X) = q_\alpha(F) = \inf_{x\in\mathbb{R}} \{F(x) \geq \alpha\}\,.$$

The upper α-quantile is defined as

$$\begin{aligned} x^{(\alpha)} = q^\alpha(X) = q^\alpha(F) &= \inf_{x\in\mathbb{R}} \{F(x) > \alpha\} \\ \text{or} \quad &= \sup_{x\in\mathbb{R}} \{F(x) \leq \alpha\}\,. \end{aligned}$$

The lower α-quantile function $\inf_{x\in\mathbb{R}} \{F(x) \geq \alpha\}$ is usually seen as the quantile function which is a special case of the generalized inverse, this generalized inverse is further discussed in (Embrechts and Hofert, 2010) , and is defined as follows.

Given an increasing function

$$T : \mathbb{R} \to \mathbb{R},$$

then the generalized inverse of T is defined by

$$T^{\leftarrow}(y) := \inf_{x\in\mathbb{R}} \{T(x) \geq y\}\,,$$

with the convention $\inf\{\emptyset\} = \infty$. If $T : \mathbb{R} \to [0,1]$ is a distribution function then $T^{\leftarrow}$ defined above is called the quantile function of T and it calculates the lower quantiles of the distribution.

Independent and identically distributed assumption

A common assumption is that the returns are i.i.d. (independent and identically distributed) over time. This assumption comes down to the following. Let X and Y be independent and identically distributed. The independence assumption implies that there is no correlation between the two random variables, so $\mathrm{Cov}(X, Y) = 0$. The identical distribution assumption implies that the two random variables have the same distribution, so $X \sim Y$. From this follows that

$$\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y) + 2\,\mathrm{Cov}(X,Y) = \sigma^2(X) + \sigma^2(Y) = 2\sigma^2(X).$$

This can be generalized to n random variables. With the aggregation over time property of log returns discussed in Section 4.2.2 we get that if $r(t)$ is a random variable for any t and the log returns are i.i.d. we have

$$\sigma^2(r^k(t)) = \sigma^2(\sum_{j=0}^{k-1} r(t-j)) = \sum_{j=0}^{k-1} \sigma^2(r(t-j)) = k\sigma^2(r(t)).$$

From this follows the square-root-of-time rule which will be discussed in Section 5.4. A reason why we use log or linear returns instead of the portfolio value $P(t)$ itself or the absolute return $D(t) = P(t) - P(t-1)$ is the portfolio value or the absolute return seriously lack the i.i.d. property. The portfolio value today is highly dependent on the portfolio value yesterday. Also the variance will be greater if the portfolio value is high. The absolute return does not give the price change relative to a given value. A 10 euro return is high if the value of the portfolio is 100, but fairly low if the value is 100.000, therefore a portfolio with value 100.000 is expected to have higher absolute returns than a portfolio with value 100. Besides being non-identical because of this, the absolute return is also non-comparable, if 2 portfolios have the same absolute return of 10 euro, with one portfolio value equal to 100 and the other portfolio value equal to 100.000, it's not possible to say which portfolio performed better by just looking at the absolute return of 10.

For linear returns and log returns the assumption of i.i.d. is roughly plausible; there have been many objections, however, including returns showing signs of volatility clustering: there are periods where the returns have high volatility and there are periods with low volatility, this would contradict the assumptions of an identical distribution. Or the claim is made that the returns show mean reversion. This claim is made if one believes that the stock prices will tend to move to an average price over time or, with a more optimistic view, the returns will tend to move around an average positive return. In these cases after a low return it will be more likely that a high return follows and vice versa. When returns show mean reversion then the returns are clearly not independent. Also the identical assumption is contradicted.

Returns that show signs of the above mentioned objections against being i.i.d. can be modeled with time series using a varying standard deviation for volatility clustering or a lag can be added for the mean reversion. This we will discuss in Section 6.8.2.

Normal distribution assumption

Since we assumed

$$P(t) \geq 0$$

we have

$$D(t) \geq -P(t-1)$$

and

$$R(t) = \frac{P(t)}{P(t-1)} - 1 \geq -1,$$

giving lower bounds for these returns. A normal distribution however is unbounded; hence these returns cannot be normally distributed. And, when analyzed, the portfolio value and absolute returns are often clearly not normally distributed having a high kurtosis and often a positive skew[3]. Also the linear and log returns are often not normally distributed; sometimes they have a slight skew and very often they have a higher kurtosis then normal. However, the normal distribution is, even then, often assumed for easy of computation. A frequently used alternative is assuming a student-t distribution instead of a normal distribution.

[3] A positive skew is sometimes called 'a skew to the right'. And a negative skew is called 'a skew to the left'.

Chapter 5

Active Management Risk

In this chapter we will investigate ways to quantify active management risk. Our main purpose will be to do this for the standard model, this means that we have a risk horizon of 1 year and a confidence level of 97.5%. We'll first define the active returns in Section 5.1, then we'll give some measures we can use given these active returns. Then in Section 5.3 we'll discuss some desirable properties of such a measure. In Section 5.4 we give a frequently used rule to adjust the risk horizon.

5.1 Active Returns

Let $r_b(t)$ denote the benchmark return and $r_p(t)$ denote the portfolio return. The active return is the return of the portfolio relative to a benchmark.

$$\text{Active return} = \text{return portfolio} - \text{return benchmark}.$$

So for any t, we have

$$r_a(t) = r_p(t) - r_b(t)$$

with $r_a(t)$ the active returns.

Now let us assume that the returns are known up to time $t = T$ with the time in years. For examining the risk with a risk horizon of 1 year we want to know $r_a(T+1)$. So with the returns to time T known we make a forecast for time $T+1$ keeping in mind a confidence level of 97.5%. We can do this directly by using a point forecast or we can look at the forecast distribution of $r_a(T+1)$. When the distribution is forecasted we denote for convenience $r_a(T+1) := r_a$ and we see r_a as a random variable. Furthermore, when a sample is taken from this random variable we'll denote this by r_i with $1 \leq i \leq n$. When the history for $t \leq T$ is analyzed the active returns are denoted by $r(t)$, without subscript.

5.2 Measures of active risk

5.2.1 Tracking Error

The tracking error is the main measure used for active management risk, and funds often mention their tracking error in their reports. That's why in this thesis we'll focus mainly on this active risk measure. If r_a is the forecast distribution of the active returns then we define the tracking error as the standard deviation of this distribution. Which is the square root of

$$\sigma^2_{\text{tracking error}} = \sigma^2(r_a) = \mathbb{E}\left[(r_a - \mathbb{E}(r_a))^2\right]$$

or in a sample

$$\hat{\sigma}^2_{\text{tracking error}} = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2.$$

The 97.5% risk will be $1.96 \times \sigma_{\text{trackingerror}}$ if the active return distribution is assumed to be normally distributed. The tracking error calculates the deviation around the mean of the active return. So it takes into account the downward risk but also the upward risk. This can be misleading if the active return distribution is skewed. Instead of the normal tracking error we can also look at the mean-adjusted tracking error

$$\text{mean-adjusted } \sigma^2_{\text{tracking error}} = \mathbb{E}\left[(r_a)^2\right]$$

or in sample

$$\text{mean-adjusted } \hat{\sigma}^2_{\text{tracking error}} = \frac{1}{n}\sum_{i=1}^{n}(r_i^2).$$

This calculates the deviation around zero, and hence makes the implicit assumption of an expected return of zero. Arguably this is the better assumption for the risk since it reflects the view that there is no way to structurally outperform the market.

5.2.2 Value at Risk

(Morgan and Reuters, 1996) provides a detailed description of how RiskMetrics works. This Technical Document is mainly based around the calculation of Value at Risk. Since this document Value at Risk is widely used as a measure for risk. It is mainly used for the risk in a portfolio and the definition in words says that it calculates what the potential loss over a target horizon is, given a predetermined confidence level. As an example: if the daily VaR given a confidence level of 95% equals 200 euro, then there is only 5% chance that over one day the loss will be 200 or more. Notation for this will be 95%VaR = 200 or $\text{VaR}_{0.05} = 200$. Note the way the percentages are used and note that the VaR is presented as a positive number.

Instead of portfolio risk we are, however, investigating active risk, that is, the risk of the portfolio relative to the benchmark. (Alexander, 2008c) calls this type of VaR the Benchmark VaR and (Mina and Xiao, 2001) calls it the relative VaR, However it is often clear from context which type of VaR is used. For $\alpha \in (0,1)$ the mathematical definition of VaR is

$$[(1-\alpha)\times 100\%]\,\mathrm{VaR}(r_a) = \mathrm{VaR}_\alpha(r_a) = -q_\alpha(r_a).$$

Here $q_\alpha(r_a)$ is the quantile function defined in Section 4.3. The minus is there to make the VaR positive.

Note that for a 97.5% confidence interval we'll investigate $\mathrm{VaR}_{0.025}(r_a)$. In many situations the tracking error will have a 1 to 1 correspondence with the VaR. For example if we assume a normal distribution of the active returns than VaR given a confidence level of 97.5% will just be $1.96 \times \sigma_{\text{tracking error}}$ and in general there will be a similar 1 to 1 correspondence for any symmetric distribution. However, if there is a skew in the distribution then the VaR will differ from the corresponding measure found with the tracking error.

> **Note:** *In practice this 1 to 1 correspondence is frequently used. This means that tracking errors can be and are calculated using VaR techniques and vice versa.*

An interesting modification of VaR, if there is a skew or a kurtosis not equal to zero, is given by (Favre and Galeano, 2002). If we express the normal VaR with standard deviation σ and with c_p depending on the confidence interval, as follows

$$\mathrm{VaR}_p = c_p\sigma.$$

Then (Favre and Galeano, 2002) extends this to the modified VaR, using the skewness S and kurtosis K of the distribution, by

$$\mathrm{MVaR}_p = \left[c_p + \frac{1}{6}(c_p^2-1)\mathrm{S} + \frac{1}{24}(c_p^3-3c_p)\mathrm{K} - \frac{1}{36}(2c_p^3-5c_p)\mathrm{S}^2\right]\sigma.$$

With VaR there also exists a mean-adjusted type, which is defined as

$$\text{mean VaR}_\alpha(r_a) = \mathrm{VaR}_\alpha(r_a) - \mathbb{E}(r_a).$$

.

5.2.3 Expected shortfall

The expected shortfall[1] measure is an extension of the VaR and in words it is defined as the expected loss given the fact that the VaR level is exceeded. The formula for expected shortfall (ES) is

$$\mathrm{ES}_\alpha(r_a) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(r_a)\,\mathrm{d}u$$

[1] Other names are Conditional VaR, Expected tail loss or Average VaR. However, (Alexander, 2008d) uses expected shortfall specifically when measuring the returns relative to a benchmark and Expected tail loss when measuring the returns of a portfolio.

which, in the continuous case, is equal to

$$\mathrm{ES}_\alpha(r_a) = -\mathbb{E}(r_a \mid r_a \leq q_\alpha(r_a)) = \mathbb{E}(-r_a \mid -r_a \geq \mathrm{VaR}_\alpha(r_a)).$$

For a risk manager this measure can be very useful. While VaR only measures the risk level which will not be exceeded with a given confidence, ES measures how high this risk will be if this level is exceeded. However, for our standard model the ES will not be very useful; the standard model explicitly looks at scenarios that can happen with a 97.5% confidence level, not at what will happen if we are in the 2.5% part of the scenario.

5.2.4 Other measures

Other measures are also possible. For example (Alexander, 2008d) mentions the semi-standard deviation, calculated as the square root of the semi-variance

$$\mathrm{SV}(X) = \mathbb{E}(\min(X - \mathbb{E}(X), 0)^2),$$

which calculates the variance on the condition that X is less then it's expectation.

(McNeil et al., 2005) extends this semi-variance to what he calls the lower partial moment

$$\mathrm{LPM}(k, q) = (-1)^k \int_\infty^q (x - q)^k \,\mathrm{d}F(x)$$

with $F(x)$ the distribution function. Notice that for $k = 2$ and $q = \mathbb{E}(X)$ this is the semi-variance.

5.3 Coherent risk measure

Until now we looked at measures acting on the active return distribution $r_a = r_a(T+1)$. In this section we generalize this to any active return random variable X in a measurable space $(\Omega,\ \mathcal{F},\ P)$. Where Ω is the set of all outcomes, $\mathcal{F}$ is the collection of all events that are considered and $P(A)$ is the probability that an event $A \in \mathcal{F}$ happens. A random variable X is now a function on this space, notation: $X \in L^0(\Omega,\ \mathcal{F},\ P)$ or simply $X \in L^0$. Here $L^0(\Omega,\ \mathcal{F},\ P)$ is the set of all random variables on $(\Omega,\ \mathcal{F},\ P)$ that are almost surely finite. For active risk we can see the random variable as an active position and when an outcome $\omega \in \Omega$ is realized we have the active return $X(\omega)$. On this random variable we have the measures discussed in Sections 5.2. (Artzner et al., 1999) introduces the notion of coherent risk measures, which is defined as follows: A mapping $\rho\ :\ L^0(\Omega,\ \mathcal{F},\ P) \to \mathbb{R}$ is called a coherent risk measure if the following holds

1. *Monotonicity :* if $X,\ Y \in L^0$ and $X \leq Y$ almost surely then $\rho(X) \geq \rho(Y)$ The monotonicity property states that if for any outcome the position X has less return compared to Y, then X should be considered as riskier.

2. *Subadditivity :* if $X,\ Y,\ X+Y \in L^0$ then $\rho(X+Y) \leq \rho(X) + \rho(Y)$
   The subadditivity property states that if we combine 2 positions then the risk will not be greater then the risk of the 2 positions separately. This is the diversification principle.

3. *Positive homogeneity :* if $X \in L^0$ and $\lambda \geq 0$ we have $\rho(\lambda X) = \lambda \rho(X)$
   The positive homogeneity property states that if we double the size of the position then the risk is doubled.

4. *Translation invariance :* if $X \in L^0$ and $c \in \mathbb{R}$ we have $\rho(X+c) = \rho(X) - c$
   The translation invariance property states that adding a quantity c to the position will reduce the risk by exactly c. In particular we have $\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$.

The tracking error is not coherent

The tracking error is subadditive since for the standard deviation we have

$$\begin{aligned}\sigma^2(X+Y) &= \sigma^2(X) + \sigma^2(Y) + 2\operatorname{Corr}(X,Y)\sigma(X)\sigma(Y) \\ &\leq \sigma^2(X) + \sigma^2(Y) + 2\sigma(X)\sigma(Y) \\ &= (\sigma(X) + \sigma(Y))^2\end{aligned}$$

and subadditivity follows since both are positive. The tracking error is also positive homogeneous

$$\sigma(\lambda X) = \lambda \sigma(X) \quad \text{for } \lambda \geq 0.$$

However, the tracking error is not monotone, for a counterexample we look at X and Y defined on 3 events. X gives a return of -1 in the first event which happens with probability 0.1, a return of 0 in the second event which happens with probability 0.8 and a return of 1 in the last event which happens with probability 0.1. Y gives the same returns with the only difference that in the last event the positive return is 2 instead of 1. Clearly we have $X \leq Y$, but if we calculate the tracking error we have

$$\sigma(X) = \sqrt{\sum_{i=1}^{3} p_i(x_i - \mu)^2} = \sqrt{0.1 \cdot (-1)^2 + 0.8 \cdot 0^2 + 0.1 \cdot 1^2} = \sqrt{0.2} \approx 0.45$$

and for Y we have

$$\sigma(Y) = \sqrt{0.1(-1-0.1)^2 + 0.8(0-0.1)^2 + 0.1(2-0.1)^2} = \sqrt{0.49} = 0.7.$$

So $\sigma(X) \leq \sigma(Y)$ in this case. Also the translation property fails to hold since

$$\sigma(X+c) = \sigma(X) \neq \sigma(X) - c \quad \text{if } c \neq 0.$$

Hence the tracking error is not a coherent risk measure.

