Thinking About Knowledge

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Abstract

This thesis is about logics that are concerned with reasoning about knowledge. The acquisition of knowledge can be modelled. There are several systems to formalize reasoning about knowledge. Two systems that are used for epistemic logic are intuitionistic logic and the modal logic S4. Both logics will be discussed. The two differ a lot from each other, especially in the way that statements are considered to be true. But classical logic can be reduced to intuitionistic logic and intuitionistic logic can be reduced to the modal logic S4. This is done by the tranlations that were introduced by Gödel. One of these translations will be examined explicitly. The thesis assumes some familiarity with classical logic.

Contents

1 Introduction

The logic of knowledge is called epistemic logic. Principles of reasoning about knolwedge can be formalized and these formal statements can then be verified. There are several kinds of logic that are useful to represent these statements. We will discuss some logics that are very different from each other.

An example of a logic that is used to prove statements about knowledge is intuitionistic logic. This logic was based on the ideas of intuitionism, which was founded by the Dutch mathematican and philosopher L.E.J. Brouwer round 1900, see [19]. Intuitionism is a philosophical approach to mathematics with the idea that mathematical truths are being created rather than being discovered. This approach is based on the intuitive reasoning of humans. The charactarization of intuitionism is that mathematical objects only exist if they can be constructed. Brouwer's ideas of intuitionism were still rather vague and therefore needed a formalization. In 1927 the Dutch Mathematical Association published a prize question to formalize the intuitionistic ideas. Brouwer's student A. Heyting then defined intuitionistic logic and was awarded the prize. In 1930 he also formalized the meaning of the logical operators in this logic. This logic is based on constructions of objects. This means that the validity of proofs are derived from constructions. Because of this intuitive view on truths, some assumptions of classical logic are being rejected. Hence intuitionistic logic had to be different from the classical one. As the truth of a mathematical statement can only be verified if it is intuitionistically true. If a statement is not constructed yet, it is neither true nor false. Therefore the law of excluded middle is not valid in intuitiontic logic and must be rejected. In this thesis we will not discuss predicate logic. Therefore "intuitionistic logic" should be read as intuitionistic propositional logic.

Another logic that is used for knowledge is S4. This is an example of modal logic, which contains different sorts of modalities, see [20]. It was first introduced in 1918 by C.I. Lewis and further developed by several other logicians among which S. Kripke in the 1960's. During these years various modal systems were created and improved. Modal logic is an extension of classical logic. It uses modal operators to express modalities, for example the modality of knowledge, epistemic logic. The semantics of modal logic consists of propositions including the modal operators. In epistemic logic they express both truth and knowledge. Lewis introduced the five different systems S1 up to S5. In 1943 Alban created S6 and in 1950 Halldén developed S7 and S8. All of them an extension of the former one. S4 is a type of modal logic that is very useful for reasoning about knowledge. It has some particular rules that contain the modal operators. This way statements about truth knowledge can be derived from other statements about truth and knowledge.

Even if intuitionistic logic and S4 contain very different rules, the two logic share a correspondence, as will be shown in this thesis. Intuitionistic logic is embeddable into classical modal logic by means of the Gödel translation, created by K. Gödel. This translation is possible, because the modal language is sufficiently rich. Gödel noticed the resemblance of the semantics that were used for the logics. He also realized how statements of intuitionistic logic can be interpreted as statements of S4. By means of the Gödel translation statements in intuitionistic logic can be converted into statements with classical connectives.

The research on different epistemic logics is very relevant for the study of artificial intelligence. This topic was first studied for philosophical purpose, but since the 1980's epistemic logic has been studied by computer scientists too. Thinking about the way human reason about knowledge is a requisite for understanding human intelligence, as this enables us acting and thinking. Therefore research on knowledge is necessary for developing a simulation on a computer, because this way we can reason about the knowledge of agents. To build a computer simulation of a human being (i.e. an agent or robot), the builder might want to add some knowledge in the form of data to the robot. The robot must be able to think about the truth of different statements. But he also has to evaluate the knowledge it has acquired. This thinking must be based on formal definition, rather than intuitive ideas. Therefore epistemic logic is very useful for the building of robots and agents and therefore an interesting topic in artificial intelligence, see, [9].

Thus intuitionistic logic and S4 are two very different logics and both will be discussed intensively. This paper is organized as follows. In Section 2 we will go through the concepts of intuitionistic logic and we will present a proof system, called natural deduction. This system is used for classical logic, but it suits even better for intuitionistic logic. Also Kripke models that form a consistent and complete semantics for intuitionistic logic are presented. Section 3 contains an introduction to modal logic and a description of S4. Kripke models for S4 are clarified and the differences between these models for S4 and intuitionistic logic are shown. In Section 4 both logics will be compared and it will be demonstrated how intuitionistic logic can be interpreted in S4. In Section 5 we evaluate the Gödel translation and discuss the importance of it. The translation has consequences for both philosophers and mathematicians. We will review if the translation is only interesting in a philosophical way or also for a technical purpose. In this paper we will use the letters ϕ, ψ and χ for formulae, whereas the letters p and q range over propositional variables.

