A Regime Switching Jump-Diffusion Model and its Application to Credit Risk and Option Pricing

by

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"Complete realism is clearly unattainable, and the question whether a theory is realistic enough can be settled only by seeing whether it yields predictions that are good enough for the purpose in hand or that are better than predictions from alternative theories."

Milton Friedman 1953
(Nobel Prize Laureate in Economics 1976)

Abstract

A vital issue in financial mathematics is the choice of an appropriate model for the financial market. It is well-known that the famous Black-Scholes model does not perform well in reality. The aim of this thesis is to extend the Black-Scholes model in order to improve its empirical performance. Our motivation originates from credit risk and option pricing. We start with credit risk and in particular with the valuation of risky debt. The approach initiated by Merton [57] will be adapted to the extended model. We will see that the obtained credit spreads are consistent with historical data. Further, several calibration methodologies for estimating the model parameters will be discussed. Due to the fact that calibration is feasible, we are able to use the model for practical purposes such as the calculation of probabilities of default. As a test example, the model will be applied to a real firm that has been in financial distress recently. In the second part of this thesis we will focus on option pricing. The volatility smiles produced by the model for short as well as for long maturities are adequate. Moreover, we will propose two different ways of option pricing: an analytical way and by means of the FFT algorithm. The last technique has the advantage that it can be extended to price more complex options, e.g., exotic options.

Keywords: Regime Switching Jump-Diffusion ⋆ Credit Risk ⋆ Calibration ⋆ MLE ⋆ EM ⋆ Inverse Problem ⋆ Probabilities of Default ⋆ Option Pricing ⋆ SA ⋆ FFT

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Chapter 1

Introduction

Over the last few years the importance of finance in our society has grown significantly. The bankruptcy of the Lehman Brothers in 2008 and Greece's default in 2012 are probably the two most controversial examples of the current global financial crisis. A lot of factors have contributed to these developments, think of political irresponsibility, excessive greed and bad regulation, but also the abuse of present mathematical models. By 2007, the international financial system was trading derivatives valued at one quadrillion $\left(10^{15}\right)$ dollars per year. This is ten times the total worth of all products made by the world's manufacturing industries over the last century (Stewart [65])! By a derivative we mean an agreement between two parties that has a value determined by the price of something else. Examples of derivatives are credit default swaps (CDSs) and options. This world of derivatives is the main source of the current financial crisis and it was the famous Black-Scholes (BS) model that made this world possible. It is well-known that the BS model performs badly in reality. This is partially due to some unrealistic assumptions such as constant volatility and continuity of the process. In this thesis we extend the original BS model in order to improve its empirical performance. The resulting model is consistent with historical data and it completely replaces the Black-Scholes model in some situations. At the same time, this model inherits important benefits, for instance, analytical tractability, fast computation of option prices and feasible calibration. Calibrating the model and its applications to credit risk and option pricing are the most important subjects we will discuss.

There are two main reasons for us to introduce this model: (1) In credit risk, the original BS model systematically underestimates credit spreads (especially for short maturities). (2) From an option pricing perspective, observed phenomena such as the volatility smile are not captured by the BS model. In this chapter we will explain this motivation in more detail.

Credit Risk
Let us first focus on credit risk, that is, the risk associated with a borrower going into default. A vital notion in credit risk is the probability of default[1] (PD), which is simply the probability that a certain financial institution goes into default. There exist two types of probabilities: the real-world and the risk-neutral probability. The real-world probability is implied from historical data and it is often called the historical or physical probability. Real-world probabilities are used in scenario analysis like the computation of potential future losses from defaults and in the calculation of bank capital requirements under, for instance, Basel III. On the other hand, risk-neutral probability is simply an auxiliary mathematical concept that is used to price financial instruments such as derivatives. In fact, once we have determined the risk-neutral probabilities, any financial instrument can be valued by taking its discounted expected payoff regardless of the specific risk preferences of the investor. The existence of such a risk-neutral measure is required in order to avoid arbitrage opportunities. (An arbitrage is the practice of taking advantage of a price difference between two or more markets.) Moreover, this risk-neutral measure must be equivalent to the physical one, i.e., they must define the same set of scenarios. Another important concept is credit spread. The credit spread reflects the additional net yield an investor can earn from a financial instrument with more credit risk relative to one with less credit risk. Thus, a low credit spread with a risk-free bond represents a relatively low probability of default. We can now move on to credit risk modeling.

There are two basic approaches to modeling credit risk: structual and reduced-form approach. In the structural approach, we make explicit assumption about the dynamics of a firm's assets value[2], its capital structure, debt and shareholders. The event of default occurs when the firm's value drops below a certain threshold level such as the value of its debt or some percentage of it. Structural models have been studied for the first time by Black and Scholes [10] and Merton [57]. These models were later extended by, among others, Black and Cox [9] and Longstaff and Schwartz [50]. The big advantages of the structual approach are that the models provide an intuitive economic interpretation and the inputs and outputs of the models are in terms of understandable economic variables, e.g., asset volatility and market's assessment of a firm's value. An alternative approach is the so-called reduced-form approach. The main difference between the two is the interpretation (or definition) of the event of default. Reduced-form models treat default as an unpredictable Poisson event involving a sudden loss in the market value. This approach is adopted by Duffie and Singleton [24], Jarrow and Turnbull [44], Madan and Unal [53] and others. Probably its most attractive property is its mathematical tractability. Jarrow and Protter [43] argue further that the reduced-form approach is

[1]Sometimes also called the "default probability".
[2]For simplicity, we will frequently use the term "firm's value".

more appropriate because in the structural case, one is unlikely to know exactly the default point, that is, the threshold level. Nevertheless, reduced-form models do suffer from weaknesses as well. One of the weaknesses is the fact that the relation between the firm's value and the default event is not immediately clear. Another drawback is the implication that firms can only default "by surprise", which is problematic since a default can also occur because of slow but steady declines in the firm's value (hence "expectedly"). Based on these arguments, we think that the reduced-form approach is not realistic enough and therefore in the sequel of this thesis, we will solely focus on structural models.

Before we do that, let us make the following observation. Suppose that firms can only default expectedly. This means that if we observe a default we immediately know that the default has occured due to gradual declines in the firm's value. Firms that are not currently in financial distress must therefore have zero probability of default on very short-term debt. And so the firm's short-term debt must have zero credit spreads as well. However, this implication contradicts the empirical data shown in Figure 1.1![3] From this

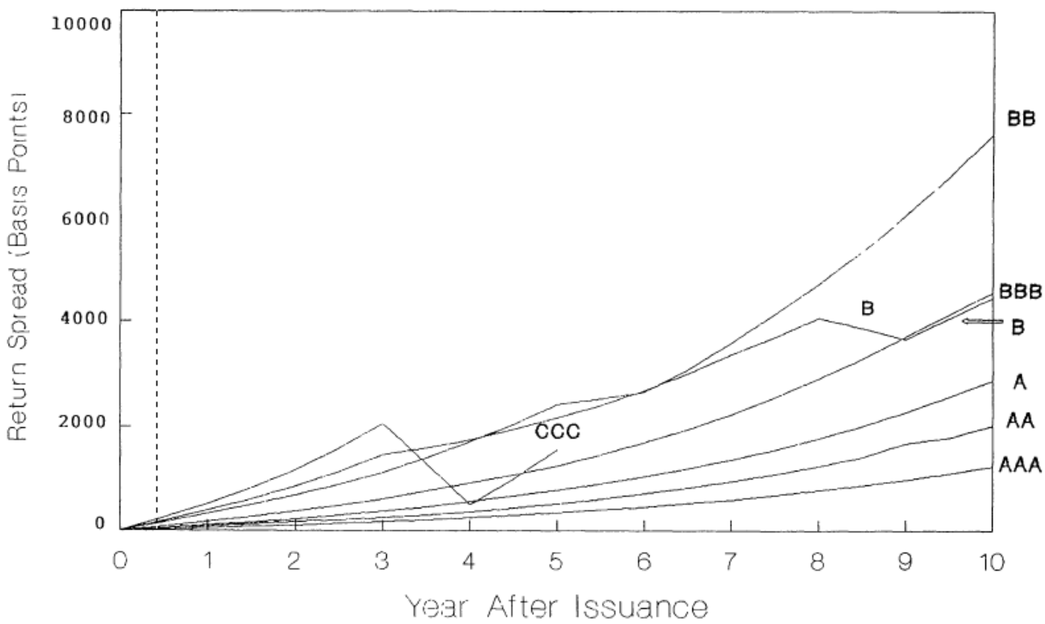

Figure 1.1: Credit spread term structures for various corporate bonds.

picture we see that on short maturities, e.g., a few weeks, corporate bonds[4] that even have an investment grade (AAA-BBB) have nonzero credit spreads. (An investment grade indicates that a corporate bond has a relatively low probability of default.) So we may conclude that in a proper credit risk model a default can occur in two ways: (1) by surprise because of a sudden loss in the market value and (2) expectedly due to gradual

[3]This figure can be found in [1].

[4]The definition of a corporate bond is provided at the begin of Chapter 3.

declines in the firm's value.