VaR is not coherent

Let $\alpha \in (0, 1)$, then VaR_α satisfies the monotonicity property since for quantiles we have

$$q_\alpha(X) \leq q_\alpha(Y) \quad \text{if} X \leq Y$$

and hence

$$\mathrm{VaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(Y) \quad \text{if} X \leq Y.$$

VaR_α is also positive homogeneous since

$$q_\alpha(\lambda X) = \lambda q_\alpha(X) \quad \text{for } \lambda \geq 0$$

and VaR_α is translation invariant since

$$q_\alpha(X + c) = q_\alpha(X) + c \quad \text{for} \lambda \geq 0.$$

However, VaR_α is not subadditive, for suppose X and Y are independent but X and Y both gives a return of -1 with a probability of 0.04 and zero return otherwise. Then the 5% VaR for X and Y separately is equal to

$$\mathrm{VaR}_{0.05}(X) = \mathrm{VaR}_{0.05}(Y) = 0,$$

however the 5% VaR for holding both X and Y at the same time equals

$$\mathrm{VaR}_{0.05}(X + Y) = 1.$$

Since there exists 0.0016 chance that both will give a return of -1 and a 0.0784 chance that at least one of them has a return of -1. So in this example we have

$$\mathrm{VaR}_{0.05}(X + Y) > \mathrm{VaR}_{0.05}(X) + \mathrm{VaR}_{0.05}(Y).$$

From this follows that VaR is also not a coherent risk measure. However, this counterexample seems fabricated and only has 2 discrete random variables. If we take a large and arbitrary collection of random variables, we will see that the VaR measure often is subadditive.

Expected shortfall is coherent

Expected shortfall, on the other hand, is a coherent risk measure. To see this we use the definition of ES in Section 5.2.3. Let $\alpha \in (0, 1)$, then for monotonicity we have

$$X \leq Y \Rightarrow \mathrm{VaR}_u(X) \geq \mathrm{VaR}_u(Y) \quad \text{for } 0 \leq u \leq \alpha$$

since VaR is monotone, and hence

$$\mathrm{ES}(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_u(X) \, \mathrm{d}u \geq \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_u(Y) \, \mathrm{d}u = \mathrm{ES}(Y).$$

Translation invariance we also get from the translation invariance property of VaR,

$$\begin{aligned}\mathrm{ES}(\lambda X) =&\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(\lambda X)\,\mathrm{d}u\\ =&\frac{1}{\alpha}\int_0^\alpha \lambda\mathrm{VaR}_u(X)\,\mathrm{d}u\\ =&\lambda\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)\,\mathrm{d}u\\ =&\lambda\mathrm{ES}(X)\end{aligned}$$

and positive homogeneity follows in the same way

$$\begin{aligned}\mathrm{ES}(X+c) &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X+c)\,\mathrm{d}u\\ &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)-c\,\mathrm{d}u\\ &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)\,\mathrm{d}u \;-\; \frac{1}{\alpha}\int_0^\alpha c\,\mathrm{d}u\\ &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)\,\mathrm{d}u \;-\; c\\ &= \mathrm{ES}(X)-c.\end{aligned}$$

For the subadditivity of Expected Shortfall we refer to a detailed proof given by (Acerbi and Tasche, 2002).

5.4 Square-root-of-time rule

In our standard model we typically have a 1 year risk horizon and therefore we want 1 year (or annual) risk measures. The 1 year risk measure is also often preferred since then risk measures can easily be compared. However, returns on short horizons often have better statistical properties and therefore the risk estimation will be more precise. Also data-availability plays a role. For these reasons the tracking error and VaR are usually estimated using a 1 day risk horizon instead of a 1 year risk horizon and afterwards they are scaled to a 1 year risk horizon. There are sophisticated ways to do this, but all procedures rely on assumptions made for the returns. In this section we'll only present the simple square-root-of-time rule. This procedure makes the assumption that the active returns are i.i.d. and relies on the time aggregation property discussed in Section 4.2.2. We have seen in Section 4.3 that if $r_a(T+1)$ and $r_a(T+2)$ are i.i.d. then

$$\sigma^2(r_a(T+1)+r_a(T+2)) = 2\sigma^2(r_a(T+1)) = 2\sigma^2(r_a).$$

This can be extended to s active returns to get for the s time units ahead active return $r_a(T+s)$

$$\sigma^2(r_a(T+s)) = s\sigma^2(r_a),$$

this gives the rule

$$\sigma(r_a(T+s)) = \sqrt{s}\sigma(r_a).$$

For the VaR case, if we assume we have the 1 on 1 correspondence discussed in Section 5.2 which gives $\mathrm{VaR}_\alpha(r_a) = c_\alpha\,\sigma(r_a)$, with c_α depending on the distribution and the confidence level, we get the rule

$$\mathrm{VaR}(r_a(T+s)) = \sqrt{s}\mathrm{VaR}(r_a).$$

This rule is often used as a simple approximation for the 1 year tracking error or 1 year VaR.

Chapter 6

Methods for getting the active return distribution

In this Chapter we look at the methods to derive the active return distribution. This distribution is needed to calculate the tracking error and the VaR or other measures discussed in Chapter 5. The methods we introduce are usually applied to portfolio returns, but we will use it for active returns. First we give some notation, and then we'll discuss the methods.

6.1 General setting and notation

We begin with the general setting for portfolio returns and some notation before we introduce the methods. Here we make the distinction between finding the distribution of the portfolio value, i.e. the profit-loss distribution and finding the distribution of the portfolio return using a factor model.

Profit loss distribution

Let $P(t) \geq 0$ be the portfolio value at time t and let $P(T)$ be the current portfolio value. Now our objective is to predict the portfolio value 1 time unit ahead. So we'll assume that at time T the portfolio value is observable and a forecast is made for the distribution $P := P(T+1)$, using the same notation for the forecasted distribution as in Section 5.1. If we have forecasted the distribution P then the profit-loss distribution or P&L-distribution is defined as

$$\text{P\&L} = P(T+1) - P(T)(= D(T+1))$$

If we have this P&L distribution we can calculate the tracking error and the VaR of this portfolio. Note that this P&L distribution is just the absolute return of the portfolio.

Risk factors

To make a forecast of the portfolio value P, we'll assume that the portfolio value is a function f of certain risk factors, which are values like

- interest rates
- stock prices
- foreign exchange rates
- spreads
- etc.

The risk factors are denoted by $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))'$, so we have

$$P(t) = f(\mathbf{X}(t)).$$

In order to simplify the forecasting process we want a linear function. Therefore the risk factors are usually given with a certain sensitivity $\theta = (\theta_1, \ldots, \theta_n)'$, this sensitivity explains the relation between a risk factor and the portfolio, so we have

$$P(t) = f(\mathbf{X}(t)) = \sum_{i=1}^{n} \theta_i X_i(t).$$

For more on risk factors and risk factor sensitivities see (Alexander, 2008b). For equity portfolios we will generally already have a linear function of stock prices[1]. For n stocks we have

$$P(t) = \sum_{i=1}^{n} \theta_i X_i(t)$$

with θ_i, the sensitivity, denoting the number of holdings of stock i with price $X_i(t)$. For the profit-loss distribution we then get

$$\text{P\&L} = \sum_{i=1}^{n} \theta_i X_i(T+1) - \sum_{i=1}^{n} \theta_i X_i(T).$$

If we now have forecast distributions of stock prices $X_i := X_i(T+1)$ we can get a forecast distribution of the profit-loss distribution.

If f is not linear we can use a linearization of f using the gradient g of f.

$$g(x) = (f_1(x), \ldots, f_n(x))' \quad \text{with } f_i(x) = \frac{\partial f(x)}{\partial x_i}.$$

With this gradient we have the linear Taylor approximation

$$f(x) \approx f(0) + x'g(0) = \sum_{i=1}^{n} x_i f_i(0).$$

In this linearized setting, we have x_i as risk factors and $f_i(0)$ as the sensitivities.

[1]Foreign exchange rates can be included when the portfolio also invests in equity valued in foreign currencies and interest rates can be included when the discounted profit-loss distribution is examined.

Factor model

In this setting we look at a factor model which explains the portfolio return[2]. The factor model is a generalization of the capital asset pricing model (CAPM) which is as follows

$$\mathbb{E}(r_i) - r_f = \beta_i(\mathbb{E}(r_b) - r_f)$$

where $\mathbb{E}(r_i)$ is the expected return of asset i, r_f is the risk free return, $\mathbb{E}(r_b)$ is the expected return of the market or benchmark and β_i is the sensitivity of the expected asset return to the expected market return[3]. From this we get the relation

$$\mathbb{E}(r_i) = \alpha + \beta_i \mathbb{E}(r_b).$$

This relation we generalize to the single factor model, where X denotes the return on a risk factor,

$$r_i(t) = \alpha + \beta_i X(t) + \varepsilon_i(t) \quad \varepsilon_i(t) \sim \text{i.i.d.}\mathcal{N}(0,\ \sigma_i^2).$$

We also assume that $X(t)$ is independent to $\varepsilon_i(t)$. Note that this is just a linear regression. For a k-multiple factor model we get

$$r_i(t) = \alpha + \sum_{j=1}^{k} \beta_{ij} X_j(t) + \varepsilon_i(t) \quad \varepsilon_i(t) \sim \text{i.i.d.}\mathcal{N}(0,\ \sigma_i^2).$$

Here α is the expected return relative to the return of the risk factor, X_j is the return on the j-th risk factor, β_{ij} is the sensitivity of risk factor j to asset i and $\varepsilon_i(t)$ is the error term, also called the specific return, which is independent to $X_j(t)$.

This regression can be made for any asset return $r_i(t)$. For our interest we will look at the portfolio return or the active return. And the risk factors will, for a simple equity fund, be the stock returns, but in general any risk factor can be used. Our model for active returns can now be formulated as

$$r_a(t) = \alpha + \sum_{j=1}^{k} \beta_j X_j + \varepsilon(t) \quad \varepsilon(t) \sim \text{i.i.d.}\mathcal{N}(0,\ \sigma^2).$$

Now when we forecast the distribution of r_a we see that this distribution depends on the distribution of the risk factors and of the distribution of the error term. In risk terminology this is usually called systematic risk or market risk and specific risk respectively.

Notice that this factor model is also a linearization of a possible non-linear function

$$r_a(t) = f(\mathbf{X}(t)).$$

But this time we use a linear regression to get the linearization.

[2]The return can be linear as well as log, but as stated earlier we will use the log return.

[3]If β is measured as the sensitivity of a stock to a benchmark then this is called the beta of a stock and α is called the alpha of a stock.

In general

There are many variations to these models, but the process remains the same, namely

- We have a risk factor mapping function $r_a(t) = f(\mathbf{X}(t))$
- If this mapping is non-linear, then linearize this function using an approximation or a regression.
- Estimate the distribution of the risk factors or the risk factor returns and, when regressing, the specific returns.
- With these distributions, calculate the distribution of the active returns $r_a(t)$
- Make a forecast of the distribution r_a at time $T+1$.

When we have the forecasted distribution of r_a we can calculate the tracking error and the VaR.

6.2 Ex-post and ex-ante methods

There is a distinction of getting the active return distribution ex-post or ex-ante.

Ex-post

In the ex-post setting, which stands for ‘after the event’, we look directly at active returns that happened in the past. In this setting we have the method named

- **Empirical Method**.

In this method we're not really forecasting the active return distribution, but we're using the empirical distribution over some history. We will see that this is a very easy and straightforward method and works good to show the performance of a fund, however, if we're measuring risk this is not a very useful method because a good performance in the past is not a guarantee for a good performance in the future. It is possible that a fund has a strategy that worked well for past events, but performs badly in some market conditions.

Ex-ante

Ex-ante stands for ‘before the event’; here we make a prediction of what the active return distribution will be in the future. Historic data is used to measure the market conditions in the past and with these measurements a prediction is made for the market condition in the future and through a mapping procedure the active return is predicted. We will make the distinction between the following methods:

- **Parametric linear method** (or **Variance-Covariance method**)

This method assumes that the active return distribution is a linear function or a linearized function (linear approximation) of the risk factors or risk factor returns. With this linear function and with the covariance matrix (also called variance-covariance matrix) of the risk factors an analytic expression is given for the Value at risk or the tracking error.

Terminology:

The terminology used for the first method has not been consistent over the years since it was introduced by (Morgan and Reuters, 1996). They call it the ***parametric method*** *since it assumes that the data comes from a fixed distribution. However in the Monte Carlo method simulations are drawn from a fixed distribution and this method can therefore also be called parametric.*

Also the name ***variance-covariance method*** *is given emphasizing the link to the portfolio theory started by (Markowitz, 1952), he was one of the first who used the variance-covariance matrix in economic situations. The core of the calculation in the parametric linear method relies on this variance-covariance matrix. (McNeil et al., 2005), (Crouhy et al., 2000) and many other papers still use this name. Critique of this term by (Holton, 2003) and (Alexander, 2008d) is that the Monte Carlo method is also based on the variance-covariance matrix. They propose to call it the* ***linear method****, emphasizing the linear risk factor mapping which is needed to calculate the risk measures. This naming extends to the quadratic method, which uses a quadratic function. This term, as opposed to the term delta method, seems to exclude the case were we have an approximation.*

Some works like (Jorion, 2000) and others use the term ***delta-normal method*** *when the method explicitly uses an approximation and assumes a normal distribution. This naming extends to the delta-gamma method, where we use a quadratic approximation of the risk factor function. This term, however, seems to exclude the case were we have a pure linear function and not an approximation.*

In general many combinations of parametric, analytic, delta, normal, linear, variance, covariance, modeling, etc. have been used. We will use the term parametric linear method since for equity funds the mapping will often be linear.

- **Historical simulation method**

This method uses historical realizations of risk factors to derive the active return distribution. For every realization of the risk factors the active return is calculated given the (active) position of a fund. Notice the difference between the empirical and the historical simulation method: while the empirical method only uses the realizations of the active returns in the past, it implicitly includes changing fund positions in calculating the active returns. The historical simulation method, however, looks at the realizations of the risk factors and the active returns are calculated with the current position of the fund.

- **Monte Carlo method**

This method generates possible realizations of risk factors to derive the active return distribution. Using the covariance matrix of the risk factors or other assumptions on the distribution a large sample is randomly generated from the risk factor distributions. With this generated sample the active return is calculated in the same way as in the historical simulation.