2 Intuitionistic logic

2.1 Basic principles

Intuitionistic logic is the logic that is used for intuitionism, which is an approach to mathematics. The idea of this approach is that mathematical objects only exist when they have been constructed. In intuitionism mathematical objects only exist if they can be constructed in the future. It is based on the human experience of knowledge that is obtained over time. Information is obtained over time and this knowledge will not be lost.

Intuitionistic logic differs from classical logic in its interpretation of what it is for statements to be true. In classical logic statements are either true or false, even if it has not been proved yet. In intuitionistic logic this is not the case. The truth of a statement is constructive. It is based on a proof. Therefore, aside from being true or false, statements can also be undecided. There is a lack of proof for its truth.

Intuitionistic logic uses the same connectives as classical logic:

- negation: $\neg$
- conjunction: $\wedge$
- disjunction: $\vee$
- implication: $\rightarrow$
- equivalence: $\leftrightarrow$

Thus an atom is considered true if we have a proof for it. Below here is a list of the logical connectives that shows how proofs of composite statements can be constructed from proofs of there parts.

- $p \wedge q$: p is proved and q is proved
- $p \vee q$: p is proved or q is proved
- $p \rightarrow q$: a construction is provided that converts every possible proof of p into a proof of q

- $\neg p$: this can be converted into $p \rightarrow \perp$. This means: every possible proof of p results into a proof of a contradiction

All formulas that are provable in intuitionistic logic are also provable in classical logic. But intuitionistic logic is a restriction of classical logic. One of the main principles that are not valid in intuitionistic logic is the law of excluded middle. After all, if a statement has not been constructed yet, it is neither true nor false. This results into many other principle that are valid in classical logic, but not in intuitionistic logic, for example double elimination principle, which will be further explained later.

The disjunction property is that if $\phi \vee \psi$ is derivable, then ϕ is derivable or ψ is derivable. But in intuitionistic logic a statement can only be true if it is there is a proof that it is true. If a statement cannot be proved, the complement cannot automatically be inferred. In classical logic the axiom $p \vee \neg p$ is a tautology, but the axiom is not valid in intuitionistic logic. Another axiom that intuitionistic logic does not inherit from classical logic is double negation elimination principle: $\neg\neg p \rightarrow p$. The reason why this axiom is not valid is because if a state makes $\neg\neg p$ true, it is simply because $\neg p$ cannot be proved. But that does not imply that p can be constructed. So p does not have to be valid in this state. Therefore $\neg\neg p \rightarrow p$ is not a valid axiom in intuitionistic logic. On the other hand, $p \rightarrow \neg\neg p$ is a valid axiom. This is because as soon as p is proved in some state, $\neg p$ cannot be proved anymore, so $\neg\neg p$ is valid too. The fact that the law of the excluded middle and the double elimination principle are not valid in intuitionistic logic, implies that several other axioms are not valid either.

An example for this is De Morgan laws. These laws convert propositions that include a disjunction into propositions that include a conjunction and vice versa: $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ and $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$. One of these four implications is invalid in intuitionistic logic: $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$.

Peirce's law can be thought of the law of excluded middle in a different form. This proposition is also not valid in intuitionistic logic. Peirce's law is as follows: $((p \rightarrow q) \rightarrow p) \rightarrow p$. The constructive invalidity of these propositions will be proved later.

2.2 Natural deduction

A useful way to prove propositions in classical logic is using truth tables. These tables consist of all possible combinations of assignments of truth values to the atoms. For every single valuation it can be decided if the formula holds. If the formula holds for all valuations, the formula is valid. If it holds for at least one valuation, the formula is satisfiable. In classical logic atoms have only two possible truth values: true and false. In intuitionistic logic there can not be given any finite truth tables, because there is no excluded middle. Therefore truth tables is not a valid system to decide the satisfiability or validity of propositions.

Another system that is used to verify statements in classical logic is natural deduction. This proof system was introduced by Gentzen in 1935. The idea is that conclusions are derived from assumptions. This system is based on the natural form of reasoning. It contains introduction and elimination rules for the connectives: conjunction, disjunction and implication. The introduction rules indicate how is verified that the proposition with one of the connectives is true. The elimination rules tell what statements can be derived from the truth of the propositions with one of the connectives. One starts with several premises that are assumed to be true. From these statements other statements are derived, using the introduction and elimination rules. Eventually one will come to the conclusion.

These are the introduction rules (I) and the elimination rules (E) of natural deduction:

- $\wedge I$: If ϕ is true and ψ is true, then $\phi \wedge \psi$ is true
- $\wedge E$: If $\phi \wedge \psi$ is true, then ϕ is true and ψ is true
- $\vee I$: If ϕ is true or ψ, then $\phi \vee \psi$ is true
- $\vee E$: If $\phi \vee \psi$ is true and χ can be derived from ϕ and from ψ, then χ is true
- $\rightarrow I$: If $\phi \rightarrow \psi$ is true and ϕ is true, then ψ is true
- $\rightarrow E$: If ψ is verified from ϕ, then $\phi \rightarrow \psi$ is true
- $\neg I$: If ϕ leads to $\perp$, then $\neg\phi$ is true

- $\neg E$: If ϕ is true and $\neg\phi$ is true, $\bot$ is true
- $\bot E$: If $\bot$ is true, then any proposition χ can be derived

Natural deduction is also a good system for intuitionistic logic. Only the rules that are allowed to derive new statements are slightly different from the rules in classical logic. All the rules that are enumerated above are valid in intuitionistic logic. But classical logic contains an extra rule that is not valid in intuitionistic logic: reduction to the absurd. This rule encompasses that if ϕ implies $\bot$, $\neg\phi$ is derived. This rule can not be inherited from classical logic when using the natural deduction system in intuitionistic logic.