Now let us consider the very first structural model: the Black-Scholes model. In this model, for a given firm, the firm's value V is modeled by the following diffusion process

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \tag{1.1}$$

where μ is the so-called drift (rate), σ the volatility and the process $\{W_t\}$ denotes a standard Brownian motion. The drift rate of a stochastic process indicates the mean change per unit time. And the volatility of a process is roughly speaking a measure of how uncertain we are about the future movements of the process. For instance, higher volatility represents more uncertainty. Any stochastic process $\{V_t\}$ that satisfies equation (1.1) is said to follow a (standard) geometric Brownian motion. Merton was the first one to apply this model to credit risk. In particular he investigated pricing risky debt by modeling the evolution of the firm's value using the BS model. Due to the assumption that the firm's value dynamics follows a diffusion process, which is a continuous process, a sudden drop in the firm's value is impossible. Hence, firms can never default by surprise. Therefore the PDs and observed credit spreads, especially for short maturities, are systematically underestimated. One way to avoid this problem is by introducing some extra randomness in the model. For instance, we can add a jump process to the diffusion model. This was done by Zhou who used the jump-diffusion model that was developed by Merton in [58]. We will denote this model by JDM, where the letter M stands for Merton. General jump-diffusion models will be denoted by JD.

Actually, Zhou [72, 73] was the first one to model the firm's value dynamics using Merton's model. The JDM model is given by

$$dV_t = \mu V_t dt + \sigma V_t dW_t + (Y_t - 1)V_t dN_t, \tag{1.2}$$

where the process $\{N_t\}$ represents a Poisson process with rate λ_Y and Y is just some random variable such that it has a probability measure with compact support and $Y \geq 0$. The idea behind the model is the following. There are two types of information that can change the firm's value: (1) Information that causes marginal changes in the firm's value, e.g., temporary imbalance between supply and demand or changes in the economic outlook. This type of information is modeled by the "diffusion part" of the process. (2) Information that has more than marginal effect on the value. This kind of information will be important and specific to the firm or its industry. It is natural to expect that important information arrives only at discrete points in time and therefore these arrivals are modeled by a Poisson process with rate λ_Y. Now we can rewrite (1.2) as

$$dV_t = \begin{cases} \mu V_t dt + \sigma V_t dW_t, & \text{if the Poisson event does} \\ & \text{not occur,} \\ \mu V_t dt + \sigma V_t dW_t + (Y_t - 1)V_t, & \text{if the Poisson event occurs.} \end{cases}$$

Since we are dealing with a Poisson process, the arrivals are assumed to be independent and the inter-arrival times are exponentially distributed. So, if we fix t, the probability that an event (a jump) occurs during a small time interval dt can be written as

$$\begin{aligned}
&\text{Prob}\{\text{the event does not occur during } (t, t + dt)\} = 1 - \lambda_Y dt + o(dt), \\
&\text{Prob}\{\text{the event occurs once during } (t, t + dt)\} = \lambda_Y dt + o(dt), \\
&\text{Prob}\{\text{the event occurs more than once during}(t, t + dt)\} = o(dt),
\end{aligned}$$

where $o(\cdot)$ is an asymptotic order symbol. This is because the number of arrivals in a time interval $(t, t + dt)$ follows a Poisson disbribution with parameter $\lambda_Y dt$ and hence

$$\text{Prob}\{\text{the event occurs once during } (t, t + dt)\} = \frac{e^{-\lambda_Y dt}(\lambda_Y dt)^1}{1!} = \lambda_Y dt + o(dt).$$

Furthermore, we assume that if a jump occurs, the firm's value goes from V_t to $V_t Y_t$ and so the percentage change equals $\frac{dV_t}{V_t} = \frac{V_t Y_t - V_t}{V_t} = Y_t - 1$. Due to this fact, the firm's value process is not continuous anymore which implies that a firm can default by surprise. The JDM model can generate various shapes of credit spread which are also observed in the market. But even though Zhou achieved more realistic results, the assumption that the volatility and drift of the firm's value process are both constant is still unrealistic; certainly in the current financial climate.

Krystul, Bagchi and Bouman [48] focussed precisely on this weakness of the (jump-) diffusion model. They considered a diffusion model whereby the volatility and drift of the process may switch from one value to another depending on the current state of the economy ("good" or "bad"), which is modeled by a continuous-time Markov process (the continuous-time version of a Markov chain). Note that this is another way of introducing more randomness in the model. Their model, the regime switching geometric Brownian motion (RSM) model, is given by

$$\begin{aligned}
dV_t &= \mu_t V_t dt + \sigma_t V_t dW_t \\
P_{\theta_{t+dt}|\theta_t}(e_j|e_i) &= \lambda_{ij} dt + o(dt), \quad i \neq j.
\end{aligned}$$

Observe that the drift μ_t and volatility σ_t are indeed time-dependent (for a proper formulation see Chapter 2). In virtue of the simplification that the Markov process has only two states, analytical expressions for estimation of the PDs and pricing of defaultable corporate bonds can be derived. Unfortunately, the incorporated regime switching does not provide enough flexibility (randomness), since the credit spreads for very short maturities are underestimated (see Chapter 3, Section 3.3 and Figure 3.1).

We have now discussed three structural models: BS, JDM and RSM. For us the most important properties of these models are provided in Table 1.1 below. As mentioned

model	realistic credit spreads/PDs	non-constant parameters
BS	✗	✗
JDM	✓	✗
RSM	✗	✓

Table 1.1: Properties of the BS, JDM and RSM model.

before, the motivation for the model we will construct originates partially from credit risk. If we look at Table 1.1, we might be tempted to combine the JDM and the RSM model in order to acquire a better credit risk model. And this is exactly what we did! The resulting model, called the regime switching jump-diffusion (RSJD[5]) model, is given by

$$dV_t = \mu_t V_t dt + \sigma_t V_t dW_t + (Y_t - 1)V_t dN_t \tag{1.3}$$

$$P_{\theta_{t+dt}|\theta_t}(e_j|e_i) = \lambda_{ij}dt + o(dt), \quad i \neq j. \tag{1.4}$$

In other words, we extend the BS model by including two jump processes: one for the firm's value (the jump-diffusion part) and one for the drift and volatility (the regime switching part). (Again, see Chapter 2 for a rigorous formulation.) A sample path of the RSJD model is presented in Figure 1.2. We see that in this case the regime switching occurs around the point 0.46. In fact, at this point the volatility jumps from 0.1 to 0.3, while the drift rate remains the same. In Chapter 3, Section 3.3 we will see that the RSJD model indeed produces better credit spreads than the JDM and RSM model. Beside a formula for the credit spreads, we will derive an analytical expression for the firm's debt value and we will estimate the PDs. Moreover, the RSJD also inherits some important properties from the JDM and RSM model as well, for example, the ability to produce various shapes of credit spread and analytical tractability.

Now the first part of our motivation is underpinned more mathematically, we need to devise how to use the model in practice before we can continue with the second part. Depending on the particular situation, we have to find proper values for the model parameters. This is called calibration. Obviously, this procedure requires real data.

[5]Actually, one should expect the abbreviation to be RSJDM instead of RSJD, but we think that it is notationally convenient to write RSJD.

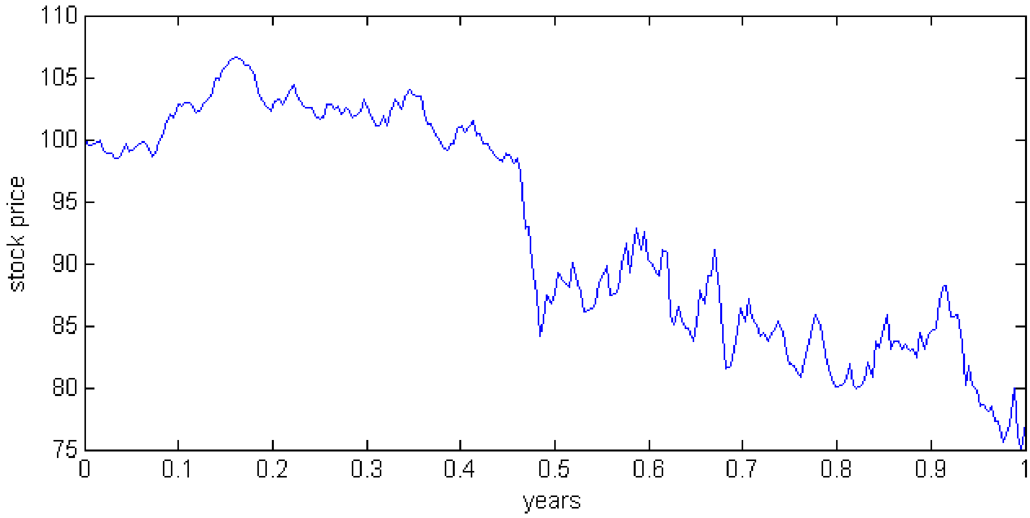

Figure 1.2: A sample path of the RSJD model.

Maximum-likelihood estimation (MLE), expectation-maximization (EM) algorithm and inverse problem are the calibration methodologies which we will discuss. Observe that this is an important point since parameter estimation becomes increasingly difficult as models become complex (Timsina [67]). Once the model is calibrated, we can apply it to credit risk and option pricing.