6.3 Used dataset

For illustrative purpose we have the following dataset, see Figure 6.1, with the daily time series of 5 stocks, $S_1(t)$ to $S_5(t)$, from the MSCI Europe benchmark and the time series of the benchmark itself, denoted by $B(t)$. With these 5 stocks

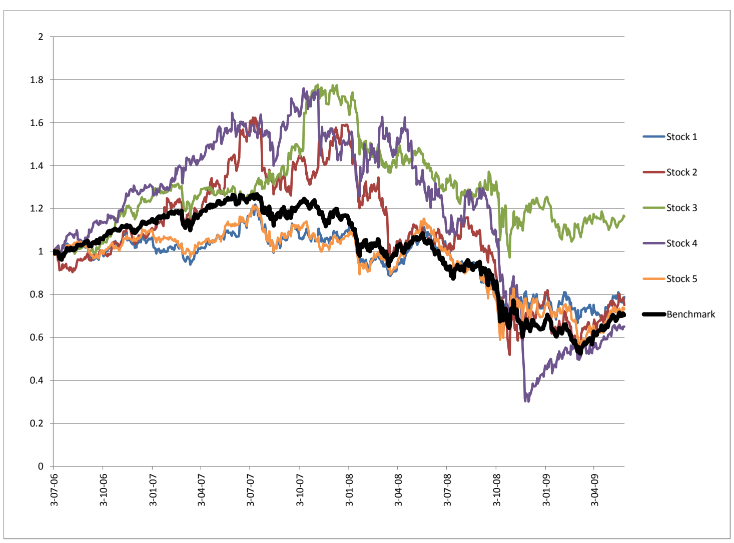

Figure 6.1: The scaled down stocks and benchmark time series.

we'll construct a hypothetical portfolio. Since the MSCI Europe has around 500 stocks, holding 5 stocks will replicate only a small part of the benchmark, therefore we expect a high tracking error. To support this we look at Figure 6.2 taken from (Fabozzi and Focardi, 2004), with only 5 stocks we can expect a tracking error above 10%.

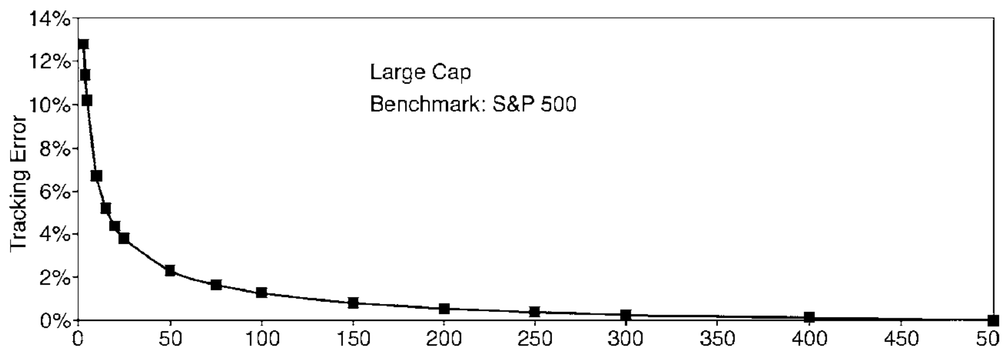

Figure 6.2: Tracking Error versus the Number of Benchmark Stocks in the Portfolio.

Our data covers 3 years of history running from 01-07-2006 to 30-06-2009, this gives 764 data points and 763 returns. On average this is $254\frac{1}{3}$ returns per year[4]. The returns are denoted by $r_i(t)$ with $1 \leq i \leq 5$ for the stocks and $r_b(t)$ for the benchmark, with t ranging from 1 to 763.

6.4 Empirical method

To get a good idea about how tracking error and Value at Risk are calculated we first look at the empirical method before we look at the other methods. Here we don't look at the risk factors, but we look directly at the active returns, that is, the portfolio returns minus the benchmark returns. However, we don't have an actual portfolio in our dataset. Therefore, we'll construct a hypothetical portfolio by using the buy-and-hold strategy, which means that at the beginning one share of each stock, 1 to 5 is bought and there are no further transaction made in these 3 years. So, we have that our portfolio equals

$$P(t) = \sum_{i=1}^{5} S_i(t)$$

With this constructed portfolio, see Figure 6.3, the active returns can then be calculated as

$$r_a(t) = r_p(t) - r_b(t).$$

[4]Not all 365 days in a year are trading days, on average a year consists of 252 trading days.

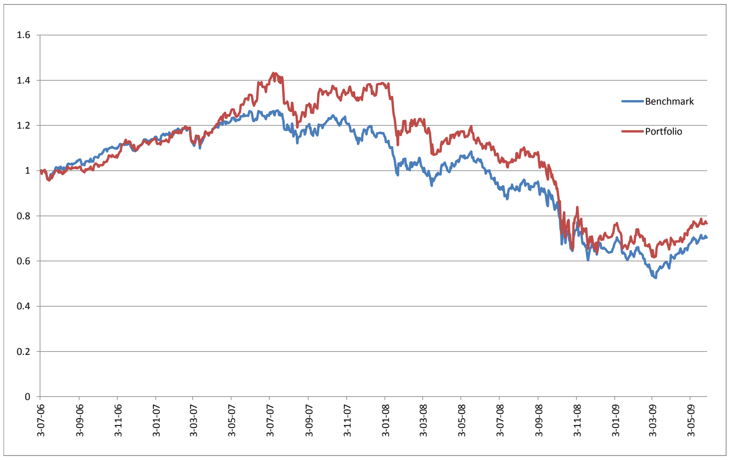

Figure 6.3: The scaled down portfolio and benchmark time series.

Note that in general we have

$$r_p(t) = \ln P(t) - \ln P(t-1) \neq \sum_{i=1}^{5} r_i(t).$$

These active returns give an empirical distribution as depicted in the histogram, see Figure A.1 in Appendix A.2. With these active returns we calculate the (daily) tracking error as

$$\text{daily } \sigma(r_a) = \sqrt{\frac{1}{n-1}\sum_{i=1}^{763}(r_a(i) - \bar{r}_a)^2} = 0.873\%.$$

The annual tracking error we get using the square-root-of-time rule

$$\sigma(r_a) = \sqrt{254\frac{1}{3}} \cdot 0.873\% = 13.92\%.$$

The daily 97.5% VaR we get, using the percentile function of Excel[5] which approximates the 0.025% quantile,

$$\text{daily VaR}_{0.025\%} = 1.740\%.$$

[5] This percentile function is somewhat different from the quantile function discussed in Section 4.3.

Again using the square-root-of-time rule, the annual 97.5% VaR is

$$\mathrm{VaR}_{0.025\%} = \sqrt{254\frac{1}{3}} \cdot 1.74\% = 27.75\%.$$

Notice that this is close to the 97.5% VaR calculated using the standard deviation

$$1.96 \cdot 13.92\% = 27.28\%$$

6.5 Parametric linear method

Model

As mentioned earlier the parametric linear method is based on the assumption that the mapping function is a linear function (or a linearized function). Therefore we can write $r_a = f(\mathbf{X}) = \theta'\mathbf{X} = \sum_{i=1}^{n} \theta_i X_i$. With $\mathbf{X} = (X_1, \ldots, X_n)'$ the risk factors and $\theta = (\theta_1, \ldots, \theta_n)'$ the sensitivities. If we assume a distribution for $\mathbf{X}$, for example the multivariate normal distribution: $\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$, we have that $r_a \sim \mathcal{N}(\theta'\mu, \theta'\Sigma\theta)$. Where in practice the μ and Σ need to be estimated and μ is often assumed to be zero. These estimations can be made using the formulas in Section 4.3. And with these estimations we have

$$r_a \sim \mathcal{N}(\theta'\hat{\mu}, \theta'\hat{\Sigma}\theta).$$

From this we get that the tracking error equals

$$\sigma(r_a) = \sqrt{(\theta'\hat{\Sigma}\theta)}$$

and using that the distribution is normal the 97.5%VaR equals

$$\mathrm{VaR}_{0.025} = -q_{0.025} = 1.96\sqrt{(\theta'\hat{\Sigma}\theta)} - \theta'\hat{\mu}.$$

In general with the assumption on the distribution of $\mathbf{X}$ we can try to get an exact formula for the tracking error or the VaR. This will however not work for all distributions.

Example

In our example we use the dataset from Section 6.3. We want to calculate the risk that is taken over a risk horizon from $T = 763$ to $T + 1$. To calculate this risk we again have to construct a hypothetical portfolio, we do this by assuming that on time T the portfolio has certain weights for every stock and that these weights are kept constant, a constant-weighting strategy. For our portfolio we take these weights to be the last buy-and-hold position on 30-06-2009,

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5
Price	38.48	49.16	16.12	25.74	16.85
Weight	$\omega_1 =$ 26.29%	$\omega_2 =$ 33.59%	$\omega_3 =$ 11.01%	$\omega_4 =$ 17.59%	$\omega_5 =$ 11.51%

With these weights we have as risk factor mapping function, formulating the straightforward approach,

$$r_a = f(\mathbf{X}(t)) = f(r_1, \ldots, r_5, r_B) = \omega_1 r_1 + \ldots + \omega_5 r_5 - r_B.$$

The sample covariance matrix $\hat{\Sigma}$ of the dataset is

$$\begin{pmatrix} 0.046\% & 0.040\% & 0.023\% & 0.017\% & 0.041\% & 0.029\% \\ 0.040\% & 0.076\% & 0.028\% & 0.017\% & 0.040\% & 0.036\% \\ 0.023\% & 0.028\% & 0.028\% & 0.013\% & 0.023\% & 0.021\% \\ 0.017\% & 0.017\% & 0.013\% & 0.093\% & 0.015\% & 0.016\% \\ 0.041\% & 0.040\% & 0.023\% & 0.015\% & 0.050\% & 0.029\% \\ 0.029\% & 0.036\% & 0.021\% & 0.016\% & 0.029\% & 0.027\% \end{pmatrix}$$

So with $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, -1)'$ and assuming normality around zero, we have that r_a is distributed as

$$r_a \sim \mathcal{N}(0, \omega' \hat{\Sigma} \omega).$$

We calculate the annual tracking error, while using the square-root-of-time-rule, as

$$\sigma(r_a) = \sqrt{(\omega' \hat{\Sigma} \omega)} \cdot \sqrt{254\frac{1}{3}} = 0.865\% \cdot \sqrt{254\frac{1}{3}} = 13.80\%.$$

The VaR we can only calculate using the assumed distribution, which in this case is the normal distribution, and hence

$$\mathrm{VaR}_{0.025} = 1.96\sigma(r_a) = 27.05\%.$$

These results are very close to the results calculated in Section 6.4.

Advantages, disadvantages and extensions

The parametric linear method gives a quick way to calculate the tracking error and the VaR; furthermore it gives the tracking error and VaR in an explicit formula. The downside to this method is that we need quite strong and often unrealistic assumptions of which only some can be relaxed. The assumption of a normal distribution can be relaxed to a student-t distribution or elliptical distribution (of which the student-t is a special case). But even with these distributions we must assume that the covariance matrix is fixed.

Furthermore we assume that the risk factor mapping is a linear function, this assumption is only accurate if the time horizon is small. We can extend the parametric linear method to quadratic functions; this is also called the delta-gamma method. For quadratic functions there are also explicit formulas, see for examples (Holton, 2003). This quadratic function method can give a better approximation.

6.6 Historical simulation

Model

This method is based on the assumption that history will repeat. Let $r_a = f(\mathbf{X})$ be the risk factor mapping and let

$$X(t) \quad \text{for } T-k+1 \leq t \leq T$$

be k realizations of the risk factors in the past. These realizations of risk factors we can map to active returns using the risk factor mapping. So $f(\mathbf{X}(t))$ gives the active return that will be experienced over the next period if the risk factor change is $\mathbf{X}(t)$. This will give a sample of k active returns. Now we can proceed like in the empirical method and compute VaR with a quantile of this sample and we can calculate the tracking error as the standard deviation of the sample.

Example

For the historical simulation example we use the same hypothetical portfolio, using the current weights of

$$\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, -1)' = (26.29\%, 33.59\%, 11.01\%, 17.59\%, 11.51\%, -1)'.$$

With these weights we calculated active returns using the $1 \leq t \leq 763$ realizations of the risk factors

$$r_a(t) = \omega_1 r_1(t) + \ldots + \omega_5 r_5(t) - r_b(t).$$

These active returns are now distributed as follows, see the histogram in Figure A.2 in Appendix A.2. We calculate the annual tracking error as the standard deviation of these active returns and using the square-root-of-time-rule we get

$$\sigma(r_a) = \sqrt{\frac{1}{n-1}\sum_{i=1}^{763}(r_a(i) - \bar{r}_a)^2} \cdot \sqrt{254\frac{1}{3}} = 0.878\% \cdot \sqrt{254\frac{1}{3}} = 14.00\%$$

and we have

$$1.96 \cdot \sigma(r_a) = 27.44\%$$

The annual 97.5% VaR of these active returns is

$$\mathrm{VaR}_{0.025} = 1.805\% \cdot \sqrt{254\frac{1}{3}} = 28.76\%.$$

Again notice the difference between the empirical method and the historical simulation method, while the empirical method calculates the active return for every position in the past, the historical simulation method only uses the current position to calculate active returns. In this setting, however, the results differ only slightly. But if we'll use more stocks and a more active position then it can differ more.

Advantages, disadvantages and extensions

The main advantage of the historical simulation method is that there's no need to make any assumption about or calculations of the distribution of the risk factors. Characteristics of the active return distribution, such as volatilities and correlations are captured implicitly in the realizations of the risk factor changes. However this method is highly dependent on which data set is used. Extreme events such as market crashes or other outliers can seriously distort the historical simulation risk measures. Especially the VaR for high quantiles is sensitive to such events.

Another problem is the lack of available data, one year of data corresponds to only 252 data points, the number of trading days on average. This means that we need a few years of data to get a good measure for the daily tracking error and the daily VaR. If we use one year of history the 97.5% VaR measure will only be based on the 6-th and/or 7-th lowest return, since 2.5% of 252 is 6.3. For weekly or monthly measures we need even more years. If we use more years of data, then there's a greater chance that we include years with 'polluted data' because of structural changes in the market[6]. It's not very clear if and how we can use such data for predicting the risk which is taken now.

A way to partly deal with these drawbacks is using exponential weighting of the returns instead of equally weighting. We do this by including a decay factor that gives more weight to the more recent observations. (Alexander, 2008d) gives another method, volatility weighting: here the weights for the returns are dependent on the volatility, this gives the adjusted returns

$$\tilde{r}(t,T) = (\frac{\hat{\sigma}_T}{\hat{\sigma}_t})r(t)$$

with $\hat{\sigma}_T$ the volatility estimated at the current time and $\hat{\sigma}_t$ the volatility estimated at time $t < T$.