2.3 Kripke models

The semantics that are used for intuitionistic logic is convenient and simple. You may think of a mathematician who extends his knowledge in nondeterministic time. He has a set of objects that he has constructed. At any moment he can choose to stop or to continue creating new objects. The mathematican also has a set of statements about the objects. In every stage he observes his knowledge about the objects and adjusts his set of statements. His current statements are all considered true at the moment. When going to a next moment he can choose from various stages. These stages are the possible worlds.

As stated before, natural deduction is a useful system for proving the validity of a formula. To prove the invalidity of a formula, Kripke semantics is very helpful. This was introduced by Saul Kripke in the late 1950s and the early 1960s, see [18]. The Kripke models were first considered for modal logic, but later also used for intuitionistic logic.

When one wants to prove that a formula is invalid, a counter model can be created. This model is called a Kripke model. The model consists of stages that represent the possible worlds. One of these is stages, the root, encompasses the current situation. The stage has a set of objects that have been constructed so far. And it includes a set of statements about these objects that are recognised as true at that moment. So the first stage represents the present with all the knowledge that is obtained so far.

There are other stages that show what can happen in the future. The stages are connected by branches. These branches can be seen as various choices for objects that can be constructed. The obtained knowledge is represented by atoms. All worlds in the model are accessible from the root via the branches. The atoms in the worlds show what can be proved in the future. The idealization of this semantics is that the amount of information will never decrease. Knowledge can be obtained, but cannot be lost.

Kripke models are very helpful to see whether statements of intuitionistic logic are valid or invalid. A model that shows a contradiction for a statement, is a counter model for this statement.

A Kripke model is a triple $\langle K, \leq, V \rangle$ where:

- K is a set of worlds
- $\leq$ is a partial ordered relation on K
- V is a function on worlds that takes a set of atoms that are known in these worlds, where $V(k)$ stands for the set of atoms

These worlds assign truth values to statements. If p is true in node k we write: $k \Vdash p$. This denotes k forces p. In every world new statements can be proved true. $\leq$ can be seen as an accessibility relation between the elements of K and tells what worlds can be reached from a particular world. The model is a structure of worlds with a certain hierarchy. So $k \leq \ell$ means that the world ℓ is accessible from the world k. And ℓ inherits all the formulas that are valid in k. So the relation $\leq$ implies what formulas are inherited in the several worlds. The atoms are persistent, which means that if $k \leq \ell$ then $V(k) \subseteq V(\ell)$. The relation is reflexive, transitive and antisymmetric. It is partially ordered, instead of linearly ordered, as worlds can have more than one accessible world and can also end any time. We say that ℓ is the immediate successor of k if $k \leq \ell$ and there is no world m for which $k \leq m$ and $m \leq \ell$. With other words, m is not in between the worlds k and ℓ. We use the notation $k \Vdash \phi$ when ϕ is valid in every model of K.

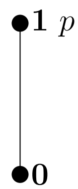
These rules are valid in intuitionistic logic: For all k in K:

- $k \Vdash p$ iff $p \in V(k)$
- $k \Vdash \phi \wedge \psi$ iff $k \Vdash \phi$ and $k \Vdash \psi$
- $k \Vdash \phi \vee \psi$ iff $k \Vdash \phi$ or $k \Vdash \psi$
- $k \Vdash \phi \to \psi$ iff for all ℓ such that $k \leq \ell$, if $\ell \Vdash \phi$ then $\ell \Vdash \psi$
- $k \Vdash \perp$ never occurs

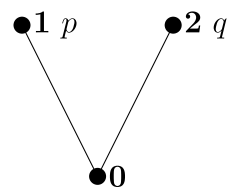
One can show that if a statement has been established in $k \in K$, then for all $\ell \in K$ where $k \leq \ell$, that statement will be true in ℓ. This theorem means that the accessible states contain knowledge that is possible in the future and the knowledge that is obtained in previous states.

Below is an example of a Kripke model. The world 0 is the root. World 1 is directly accessible from 0. In 1 the atom p is true. This model is a counter model for the formula $p \vee \neg p$. In world 0 neither p nor $\neg p$ is proved. Then $p \vee \neg p$ is not always true and therefore this formula is not a tautology in intuitionistic logic.

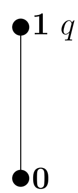
Counter model for $p \vee \neg p$:



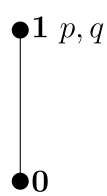
Counter model for $\neg(p \wedge q) \to (\neg p \vee \neg q)$:



Counter model for $((q \to p) \to q) \to q$:



Counter model for $(p \to q) \to (\neg p \vee q)$



3 System S4

3.1 Modal logic

Modal logic is an extension of classical logic that uses modal operators. These operators express the modality of statements. They qualify the truths of judgements. The modalities are necessity (using the box operator: $\Box$) and possibility (using the diamond operator: $\Diamond$). For example the proposition "it is necessary that p" implies the proposition "it is possible that p", but not vice versa.