Option Pricing

As mentioned before, the second part of our motivation is option pricing. Option pricing is the very first and the most important application of the BS model. However, similar to the credit risk case, the model insufficiently explains important empirical properties. One of these properties is the so-called volatility smile. The volatility smile is a plot of the implied volatility of an option as a function of its strike price. (The implied volatility is the volatility that, when used in a given pricing model, provides a theoretical option price equal to the current market price of the option.) According to the BS model the implied volatility must be constant regardless of the specific strike price. This is certainly not true in reality as one can see in Figure 1.3.[6] We should note that, depending on the particular market, the volatility smile does not have to look like the one shown in Figure 1.3, but this shape is probably the most common one (see Appendix A.1).

Two ways to fit the volatility smile are: including jumps into the model or assuming a non-constant volatility. It turns out that jump-diffusion models explain the volatility smile only for short maturities (Tankov and Voltchkova [66]). On the other hand, many of the stochastic volatility models, such as Heston model, perform satisfactorily solely for long maturities (Cizek, Härdle and Weron [17]). We will see that the RSJD model

[6] This figure can be found in [38].

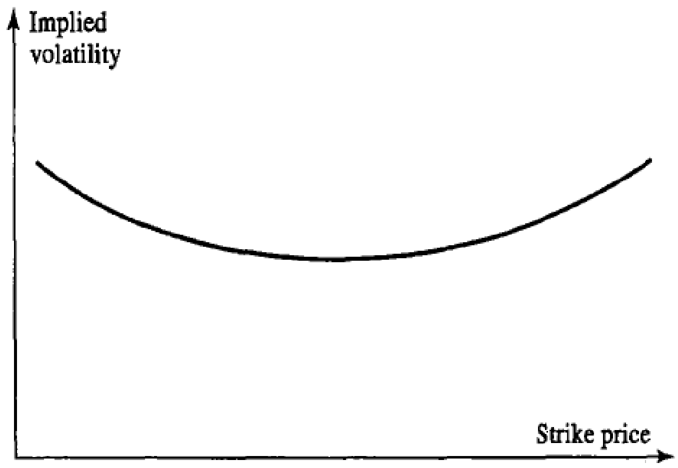

Figure 1.3: Volatility smile for foreign currency options.

is able to fit the volatility smile for short maturities as well as for long maturities.

Merton [58] was the first one to price options using a JD model and nowadays these models are widely used because of their adequate description of stock price fluctuations. Specific reasons why JD models, also for option pricing, are more preferable than diffusion models (such as the BS model) are: (1) Assuming the BS model, the probability that a stock moves by a large amount over a short period of time is very small.[7] This is empirically inconsistent. In JD models this problem is solved by adding a jump component. (2) The market described by JD models is incomplete. Incomplete markets are more desirable than complete markets which are described by diffusion models (see Appendix A.2 for more details). (3) And the most important argument is probably simply the presence of jumps in observed prices. Our model, the RSJD model, has even one more advantage than regular JD models, namely, that the volatility and the drift do not remain constant over time, which is also an empirical fact.

Certainly the BS model would not be so popular if it had no benefits at all. One of the strengths of the BS model is that it does not only provide analytical expressions for the price of put and call options, but also for a number of exotic options (e.g. barrier options) as well. For most complex models, deriving analytical pricing formulae is very difficult, if even possible. Hence, the prices need to be determined numerically. A standard technique to do that is the so-called fast Fourier transform (FFT). We will see that for our model we are able to derive an analytical expression for the price of European-style put and call options. Moreover, this price will also be computed using the FFT.

[7]Unless, of course, one takes an unrealistically large volatility.

Based on the number of publications, we might conclude that jump-diffusion models with regime switching are probably not the most popular models in the literature. Finance and electricity prices seem to be their main application areas. We briefly mention some notable publications. We begin with Yin, Song and Zhang [70] who derived a numerical solution for the general case when the Markov process has n states. A similar model has been applied to stochastic optimal control in [71] by Zhang, Elliot and Siu. Jackson, Jaimungal and Surkov [41] considered a even more generic model. They studied regime switching Lévy processes[8] which they applied to pricing options using a new Fourier Space Time-stepping (FST) algorithm. In [37], Huang, Forsyth and Labahn looked at a regime switching stochastic process where the jump size of the jump-diffusion part is deterministic. This method is then applied to pricing American options. Finally, Fuh and Lin [32] considered a Markov jump-diffusion model, where the volatility is assumed to be independent of the Markov process. They derived a closed-form pricing formula for European call options. A common feature of almost all these models is that the calibration procedure is very hard to carry out, if even possible.

To summarize: In this thesis we will extend the classical Black-Scholes model in order to improve its empirical performance. The new model preserves important properties of the original one such as analytical tractability. Our motivation originates from credit risk and option pricing. We show that the extended model performs well when it is applied to these two areas. More importantly, practical implementation is possible due to the fact that calibration is feasible. This means that in practice, the BS model can be replaced by the more realistic RSJD model. Moreover, based on existing literature and our findings this model can be developed even further to price more complex and even exotic options.

The remainder of this thesis is organized as follows. First, the mathematical formulation for the model is provided in Chapter 2. We choose a risk-neutral measure $\mathbb{Q}$ and derive the solution for the model. In Chapter 3 we focus on credit risk and in particular on pricing risky debt. The approach by Merton is explained and after that adapted to our situation. A formula for the value of the firm's debt is derived. We end this chapter with two credit spread term structures produced by various models. A number of calibration methods are described in Chapter 4. At the end of this chapter, the PDs are computed and a practical example is presented. In Chapter 5 we consider option pricing. We present some volatility smiles according to different models. An analytical pricing formula is derived. We compare the prices produced by the analytical expression to those obtained by the FFT algorithm. Some concluding remarks and suggestions for future work are included in Chapter 6.

[8] An example of a Lévy process is the jump-diffusion process.

Chapter 2

Model Description

In this chapter we introduce some handy notation and we formulate the RSJD model in mathematical terms. The solution for the model is derived. After that, a risk-neutral probability measure $\mathbb{Q}$ is chosen and the model is reformulated under $\mathbb{Q}$.

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, where $\mathbb{P}$ represents a real-world probability measure. Let $\{W_t\}$ be a standard Brownian motion and $\{N_t\}$ a homogeneous Poisson process with rate λ_Y. The process $\{\theta_t\}$ is a continuous-time Markov process with values in $\mathbb{M} := \{\theta_1, \theta_2, \ldots \theta_M\}$. Without loss of generality, we can restrict the state space of $\{\theta_t\}$ to a finite set of unit vectors $\{e_1, e_2, \ldots, e_M\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^M$, i.e., all coordinates are zero except for the i^{th} one. For $i \neq j$, the jump time from state e_i to state e_j is exponentially distributed with parameter λ_{ij}. We assume that $\{\theta_t\}$, $\{N_t\}$ and $\{W_t\}$ are mutually independent and defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let us consider a financial model consisting of a risk-free asset and some risky asset. The risk-free asset is either a bank account or a risk-free bond. The instantaneous (risk-free) market interest rate $\{r(t, \theta_t)\}$ is given by

$$r_t := r(t, \theta_t) =< r, \theta_t >,$$

where $r := (r_1, r_2, \ldots, r_M)$ with $r_i > 0$ for each $i = 1, 2, \ldots, M$, and $< \cdot, \cdot >$ denotes the inner product in $\mathbb{R}^M$. The price process $\{B_t\}$ of the bank account (or the risk-free bond) is now described by

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

We assume that the drift rate $\{\mu_t\}$ and the volatility $\{\sigma_t\}$ of the risky asset V also depend on $\{\theta_t\}$:

$$\mu_t := \mu(t, \theta_t) =< \mu, \theta_t >, \quad \sigma_t := \sigma(t, \theta_t) =< \sigma, \theta_t >,$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_M)$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_M)$ with $\sigma_i > 0$ for each $i = 1, 2, \ldots, M$. Furthermore, the dynamics of the firm's assets value process $\{V_t\}$ is given by the following regime switching jump-diffusion (RSJD) process[1]

$$dV_t = (\mu_t - \lambda_Y \kappa) V_t dt + \sigma_t V_t dW_t + (Y_t - 1) V_t dN_t \tag{2.1}$$

$$P_{\theta_{t+dt}|\theta_t}(e_j|e_i) = \lambda_{ij} dt + o(dt), \quad i \neq j, \tag{2.2}$$

with dt a small time interval, $o(\cdot)$ an asymptotic order symbol and $\kappa = \mathbb{E}(Y_t - 1)$. The jump size process $\{Y_t\}$ is lognormal distributed, such that for each t, $\ln(Y_t) \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, where $\Pi \sim \mathcal{N}(a, b)$ means that Π is normally distributed with mean a and variance equal to b. The random variable $Y_t - 1$ represents the percentage change in the firm's value caused by a Poisson event (i.e., $dN_t = 1$) in a small time interval. We assume the Poisson events to be independent and the process $\{Y_t\}$ to be independent of the processes defined above.