6.7 Monte Carlo method

Model

For the Monte Carlo method we simulate possible risk factors $\mathbf{X}^1, \ldots, \mathbf{X}^N$ for large N (e.g. 1.000 or 10.000). These simulations are generated from the multivariate distribution of the risk factors which is estimated using historical data. So suppose we have estimated the distribution of the risk factors $\mathbf{X} = (X_1, \ldots, X_n)$ to be $\mathbf{X} \sim \mathcal{N}(\hat{\mu}, \hat{\Sigma})$, where, for simplicity we assume the distribution to be normal. Now we need to generate risk factors with the same distribution. First of all, in any mathematical or statistical program there's a way to generate random numbers and there is an inverse of the normal probability density function. With this we can generate Z_1 to Z_n, with $Z_i \sim \mathcal{N}(0,1)$

[6]A nice example is the introduction of the euro in 1999.

i.i.d. and hence $\mathbf{Z} = (Z_1, \ldots, Z_n)$ is distributed as

$$\mathbf{Z} \sim \mathcal{N}(0, I),$$

with I the $(n \times n)$ identity matrix. Since we have for any $(n \times n)$ matrix C, that

$$C'\mathbf{Z} \sim \mathcal{N}(0, C'C),$$

our aim is now to find a matrix C such that

$$C'C = CC' = \hat{\Sigma}.$$

Note that if we need to do this when $\hat{\mu} \neq 0$ the matrix C doesn't change. We will discuss two methods for finding such a matrix C that decomposes the covariance matrix $\hat{\Sigma}$, both methods use the fact that a covariance matrix is symmetric and positive definite. An $(n \times n)$ matrix A is positive definite if

$$x'Ax > 0 \quad \forall x \in \mathbb{R}^n \text{ non-zero.}$$

Cholesky

(Golub and van Loan, 1996) prove that an $(n \times n)$ matrix A has a unique factorization into a lower triangular $(n \times n)$ matrix L with strictly positive diagonal elements and its transpose L' an upper triangular matrix, such that

$$A = LL'$$

This is called the Cholesky decomposition of A. An algorithm for finding L is easily found by writing out the equation above and using that the matrix is symmetric and positive definite. For a (3×3) matrix we have

$$\begin{aligned}
\begin{pmatrix} a_{11} & & \text{(symmetric)} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix} \\
&= \begin{pmatrix} l_{11}^2 & & \text{(symmetric)} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}
\end{aligned}$$

and we obtain from this equation, the formulas

$$\begin{aligned}
l_{11} &= \sqrt{a_{11}} \\
l_{21} &= \frac{a_{21}}{l_{11}} \\
l_{22} &= \sqrt{a_{22} - l_{21}^2} \\
l_{31} &= \frac{a_{31}}{l_{11}} \\
l_{32} &= \frac{a_{32} - l_{31}l_{21}}{l_{22}} \\
l_{33} &= \sqrt{a_{33} - l_{31}^2 - l_{32}^2}.
\end{aligned}$$

This can easily be extended to the general formulas for $(n \times n)$ matrices

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$$
$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{l_{jj}} \quad \text{for } i > j.$$

With this Cholesky decomposition we have decomposed $\hat{\Sigma} = CC'$ using a lower triangular matrix, this will make the calculation of $C'\mathbf{Z}$ simple.

Spectral decomposition or eigen decomposition

An n-dimensional eigenvector e of a square $(n \times n)$ matrix A and its corresponding eigenvalue d are defined by the property

$$Ae = de.$$

Now if A is symmetric then we have n eigenvalues, not necessarily distinct, which are all positive if A is positive definite. We order the eigenvalues from largest to smallest. That is, $d_1 \geq d_2 \geq \ldots \geq d_n$. Since A is symmetric the corresponding eigenvectors e_1 to e_n can be chosen to be orthogonal to each other and have norm one. Let

$$D = \operatorname{diag}(d_1, \ldots d_n)$$

be the diagonal matrix with the eigenvalues on its diagonal, we have the obvious property for diagonal matrices that $D^T = D$. Let $E = (e_1, \ldots, e_n)$ be the orthogonal matrix with the eigenvectors as columns, an orthogonal matrix satisfies the property that

$$E' = E^{-1}.$$

Now since we have $Ae_i = d_i e_i$ for all $1 \leq i \leq n$, we have that

$$AE = A(e_1, \ldots e_n) = (Ae_1, \ldots, Ae_n) = (d_1 e_1, \ldots, d_n e_n) = ED.$$

From this equality we get the spectral or eigen decomposition

$$A = EDE'.$$

Because A was assumed to be positive definite the eigenvalues are all positive which means that the square root of the diagonal matrix is well-defined $\sqrt{D} = \operatorname{diag}(\sqrt{d_1}, \ldots \sqrt{d_n})$, such that $D = \sqrt{D}\sqrt{D}$. From this we can get the decomposition we want for the covariance matrix, if we set $A = \hat{\Sigma}$, this decomposition is

$$A = EDE' = E\sqrt{D}\sqrt{D}E' = E\sqrt{D}(\sqrt{D})'E' = (E\sqrt{D})(E\sqrt{D})'.$$

The Choleksy decomposition is more straight forward and is easier to use in the calculation then the eigen decomposition, the disadvantage to the Cholesky decomposition is that it fails if the covariance matrix is not positive-semidefinite, because then the algorithm tries to take the root of a negative number. In theory every covariance matrix is positive-semidefinite, but in practice when using big matrices a simple miscalculation or using a hypothetical covariance matrix, can lead to a non positive-semidefinite covariance matrix so that the Cholesky decomposition fails. It's not easy to find a workaround for this problem. For the Eigen decomposition this can be done easily by setting any negative eigenvalue to zero.

Now with the simulations generated using the Cholesky decomposition or the eigendecomposition of the covariance matrix, we can, in the same way as in the historical simulation method, calculate the losses that will occur, using these generated risk factors, by $r_i = f(\mathbf{X}^i)$ for $i = 1, \ldots, N$. And again, VaR can be calculated with the quantile and the tracking error as the standard error of the sample.

Example

For the Monte Carlo method, with 1000 simulations, we use the same portfolio as for the parametric and historical simulation methods. First of all we generate 6×1000 random drawings from a normal distribution, 1000 for each stock and 1000 for the benchmark. We've already seen the covariance matrix Σ for the parametric linear method in Section 6.5. Then using the Cholesky decomposition C of Σ

$$\begin{pmatrix} 0.021 & 0 & 0 & 0 & 0 & 0 \\ 0.019 & 0.020 & 0 & 0 & 0 & 0 \\ 0.011 & 0.004 & 0.012 & 0 & 0 & 0 \\ 0.008 & 0.001 & 0.003 & 0.029 & 0 & 0 \\ 0.019 & 0.002 & 0.001 & 0.000 & 0.011 & 0 \\ 0.014 & 0.005 & 0.004 & 0.001 & 0.001 & 0.007 \end{pmatrix}$$

or the eigen decomposition D of Σ

$$\begin{pmatrix} 0.0013 & 0.0058 & 0.0021 & 0.0066 & 0.0040 & 0.0189 \\ 0.0010 & -0.0004 & 0.0026 & -0.0119 & 0.0071 & 0.0237 \\ 0.0019 & -0.0007 & -0.0104 & 0.0007 & 0.0019 & 0.0129 \\ 0.0002 & -0.0001 & 0.0006 & -0.0007 & -0.0267 & 0.0146 \\ 0.0004 & -0.0053 & 0.0031 & 0.0081 & 0.0046 & 0.0193 \\ -0.0055 & 0.0008 & -0.0024 & 0.0002 & 0.0022 & 0.0153 \end{pmatrix}$$

we can transform these 6×1000 drawings from a normal distribution to 6×1000 drawings $r_i(1)$ to $r_i(1000)$ for $i = 1, \ldots, 5, b$ from a distribution $\mathcal{N}(0, \Sigma)$, these drawings represent the simulated risk factors. Now with the portfolio position we calculate the active returns by

$$r_a(n) = \omega_1 r_1(n) + \ldots + \omega_5 r_5(n) - r_b(n) \quad \text{for } 1 \leq n \leq 1000.$$

These active returns depend, of course, on the generated drawings from the normal distribution. Also, using the Cholesky decomposition or the eigen decomposition leads to (slightly) different, but similar results. For one of the simulation done, using the Cholesky decomposition, the active returns are distributed as follows, see the histogram in Figure A.3 in Appendix A.2. We calculate the annual tracking error as the standard deviation of these active returns, while using the square-root-of-time-rule,

$$\sigma(r_a) = \sqrt{\frac{1}{n-1}\sum_{i=1}^{1000}(r_a(i) - \bar{r}_a)^2} \cdot \sqrt{254\frac{1}{3}} = 0.828\% \cdot \sqrt{254\frac{1}{3}} = 13.20\%$$

and we have

$$1.96 \cdot \sigma(r_a) = 25.88\%.$$

The annual 97.5% VaR of these active returns is

$$\mathrm{VaR}_{0.025} = 1.581\% \cdot \sqrt{254\frac{1}{3}} = 25.21\%.$$

Advantages, disadvantages and extensions

This is the most versatile method. We can assume any distribution for the risk factors and any function can be used. But we still rely on estimations of the risk factor distributions. The main drawback is that if N is chosen to be a large number and the risk factors and function is complex the computation time can become very large. And although with extensions this method seems to be able to model the active return distribution very accurately, the results are only 'as good as the model'.

This method can and has been extended in many ways. See (Jäckel, 2002) or (Glasserman, 2003) for a rich analysis on the techniques that can be used to enhance the Monte Carlo method.

6.8 Further extension

Here we cover some further extensions with regard to calculating the volatilities, covariances and sensitivities.

6.8.1 Principal components analysis

Not all risk factors influence the active return in the same way. Some risk factors explain more of the variance in active returns than other risk factors. Principal component analysis is a way to order the risk factors according to their influence. Furthermore it can be used to reduce the dimension of the covariance matrix used in the parametric linear model and the Monte Carlo model. Here we'll discuss the principal components analysis, for an elaborate example and application see (Alexander, 2008a).

Let $\mathbf{X} = (X_1, \dots X_n)'$ the $(n \times t)$ matrix with n risk factors and $X_i = (X_i(1), X_i(2), \dots, X_i(t))$ the t realizations of these risk factors, for simplicity we assume a zero mean. Let $\hat{\Sigma}$ be the covariance matrix of $\mathbf{X}$. The principal components analysis also uses the spectral decomposition discussed in Section 6.7. So we have

$$\hat{\Sigma} = EDE',$$

with E and D as in Section 6.7. The principal components matrix P of $\mathbf{X}$ is now defined by

$$P = E'\mathbf{X}$$

and the j-th principal component is defined as the j-th row of this matrix, given by

$$p_j = e_j'\mathbf{X}$$

With e_j the j-th eigenvector. This matrix has the property that

$$\mathrm{Cov}(P) = E' \,\mathrm{Cov}(\mathbf{X})E = E'\hat{\Sigma}E = E'EDE'E = D.$$

Here we used, $E' = E^{-1}$ and hence $EE' = E'E = I$ the $(n \times n)$ identity matrix. This covariance matrix means that the principal components of $\mathbf{X}$ are uncorrelated to each other and that the variance of p_j is equal to d_j. Since the eigenvalues d_j are ordered from largest to smallest, the principal components are ordered by variance from large to small. It can be shown that if we maximize the variance of $p_1 = y_1\mathbf{X}$ over all y_1 with norm one, then this maximum will be the first eigenvector e_1 corresponding to the largest eigenvalue. And the second eigenvector e_2 maximizes $p_2 = y_2\mathbf{X}$ over all y_2 with norm one and orthogonal to y_1. And so on to the last eigenvector e_n.

Now, since $P = E'\mathbf{X}$ and $E' = E^{-1}$ we can define the principal component representation of $\mathbf{X}$ as

$$\mathbf{X} = EP.$$

If we work out this multiplication, we have for each risk factor X_i with $1 \leq i \leq n$ the representation

$$X_i = e_{1i}p_1 + e_{2i}p_2 + \dots + e_{ni}p_n$$

here we use the notation for the j-th eigenvector as

$$e_j = (e_{j1}, e_{j2}, \dots, e_{jn})'.$$

The principal components reduction is now as follows: since we ordered the eigenvalues, the principal components p_i are ordered by variance, from largest to smallest. So a major part of the variance is explained by the first k principal components, usually k is taken small like 3 or 5, and therefore the risk factor X_i can be approximated by

$$X_i \approx e_{1i}p_1 + \dots + e_{ki}p_k.$$

And in matrix notation

$$\mathbf{X} \approx E_kP_k$$

With E_k the adjusted $(n \times k)$ matrix with the first k eigenvectors as columns and P_k the adjusted $(k \times t)$ matrix with the first k principal components as rows. And from this approximation of $\mathbf{X}$ we get an approximation of $\hat{\Sigma}$ by

$$\hat{\Sigma}^* = \mathrm{Cov}(E_k P_k) = E_k \,\mathrm{Cov}(P_k) E_k^T = E_k D_k E_k^T$$

with $D_k = \mathrm{diag}(d_1, \ldots, d_k)$. We can use this approximation of $\hat{\Sigma}$ to simplify the calculations, without loosing much of the underlying information.

6.8.2 Using time series

Random walk model

Until now we made the approach that the risk factor returns are randomly drawn from some overlying distribution and that their mean and variance is fixed over time. This comes down to the assumption that the risk factor distributions are unconditional, which means that if $(X_t)_{t \in \mathbb{N}}$ is a time series that the distribution $F_{\mathbf{X}_{t+1}|\mathcal{F}_t} = F_X$ for all t with $\mathcal{F}_t$ the sigma field (the information available on time t). In the conditional case this equality does not necessarily hold. This means we can have a varying mean and covariance matrix over time in the conditional case. In this section we will further investigate this using as basis the random walk model with drift[7] for the time series $(X_t)_{t \in \mathbb{N}}$ which is

$$X_t = c + X_{t-1} + \varepsilon_t \quad \varepsilon_t \sim \text{i.i.d.}\mathcal{N}(0, \sigma^2).$$

The property that ε_t is distributed as $\mathcal{N}(0, \sigma^2)$ and i.i.d. is usually referred to as ε_t following a white noise process. A white noise process can be described by the properties

$$\mathbb{E}[\varepsilon_t] = 0 \quad \text{for all } t,$$

$$\mathbb{E}[\varepsilon_t^2] = \sigma^2 \quad \text{for all } t,$$

$$\mathrm{Cov}(\varepsilon_s, \varepsilon_t) = \mathbb{E}[\varepsilon_s \varepsilon_t] = 0 \quad \text{for all } s \neq t.$$

If in this model X_t are the log prices of a stock, then we have for the log return the basic model

$$r_t = X_t - X_{t-1} = c + \varepsilon_t$$

with ε_t a white noise process. Notice that in this basic model the return is independent on past returns, and all the returns are i.i.d. normal distributed. This boils down to the unconditional approach where the risk factor returns are randomly drawn from a normal distribution

[7] In all the models introduced in this section we include a drift, which is just a constant c in the model. In many works, however, this constant is omitted or assumed to be zero for simplicity.

Stylized facts

Before we extend this model we first look at some stylized facts regarding daily return series, these are empirical findings that generally hold for a majority of daily return series. (McNeil et al., 2005) state these as follows, in their work also a motivation can be found,

- Return series are not i.i.d. although they show little serial correlation.
- Series of absolute or squared returns show profound serial correlation.
- Conditional expected returns are close to zero.
- Volatility appears to vary over time.
- Return series are leptokurtic or heavy-tailed.
- Extreme returns appear in clusters.

With these 'facts' in mind, we can enhance our basic model for return series $(r_t)_{t\in\mathbb{N}}$.

Autoregressive-moving average models

Autoregressive-moving average models, or ARMA models, focus mainly on data where the return series show signs of autocorrelation or serial correlation. First of all, there is the autoregressive model that makes the returns dependent on past returns. An autoregressive model of order p, denoted as AR(p), with ε_t a white noise process, is written as

$$r_t = c + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + \varepsilon_t.$$

Next we can make the returns dependent on past errors ε_t. This is called the moving average model of order q, denoted as MA(q), with ε_t a white noise process,

$$r_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}.$$

These two models can also be combined to the autoregressive moving average model with p autoregressive components and q moving average components. This is denoted as ARMA(p, q), with ε_t a white noise process,

$$r_t = c + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}.$$

Notice that ARMA($p, 0$) = AR(p) and ARMA($0, q$) = MA(q).

Autoregressive conditional heteroskedasticity model

Until now we have assumed that ε_t has constant variance σ^2 over time. The generalized autoregressive conditional heteroskedasticity (GARCH) model, introduced by (Bollerslev, 1986), assumes a time-varying variance σ_t^2. With a

GARCH(p, q) model this variance is modeled with p error terms and q variance terms as follows

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \varepsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_q \sigma_{t-q}^2.$$

If we set $q = 0$, then we have GARCH$(p, 0)$ = ARCH(p), which is the ARCH model, or autoregressive conditional heteroskedasticity model, introduced by (Engle, 1982). The ε_t in these models often come from the basic model

$$r_t = \mu + \varepsilon_t$$

But also an ARMA(p, q) model, including the possibilities $p = 0$ or $q = 0$, can be used.