$\Box p$ means: p is necessarily true $\Diamond p$ means: p is possibly true $\Diamond$ is equivalent to $\neg\Box\neg$, because 'possibly true' means 'not necessarily not true'.

The concenption of modal logic lead to a whole new set of judgements about what statements implied others. If p and q are necessarily true, then the proposition $p \wedge q$ is necessarily true and vice versa. And if p or q is necessarily true, then $p \vee q$ is necessarily true. But the reverse is not valid. Another example for a consequence is: If $p \rightarrow q$ is necessary, then if p is possible, so is q. An important rule in modal logic is the Necessitation rule, which says that if ϕ is a theorem in the logic, then so is $\Box\phi$. This means that if a statement is a tautology, say $p \vee \neg p$, this statement is necessarily true. This implies that $\Box(p \vee \neg p)$ is also valid.

An example of modal logic is epistemic logic. This logic uses the modality operators to indicate the knowledge of statements. The box operator expresses the knowledge of a proposition. As the diamond is equal to the negation of the box before the negation of a statement, the diamond expresses that it is not known that the statement is not true.

$\Box p$ means: p is known.
$\Diamond p$ means: not p is not known to be true.

Thus $\Diamond p$ means that p is compatible with the present state of obtained knowledge. When modal logic is used for reasoning about knowledge, the set of rules in this logic depend on our intuitionistic ideas of obtaining knowledge.

3.2 Definition of system S4

There are several systems in modal logic. One of these is called S4 and is often used to verify statements in epistemic logic. S4 inherits all the rules of classical logic and includes several other rules. The logic also contains the Necessitation rule. If some statement is a tautology, one will know this statement: $\vdash (p \vee \neg p) \Rightarrow \vdash \Box(p \vee \neg p)$. This means that it is assumed that everybody knows the axioms that are always true. Apart from these rules, S4 includes some other rules:

- K: $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$ *This is the distribution rule. It means that if Mary knows* $\phi \rightarrow \psi$*, then if Mary knows* ϕ*, Mary knows* ψ*.*
- T: $\Box\phi \rightarrow \phi$ *If Mary knows* ϕ*,* ϕ *must be true.*
- 4: $\Box\phi \rightarrow \Box\Box\phi$ *If Mary knows* ϕ*, Mary knows that she knows* ϕ*.*

Rule 4 is called the positive introspection axiom. It means that one knows that one knows what one knows. S4 can be used for several kinds of modal logic and is among others to prove statements in epistemic logic. Another system that is often used for epistemic logic is called S5. This system obtains the rules of S4, but is extended by the axiom: $\neg\Box\phi \rightarrow \Box\neg\Box\phi$. This axiom is called the negative introspection axiom. It means that one knows that one does not know what one knows. System S5 is often used by computer scientists, but not by philosophers as they suppose that humans do not always know what they do not know.

To decide the validity of statements about knowledge, one can use a system made for modal logic. This system resembles the natural deduction system of classical logic, but includes also the rules of S4 (or one of the other logics if that is preferred). The new rules in S4 enable eliminating and introducing boxes and diamonds as well.

Not only do we want to prove the validity of statements, we also need models to visualize the validity of statements. Kripke models are also used for modal logics like S4. Again the model exists of world that are accessible from the root world. In this case the box, which means necessarily true, holds that the proposition is true in all accessible worlds. Whereas the diamond, which means possibly true, means that the proposition is true in at least one accessible world. So $\Box p$ is true in world w if and only if p is true in all worlds that are directly reachable from world w.

3.3 Kripke models for S4

The semantics for S4 also consists of Kripke models. These models can among others show the soundness and completeness of S4. These models are very similar to the Kripke models for intuitionistic logic. Only the nodes do not correspond to time stadia, but to possibilities. The frames consist of a set of worlds and each pair of worlds can have a accessibility relation or not. This relation is indicated by an R. Thus $k\ R\ \ell$ means that world ℓ is accessible from world k.

In each world truth values are assigned to all propositons. If p is true in world k, we write: $k \vDash p$. If a propositional variable p is necessarily true in world k (i.e. $k \vDash \Box p$), it means that p is true in all accessible world from k. And p is possibly true in world k (i.e. $k \models \Diamond p$), it means that p is true in at least one accessible world from k. If a world does not have any accessible worlds, $\Box p$ holds for any atom p.

A Kripke model is a tripler $\langle K, R, V \rangle$ where:

- K is a non-empty set of worlds
- R is a binairy relation between worlds that shows which worlds are accessible from other worlds
- V is a function that determines which propositions are true in a possible world

In contrast to Kripke models of intuitionistic logic, there is no hierarchy in these models for S4. Worlds can also have an accessibility relation to theirselves. We write: wRw if w is accessible from w.