Define $Z_t := \ln(V_t/V_0)$ to be the logarithmic return from V over the interval $[0, t]$ for each t. Then, the solution to (2.1-2.2) can be expressed as

$$V_t = V_u \exp(Z_t - Z_u), \tag{2.3}$$

with Z_t satisfying

$$Z_t = \int_0^t \left(\mu_s - \lambda_Y \kappa - \frac{1}{2}\sigma_s^2 \right) ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \ln Y_i, \tag{2.4}$$

where the sum is interpreted as zero if $N_t = 0$. The derivation of formulae (2.3-2.4) is provided in Appendix A.3. Let $\{\mathcal{F}_t^\theta\}$ and $\{\mathcal{F}_t^Z\}$ be the $\mathbb{P}$-augmentation of the natural filtrations generated by $\{\theta_t\}$ and $\{Z_t\}$, respectively. For each $t \in [0, T]$, define $\mathcal{G}_t := \mathcal{F}_t^\theta \vee \mathcal{F}_t^Z$ to be the smallest σ-algebra containing $\mathcal{F}_t^\theta$ and $\mathcal{F}_t^Z$.

The market described by a jump-diffusion process is in general[2] incomplete (see Appendix A.2 for the definition). Hence, we have to choose a risk-neutral probability measure $\mathbb{Q}$. To this end, we adopt the approach used by Elliot et al. [26]. Their choice, the regime switching generalized Esscher transform, is a generalization of the widely known Esscher transform, which is often used for the pricing of derivatives in incomplete markets.[3] More precisely: choose $A \in \mathcal{F}$ and put $\mathbb{Q}(A) = \int_A \phi_t(x) \mathbb{P}(dx)$, where

[1]Notice the difference with (1.3) where the term $-\lambda_Y \kappa V_t dt$ is missing. As we will see later in this chapter, including this term avoids unnecessary technicalities.

[2]There are examples of jump-type processes that describe a complete market. However to achieve this, one needs to make additional assumptions such as deterministic jump amplitude and a single jump process. For an example, see [22].

[3]The issue of choosing a risk-neutral probability measure is discussed in Appendix A.4.

the Radon-Nikodym derivative is given by

$$\phi_t = \exp\left(\int_0^t \frac{r_s - \mu_s + \lambda_Y \kappa}{\sigma_s} dW_s - \frac{1}{2}\int_0^t \left(\frac{r_s - \mu_s + \lambda_Y \kappa}{\sigma_s}\right)^2 ds\right).$$

Due to Girsanov's Theorem, the process $\tilde{W}_t = W_t + \int_0^t (\frac{r_s - \mu_s + \lambda_Y \kappa}{\sigma_s}) ds$ defines a standard Brownian motion with respect to $\mathcal{G}_t$ under $\mathbb{Q}$. Define $\tilde{\mathcal{G}}_{t,s} := \mathcal{F}_t^\theta \vee \mathcal{F}_s^Z$ for any $s, t \in [0,T]$ with $s \leq t$. Knowning that for $t \in [0,T], s \in [0,t]$ the discounted process $\{e^{\{-\int_0^t r_s ds\}} V_t\}$ is a martingale under $\mathbb{Q}$ with respect to $\{\tilde{\mathcal{G}}_{t,s}\}$, the dynamics of the firm's value under $\mathbb{Q}$ is now given by

$$dV_t = (r_t - \lambda_Y \kappa) V_t dt + \sigma_t V_t d\tilde{W}_t + (Y_t - 1) V_t dN_t \tag{2.5}$$

$$P_{\theta_{t+dt}|\theta_t}(e_j|e_i) = \lambda_{ij} dt + o(dt), \quad i \neq j. \tag{2.6}$$

Observe that the measure $\mathbb{Q}$ is chosen in such a way that the parameters of the Poisson and jump process remain the same. The difference between formulae (2.5-2.6) and those for the regime switching model found in [48], are the additional terms $(Y_t - 1)V_t dN_t - \lambda_Y \kappa V_t dt$. In this expression we recognize a compensated Poisson process, since $\mathbb{E}_{\mathbb{Q}}[(Y_t - 1)dN_t] = \kappa \mathbb{E}_{\mathbb{Q}}(dN_t) = \kappa \lambda_Y dt$. In the final equality we used the fact that in a small time interval dt, a jump occurs with probability $\lambda_Y dt$. Because the expectation of the additional increments together is zero, adding these terms do not spoil the martingale property.[4] Further, we should notice that the solution to (2.5-2.6) is obtained by substituting r_t for μ_t in (2.3-2.4).

Hence, at any time $t \in [0,T]$, the price of an European-style option on V with payoff $\Phi(V_T)$ at maturity T, is given by

$$\Phi(t,T,V_t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) \Phi(V_T) \middle| \tilde{\mathcal{G}}_{T,t,\theta_{t:T}}\right],$$

where $\tilde{\mathcal{G}}_{T,t,\theta_{t:T}}$ denotes the σ-algebra $\tilde{\mathcal{G}}_{T,t}$ that depends on $\theta_{t:T} := \{\theta_s, t \leq s \leq T\}$ which is a random path of the Markov process $\{\theta_t\}$ in the time interval $[t,T]$. For $i = 1, 2, \ldots, M$, we define $\eta_i(t,T) = \int_t^T < e_i, \theta_s > ds$ to be the amount of time state e_i has been occupied by θ during the interval $[t,T]$. Denote by $p_{\eta_{1:M}(t,T)}(s_{1:M})$ the joint conditional probability density of the occupation times $\eta_{1:M}(t,T) := (\eta_1(t,T), \eta_2(t,T), \ldots, \eta_M(t,T))$, where $s_{1:M} = (s_1, s_2, \ldots, s_M)$. Now the price of a derivative on V at time t with maturity T can be expressed as

$$\Phi(t,T,V_t) = \int_{\mathbb{R}^M} \exp\left(-\int_t^T r_s ds\right) \Phi(V_T) p_{\eta_{1:M}(t,T)}(s_{1:M}) ds_{1:M}.$$

[4] See Chapter 3, Section 3.1.3 of [52].

To summarize: In this chapter we have chosen a risk-neutral probability measure $\mathbb{Q}$. After that the RSJD model has been reformulated under this measure and a solution to the model has been derived. We proceed by considering the first application area: credit risk, and in particular pricing risky debt.

Chapter 3

Credit Risk: Pricing Risky Debt

As mentioned in Chapter 1, a part of the motivation for the RSJD model originates from credit risk. In this chapter we will focus on this subject and in particular on pricing risky debt. We apply the approach by Merton [57], who was the first one to systematically develop a pricing theory for corporate bonds when there is a significant probability of default. By a corporate bond, or simply a bond, we mean a formal contract to return borrowed money at a predetermined date (maturity) with interest (coupon) at fixed intervals. First, we give a short historical introduction. After that the problem is adapted to our model and we derive a formula for the value of the firm's debt. We prove Proposition 2 which will be used to derive an expression for the probabilities of default in Chapter 4, Section 4.4. At the end of this chapter, credit spread term structures produced by various models are discussed.

3.1 Historical Background

In his original paper [57], Merton made the following assumptions:

- There are no taxes, indivisibilities, bankruptcy costs, transaction costs or agency costs.
- Every individual acts as if he can buy or sell as much of any security as he wishes without affecting the market price.
- There exisits an exchange market for borrowing and lending at the same rate of interest r.
- Individuals may take short positions in any security, including the riskless asset, and receive the proceeds of the sale.
- Trading takes place continuously.
- The firm's value dynamics follows the diffusion process from (1.1).

We examine the simplest case of corporate debt pricing and hence we suppose that there exists a firm with a single liability which carries a promised final payoff L. The amount L must be paid to the bondholders (debtholders) on a predetermined date T. We assume that the firm can only default at maturity T. The firm defaults if, at maturity T, the total firm's value V_T is less than the promised payoff L. In this case the bondholders receive the amount V_T, i.e., they immediately take over the company (and the shareholders receive nothing). Otherwise, the liability L is fully repaid. In practice, it is very difficult to determine the point of default. An elaborate research on this subject has been done by Davydenko [23]. Based on this study, in Chapter 4 we choose the threshold level to be equal to $K = 0.8L$ instead of L.

The design described above can be seen as a defaultable claim on the total firm's value V with payoff at maturity T given by

$$D(T) = V_T \mathbf{1}_{\{V_T<L\}} + L\mathbf{1}_{\{V_T \geq L\}} = \min(V_T, L) = L - (L - V_T)^+.$$

One can interpret this as the difference of a default-free zero-coupon bond[1] with face value L and the value of an European put option written on V, with expiration date T and strike price equal to L. At any time $t \in [0, T]$, the firm's debt value can be expressed as

$$D(t,T) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[L - (L - V_T)^+ \middle| \mathcal{F}_t^d\right] = e^{-r(T-t)}L - P(t, T, V_t, L), \tag{3.1}$$

where $P(t, T, V_t, L)$ is the price of the put option at time t and $\mathcal{F}_t^d$ denotes the σ-algebra generated by the diffusion process (1.1) up to time t. For clarity, we will often write $P(t, T, V_t)$ instead of $P(t, T, V_t, L)$. (Similarly, if we want to suppress the dependence on V_t, we will write $P(t, T, L)$.) Since we made the assumption that the firm's value process follows the geometric Brownian motion from (1.1), the put (option) price is given by the Black–Scholes formula. Hence we get

$$P(t, T, V_t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[(L - V_T)^+ \middle| \mathcal{F}_t^d\right] = Le^{-r(T-t)}\mathcal{N}(-d_2) - V_t\mathcal{N}(-d_1), \tag{3.2}$$

where

$$d_1 = \frac{\ln \frac{V_t}{L} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Observe that $\mathcal{N}$ denotes the standard normal cumulative distribution function and since we work under the risk-neutral measure $\mathbb{Q}$, the term μ in (1.1) is replaced by r (an application of Girsanov's theorem).