Chapter 7

Implementing active management risk in the standardmodel

In this chapter we'll discuss the possible ways there are of implementing an active management risk component in the standard model. We'll focus mainly on an active management risk component when investing in developed market equities. First we'll give 3 implementation methods that are already known, then we'll discuss all the important elements that should be considered when implementing active management. Lastly we'll see what the impact is on the required funding ratio, using some artificial fund positions.

7.1 Possible ways of implementing active management risk

There are already some implementation methods in use. These methods can be divided into two main categories. Namely,

Category 1

Adding an extra risk element in the standard model. This will form an extra risk element $\mathbf{S_7}$ or $\mathbf{S_{10}}$, depending on if liquidity, concentration and operational risk are included as risk elements. When this risk element is valued, using the tracking error or the VaR or another type of valuation[1], it can be taken into the square root formula.

[1]Not necessarily a risk measure has to be used, we'll see this in Section 7.2.2.

Category 2

Incorporating an active management component to the equity risk element $\mathbf{S_2}$ and specifically to the risk element for developed market equities $\mathbf{S_{2A}}$, which is 25% in the standard model. With the measure of the active risk this 25% can be raised (or lowered) to give a buffer against equity risk plus active management. This adjustment is usually done by adding the active management risk volatility to the benchmark volatility, keeping in mind some correlation. From this we get an adjusted $\mathbf{S'_{2A}}$.

Now we'll give the three methods in use, or suggested to be used. All three methods use the ex-ante tracking error, but as we've seen in Section 5.2, this ex-ante tracking error can also be calculated using the VaR technique. For easy of discussion we'll from now on focus only on the tracking error.

Method A

This method belongs to the first category and the formula for the extra risk element $\mathbf{S_7}$ is as follows

$$\mathbf{S_7} = -\sum_{bc} \text{Amount}_{bc} \cdot (\Phi^{-1}(1-\alpha) \cdot \mathbf{TE}_{bc} - \mathbf{TER}_{bc})$$

with

$$\begin{aligned}
\text{Amount}_{bc} &: \text{the amount invested in equities} \\
\Phi^{-1}(1-\alpha) &: \text{the inverse of the normal cumulative distribution function,} \\
& \quad \text{for } \alpha = 0.975 \text{ we have } \Phi^{-1}(1-\alpha) = -1.96 \\
\mathbf{TE}_{bc} &: \text{the tracking error of the amount invested in equities} \\
\mathbf{TER}_{bc} &: \text{the total expense ratio of the amount invested in equities.}
\end{aligned}$$

We can simplify this formula to

$$\mathbf{S_7} = \sum_{bc} \text{Amount}_{bc} \cdot (1.96\mathbf{TE}_{bc} + \mathbf{TER}_{bc}).$$

This sum should be interpreted as follows: suppose 20% of the total capital is invested in developed market equities with a tracking error of 4% and a TER of 0.5% and 10% of the total capital is invested in developed market equities with a tracking error of 6% and a TER of 1% then the risk element for developed market equities is

$$\mathrm{S_{2A}} = 25\% * 30\% = 7.5\%$$

and the active management risk is

$$\mathbf{S_7} = 20\% * (1.96 \cdot 4\% + 0.5\%) + 10\% * (1.96 \cdot 6\% + 1\%) = 2.94\%.$$

Notice that in this calculation the correlation between investments with different tracking errors and total expense ratio is assumed to be 0.

The square root formula for the required funding ratio discussed in Section 3.8 will in this method change to

$$\mathbf{S_t} = \sqrt{\mathbf{S_1}^2 + \mathbf{S_2}^2 + 2\rho_1\mathbf{S_1}\mathbf{S_2} + \mathbf{S_3}^2 + \mathbf{S_4}^2 + \mathbf{S_5}^2 + \mathbf{S_6}^2 + \mathbf{S_7}^2 + 2\rho_2\mathbf{S_2}\mathbf{S_7}}.$$

Here ρ_1 is the usual correlation between $\mathbf{S_1}$ and $\mathbf{S_2}$, which is 0.5, and ρ_2 is the correlation between $\mathbf{S_2}$ and $\mathbf{S_7}$. In this method the assumption is made that the active management risk element is uncorrelated to the equity risk element, this means $\rho_2 = 0$, when analyzing this method we will, however, also investigate the possibility of other values for ρ_2. In this square root formula the active management risk element is also uncorrelated to the other risk elements.

Method B

The next method belongs to category 2 and uses three steps. For the first step the volatility (or standard deviation) of the benchmark is needed, which we'll denote by $\sigma_{\text{benchmark}}$. And we need the volatility around the benchmark, which is the tracking error: $\sigma_{\text{tracking error}}$. Now the volatility for the active investment is calculated using the formula

$$\sigma^2_{\text{active investment}} = \sigma^2_{\text{benchmark}} + \sigma^2_{\text{tracking error}} + 2\rho\sigma_{\text{benchmark}}\sigma_{\text{tracking error}}$$

with ρ the correlation between the benchmark and the tracking error, in this method this correlation is assumed to be zero. But again, we'll also analyze other values.

In the next step we calculate $\mathrm{F_a}$, which is called the active management factor, by the formula

$$\mathrm{F_a} = \frac{\sigma_{\text{active investment}}}{\sigma_{\text{benchmark}}}.$$

In the last step the adjusted scenario $\mathrm{D_{adjusted}}$ is calculated from the scenario in the standard model $\mathrm{D_{standard}}$, which is 25% for developed market equities, by

$$\mathrm{D_{adjusted}} = \mathrm{F_a} \cdot \mathrm{D_{standard}}.$$

With this adjusted scenario the risk element $\mathbf{S'_{2A}}$ is calculated and the adjusted required funding ratio follows from the square root formula.

Method C

Another method that also falls under category 2, where we again use the factor 1.96 for the 97.5% confidence interval of a normal distribution, can be formulated as follows

$$\mathrm{D_{adjusted}} = 1.96\sqrt{(\frac{\mathrm{D_{standard}}}{1.96})^2 + \sigma^2_{\text{tracking error}} + 2\rho(\frac{\mathrm{D_{standard}}}{1.96})\sigma_{\text{tracking error}}}.$$

Where ρ is the correlation between the tracking error and the scaled down standard scenario $\mathrm{D}_{\text{standard}}$, which is 25% for investments in developed market equities. In this method ρ is assumed to be 0.5.

Although this method looks different from Method B, these methods are very similar. If we take $\sigma_{\text{benchmark}} = \frac{\mathrm{D}_{\text{standard}}}{1.96}$ we have for method B

$$\begin{aligned}
\mathrm{D}_{\text{adjusted}} &= \mathrm{F}_{\mathrm{a}} \cdot \mathrm{D}_{\text{standard}} \\
&= \frac{\sigma_{\text{active investment}}}{\sigma_{\text{benchmark}}} \cdot \mathrm{D}_{\text{standard}} \\
&= \frac{\sigma_{\text{active investment}}}{\mathrm{D}_{\text{standard}}/1.96} \cdot \mathrm{D}_{\text{standard}} \\
&= 1.96\sigma_{\text{active investment}}
\end{aligned}$$

with $\sigma_{\text{active investment}}$ defined in Method B. For Method C we have the same formula for $\mathrm{D}_{\text{adjusted}}$ if $\sigma_{\text{benchmark}} = \frac{\mathrm{D}_{\text{standard}}}{1.96}$.

7.2 Considerations

Before we discuss these ways of implementing an active risk component, we first discuss some elements that should be considered. First of all, we've seen in Section 2.2 that the standard model is based on some basic principles. For active management risk these principles could play a role when deciding on the choice of implementation. We will use the numbering of these principles used in that section.

7.2.1 Correlation

As we've seen in Section 3.8 and Section 3.9 there are correlations between risk elements that should be taken into account. If we choose category 1 should there be a correlation between the active management risk element $\mathbf{S_7}$ and the equity risk element $\mathbf{S_2}$? And in category 2, when the 25% scenario for developed market equities is raised, what should be the correlation between the active management risk and the benchmark risk? These questions boil down to the general question: "Are active returns and benchmark returns correlated?" And more importantly are they correlated when, in extreme events, we have negative benchmark returns?

This is a hard question. If we look back at Section 3.10 we see that there can be different goals for active management. With 2 opposite goals being:

- Reducing the risk by lowering the volatility of the benchmark.
- Trying to outperform the benchmark by selecting volatile stocks, if the benchmark goes up (positive return) then these stocks tend to go up more. But there exists the risk of underperforming the benchmark when the benchmark goes down.

If in the first goal the benchmark return is positive our strategy aims at a lower portfolio return and hence a negative active return, since $r_a = r_p - r_b$. And if the benchmark return is negative our strategy aims at a higher portfolio return and hence a positive active return. This would lead to a negative correlation between the benchmark return and the active return.

For the second goal we have the opposite. If a benchmark return is positive the fund wants to outperform it, giving an active return that is positive. If, on the other hand, the benchmark return is negative the fund has the risk of underperforming, making the active return also negative. This would lead to a positive correlation.

So what we see is that a fund can use an active strategy to manipulate the correlation between the benchmark return and the active return. Therefore it will be hard to say what effect this strategy has on the correlation.

Another complication is stated in the 9-th principle of Section 2.2. Suppose it can be shown that the correlation between benchmark returns and active returns for a fund is on average 0.1. How can we tell if this correlation holds in extreme events when the benchmark returns are highly negative?

We can, however, distinguish between a few possibilities for this correlation, of which some are in use as we've seen in Section 7.1. We'll discuss some possibilities there are, with ρ being the correlation between benchmark returns and active returns,

$-1 \leq \rho < 0$**:**

Although strategies do exist that would have a negative correlation, there is no guarantee that this will hold in extreme events. Furthermore, the main reason that the standard model is revised is that the required funding ratio calculated with the current standard model is too low and therefore an active management component should raise this required funding ratio. With a negative correlation there exists a possibility that the required funding ratio is lowered.

$\rho = 0$**:**

This assumption is made in Method A and Method B. It comes from the view that the correlation will fluctuate over time, sometimes it will be positive, sometimes negative, but on average it will be zero. Starting from this point, the correlation can be altered for funds that have a very different strategy which would suggest an correlation above or below 0.

$0 < \rho < 1$**:**

Method C makes this assumption, with a correlation of 0.5. This view is bit more conservative; the average correlation between benchmark returns and active returns tends to be slightly positive for most pension funds and it can be taken into account that in extreme events the correlation can become higher, see principle 9, and that therefore a fund should take into account the possibility of a correlation above 0. Also the risk that is added by active management is not very transparent, see principle 4. Therefore, depending on how conservative the view on this risk is, the correlation is assumed to be somewhere between 0 and 1 and preferably it is rounded to a simple fraction like $\frac{1}{2}$, $\frac{1}{4}$ or $\frac{3}{4}$, see principle 6.

$\rho = 1$**:**

An extremely conservative possibility is setting the correlation equal to 1. This comes down to just adding the active risk component to the equity risk or adding the tracking error to the standard deviation of the benchmark. This option has the disadvantage that it very well could make funds very careful in their active management and even could stimulate funds to stop managing actively, which would lessen their diversification, see principle 10.

In this thesis we focused on active management risk when investing in developed market equities. But active management risk also plays a part in other types of investments as we've seen in Section 3.10. This means that we can have a tracking error for bonds, for developed market equities, for emerging market equities, etc. and on top of that there can be a dynamic or tactical asset allocation. Furthermore, it is not uncommon to have multiple managers for investing in developed market equities, each one of them investing a part of the capital with its own strategy, giving multiple tracking errors[2]. What then will be the total tracking error of a portfolio? Between all these tracking error there will be correlations and it's hard to say what will happen in extreme events.

7.2.2 Which tracking error?

We've seen that all three methods in Section 7.1 use the annual ex-ante tracking error[3] to calculate an active management risk component. However, this ex-ante tracking error is not a measure that remains constant over time; if the (active) position changes, then also the ex-ante tracking error changes. Should we use an average ex-ante tracking error over time? Furthermore, there are many different ways of calculating this tracking error, for instance we must choose one of the 3 different methods discussed in Chapter 6, we also need to choose between linear returns and log returns, like we discussed in Chapter 4, and we must make assumptions on their distributions. A fund can also differ in using

[2]This we saw in the example for method A, were a correlation of 0 is assumed between these different managers.

[3]The ex-ante measure is used since we're measuring the risk of what will happen over a 1-year horizon and not what did happen in the past.

a history with daily, weekly or monthly returns covering a past of 1 month, 1 year, 3 years, etc. And when using the Monte Carlo method, the number of simulations can vary. Typical for a pension fund of Syntrus would be using the Monte Carlo method with a history of weekly returns over 3 years and with around 5000 or 10000 simulations.

A solution for a varying tracking error is using the maximum tracking error. This maximum tracking error is usually given in the policy guideline plan of a pension fund. A fund manager may not take on a strategy that exceeds this maximum tracking error. So if on one day the calculated ex-ante tracking error is above this maximum tracking error, then the manager should change his investment positions. The advantage of using the maximum tracking error is that it's a very constant element; typical for a Syntrus fund is a maximum tracking error between 3% and 6% for equities and it only changes after a year or more. A disadvantage is that it's not really a tracking error, but already an extremum of possible tracking errors and already takes into account a certain confidence level, not necessarily 97.5%. If we treat the maximum tracking error as a standard deviation, which is done in all three methods discussed in Section 7.1, the risk will be overestimated. We could of course adjust this maximum tracking error in some way to get a useful standard deviation. But in Appendix A.1 we show that the distribution of the estimator for the tracking error only approaches normality for large samples, for small samples this distribution is not normal so the extremum in the form of the maximum tracking error is hard to manipulate.

A solution for the many varying ways to calculate tracking error is somewhat harder. Often the tracking error is reported, while giving only little information about the way it has been calculated and what assumptions have been made. A possibility for DNB could be to prescribe the exact method, including the amount of history and the number of simulations etc., to be used.

7.2.3 TER and alpha

In method A we see that the total expense ratio (TER_{bc}), which is calculated as the total costs made to actively manage the amount invested in equities divided by the total amount invested in equities, is also included in the formula. And more specifically it is added to the risk without a factor. So extra costs lead to extra risk, or more precisely, to extra required funding ratio.

In none of the methods the alpha is included. Remember that α is the constant in the regression of the active return against the individual asset returns or market return. This alpha can be seen as the (ex-ante) expected active return. If a fund tries to outperform the benchmark then of course they want this alpha to be positive and it can simply choose an active strategy with positive alpha. Should this alpha be included in the formula for the active risk element?

If we look again at the principles, then the 9-th principle which says that "the determination of a parameter should be consistent with the determination of the other parameters" should be considered here. Remember the way the parameters for equity risk were obtained in Section 3.2. The volatility of the benchmark was estimated to be approximately 17% and a normal distribution was assumed around an expected return of 8%, this gave the $25\% \approx 8\% - 1.96 \cdot 17\%$ parameter for $\mathbf{S_{2A}}$. So in line with this calculation the alpha should be included.

It should be noted that the TER and the alpha are usually quite small, around 1%. So including these elements in the calculation of $\mathbf{S_7}$ will not give a significant change in this risk element. And hence according to the 5-th principle these elements can be excluded in the calculation. Another argument for leaving out the TER and the alpha is that it is possible that they cancel out. Ideally with extra costs one can expect a higher alpha[4], so they have an opposite effect making the total effect even less significant.