To sum up, these are the rules that hold for Kripke models for S4:

- $k \vDash p$ iff p is true in k
- $k \vDash \neg\phi$ iff $k \nvDash \phi$
- $k \vDash \phi \wedge \psi$ iff $k \vDash \phi$ and $k \vDash \psi$
- $k \vDash \phi \vee \psi$ iff k$\vDash \phi$ or $k \vDash \psi$
- $k \vDash \Box\phi$ iff for all ℓ such that $k\ R\ \ell$, $\ell \vDash \phi$
- $k \vDash \Diamond\phi$ iff for some ℓ such that $k\ R\ \ell$, $\ell \vDash \phi$

The accessibility relation for the system S4 has the properties reflexivity and transitivity. The reflexive property means that all worlds are accessible from themselves. Hence $k\ R\ k$ for every world k. This means that $\Box\phi \rightarrow \phi$ holds. Whereas the transitivity property implies that if $k\ R\ \ell$ and $\ell\ R\ m$ then $k\ R\ m$. This property corresponds to the rule $\Box\phi \rightarrow \Box\Box\phi$.

4 Gödel translation

4.1 Relation of intuitionistic logic and S4

Intuitionistic logic and the modal logic S4 are two very different kinds of logics. But Kurt Gödel provided a translation to embed one into another, see [14]. According to Gödel intuitionistic logic can be interpreted into classical modal logic. This can be done by the Gödel translation, also sometimes called the Gödel-McKinsey-Tarski translation. He also showed how classical first-order logic can be embedded into intuitionistic first-order logic using the Gödel-Gentzen negative translation. The main property of this translation has the following form: $\Gamma \vdash_{\mathrm{c}} \phi$ iff $\Gamma^{\neg\neg} \vdash_{\mathrm{i}} \phi^{\neg\neg}$. In this thesis this translation will not be discussed.

The Gödel translation, which is called τ, has the property that it can convert every intuitionistic propositional formula ϕ into a modal propositional formula ϕ^τ such that: $\Gamma \vdash_{\mathrm{i}} \phi$ iff $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau$. The formula $\Gamma \vdash_{\mathrm{i}} \phi$ is used for entailment in intuitionistic logic and $\Gamma \vdash_{\mathrm{S4}} \phi$ is used for entailment in S4. Γ^τ means that the translation is applied to every formula in Γ. The symbol $\vdash$ is used to indicate derivability: we write $\phi \vdash \psi$ for ψ is derived from ϕ.

Gödel created a mapping from formulae of intuitionistic logic into formulae of S4:

- $p^\tau = \Box p$ for any propositional variable p
- $\bot^\tau = \bot$
- $(\phi \wedge \psi)^\tau = \phi^\tau \wedge \psi^\tau$
- $(\phi \vee \psi)^\tau = \phi^\tau \vee \psi^\tau$
- $(\phi \rightarrow \psi)^\tau = \Box(\phi^\tau \rightarrow \psi^\tau)$
- $(\neg\phi)^\tau = \Box\neg\phi^\tau$

To prove that the formula $\Gamma \vdash_{\mathrm{i}} \phi$ iff $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau$ holds, both directions of the formula must be proved.

4.2 From intuitionistic logic to S4

In this subsection we will prove $\Rightarrow$: If $\Gamma \vdash_{\mathrm{i}} \phi$ then $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau$. This proof is done by induction on the proofs for every logical connective.

Lemma 4.1. $\vdash_{\mathrm{S4}} \phi^\tau \to \Box\phi^\tau$.

This Lemma is called the self necessitating rule. If a formula is valid, it is valid in every possible world. So the formula is necessarily valid ϕ stands for any formula containing the logical connectives. This Lemma will be proved by induction on ϕ for the following formulae:

Proof. First the base axioms will be proved:

We prove $\vdash \bot^\tau \to \Box\bot^\tau$:
Since $\bot^\tau =_{\mathrm{def}} \bot$ and $\vdash \bot \to \Box\bot$ ($\bot$ elimination), $\vdash \bot^\tau \to \Box\bot^\tau$.

We prove $\vdash p^\tau \to \Box p^\tau$:
Since $p^\tau =_{\mathrm{def}} \Box p$ and $\vdash \Box p \to \Box\Box p$ (rule in S4), $\vdash p^\tau \to \Box p^\tau$.

Now the other formulae will be proved:

First induction step: $\vdash (\phi \wedge \psi)^\tau \to \Box(\phi \wedge \psi)^\tau$. Suppose $\vdash \phi^\tau \to \Box\phi^\tau$ and $\vdash \psi^\tau \to \Box\psi^\tau$, then $\vdash \phi^\tau \wedge \psi^\tau \to \Box\phi^\tau \wedge \Box\psi^\tau$. We need to prove $\vdash \phi^\tau \wedge \psi^\tau \to \Box(\phi^\tau \wedge \psi^\tau)$. This can be derived from $\vdash \phi^\tau \wedge \psi^\tau \to \Box\phi^\tau \wedge \Box\psi^\tau$ and from $\vdash \Box\phi^\tau \wedge \Box\psi^\tau \to \Box(\phi^\tau \wedge \psi^\tau)$ (rule in modal logic). And given $(\phi \wedge \psi)^\tau =_{\mathrm{def}} \phi^\tau \wedge \psi^\tau$ and $\vdash \phi^\tau \wedge \psi^\tau \to \Box(\phi^\tau \wedge \psi^\tau)$, we thus have proved $\vdash (\phi \wedge \psi)^\tau \to \Box(\phi \wedge \psi)^\tau$.