[1] Zero-coupon bonds do not make periodic interest payments (the coupons). At maturity, the investor of the bond receives its face value.

Unfortunately there are two problems when one tries to implement Merton's approach for practical purposes. Namely, we see that equation (3.2) depends on the firm's value V and volatility σ which are unobservable. Hence, identification of these quantities is needed. It turns out that estimating the firm's value V is very difficult to do, especially for our model. Nevertheless, this is done in Chapter 4, Section 4.1 where equity prices are used which are observable for public firms. The value of σ is obtained by calibration (see Chapter 4).

This approach by Merton is later extended by, among others, Black and Cox [9] and Geske [34]. Black and Cox assumed that, at any time $t \in [0, T)$, the firm's bondholders have the right to force the firm to bankruptcy and take it over as soon as the firm's value drops below a certain time-dependent deterministic barrier $Le^{-\gamma(T-t)}$, for some γ. In other words, default may occur at any time. At time $t = T$, they assumed the original Merton's approach. Black and Cox successfully provided closed-form analytical expressions for their model. One year later, Geske [34] also developed an extension to the approach initiated by Merton. He allowed the simultaneous existence of multiple debt issues that can differ in maturity, coupon size and seniority. Despite of the increased complexity of the considered model, Geske derived closed-form formulae as well.

3.2 Extension of Merton's Approach

Let us focus on the original Merton's approach which we will adapt to our situation. Hence, instead of the last assumption from Section 3.1 we suppose that the firm's value process V follows the regime switching jump-duffision model (2.5-2.6) under the risk-neutral measure $\mathbb{Q}$. Recall from Chapter 2 that the market described by this model is incomplete. The measure $\mathbb{Q}$ is chosen using the regime switching generalized Esscher transform in the same way as we did in Chapter 2. In the sequel of this chapter we will solely work under this risk-neutral measure since we are concerned with pricing. We begin with a simple model where the continuous-time Markov process $\{\theta_t\}$ assumes only two values, i.e., $\theta_t \in \mathbb{M} = \{e_1, e_2\}$. By making this simplification we are able to derive analytical expressions. The initial state of the Markov process is always e_1 and it can switch to the absorbing state e_2 with a certain constant rate λ. Depending on the current state of the economy, the volatility $\{\sigma_t\}$ can change from its initial value σ_1 to σ_2, where $\sigma_2 > \sigma_1$ because the initial state of the economy is assumed to be "good". The same holds true for the drift rate $\{\mu_t\}$.

If we now try to compute the firm's debt value by taking a conditional expectation as in (3.1), we need to condition on the σ-algebra $\mathcal{G}_t$, since the firm's value at maturity V_T does not only depend on the evolution of the diffusion process but also on that of the two jump processes. Thus, for any $t \in [0,T]$, we have

$$D(t,T) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[L - (L - V_T)^+|\mathcal{G}_t] = e^{-r(T-t)}L - P(t,T,V_t). \tag{3.3}$$

In order to derive a closed-form expression for the put price $P(t,T,V_t)$, we adapt the Black-Scholes formula. This is done in the proof of Proposition 1 below.

Denote by $\tau := \inf\{t > 0 : \theta_t = e_2\}$ the first jump time of the Markov process $\{\theta_t\}$. In addition, since all expectations will be taken with respect to the risk-neutral measure $\mathbb{Q}$, it is convenient to write $\mathbb{E}$ for $\mathbb{E}_{\mathbb{Q}}$. In the next proposition we derive a formula for the firm's debt value.

Proposition 1. *On the set $\{\tau > t\}$, we have:*

$$\begin{aligned}
D(t,T) &= e^{-r(T-t)}L \\
&- \sum_{i=0}^{\infty}\Bigg(\int_0^{T-t}\Big(Le^{-r(T-t)}\mathcal{N}(d_1(T-t,s)) - V_0 e^{-\lambda_Y\kappa T + i\mu_Y + \frac{1}{2}i\sigma_Y^2}\mathcal{N}(d_2(T-t,s))\Big) \\
&\quad \times \lambda e^{-\lambda s} ds + e^{-\lambda(T-t)}\Big(Le^{-r(T-t)}\mathcal{N}(d_1(T-t,T-t)) - V_0 e^{-\lambda_Y\kappa T + i\mu_Y + \frac{1}{2}i\sigma_Y^2} \\
&\qquad \times \mathcal{N}(d_2(T-t,T-t))\Big)\Bigg)\frac{e^{-\lambda_Y(T-t)}(\lambda_Y(T-t))^i}{i!},
\end{aligned}$$

where for every $t \in [0,T]$

$$\begin{aligned}
d_1(T-t,s) &= \left(\sigma_1^2 s + \sigma_2^2(T-t-s) + i\sigma_Y^2\right)^{-1/2} \\
&\quad \times \left(\ln\frac{L}{V_0} - (r - \lambda_Y\kappa)(T-t) + \frac{1}{2}\left(\sigma_1^2 s + \sigma_2^2(T-t-s)\right) - i\mu_Y\right)
\end{aligned}$$

and

$$d_2(T-t,s) = d_1(T-t,s) - \sqrt{\sigma_1^2 s + \sigma_2^2(T-t-s) - i\sigma_Y^2}.$$

Proof. For the sake of notational convenience, we put $t = 0$. For $t \in (0,T]$, generalization follows through substituting T by $T-t$. If the process $\{\sigma_t\}$ is not depending on $\{\theta_t\}$ and deterministic, and if there is no jump in the model, the price of an European put at time $t \in [0,T]$ is given by

$$P(t,T,V_t) = e^{-r(T-t)}\mathbb{E}[(L - V_T)^+|\mathcal{G}_t] = Le^{-r(T-t)}\mathcal{N}(-d_2) - V_t\mathcal{N}(-d_1), \tag{3.4}$$

where

$$d_1 = \left(\int_t^T \sigma_u^2 du\right)^{-1/2} \left[\ln \frac{V_t}{L} + (r - \lambda_Y \kappa)(T-t) + \frac{1}{2}\int_t^T \sigma_u^2 du\right],$$

$$d_2 = d_1 - \left(\int_t^T \sigma_u^2 du\right)^{1/2}.$$

In order to determine the price of the put option in our case, we take twice an expectation: once over N_T and the second time over $\eta_1(0,T)$. Thus

$$\begin{aligned}
\mathbb{E}[(L-V_T)^+|\mathcal{G}_t] &= \mathbb{E}[\mathbb{E}[(L-V_T)^+|\mathcal{G}_t, N_T]|\mathcal{G}_t] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{E}[(L-V_T)^+|\mathcal{G}_t, N_T, \eta_1(0,T)]|\mathcal{G}_t, N_T]|\mathcal{G}_t] \\
&= \sum_{i=0}^{\infty} \mathbb{E}[\mathbb{E}[(L-V_T)^+|\mathcal{G}_t, N_T = i, \eta_1(0,T)]|\mathcal{G}_t, N_T = i] \frac{e^{-\lambda_Y T}(\lambda_Y T)^i}{i!} \\
&= \sum_{i=0}^{\infty} \mathbb{E}[\mathbb{E}[(L-V_T)^+|\mathcal{G}_t, N_T = i, \eta_1(0,T)]|\mathcal{G}_t] \frac{e^{-\lambda_Y T}(\lambda_Y T)^i}{i!} \\
&= \sum_{i=0}^{\infty} \Bigg(\int_0^T \mathbb{E}[(L-V_T)^+|\mathcal{G}_t, N_T = i, \eta_1(0,T) = s]\lambda e^{-\lambda s} ds \\
&+ \mathbb{E}[(L-V_T)^+|\mathcal{G}_t, N_T = i, \eta_1(0,T) = T] e^{-\lambda T} \Bigg) \frac{e^{-\lambda_Y T}(\lambda_Y T)^i}{i!}, \qquad (3.5)
\end{aligned}$$

where the last equality is obtained by an application of the the probability density function of $\eta_1(0,T)$, that is given by $p_{\eta_1(0,T)}(s) = e^{-\lambda s}\left(\lambda \mathbf{1}_{[0,T)}(s) + \delta(T-s)\right)$. A derivation of this formula is provided in [48]. Based on (2.3-2.4), observe the following[2][3]

$$\begin{aligned}
\ln(V_T/V_0) &= \int_0^T \left(r - \lambda_Y \kappa - \frac{1}{2}\sigma_s^2\right) ds + \int_0^T \sigma_s dW_s + \sum_{i=1}^{N_T} Y_i \\
&\stackrel{d}{=} (r - \lambda_Y \kappa)T - \frac{1}{2}\left(\sigma_1^2 \eta_1(0,T) + \sigma_2^2(T - \eta_1(0,T))\right) \\
&+ \sqrt{\sigma_1^2 \eta_1(0,T) + \sigma_2^2(T - \eta_1(0,T))} W_1 + \sum_{i=1}^{N_T} \ln Y_i,
\end{aligned}$$

where the property $W_T \stackrel{d}{=} \sqrt{T} W_1$ is used in the final step. For $i \in \mathbb{N}$ and $s \in (0,T]$, write $\hat{\mu}_{i,s} = (r - \lambda_Y \kappa)T - \frac{1}{2}(\sigma_1^2 s + \sigma_2^2(T-s)) + i\mu_Y$ and $\hat{\sigma}_{i,s}^2 = \sigma_1^2 s + \sigma_2^2(T-s) + i\sigma_Y^2$.