7.2.4 Consistent

We just mentioned that the determination of an active management risk element should be consistent with the determination of the other elements, and in particular with the determination of the developed market equity risk element $\mathbf{S_{2A}}$, which was done by the expected return minus 1.96 times the benchmark volatility. In method A the expected active return (or alpha) is not included in the calculations of $\mathbf{S_7}$, but otherwise this method seems 'in line' with the calculation of $\mathbf{S_{2A}}$.

The other 2 methods on the other hand are not in line with the calculation of $\mathbf{S_{2A}}$. In method B a factor is calculated using the current volatility of the benchmark, not necessarily 17%, and the tracking error. But then the 25% scenario is multiplied with this active investment volatility factor. In line with the calculation of $\mathbf{S_2}$ this method could be adjusted to

Method B'

The active volatility is

$$\sigma^2_{\text{active investment}} = \sigma^2_{\text{benchmark}} + \sigma^2_{\text{tracking error}} + 2\rho\sigma_{\text{benchmark}}\sigma_{\text{tracking error}}$$

with $\rho = 0$. The active management factor is then

$$\mathrm{F_a} = \frac{\sigma_{\text{active investment}}}{\sigma_{\text{benchmark}}}.$$

And the adjusted scenario $\mathrm{D_{adjusted}}$ is

$$\mathrm{D_{adjusted}} = -(8\% - 1.96 \cdot (\frac{\mathrm{D_{standard}} + 8\%}{1.96}) \cdot \mathrm{F_a}).$$

We emphasize that we calculate the factor using the current benchmark volatility instead of the for $\mathbf{S_{2A}}$ assumed volatility of $\frac{\mathrm{D_{standard}}+8\%}{1.96} \approx 16.837\%$. Method C can be adjusted in a similar way.

[4]Although some studies claim that actively managed funds more often underperform a benchmark then outperforming it, meaning that they have a negative alpha and a positive TER.

Method C'

$$\mathrm{D_{adjusted}} = -(8\% - 1.96 \cdot \sigma_{\text{active investment}})$$

with $\sigma_{\text{active investment}}$ equal to

$$\sqrt{(\frac{\mathrm{D_{standard}} + 8\%}{1.96})^2 + \sigma^2_{\text{tracking error}} + 2\rho(\frac{\mathrm{D_{standard}} + 8\%}{1.96})\sigma_{\text{tracking error}})}.$$

This is again equal to a method B' type, now with a benchmark volatility of $\frac{\mathrm{D_{standard}}+8\%}{1.96}$ and a correlation $\rho = 0.5$ instead of 0.

7.2.5 Impact and the 'look-through principle'

In the 3 methods we've discussed the tracking error gives but a minor factor in the calculation of the required funding ratio. Often the impact on the required funding ratio is not more then 1%, if the tracking error is not too large. If we look at principle 5 then we could choose to ignore this risk element.

On the other hand (DNB, 2011) takes into account the look-through principle[5]. This says that for every investment it should be clear what the risks are. With more data on tracking errors and total expense ratios the risk of active investment can better be analyzed. Hence, one can choose to include tracking error in the standard model for this reason.

7.3 What is the impact?

To give an indication of the impact of including an active management risk element we calculate the impact active management risk has on the required funding ratio for some asset allocations and tracking errors. For this we will use the mean risk elements calculated by (DNB, 2011)

Risk element	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$	$\mathbf{S_4}$	$\mathbf{S_5}$	$\mathbf{S_6}$
Mean risk	8.9%	14.9%	2.3%	1.1%	1.1%	3.5%

This mean was taken over all the funds in their analysis covering about 90% of the pension funds in terms of technical provision. And it gives a required funding ratio of 21.3%.

Assumptions for calculating the impact with category 1

First we look at the square root formula for a category 1 implementation

$$\mathbf{S_t} = \sqrt{\mathbf{S_1}^2 + \mathbf{S_2}^2 + 2\rho_1\mathbf{S_1}\mathbf{S_2} + \mathbf{S_3}^2 + \mathbf{S_4}^2 + \mathbf{S_5}^2 + \mathbf{S_6}^2 + \mathbf{S_7}^2 + 2\rho_2\mathbf{S_2}\mathbf{S_7}}.$$

We will simplify this model to

$$\mathbf{S_t} = \sqrt{\mathbf{S_1}^2 + \mathbf{S_2}^2 + 2\rho_1\mathbf{S_1}\mathbf{S_2} + \mathbf{S_{3-6}}^2 + \mathbf{S_7}^2 + 2\rho_2\mathbf{S_2}\mathbf{S_7}}$$

[5]In Dutch called 'Doorkijkbeginsel'.

and we calculate $\mathbf{S_{3-6}}$ from the means reported by (DNB, 2011) as

$$\mathbf{S_{3-6}} = \sqrt{\mathbf{S_3}^2 + \mathbf{S_4}^2 + \mathbf{S_5}^2 + \mathbf{S_6}^2} \approx 4.5\%.$$

For $\mathbf{S_1}$ we also use the mean reported by (DNB, 2011) to be 8.9%. For $\mathbf{S_2}$ we will look at 3 types of investment strategies. Type 1 with 25% in equities, type 2 with 50% in equities and type 3 with 75% in equities. We assume that all these investments in equity are in developed market equities $\mathbf{S_{2A}}$, with a downward scenario of 25%. For many funds the majority of their investment in equities is in developed market equities and the effect of having a small part of the investments in riskier equities like emerging markets or private equity will be, roughly, diversified away by the correlation of 0.75 between these equities. For $\mathbf{S_7}$ we will vary the tracking error between 0%, 2%, 4% and 8%. And the Total expense ratio will vary between 0%, 1.5% and 3%. For the correlation between active risk and equity risk we will take $\rho_2 = 0$, $\rho_2 = 0.5$ or $\rho_2 = 1$.

Assumptions for calculating the impact with category 2

For category 2 we take a similar approach. Now we have the simplified square root formula with the adjusted $\mathbf{S_2'}$,

$$\mathbf{S_t} = \sqrt{\mathbf{S_1}^2 + \mathbf{S_2'}^2 + 2\rho_1 \mathbf{S_1}\mathbf{S_2'} + \mathbf{S_{3-6}}^2}.$$

Again we take $\mathbf{S_{3-6}} \approx 4.5\%$ and $\mathbf{S_1} = 8.9\%$. For calculating $\mathbf{S_2'}$ we also look at the 3 types of investments with 25%, 50% and 75% developed market equities. And we take the tracking errors to be 0%, 2%, 4% and 8%. The benchmark volatilities, for Method B, we set equal to 14%, 17% and 20%. And the correlation between the tracking error and the benchmark volatility is again 0, 0.5 or 1. The total expense ratio is not used in these methods.

Results

Since there are many variables we only represent a part of the results, namely we'll show the added required funding ratio with varying tracking error and varying investments in equity for Method A with a TER of 1.5%[6]; changing the TER only gives a small change in added required funding ratio. The results for this method are shown in Figures A.4, A.5 and A.6 in Appendix A.3.1. Also we'll show the added required funding ratio for Method B with a benchmark volatility of 17%; changing the benchmark volatility leads to weird results. The reults for this method are shown in Figures A.7, A.8 and A.9 in Appendix A.3.2. For Method C we only show the added required funding ratio for the adjusted type; the non-adjusted type is similar to a Method B type with benchmark volatility $\frac{25\%}{1.96} \approx 12.755\%$ and we leave out the adjusted Method B, since with a benchmark volatility of $\frac{25\%+8\%}{1.96} \approx 16.837\%$ this type is similar to the adjusted

[6]For this method we assume that if we have a tracking error of 0% then also the TER is 0%.

Method C. The results for Method C' are shown in Figures A.10, A.11 and A.12 in Appendix A.3.3.

Summary of the Main Results

Method C as discussed in Section 7.1 with a correlation of 0.5 calculates the highest added required funding ratio, but this is mainly due to the correlation. If we keep the correlation fixed then all the methods give roughly similar results. There are, however, a few remarks that can be made

- **Influence of the investment type, 25%, 50% or 75% in equities.**
  It is clear that a higher investment amount in equities leads to higher added required funding ratio if the tracking error is raised. Firstly we remark that if $\mathbf{S_1}$ and $\mathbf{S_{3-6}}$ are set to zero, then the ratio between the results of 25% equity, 50% equity and 75% equity is exactly $1:2:3$, this is clear when we look at the formulas. When $\mathbf{S_1}$ and $\mathbf{S_{3-6}}$ are set to 8.9% and 4.5%, we notice that for Method A a change in investment type has the most influence, on average a ratio of approximately $1:2.83:4.82$, only changing a little if the tracking error or the correlation is changed. For the other methods these ratios are almost similar to each other and on average $1:2.29:3.60$.

- **Influence of the correlation.**
  The correlation is a great influence on the added required funding ratio. Even for small tracking errors changing the correlation from 0 to 0.5 and 1 will give very different results. Relatively this change is only mildly dependent on the investment in equities, but highly dependent on the tracking error. One other thing we can say about the influence is that it's relatively more for Method A then for the other methods. However, if we look at the influence of the correlation if smaller changes are made, like from 0 to 0.1 or 0.1 to 0.2 for example, then we only see small changes in added required funding ratio, ranging from a few basis points to at most 1% if the tracking error is 8% and 75% in equities.

- **Influence of the tracking error.**
  With a small tracking error the added required funding ratio is not very high. But if the tracking error rises to 8% then this added required funding ratio can become substantial, even more so if the correlation is high and the investment in equities is high. It should be noted that between the methods the effect of changing the tracking error is not very different and with a correlation of 0.5 or 1 the effect is almost linear; doubling the tracking error almost doubles the added required funding ratio.

Other Results

- **Influence of TER.**
  The influence of TER is minimal, ranging from only a few basis points if

the correlation is 0 and with 25% in equities to 1% with tracking error 8%, a correlation of 1 and 75% in equities.

- **Influence of the benchmark volatility.**
  The influence of the benchmark volatility, 12.755% (Method C'), 14%, 16.837% (Method C), 17% or 20%, is more significant. Relatively this influence is higher if the correlation is low. Also we can remark that the lower the benchmark volatility the higher the added required funding ratio, this is because the tracking error has relatively more influence on a 14% benchmark volatility, then on a 20% benchmark volatility. This is a weird result, since the standard model treats these benchmarks the same for the calculation of $\mathbf{S_2}$. So by investing in a benchmark with a low volatility, the active management risk is higher while the equity risk stays equal.

- **A negative correlation.**
  With a negative correlation of, for example, $\rho = -0.5$ the added required funding ratio will generally be negative. Only for large tracking errors the added required funding ratio will then be positive. For Method A with $\rho = -0.5$ the added required funding ratio becomes positive when the tracking passes $\frac{25\%}{1.96}$ or somewhat lower if the TER is positive, approximately 11% for a TER = 3%. In general for category 1 this boundary is passed if

  $$2\rho\mathbf{S_2}\mathbf{S_7} + \mathbf{S_7}^2 = 0.$$

  For Method B and Method B', with a correlation of -0.5, the added required funding ratio becomes positive when the tracking error passes the volatility of the benchmark. For Method C, with a correlation of -0.5, this is $\frac{25\%}{1.96}$ and for Method C' this is $\frac{25\%+8\%}{1.96}$. In general for category 2 this boundary is passed if

  $$\sigma^2_{\text{tracking error}} + 2\rho\sigma_{\text{benchmark}}\sigma_{\text{tracking error}} = 0.$$

  These tracking error boundaries are high; for a pension fund a tracking error above 10% is rare. And even if a fund has a tracking error above such a border, then the impact will be low. The negative correlation could be set to $\rho = -0.1$ or $\rho = -0.2$, giving some lower boundaries. This, however, does not give much different results than the results with $\rho = 0$.

- **Influence of the other risk elements.**
  A change in the parameters $\mathbf{S_1}$ and $\mathbf{S_{3-6}}$ leads to small changes in the added required funding ratio. In general we have that the larger these parameters the smaller the added required funding ratio is and vice versa, this can be explained that the required added funding ratio is relative to the funding ratio calculated with 0% tracking error which is dependent on these 2 parameters. For changing $\mathbf{S_1}$ there is an additional effect in category 2 methods, which comes from the component $2\rho_1\mathbf{S_1}\mathbf{S_2'}$. If $\mathbf{S_1}$ is raised then also the effect of an higher tracking error is raised via this component. This gives a small dampening effect on the first effect.

Chapter 8

Discussion

Now we'll discuss some conclusions we've done throughout this thesis. We'll also make some suggestions for future research.

8.1 Conclusions

In this thesis we've focused first on methods for calculating and quantifying active management risk and then on ways to implement this risk element in the standard model.

Quantifying active management risk

We've seen that there are many varying methods to calculate active management risk. Therefore it is important to be very clear about which measure is used, which method is used to come to a distribution, which assumptions are made, how much history is used, etc. when reporting the active management risk. A pension fund is free to use any of the available options, however from the DNB point of view it is important that the calculation of active management risk is done in an uniform way, because only then it is meaningful to compare fund performances. It is advisable for the DNB to come up with certain rules and guidelines, so that every fund calculates its risk in a similar way. These guidelines can include some of the following recommendations

- Since the majority of the available programs to calculate risk use the log return it is advisable to use the log return for the calculations.

- It is preferable to use daily or weekly returns and calculate the active risk using a 1 or 7 day risk horizon. This is because there are more of these returns, which improves the estimation, and these returns are less skewed and have a mean very close to zero.

- A history of around 3 years is preferred. This gives enough data to make meaningful estimation and using a larger history can give data which is not relevant for today's market.

- Although normality is the standard assumption for the distribution of the returns it is often not a valid one, this is mainly due to the high kurtosis. If it is possible to use a t-distribution instead then this can lead to better results.

- Although the tracking error and VaR are not coherent risk measures, they are more useful than other measures if we want to implement active management risk in the standard model.

- These measures will change if the investment position of a fund changes, therefore a mean of tracking errors over, for example, 60 days can be used to give more stable results.

- It is possible to quantify active management risk using for example the maximum tracking error, which is not a measure but a given maximum for the tracking errors. This has the advantage that it gives stable results.

- In almost all cases one should use an ex-ante method to calculate the risk. However, if a fund has a very passive strategy like index tracking, then also the empirical method, which is ex-post, can be used.

- The parametric linear method gives the quickest way to calculate active management risk, however, it offers very few possibilities to extend and enhance the computation.

- The historical simulation method can give good results if the history is relevant, but if the market changes, then it's unclear if the results are useful. Furthermore, a dataset covering a large history is needed to give stable results.

- The Monte Carlo method is by far the most diverse method and can be extended in many ways. The main drawback is that computation time can become large. But in most cases this method should be preferred.

Active management risk in the standard model

If we want to implement this active management risk in the standard model, there are some methods available, which we discussed in Chapter 7. In that chapter we already discussed many aspects that should be considered and we showed the impact these methods have on the required funding ratio. Here are the main conclusions, first we'll discuss the 2 available categories, and then we'll discuss other conclusions.

Category 1

This category adds a risk element $\mathbf{S_7}$, a main advantage to this is that we isolate the active management risk part and therefore later on it will be easy to monitor the effect this risk element has. Another advantage is that it can easily add a risk component for active management in investments other then developed market equities, if we have the tracking error relative to a benchmark, the amount invested in this benchmark and optionally the total expense ratio, we can easily implement it in the calculations. Even a dynamic asset allocation could possibly also be implemented. From the results we've seen that this category gives a bit more weight on the investment type; funds with great amounts invested actively will see their required funding ratio raise relatively more if this category is used instead of category 2.