Second induction step: $\vdash (\phi \vee \psi)^\tau \to \Box(\phi \vee \psi)^\tau$. Suppose $\vdash \phi^\tau \to \Box\phi^\tau$ and $\vdash \psi^\tau \to \Box\psi^\tau$, then $\vdash \phi^\tau \vee \psi^\tau \to \Box\phi^\tau \vee \Box\psi^\tau$. We need to prove $\vdash \phi^\tau \vee \psi^\tau \to \Box(\phi^\tau \vee \psi^\tau)$. This can be derived from $\vdash \phi^\tau \vee \psi^\tau \to \Box\phi^\tau \vee \Box\psi^\tau$ and $\vdash \Box\phi^\tau \vee \Box\psi^\tau \to \Box(\phi^\tau \vee \psi^\tau)$ (valid rule in modal logic). And given $(\phi \vee \psi)^\tau =_{\mathrm{def}} \phi^\tau \vee \psi^\tau$ and $\vdash \phi^\tau \vee \psi^\tau \to \Box(\phi^\tau \vee \psi^\tau)$, we have proved $\vdash (\phi \vee \psi)^\tau \to \Box(\phi \vee \psi)^\tau$.

Third induction step: $\vdash (\phi \to \psi)^\tau \to \Box(\phi \to \psi)^\tau$. Since we know that $(\phi \to \psi)^\tau =_{\mathrm{def}} \Box(\phi^\tau \to \psi^\tau)$, we can replace $(\phi \to \psi)^\tau$ by $\Box(\phi^\tau \to \psi^\tau)$ and get $\Box(\phi^\tau \to \psi^\tau) \to \Box\Box(\phi^\tau \to \psi^\tau)$. This is perfectly valid, because of the axiom $\vdash \Box p \rightarrow \Box\Box p$ in S4.

Fourth induction step: $\vdash (\neg\phi)^\tau \rightarrow \Box(\neg\phi)^\tau$. This can be converted into $\vdash (\phi \rightarrow \bot)^\tau \rightarrow \Box(\phi \rightarrow \bot)^\tau$. And this can be proved by the third induction step.

$\Box$

And because of these induction steps, we have proved the Lemma $\phi^\tau \rightarrow \Box\phi^\tau$ for any formula ϕ.

Now we still have to prove that the formula $\Gamma \vdash_{\mathrm{i}} \phi \Rightarrow \Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau$ hold, using Lemma 4.1. This proof will be done by induction, so it will be verified that ϕ can be any propositional formula. Below are the several steps of the construction of the proof.

i We prove $\Gamma \vdash_{\mathrm{i}} \phi \Rightarrow \phi^\tau \vdash_{\mathrm{S4}} \Gamma^\tau$. Suppose we have $\Gamma \vdash_{\mathrm{i}} \phi$. Then ϕ is an element in the assumption set Γ and thus ϕ^τ is also in the assumption set Γ^τ. And since ϕ is derivable from Γ, then ϕ^τ must also be derivable from Γ^τ. Hence $\phi^\tau \vdash_{\mathrm{S4}} \Gamma^\tau$.

ii We prove the formula for conjunction introduction. Suppose we have $\Gamma \vdash_{\mathrm{i}} \phi \wedge \psi$ derived from $\Gamma \vdash_{\mathrm{i}} \phi$ and $\Gamma \vdash_{\mathrm{i}} \psi$. Given the induction hypothesis we have $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau$ and $\Gamma^\tau \vdash_{\mathrm{S4}} \psi^\tau$, hence $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau \wedge \phi^\tau$. But since we know that $\phi^\tau \wedge \psi^\tau =_{\mathrm{def}} (\phi \wedge \psi)^\tau$, we have proved $\Gamma \vdash_{\mathrm{i}} \phi \wedge \psi \Rightarrow \Gamma^\tau \vdash_{\mathrm{S4}} (\phi \wedge \psi)^\tau$.

iii We leave the proofs for the formulae for conjunction elemination, disjunction introduction, disjunction elemination and implication elemination to the reader, since they are similar to the former proof.

iv Suppose we have $\Gamma \vdash_{\mathrm{i}} \phi \rightarrow \psi$ derived from $\Gamma, \phi \vdash_{\mathrm{i}} \psi$. By the induction hypothesis we get $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau \rightarrow \phi^\tau$. And this formula can be converted into $\vdash_{\mathrm{S4}} \bigwedge \Gamma^\tau \rightarrow (\phi^\tau \rightarrow \phi^\tau)$. Applying the Necessitation rule to this formula we have $\vdash_{\mathrm{S4}} \Box(\bigwedge \Gamma^\tau \rightarrow (\phi^\tau \rightarrow \phi^\tau))$. We rewrite this to

$$\vdash_{\mathrm{S4}} \bigwedge_{\gamma \in \Gamma} \Box\gamma^\tau \rightarrow \Box(\phi^\tau \rightarrow \phi^\tau).$$

And using Lemma 4.1 this entails $\Gamma^\tau \vdash_{\mathrm{S4}} \Box(\phi^\tau \rightarrow \phi^\tau)$ and we apply $(\phi \rightarrow \psi)^\tau =_{\mathrm{def}} \Box(\phi^\tau \rightarrow \psi^\tau)$, which leads to $\Gamma^\tau \vdash_{\mathrm{S4}} (\phi \rightarrow \psi)^\tau$. Hence we have proved $\Gamma \vdash_{\mathrm{i}} \phi \rightarrow \psi \Rightarrow \Gamma^\tau \vdash_{\mathrm{S4}} (\phi \rightarrow \psi)^\tau$.