[2]The reader should be aware of the previously made assumption that the market interest rate is constant and the fact that here we work under the risk-neutral measure $\mathbb{Q}$.

[3]The notation "$\stackrel{d}{=}$" means "equal in distribution".

Then

$$
\begin{aligned}
&\mathbb{E}[(L-V_T)^+|\mathcal{G}_t, N_T = i, \eta_1(0,T) = s] \\
&= \int_{\mathbb{R}} \left(L - V_0 e^{\hat{\mu}_{i,s} + \hat{\sigma}_{i,s} x}\right)^+ \frac{1}{\sqrt{2\pi \hat{\sigma}_{i,s}^2}} e^{-\frac{x^2}{2}} \, dx \\
&= \int_{-\infty}^{y} \left(L - V_0 e^{\hat{\mu}_{i,s} + \hat{\sigma}_{i,s} x}\right) \frac{1}{\sqrt{2\pi \hat{\sigma}_{i,s}^2}} e^{-\frac{x^2}{2}} \, dx, \quad \text{where } y = \frac{\ln \frac{L}{V_0} - \hat{\mu}_{i,s}}{\hat{\sigma}_{i,s}} \\
&= L \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi \hat{\sigma}_{i,s}^2}} e^{-\frac{x^2}{2}} \, dx - V_0 e^{\hat{\mu}_{i,s}} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi \hat{\sigma}_{i,s}^2}} e^{\hat{\sigma}_{i,s} x - \frac{x^2}{2}} \, dx \\
&= L \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi \hat{\sigma}_{i,s}^2}} e^{-\frac{x^2}{2}} \, dx - V_0 e^{\hat{\mu}_{i,s} + \frac{1}{2}\hat{\sigma}_{i,s}^2} \int_{-\infty}^{y - \hat{\sigma}_{i,s}} \frac{1}{\sqrt{2\pi \hat{\sigma}_{i,s}^2}} e^{-\frac{x^2}{2}} \, dx \\
&= L \mathcal{N}(d_1(T,s)) - V_0 e^{(r - \lambda_Y \kappa)T + i\mu_Y + \frac{1}{2} i \sigma_Y^2} \mathcal{N}(d_2(T,s)),
\end{aligned}
\tag{3.6}
$$

where $d_1(T,s)$ and $d_2(T,s)$ are as stated in the proposition. A combination of (3.3-3.6) yields the desired result. $\square$

Next we compute the probability that, at time $t \in [0,T]$, the firm's value V_T is less than a certain threshold level L. Later on, we will need this expression for the calculation of the probabilities of default in Chapter 4, Section 4.4.

Proposition 2. *On the set $\{\tau > t\}$, we have:*

$$
\begin{aligned}
&\mathbb{Q}_{RSJD}(V_T < L|\mathcal{G}_t) \\
&\quad = \sum_{i=0}^{\infty} \left(\int_0^{T-t} \mathcal{N}(d_1(T-t,s)) \lambda e^{-\lambda s} ds + e^{-\lambda(T-t)} \mathcal{N}(d_1(T-t, T-t)) \right) \\
&\qquad\qquad\qquad\qquad \times \frac{e^{-\lambda_Y (T-t)} (\lambda_Y (T-t))^i}{i!},
\end{aligned}
$$

with d_1 as defined in Proposition 1.

Proof. One observes

$$
\mathbb{Q}_{RSJD}(V_T < L|\mathcal{G}_t) = \mathbb{E}[\mathbf{1}_{V_T<L}|\mathcal{G}_t] = \mathbb{E}[\mathbb{E}[\mathbb{E}[\mathbf{1}_{V_T<L}|\mathcal{G}_t, N_T, \eta_1(t,T)]|\mathcal{G}_t, N_T]|\mathcal{G}_t],
$$

from which the result follows in virtue of the proof of Proposition 1. $\square$

3.3 Credit Spread

In this section we present the credit spread term structure for two different bonds predicted by the RSJD model and we compare them to those produced by the JDM and RSM model.

Let $B(t,T)$ denote the risk-free bond at time t with maturity T. The credit spread $CS(t,T)$ is defined as the difference between a defaultable yield-to-maturity and the default-free yield-to-maturity. Hence we have

$$CS(t,T) = -\frac{\ln D(t,T)}{T-t} - -\frac{\ln B(t,T)}{T-t} = -\frac{1}{T-t}\ln\left(\frac{D(t,T)}{B(t,T)}\right), \quad T > t.$$

Observe that we are considering bonds that pay the amount L at maturity. Moreover, we made the assumption that the risk-free interest rate r is constant, i.e., $B(t,T) = Le^{-r(T-t)}$. Thus, the above expression can be simplified to

$$CS(t,T) = -\frac{1}{T-t}\ln\left(\frac{D(t,T)}{L}\right) - r, \quad T > t.$$

The credit spread shape depends on the specific bond one considers. We will choose two types of bonds and we will compare their credit spread term structures according to the JDM, RSM and RSJD model. This is done in Figures 3.1-3.2 below.

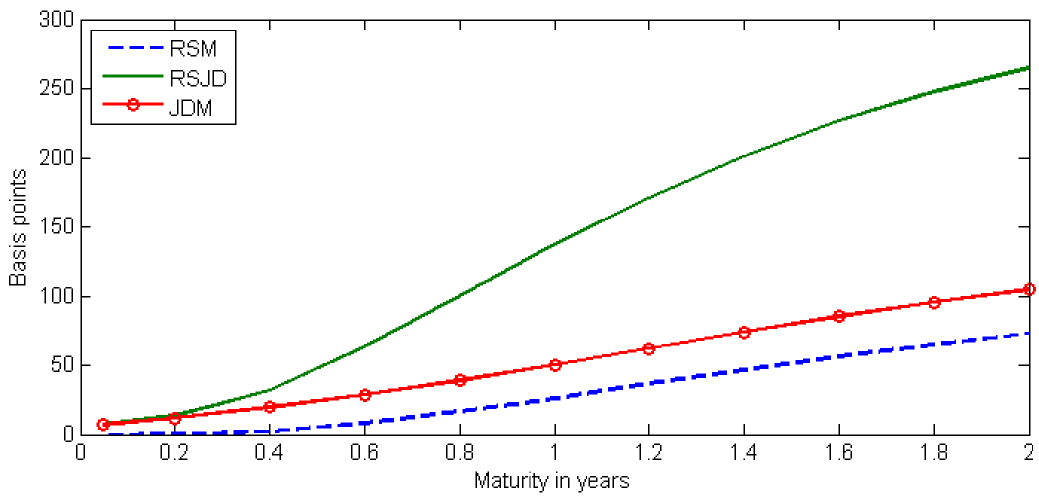

Figure 3.1: Credit spread term structure of short maturity risky bond. We used the following parameter values: $V_0 = 150$, $L = 100$, (Leverage ratio: 0.67), $r = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\lambda = 1$, $\mu_Y = 0.1$, $\sigma_Y = 0.2$ and $\lambda_Y = 1.5$.

We see that on short maturities, for instance, a few weeks, the credit spreads produced by our model are even higher than those according to the RSM and JDM model. Hence, the RSJD model definitely does not underestimate the credit spreads. In fact, the model is

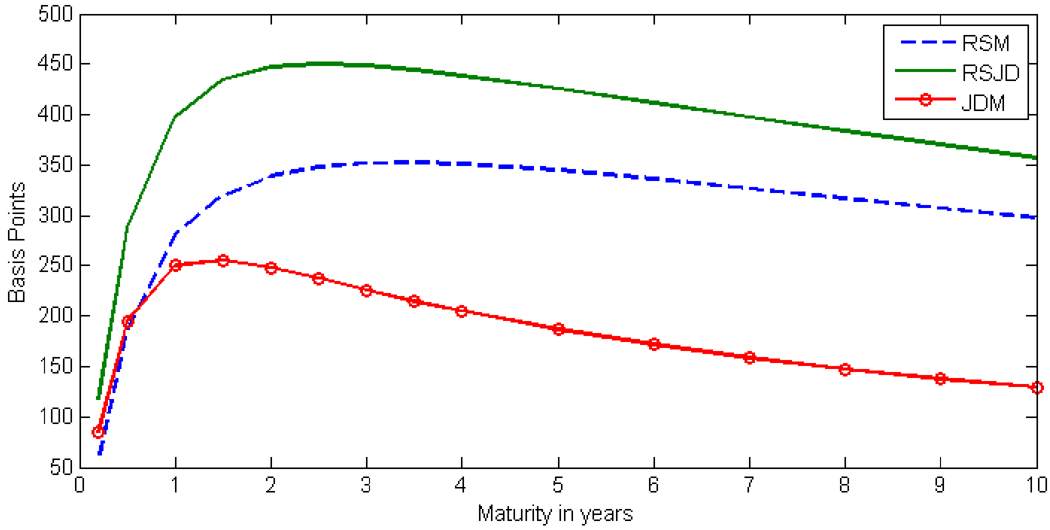

Figure 3.2: Credit spread term structure of low credit rating corporate bond. The following parameter values are used: $V_0 = 120$, $L = 100$, (Leverage ratio: 0.83), $r = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\lambda = 0.5$, $\mu_Y = 0.1$, $\sigma_Y = 0.2$ and $\lambda_Y = 0.5$.

more likely to fit empirical data better. On long maturities we see that the credit spreads produced by the RSJD are much higher than those according to the RSM and/or JDM. This is not an overestimation but a consequence of the chosen parameter set. Notice that from Figure 3.1 one also sees that the RSM model indeed underestimates the credit spreads on short maturities as mentioned previously in Chapter 1. Finally, we note that Zhou [72, 73] already showed that the JDM model is able to generate various shapes of credit spread observed in the market. Since the JDM is a special case of the RSJD model, this property remains preserved.