Category 2

This category adjusts the scenarios used in the standard model. Therefore only if the underlying calculations of these scenarios are clear these scenarios can appropriately be adjusted. Of course developed market equities fits this criteria, as we've seen, and possibly also emerging market equities, but the calculations for other investment categories are much less clear and therefore it's not really appropriate to adjust these scenarios. Also, a dynamic asset allocation will be hard to implement. The most interesting part of this category we see in Method B and B', where the volatility of the benchmark is used in the calculation. For developed market equities there are quite a few benchmarks a fund can use, but the standard model treats every benchmark similar. In this method, however, the benchmark volatility does play a role. But this gives a weird result; benchmarks with a low volatility give higher results for the active management risk, while giving the same results for equity risk. In other words, active management risk is benchmark specific, but equity risk is not. To keep this category consistent with the calculation of equity risk we therefore have to use Method C', which is equivalent to Method B' with a volatility of around 17%. Method C' does not give very different results in comparison to Method A; the influence of the correlation or the investment type is somewhat different. In this category the risk element $\mathbf{S_2}$ is increased, while in category 1 the active management risk is isolated. Therefore, for later research on the impact of active management risk using category 1 will be preferable.

Other conclusions

- We've seen that the correlation between active management risk and equity risk has a great effect on the results; the added required funding ratio is higher for $\rho = 1$ than for $\rho = 0$ and the effect of the correlation differs if different tracking errors are chosen. In the results we also see that these effects are somewhat different for the two categories. But it remains a question what correlation we should use to get closest to the 97.5% confidence interval. We should also note that for fixed tracking errors changing the correlation by 0.1 doesn't give very noticeable changes. Therefore, if we follow the principles from Section 2.2, we should round this correlation to avoid the illusion of precision. This leads to the two more evident options: a correlation of 0 or 0.5.

- If the correlation is set to 0 then there are many cases where the added required funding ratio is only a few basis points. Only with excessive positions, like 75% in equity and a 8% tracking error over these investments the added required funding ratio becomes more than 1%. Even with a correlation 0.5 there are still some positions with a small added required funding ratio. In these cases one can choose to ignore active management risk since the impact is minimal, however, even then it can be included to give more insight (look-through principle).

- Besides the correlation mentioned above, there are also the correlations between different managed investments and different types of active investments. Also for these correlation it is unclear if it should be set to zero, or if it should be set to something positive.

- Adding the total expense ratio or TER in the calculation doesn't give very significant changes in the impact. However, it could be included to give more insight in this element (look-through principle). Furthermore, it is not yet clear if the implementation of the TER, as it is done in Method A, is the right way. Does adding costs, lead to a bigger risk? Or can it also lessen the risk?

- The α of the active investment doesn't have a place in any of the methods. The reason for this is that it is not a very reliable measure and its impact will be minimal. However, like the TER, it could be included to give more insight.

8.2 Future research

The following list states some open problems that I've come across while working on this thesis.

- We've seen many ways to come to an active management risk, what exactly are the differences between their results? And is it possible to say something about the optimal way to calculate active management risk?

- The correlation between active management risk and equity risk is still one of the main unknowns. How high is this correlation on average for a standard fund? And what happens in extreme events? The same holds for the correlations between different investments.

- What effect does the TER have on the performance of the investments? Does it, in general, add to the riskiness, or does it lessen the risk?

- We've seen that the returns are often non-normal, how can we find a uniform way to adjust the standard model so that it accounts for this non-normality?
- What is the impact of applying a dynamic asset allocation? And how can we implement this in the standard model?

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Appendix

A.1 Estimated distribution of the tracking error

Supposing we are estimating the mean and the variance of a random variable X with realizations x_i and suppose the underlying distribution is normal, that is $X \sim \mathcal{N}(\mu, \sigma^2)$ and for the realizations $x_i \sim \mathcal{N}(\mu, \sigma^2)$ and i.i.d.. What, then, can we say about the estimators for the mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and for the standard deviation squared $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$?

Mean of an estimator

The mean of an estimator $\hat{\theta}$ of θ can be calculated by $\mathbb{E}(\hat{\theta})$, we already mentioned in Section 4.3 that these estimators of μ and σ^2 are unbiased, so this would mean $\mathbb{E}(\bar{x}) = \mathbb{E}(\hat{\mu}) = \mu$ and $\mathbb{E}(\hat{\sigma}^2) = \sigma^2$, we check this by evaluating[1]

$$\mathbb{E}(\bar{x}) = \mathbb{E}(\frac{1}{n}\sum_{i=1}^{n} x_i) = \frac{1}{n}\mathbb{E}(\sum_{i=1}^{n} x_i) = \frac{1}{n} n\mathbb{E}(x_i) = \mu.$$

[1] Here we use var for the variance and not for the Value at Risk, which is VaR.

And for σ^2 we have

$$\begin{aligned}
\mathbb{E}(\hat{\sigma}^2) &= \mathbb{E}\left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) \\
&= \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n}(x_i-\mu+\mu-\bar{x})^2\right) \\
&= \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n}(x_i-\mu)^2+2(\mu-\bar{x})\sum_{i=1}^{n}(x_i-\mu)+\sum_{i=1}^{n}(\mu-\bar{x})^2\right) \\
&= \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n}(x_i-\mu)^2+2(\mu-\bar{x})(n\bar{x}-n\mu)+n(\mu-\bar{x})^2\right) \\
&= \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n}(x_i-\mu)^2-2n(\mu-\bar{x})^2+n(\mu-\bar{x})^2\right) \\
&= \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n}(x_i-\mu)^2-n(\mu-\bar{x})^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbb{E}(x_i-\mu)^2-n\mathbb{E}(\bar{x}-\mu)^2\right) \\
&= \frac{1}{n-1}\left(n\sigma^2-n\mathrm{var}\left(\bar{x}\right)\right) \\
&= \frac{1}{n-1}\left(n\sigma^2-n\mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i)\right)\right) \\
&= \frac{1}{n-1}\left(n\sigma^2-n\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(x_i)\right) \\
&= \frac{1}{n-1}\left(n\sigma^2-\sigma^2\right)=\sigma^2.
\end{aligned}$$

Variance of an estimator

Using the means of these estimators we can calculate the variance of an estimator $\hat{\theta}$ by $\mathrm{var}(\hat{\theta})=\mathbb{E}(\hat{\theta}-\theta)^2$. For the estimator of μ we've seen this expression worked out in the last lines of the proof for the mean of $\hat{\sigma}$, that is,

$$\mathrm{var}(\bar{x})=\mathrm{var}(\frac{1}{n}\sum_{i=1}^{n}x_i)=\frac{1}{n^2}\mathrm{var}(\sum_{i=1}^{n}x_i)=\frac{1}{n^2}n\sigma^2=\frac{\sigma^2}{n}.$$

To get the variance of the estimator for σ^2 we first define the chi-squared distribution. Let $X_1,\ldots,X_k$ be independent random variables with normal distributions and mean 0 and variance 1, then the sum of the squares

$$Y=\sum_{i=1}^{k}X_i^2$$

is distributed according to the chi-squared distribution with k degrees of freedom, and we write $Y \sim \chi^2(k)$. The moment generating function, given without proof, of the chi-squared distribution is equal to

$$M_{\chi^2(k)}(t) = (1-2t)^{-\frac{k}{2}} \quad \text{for } t < \frac{1}{2}.$$

The moment generating function has the property that if we differentiate it i times and set $t = 0$ then we get the i-th moment μ_i, that is

$$\mu_i(\chi^2(k)) = M^{(i)}_{\chi^2(k)}(0).$$

From this we calculate the mean and variance of the chi-squared distribution as follows

$$\mu(\chi^2(k)) = \mu_1(\chi^2(k)) = M'_{\chi^2(k)}(0) = -\frac{k}{2}(1-2t)^{-\frac{k}{2}-1}(-2)|_{t=0} = k.$$

And, using that $\sigma^2(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mu_2(X) - (\mu_1(X))^2$, we get

$$\begin{aligned}
\sigma^2(\chi^2(k)) &= \mu_2(\chi^2(k)) - (\mu_1(\chi^2(k)))^2 \\
&= M''_{\chi^2(k)}(0) - M'_{\chi^2(k)}(0) \\
&= (-\frac{k}{2})(-\frac{k}{2}-1)(1-2t)^{-\frac{k}{2}-2}(-2)^2|_{t=0} \;-\; k^2 \\
&= k(k+2) - k^2 = 2k.
\end{aligned}$$

Now we try to find the variance of the estimator for σ^2. We can reformulate

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

to

$$(n-1)\hat{\sigma}^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

and we proceed as follows. Since for all i we have $x_i \sim \mathcal{N}(\mu, \sigma^2)$ we have for all i

$$\frac{x_i - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

and hence

$$Y = \sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2(n).$$

This expression we can elaborate in a way similar to the way we got the mean of the estimator of σ^2, that is

$$\begin{aligned}
\sigma^2 Y &= \sum_{i=1}^{n}(x_i-\mu)^2 \\
&= \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu)^2 \\
&= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\bar{x}) + \sum_{i=1}^{n}(\bar{x}-\mu)^2 \\
&= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-\mu)(n\bar{x}-n\bar{x}) + n(\bar{x}-\mu)^2 \\
&= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 0 + n(\bar{x}-\mu)^2 \\
&= \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2.
\end{aligned}$$

So, remembering the reformulation of $\hat{\sigma}$, we have

$$\begin{aligned}
Y = \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 &= \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2} \\
&= \frac{(n-1)\hat{\sigma}^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}.
\end{aligned}$$

We've seen that the estimator $\bar{x}$ of the mean μ satisfies

$$\mathbb{E}(\bar{x}) = \mu$$

and

$$\text{variance}(\bar{x}) = \frac{\sigma^2}{n}.$$

Furthermore, since $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ this is a sum of normal distributions and hence normal. So we have

$$\bar{x} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}).$$

From this follows that

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$$

and hence

$$\frac{n(\bar{x}-\mu)^2}{\sigma^2} \sim \chi^2(1).$$

So in the equation

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \frac{(n-1)\hat{\sigma}^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}$$

We have on the left a $\chi^2(n)$ distribution and on the right, the second term, a $\chi^2(1)$ distribution. Now, it can be shown that the first term on the right has rank $(n-1)$, and then by Cochran's theorem, which we'll not proof, the first term must have a $\chi^2(n-1)$ distribution and $\frac{(n-1)\hat{\sigma}^2}{\sigma^2}$ and $\frac{n(\bar{x}-\mu)^2}{\sigma^2}$ are independent. If we assume this independency we can also quite easily proof that $\frac{(n-1)\hat{\sigma}}{\sigma^2}$ has a $\chi^2(n-1)$ distribution using moment generating functions. We use the property that if X and Y are independent, then the moment generating function of their sum is the product of the 2 moment generating functions

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

and we use that if two distributions have the same moment generating function then the distributions are the same. Using these properties we get from our equation

$$M_{\sum_{i=1}^n(\frac{x_i-\mu}{\sigma})^2}(t) = M_{\frac{(n-1)\hat{\sigma}^2}{\sigma^2}+\frac{n(\bar{x}-\mu)^2}{\sigma^2}}(t) = M_{\frac{(n-1)\hat{\sigma}^2}{\sigma^2}}(t)M_{\frac{n(\bar{x}-\mu)^2}{\sigma^2}}(t).$$

From this follows

$$M_{\chi^2(n)}(t) = M_{\frac{(n-1)\hat{\sigma}^2}{\sigma^2}}(t)M_{\chi^2(1)}(t).$$

So for $t < \frac{1}{2}$

$$\begin{aligned} M_{\frac{(n-1)\hat{\sigma}^2}{\sigma^2}}(t) &= M_{\chi^2(n)}(t) \cdot M_{\chi^2(1)}(t)^{-1} \\ &= (1-2t)^{-\frac{n}{2}} \cdot (1-2t)^{\frac{1}{2}} \\ &= (1-2t)^{-\frac{(n-1)}{2}}, \end{aligned}$$

which is the moment generating function for $\chi^2(n-1)$. This proves that

$$\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-1).$$

Now we can finally find the variance of $\hat{\sigma}^2$. Using $\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-1)$ and using $\mathrm{var}(\chi^2(n-1)) = \sigma^2(\chi^2(n-1)) = 2(n-1)$, we get

$$\begin{aligned} \mathrm{var}(\frac{(n-1)\hat{\sigma}^2}{\sigma^2}) &= 2(n-1) \\ \frac{(n-1)^2}{\sigma^4}\mathrm{var}(\hat{\sigma}^2) &= 2(n-1) \\ \mathrm{var}(\hat{\sigma}^2) &= \frac{2}{n-1}\sigma^4. \end{aligned}$$

Asymptotics

By the central limit theorem we have asymptotically, taking the limit over k, that

$$\frac{\chi^2(k) - \mu(\chi^2(k))}{\sigma(\chi^2(k))} \xrightarrow{d} \mathcal{N}(0,1).$$

So for k large enough, we have approximately

$$\chi^2(k) \sim \mathcal{N}(k, 2k).$$

Since we have $\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-1)$, we get

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{(n-1)}\chi^2(n-1).$$

So with $\mu(\chi^2(n-1)) = n-1$ and $\sigma(\chi^2(n-1)) = 2(n-1)$ this means, that for n large enough we have

$$\hat{\sigma}^2 \sim \mathcal{N}(\sigma^2, \frac{2}{n-1}\sigma^4).$$

A.2 Histograms of the active return distribution

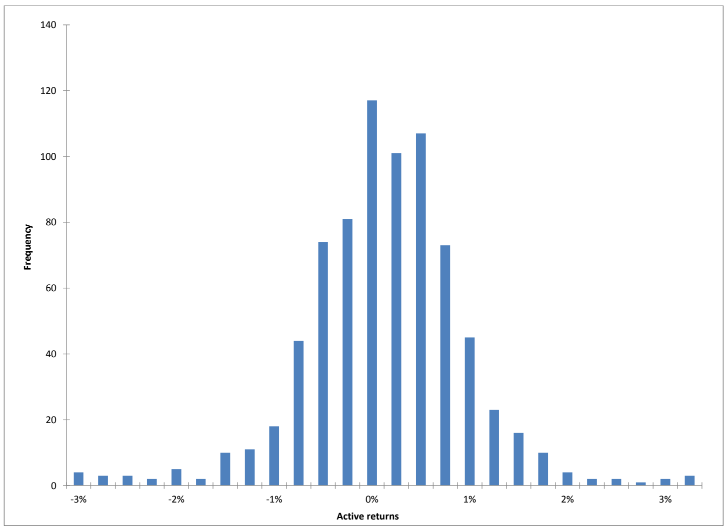

Figure A.1: The active return distribution for the empirical method.

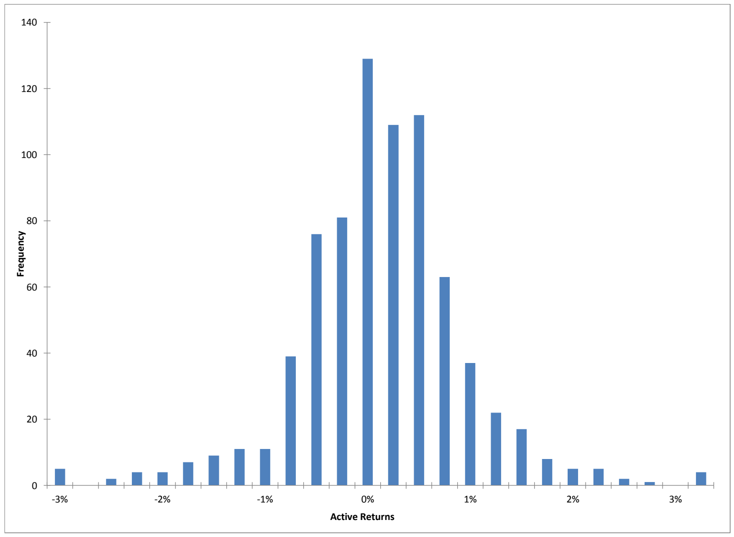

Figure A.2: The active return distribution for the historical simulation method.