Now we have proved that $\Gamma \vdash_{\mathrm{i}} \phi \Rightarrow \Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau$.

4.3 From S4 to intuitionistic logic

Now the other direction of the arrow will be proved: $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau \Rightarrow \Gamma \vdash_{\mathrm{i}} \phi$. This is done by turning the formula into $\Gamma \nvdash_{\mathrm{i}} \phi \Rightarrow \Gamma^\tau \nvdash_{\mathrm{S4}} \phi^\tau$. If the antecedent of the implication is true, the truth consequent must also be proved. By the completeness theorem it is sufficient to show that if Γ does not force ϕ in intuitionistic logic, Γ^τ will not force ϕ^τ in S4. To show that a formula is not forced, a Kripke model can be made as a counter model.

Intuitionistic logic and S4 have Kripke semantics with similar models. The difference lies in the concept of the model. For intuitionistic logic the atoms contain persistence. This means that if $k \Vdash p$ is the case, it follows that for every node ℓ for which $k \leq \ell$, $\ell \Vdash p$ holds. If the Kripke models of intuitionistic logic are converted into models for S4, the atoms still retain their persistence. In S4 the formula $k \Vdash \Box p$ simply means that p is assigned to every node ℓ that is accessible from k. Therefore if a formula holds in a Kripke model for intuitionistic logic, it also holds for a Kripke model for S4. We show $k \Vdash_{\mathrm{i}} \phi \Rightarrow k \Vdash_{\mathrm{S4}} \phi^\tau$ for any formula ϕ. This will be proved by induction to ϕ.

i We prove $k \Vdash_{\mathrm{i}} p \Leftrightarrow k \vDash_{\mathrm{S4}} \Box p$

$$\begin{aligned} k \Vdash_{\mathrm{i}} p &\Leftrightarrow \forall \ell \geq k,\ \ell \Vdash_{\mathrm{i}} p \\ &\Leftrightarrow \forall \ell \geq k,\ \ell \vDash_{\mathrm{S4}} p \\ &\Leftrightarrow k \vDash_{\mathrm{S4}} \Box p \end{aligned}$$

ii We prove $k \Vdash_{\mathrm{i}} \phi \wedge \psi \Leftrightarrow k \Vdash_{\mathrm{S4}} (\phi \wedge \psi)^\tau$. Note that "IH" stands for induction hypothesis.

$$\begin{aligned} k \Vdash_{\mathrm{i}} \phi \wedge \psi &\Leftrightarrow_{\mathrm{def}} k \Vdash_{\mathrm{i}} \phi \textit{ and } k \Vdash_{\mathrm{i}} \psi \\ &\Leftrightarrow_{\mathrm{IH}} k \vDash_{\mathrm{S4}} \phi^\tau \textit{ and } k \vDash_{\mathrm{S4}} \psi^\tau \\ &\Leftrightarrow_{\mathrm{def}} k \vDash_{\mathrm{S4}} \phi^\tau \wedge \psi^\tau \\ &\Leftrightarrow_{\mathrm{def}} k \vDash_{\mathrm{S4}} (\phi \wedge \psi)^\tau \end{aligned}$$

iii We prove $k \Vdash_{\mathrm{i}} \phi \vee \psi \Leftrightarrow k \Vdash_{\mathrm{S4}} (\phi \vee \psi)^\tau$.

$$\begin{aligned} k \Vdash_{\mathrm{i}} \phi \vee \psi &\Leftrightarrow_{def} \quad k \Vdash_{\mathrm{i}} \phi \textit{ or } k \Vdash_{\mathrm{i}} \psi \\ &\Leftrightarrow_{\mathrm{IH}} \quad k \vDash_{\mathrm{S4}} \phi^\tau \textit{ or } k \vDash_{\mathrm{S4}} \psi^\tau \\ &\Leftrightarrow_{\mathrm{def}} \quad k \vDash_{\mathrm{S4}} \phi^\tau \vee \psi^\tau \\ &\Leftrightarrow_{\mathrm{def}} \quad k \vDash_{\mathrm{S4}} (\phi \vee \psi)^\tau \end{aligned}$$

iv We prove $k \Vdash_{\mathrm{i}} \phi \to \psi \Leftrightarrow k \Vdash_{\mathrm{S4}} (\phi \to \psi)^\tau$.