Remark: Corporate bond spreads turn out to be valuable predictors for real activity, in particular at forecast horizons beyond one year (Buchmann [13]). Moreover, bond spreads are also used to compute risk-neutral probabilities of default (Anginer and Yildizhan [3]). (Risk-neutral default probabilities can be used for the valuation of credit derivatives such as CDSs (Hull [39]).)

To summarize: We have studied the first application of our model: credit risk. In particular we have discussed the approach by Merton on the subject of pricing risky debt. This approach has been adapted to the RSJD model. A formula for the firm's debt value has been derived. In the final section, credit spreads produced by various models have been plotted. Based on these results, we came to the conclusion that the credit spreads according to our model are indeed adequate. This coincides with our motivation (and expectation) provided in Chapter 1. In the next chapter we will look at a number of calibration methods for our model and the calculation of the PDs.

Chapter 4

Calibration and PDs

We present several calibration methodologies for estimating the parameters of our model. The more complicated the models become, the more effort we have to make in order to estimate the parameters. As mentioned by Jacquier and Jarrow [42]: "the estimation method becomes as crucial as the model itself"! The set of parameters one needs to find is denoted by $\mathcal{P}$, i.e.,

$$\mathcal{P} = \{\mu_1, \mu_2, \sigma_1, \sigma_2, \lambda, \mu_Y, \sigma_Y, \lambda_Y, L, K, T\},$$

where K represents the threshold level, that is, if at maturity T the firm's value V_T is less than K, we say that the firm has defaulted. The last three parameters can be estimated from the balance sheet. This choice can be quite rough depending on the complexity of the structure of the firm one considers. Throughout this thesis, we make the assumption that K equals 80% of the face value of the debt, in other words 80% of the liability L in our case (see begin of Chapter 3). But the real difficulty lies in the estimation of the remaining eight parameters. MLE, EM algorithm and inverse problem are the methods we will discuss for estimating these parameters. These methods are most commonly used for models of our kind. (In Chapter 6 we will mention two other techniques.) After that we will compute the PDs using expressions from Section 4.1. At the end of this chapter, we will present some results based on real data.

4.1 MLE

Probably the most intuitive way to calibrate our model is to use the firm's values. Alas, these are not directly observable. However, if we assume that the equity price process follows the same Markov process as the firm's value, deriving a likelihood function is feasible since the equity prices are observable for public firms. Thus suppose we observed $M+1$ equity prices E_i at time t_i, $i = 0, \ldots, M$. The likelihood function for the log equity is given by

$$\mathcal{L}^E(\mathcal{P}) = \prod_{i=1}^{M} f(\ln E_i | \ln E_{i-1}, \ldots, \ln E_0, \mathcal{P}),$$

with f the density of the log equity price and $\mathcal{P}$ as defined before, where the parameters L, K, T are assumed to be known. Rewrite this likelihood function as follows

$$\begin{aligned}\mathcal{L}^E(\mathcal{P}) &= \prod_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{N} f(\ln E_i, \theta_i = j, \theta_{i-1} = k | \ln E_{i-1}, \ldots, \ln E_0, \mathcal{P}) \\ &= \prod_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{N} P(\theta_{i-1} = k | \ln E_{i-1}, \ldots, \ln E_0, \mathcal{P}) P(\theta_i = j | \theta_{i-1} = k, \mathcal{P}) \\ &\qquad\qquad \times f(\ln E_i | \theta_i = j, \ln E_{i-1}, \mathcal{P}),\end{aligned}$$

where N represents the number of possible states of the Markov process $\{\theta_t\}$ and $\theta_i = j$ is just shorter notation for $\theta_i = e_j$. Further, $P(\theta_i = j|\theta_{i-1} = k, \mathcal{P})$ denotes the probability to go from state k at time $i-1$ to state j at time i. Given a realization ρ of $\mathcal{P}$, this probability can be immediately calculated if the time between the observed prices is known. Now fix $i \in \{1, \ldots, M\}$ and note that $f(\ln E_i|\theta_i = j, \ln E_{i-1}, \mathcal{P})$ can be expressed in terms of the density of the firm's value g, i.e.,

$$\begin{aligned} f(\ln E_i|\theta_i = j, \ln E_{i-1}, \mathcal{P}) &= g(\ln V_i|\theta_i = j, \ln V_{i-1}, \mathcal{P}) \left|\frac{\partial \ln E}{\partial \ln V}\right|^{-1}_{t=t_i} \\ &= g(\ln V_i|\theta_i = j, \ln V_{i-1}, \mathcal{P}) \frac{E_i}{V_i} \left|\frac{\partial h}{\partial V}\right|^{-1}_{t=t_i}, \end{aligned}$$

where h is the equity pricing function, $E_i = h(t_i, V_i, K_i, T_i; \mathcal{P}, \theta_i)$, and
$V_i = h^{-1}(t_i, V_i, K_i, T_i; \mathcal{P}, \theta_i)$. Since the equity price must be an increasing function of the firm's value, $\frac{\partial h}{\partial V} > 0$ holds true. What remains is to compute the conditional probability $P(\theta_{i-1} = k| \ln E_{i-1}, \ldots, \ln E_0, \mathcal{P})$:

$$\begin{aligned} &P(\theta_{i-1} = k| \ln E_{i-1}, \ldots, \ln E_0, \mathcal{P}) \\ &\qquad = \frac{\sum_{l=1}^{N} f(\ln E_{i-1}, \theta_{i-1} = k, \theta_{i-2} = l| \ln E_{i-2}, \ldots, \ln E_0, \mathcal{P})}{f(\ln E_{i-1}| \ln E_{i-2}, \ldots, \ln E_0, \mathcal{P})}. \end{aligned} \tag{4.1}$$

Notice that

$$\begin{aligned} &f(\ln E_{i-1}| \ln E_{i-2}, \ldots, \ln E_0, \mathcal{P}) \\ &\qquad = \sum_{k=1}^{N}\sum_{l=1}^{N} f(\ln E_{i-1}, \theta_{i-1} = k, \theta_{i-2} = l| \ln E_{i-2}, \ldots, \ln E_0, \mathcal{P}) \end{aligned} \tag{4.2}$$

and

$$
\begin{aligned}
f(\ln E_{i-1}, \theta_{i-1} = k, \theta_{i-2} = l | \ln E_{i-2}, \dots, \ln E_0, \mathcal{P}) \\
= P(\theta_{i-2} = l | \ln E_{i-2}, \dots, \ln E_0, \mathcal{P}) P(\theta_{i-1} = k | \theta_{i-2} = l, \mathcal{P}) \\
\times f(\ln E_{i-1} | \theta_{i-1} = k, \ln E_{i-2}, \mathcal{P}). \quad (4.3)
\end{aligned}
$$

From (4.1-4.3), one concludes that $P(\theta_{i-1} = k | \ln E_{i-1}, \dots, \ln E_0, \mathcal{P})$ is a function of $P(\theta_{i-2} = l | \ln E_{i-2}, \dots, \ln E_0, \mathcal{P})$. This implies for $i \in \{2, \dots, M\}$ that $P(\theta_{i-1} = k | \ln E_{i-1}, \dots, \ln E_0, \mathcal{P})$ can be recursively computed if the values $P(\theta_0 = l | \ln E_0, \mathcal{P})$ are given for all $l \in \{1, \dots, N\}$. Hence, if we specify the initial state of the Markov process $\{\theta_t\}$ and if we determine the equity pricing function h then the likelihood function $\mathcal{L}^E(\mathcal{P})$ can be found. As a result, the MLE of $\mathcal{P}$ is obtained by (numerically) maximizing $\mathcal{L}^E(\mathcal{P})$.