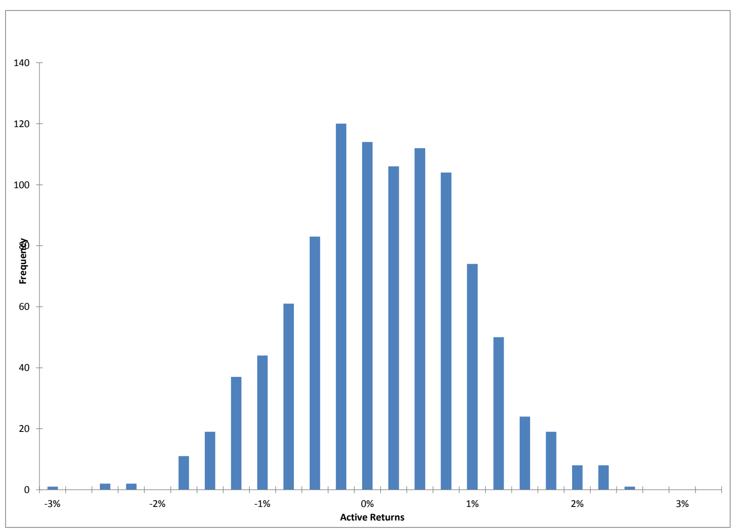

Figure A.3: The active return distribution for the Monte Carlo method.

A.3 Results

These are the results of the added funding ratio for Methods A with TER 1.5%, Method B with benchmark volatility 17% and the adjusted Method C'. Note that added RFR on the y-axis stands for added required funding ratio and on the x-axis we have the tracking error ranging from 0% to 8%. The different colours represent the different investment types of 25%, 50% or 75% in equities.

A.3.1 Method A with a TER of 1.5%

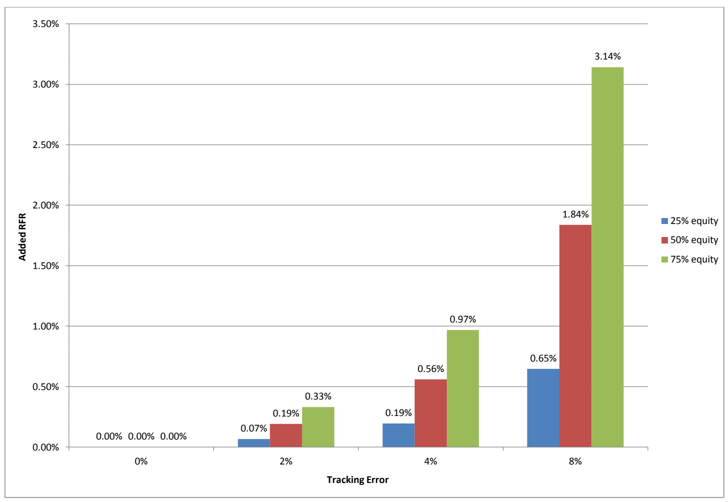

Figure A.4: The added required funding ratio for Method A with $\rho = 0$.

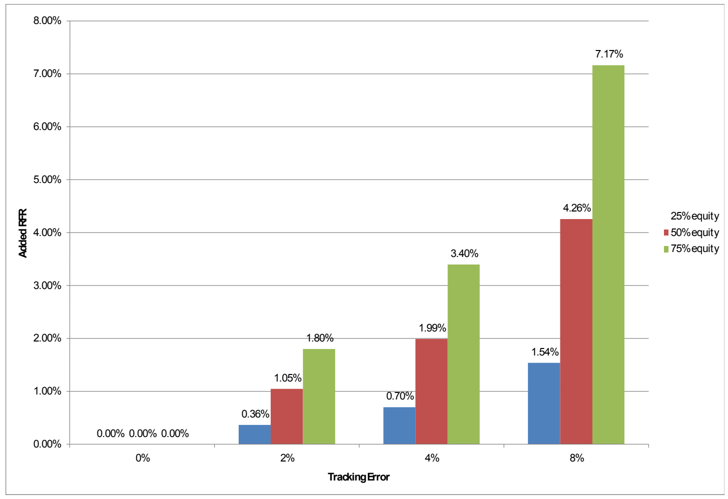

Figure A.5: The added required funding ratio for Method A with $\rho = 0,5$.

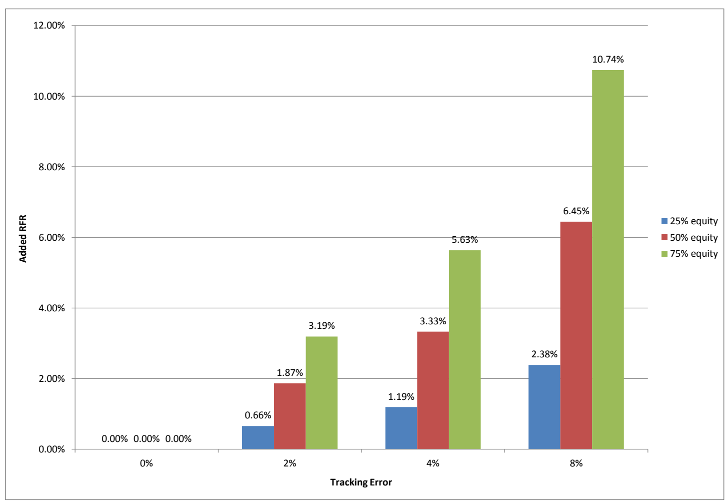

Figure A.6: The added required funding ratio for Method A with $\rho = 1$.

A.3.2 Method B with a benchmark volatility of 17%

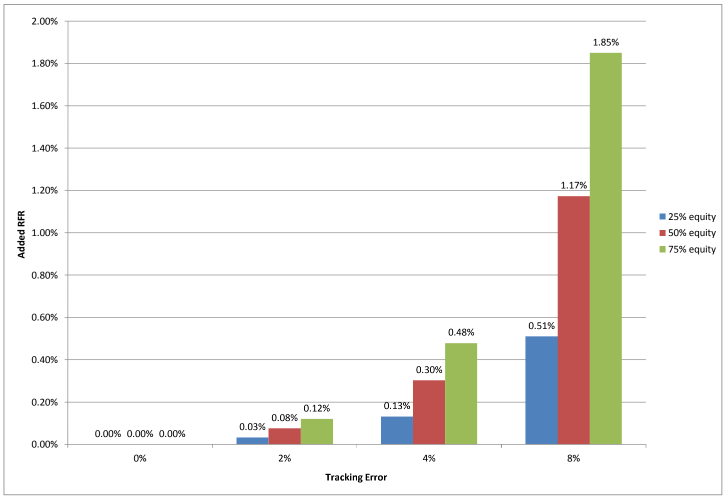

Figure A.7: The added required funding ratio for Method B with $\rho = 0$.

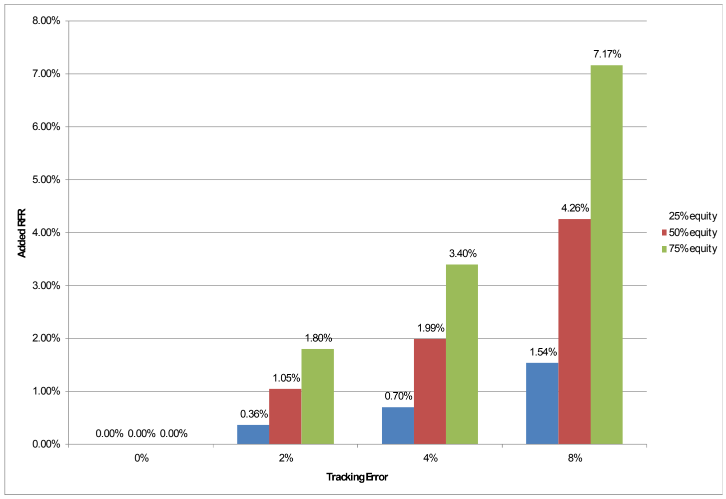

Figure A.8: The added required funding ratio for Method B with $\rho = 0,5$.

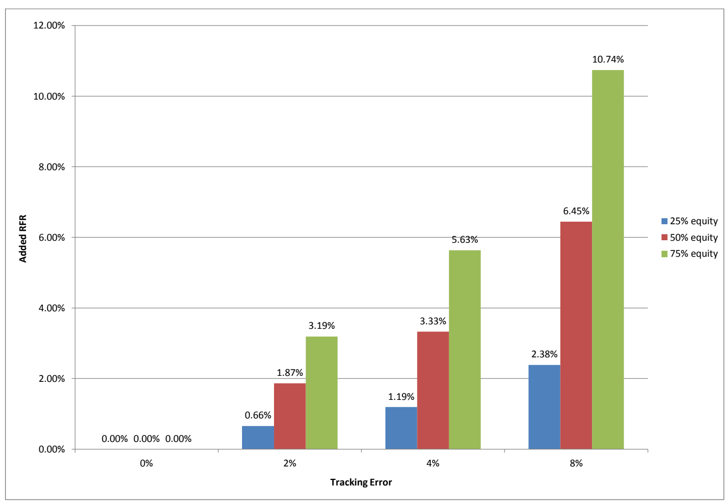

Figure A.9: The added required funding ratio for Method B with $\rho = 1$.

A.3.3 The adjusted Method C'

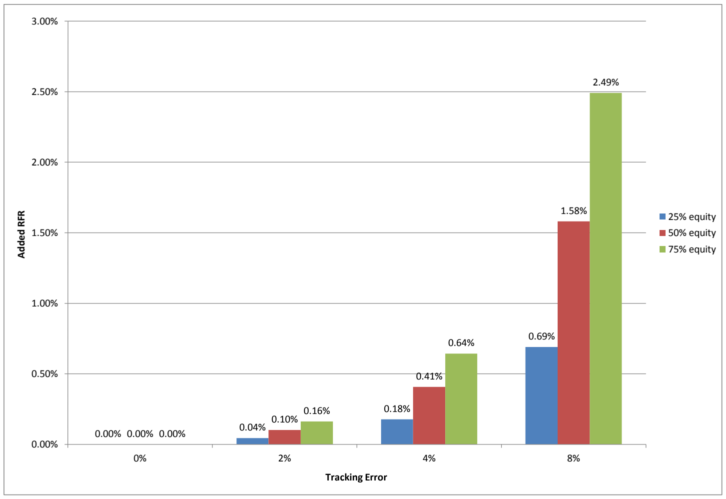

Figure A.10: The added required funding ratio for Method C' with $\rho = 0$.

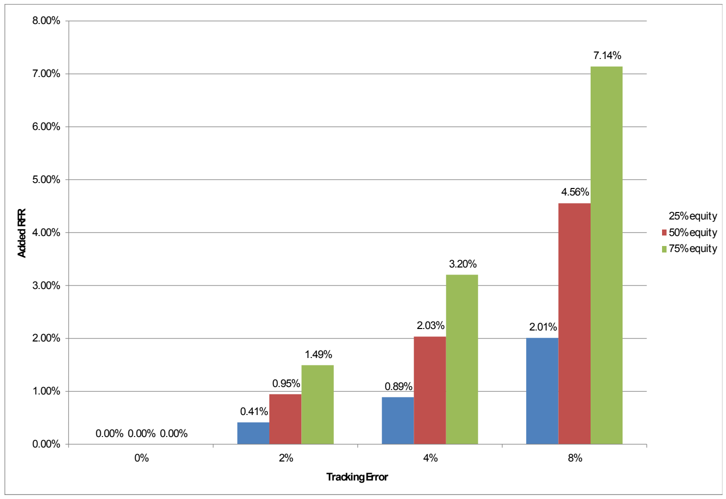

Figure A.11: The added required funding ratio for Method C' with $\rho = 0,5$.

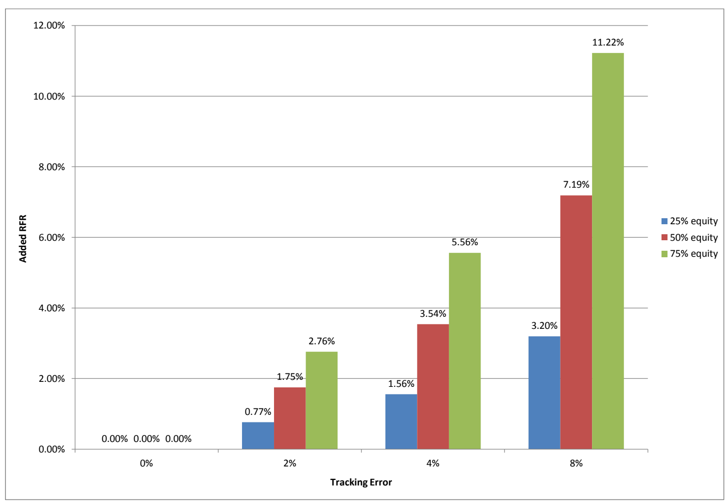

Figure A.12: The added required funding ratio for Method C' with $\rho = 1$.

A.4 Dutch version of principles

Het uitgangspunt is om het vereist eigen vermogen zodanig te bepalen dat met een zekerheid van 97.5% procent wordt voorkomen dat het pensioenfonds binnen een periode van n jaar over minder waarden beschikt dan de hoogte van de technische voorzieningen (artikel 120 PW). Idealiter wordt daartoe precies aangesloten bij het feitelijke risicoprofiel van het pensioenfonds. Dit vergt echter de investering in een intern model, wat vooralsnog slechts voor enkele pensioenfondsen is weggelegd. Vandaar dat ook een relatief eenvoudig en gemakkelijk hanteerbaar standaardmodel beschikbaar is. Dit standaardmodel is minder fijnmazig van opzet dan een intern model en bevat daarom enige prudentie. Met behulp van het standaardmodel wordt getest in hoeverre een instelling gevoelig is voor de verschillende scenario's, zoals een daling van de aandelenmarkt of een verandering in de rentetermijnstructuur. Deze scenario's zijn zo gekozen dat zij met een kans van 1 op 40 voor zullen komen. De parameters voor het standaardmodel (art. 11) zijn derhalve 'schokparameters' die gekalibreerd zijn op een risicohorizon van n jaar en een betrouwbaarheidsniveau van 97.5%: ze geven de verandering aan in de risicofactor[2] (bijvoorbeeld een daling van 25% in aandelen mature markets). De scenario's moeten aansluiten bij de genoemde zekerheid van 97.5%. Ook hier geldt dat een exacte relatie tot het betrouwbaarheidsniveau niet eenvoudig haalbaar is. Het suggereert namelijk een nauwkeurigheid die nimmer bereikt wordt. Het bepalen van scenario's die zich eens in de 40 jaar voordoen is niet eenvoudig. Dit komt vooral omdat er in veel gevallen onvoldoende historische waarnemingen zijn om dergelijke schattingen te maken; zelfs voor aandelen- en rentemarkten waar redelijk veel historische data voor beschikbaar zijn, is dit niet eenvoudig. Bovendien zijn verwachte rendementen, volatiliteit (standaarddeviatie) en correlaties geen stabiele grootheden in de tijd.

[2] In de kalibratie wordt rekening gehouden met de verwachtingswaarde van de risicofactor.