$$\begin{aligned} k \Vdash_{\mathrm{i}} \phi \to \psi &\Leftrightarrow_{\mathrm{def}} \quad \forall \ell \geq k,\ \ell \Vdash_{\mathrm{i}} \phi \Rightarrow \ell \Vdash_{\mathrm{i}} \psi \\ &\Leftrightarrow_{\mathrm{IH}} \quad \forall \ell \geq k,\ \ell \vDash_{\mathrm{S4}} \phi^\tau \Rightarrow \ell \vDash_{\mathrm{S4}} \psi^\tau \\ &\Leftrightarrow_{\mathrm{def}} \quad \forall \ell \geq k,\ \ell \vDash_{\mathrm{S4}} \phi^\tau \to \psi^\tau \\ &\Leftrightarrow_{\mathrm{def}} \quad k \vDash_{\mathrm{S4}} \Box(\phi^\tau \to \psi^\tau) \\ &\Leftrightarrow_{\mathrm{def}} \quad k \vDash_{\mathrm{S4}} (\phi \to \psi)^\tau \end{aligned}$$

We have seen that $k \Vdash_{\mathrm{i}} \phi \Leftrightarrow k \vDash_{\mathrm{S4}} \phi^\tau$ for any formula ϕ. Hence if $k \nVdash_{\mathrm{i}} \phi$ then $k \nVdash_{\mathrm{S4}} \phi^\tau$. Clearly the formula $\Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau \Rightarrow \Gamma \vdash_{\mathrm{i}} \phi$ holds. Now we have proved both directions of $\Gamma \vdash_{\mathrm{i}} \phi \Leftrightarrow \Gamma^\tau \vdash_{\mathrm{S4}} \phi^\tau$.

5 Evaluation

In this thesis we have seen how reasoning about knowledge can be formalized in a variety of ways. Through the years many logics have been created that are useful for studying knowledge. Two of them are discussed in this thesis and finally a correspondence has been shown.

The principles of intuitionistic logic were discussed and it has been clarified how it differs from classical logic. We have introduced the Kripke models that serve for the semantics for intuitionistic logic. Thereafter the idea of the modal logic S4 was explained and again Kripke models were mentioned. Both are very useful for reasoning about knowledge, but their concepts are different and therefore the logics contain different rules. Intuitionistic logic is a formalization of the mathematical approach intuitionism. Knowledge is acquired but can not be lost in later stages. S4 uses $\Box$ and $\Diamond$ as operators for knowledge. The system inherits the rules of classical logic, but extends it by modal operators.

For both logics Kripke models form a useful instrument for illustrating statements about knowledge. The models consist of nodes that represent stages. These stages contain knowledge that is obtained so far. While the Kripke models for intuitionistic logic look very similar to the ones for S4, they are interpreted in a different way. In intuitionistic logic the several nodes in a model can be seen as succesive worlds that contain constructed atoms. Every node has one or multiple accessible nodes that adopt the obtained atoms and potentially construct more atoms. The set of obtained knowledge grows and the model is therefore a dynamic picture. In S4 the nodes in a Kripke model form a set of all possible worlds. Here $\Box p$ means that the agent knows that p and $\Diamond p$ means that p is compatible with the agent's knowledge. Therefore the Kripke models used for S4 are static pictures.

Section 4 demonstrated the interpretation of intuitionistic logic to classical modal logic. Although their concepts of Kripke models differ slightly, similarities are noticeable. Thereby a comparison can be made between the models. Models from intuitionistic logic can be converted into models from S4, whereby the propositions derived by rules in intuitionistic logic are still valid in S4. This is done by the Gödel translation that included a set of rules that were needed by the translation.

As we have seen, the Gödel translation has an important advantage. By means of the Gödel tranlation, it follows that every intuitionistically valid formula is also valid in S4. Therefore the rejection of the law of excluded middle is not a genuine restriction. It only induces that the interpretation of theorem has to be changed. Thus intuitionistic appears to be nearer to S4 than to classical logic.

We wondered what kind of consequences the Gödel translation might have. For philosophers it is an interesting question what the relation is between two completely different logics that are both meant for the same matter, knowledge. Constructivists pursue a concept of truth that is very radically against the classical concepts. They stick to an idea where truths are constructed in an infinite set of worlds. They therefore do not believe in an informal correspondence of the two logics.

But the Gödel translation shows us a formal correspondence between both logics. It is very significant that both logics satisfy the same models. Due to the translation a tranfer was made of the technical view of logic. If all formulae that are valid in intuitionistic logic are valid in S4 by a translation, the logics must be very similar. The rejection of the law of excluded middle does not make a formal difference. For non-intuitionists this means that there is no informal difference either. But an intuitionist could be opposed to this analysis and think of the translation as just a formal resemblance. This means for example that if a computer system uses rules of intuitionistic logic and another system uses rules of S4, the rules can be translated for the other system.

When building an agent, we need to think of the knowledge that it contains. Agents must realize what is true and what they know and what they know that they know etc. For these agents their must be considered about what logic for knowlegde to use. And if different agents include different logics, their must be thought of a correlation between the two. If one logic can be converted into another, this is very useful for the building of agents. The translation helps us clarify the distinction between the two logics. Therefore the Gödel translation is a great utility for computer scientist and researchers that study artificial intelligence.

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