We now go back to the two-state model and suppose that equity is observed at the end of each time interval of length Δt. So, we have $P(\theta_0 = 1 | \ln E_0, \mathcal{P}) = 1$ and $P(\theta_0 = 2 | \ln E_0, \mathcal{P}) = 0$ as the Markov process starts in state e_1. The transition probabilities are given by $P(\theta_i = e_1 | \theta_{i-1} = e_1, \mathcal{P}) = (1 - \lambda \Delta t), P(\theta_i = e_2 | \theta_{i-1} = e_1, \mathcal{P}) = \lambda \Delta t, P(\theta_i = e_1 | \theta_{i-1} = e_2, \mathcal{P}) = 0$ and $P(\theta_i = e_2 | \theta_{i-1} = e_2, \mathcal{P}) = 1$. Let $\{\theta_t\}$ be constant on an interval of length Δt. This simplification should not introduce a big error since Δt is usually small, e.g., in this thesis it will be 1 day. In an interval Δt, the density of the firm's value is now given by the density of a geometric jump-diffusion process, i.e.,[1]

$$
\begin{aligned}
g(\ln V_i | \theta_i = j, \ln V_{i-1}, \mathcal{P}) \\
= \sum_{k=0}^{\infty} \frac{e^{-\lambda_Y \Delta t} (\lambda_Y \Delta t)^k}{k!} \mathcal{N} \left(\frac{\ln \frac{V_i}{V_{i-1}} - \left(\left(\mu_j - \lambda_Y \kappa - \frac{\sigma_j^2}{2} \right) \Delta t + k \mu_Y \right)}{\sqrt{\sigma_j^2 \Delta t + k \sigma_Y^2}} \right),
\end{aligned}
$$

for any $i \in \{1, \dots, M\}$. The equity pricing function $h(t_i, V_i, K_i, T_i; \theta_i, \mathcal{P})$ is obtained from the balance sheet since $E_i = V_i - D_i$, where $D_i = D(t_i, T_i, V_i, K_i; \theta_i, \mathcal{P})$. The equation $V = D + E$ or in words: *Assets = Liabilities + Owners' equity* is called the fundamental accounting equation or the balance sheet equation and it is a well-known concept from finance (Parrino and Kidwell [62]). If $\theta_i = e_1$, i.e., on the set $\{\tau > t\}$, D_i is found by Proposition 1. On the other hand, if $\theta_i = e_2$ we can either put $\sigma_1 = \sigma_2$ in Proposition 1 or use the formula from [58]. It is important to observe that the MLE is carried out under the real-world probability measure $\mathbb{P}$ since we are not dealing with pricing of derivatives (see Chapter 1). The maximization procedure can now be executed.

[1]This formula can be found in [72].

Remark: There are some issues about the MLE which are discussed in Appendix A.5.

4.2 EM Algorithm

Computing the MLE directly may be very difficult or even impossible. After all, more sophisticated techniques have been developed to make the MLE applicable. One such elaborate technique is the expectation−maximization (EM) algorithm. This algorithm is mostly used in the following two cases: the data set is incomplete or the data set contains missing values. The first situation occurs when the process one considers is not continuously observable. The second one occurs when the computation of the likelihood function can be simplified if we assume the existence of some additional, but missing or latent, parameters (Bilmes [7]).

Next, we present the EM algorithm. Suppose we have a data set of M observations $\mathcal{R} = (R(1), R(2), \ldots, R(M))$ with a density $p(R|\rho)$. These data vectors are assumed to be i.i.d. and the unknown vector ρ, from the space $\mathcal{P}$, parametrizes the specific distribution of the observations. We call the set $\mathcal{R}$ the *incomplete data set*. Further, we need the existence of a *complete data set* $\mathcal{X}$ such that $\mathcal{X} = (\mathcal{R}, \mathcal{S})$, where $\mathcal{S}$ denotes the additional data, referred to as the unobservable data. Given the incomplete data set $\mathcal{R}$ and a parameter vector ρ, $q(\mathcal{S}|\mathcal{R}, \rho)$ is the conditional density of the unobservable data $\mathcal{S}$. Applying the Bayes' rule gives us

$$q(\mathcal{R}|\rho) = \frac{q(\mathcal{R}, \mathcal{S}|\rho)}{q(\mathcal{S}|\mathcal{R}, \rho)},$$

where $q(\mathcal{R}, \mathcal{S}|\rho) = q(\mathcal{X}|\rho)$ represents the conditional joint density of $(\mathcal{R}, \mathcal{S})$ given ρ. If we take the natural logarithm on both sides, we obtain

$$\ln q(\mathcal{R}|\rho) = \ln q(\mathcal{R}, \mathcal{S}|\rho) - \ln q(\mathcal{S}|\mathcal{R}, \rho). \tag{4.4}$$

On the LHS, we recognize the log-likelihood of the parameter vector ρ given the observations $\mathcal{R}$, i.e., $\ln q(\mathcal{R}|\rho) = \sum_{i=1}^{M} \ln q(R_i|\rho) = \ln \mathcal{L}(\rho|\mathcal{R})$. So maximizing $\ln \mathcal{L}(\rho|\mathcal{R})$ is equivalent to maximizing the RHS of (4.4). Unfortunately, this is impossible since $\mathcal{S}$ is unknown. Therefore we are forced to take conditional expectations. The condition in the expectation would be the density of the unobservable data $\mathcal{S}$ given some specific parameter vector ρ_0 drawn from $\mathcal{P}$. It is convenient to denote this condition simply by ρ_0. Hence

$$\begin{aligned} \ln q(\mathcal{R}|\rho) &= \mathbb{E}[\ln q(\mathcal{R}, \mathcal{S}|\rho)|\rho_0] - \mathbb{E}[\ln q(\mathcal{S}|\mathcal{R}, \rho)|\rho_0] \\ &\equiv \mathbb{E}_0[\ln q(\mathcal{R}, \mathcal{S}|\rho)] - \mathbb{E}_0[\ln q(\mathcal{S}|\mathcal{R}, \rho)] \\ &\equiv Q(\rho, \rho_0) - H(\rho, \rho_0). \end{aligned} \tag{4.5}$$

Let us first maximize $Q(\rho, \rho_0)$ over ρ. We call the maximizer ρ_1. Trivially $Q(\rho_1, \rho_0) \geq Q(\rho_0, \rho_0)$, but it turns out that $H(\rho_1, \rho_0) \leq H(\rho_0, \rho_0)$ holds true as well. Hence, instead of miximizing the difference in (4.5) we merely need to find the ρ that maximizes $Q(\rho, \rho_0)$. Now, if we substitute ρ_1 for ρ_0 and iterate the process above, $\ln q(\mathcal{R}|\rho)$ will actually monotonically converge to a stationary solution. For more technical details and proofs about the EM algorithm, we recommend reading Chapter 3 in [56].

Continuous obsevations of the firm's assets value V_t are not very usual, if ever, available. Thus discretizing the process $\{V_t\}$ seems to be a logical step to do. Indeed, for $n \in \{1, 2, \ldots, T\}$ we define $R(n)$ to be the continouously compounded return of V_t over the unit time interval $(n-1, n]$, i.e.,

$$R(n) = \ln V_n - \ln V_{n-1}.$$

If time is measured in days, the $R(n)$ are just the daily returns. We assume that the jumps of the Markov process $\{\theta_t\}$ only occur at the end of a time interval. Put $J := \ln Y$, then from (2.3-2.4) we see that for $n \in \{1, 2, \ldots, T\}$

$$R(n) = Z(n) + \sum_{k=1}^{N(n)} J_k(n), \tag{4.6}$$

where $Z(n)$ are independent normal random variables with mean $\nu := \left(\mu_1 - \lambda_Y \kappa - \frac{1}{2}\sigma_1^2\right) + \frac{\lambda}{T}\left(\mu_2 - \mu_1 + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)\right)$ and variance $\tau^2 := \sigma_1^2 + \frac{\lambda^2}{T^2}\left(\sigma_1^2 + \sigma_2^2\right)$. The $N(n) = N_n - N_{n-1}$ are independent Poisson random variables with mean λ_Y giving the number of events in the intervals $(n-1, n]$. And, the $J_k(n)$ represent the $N(n)$ jumps that occur in $(n-1, n]$ and they are independent normal random variable with mean μ_Y and variance σ_Y. All these processes are mutually independent and the sum in (4.6) is by convention equal to zero if $N(n) = 0$. In the EM algorithm, the set of these returns will form the incomplete data set, i.e., $\mathcal{R} = (R(1), R(2), \ldots, R(T))$. Now given the $N(n)$, the returns $R(n)$ are independent normal random variables with mean $\nu + \mu_Y N(n)$ and variance $\tau^2 + \sigma_Y^2 N(n)$. This implies that the $R(n)$ are i.i.d. with distribution function

$$F(x) = \sum_{k=0}^{\infty} e^{-\lambda_Y} \frac{\lambda_Y^k}{k!} \mathcal{N}\left(\frac{x - \nu_k}{\tau_k}\right), \quad \text{for } x \in \mathbb{R},$$

where $\nu_k = \mu_Z + k\mu_Y$ and $\tau_k^2 = \sigma_Z^2 + k\sigma_Y^2$. Here, $\mathcal{N}$ represents the standard normal distribution function.

Now, applying the approach by Duncan et al. [25], for $n \in \{1, 2, \ldots, T\}$ the complete data set $\mathcal{X} = (X(1), X(2), \ldots, X(T))$ is defined as

$$X(n) = \begin{cases} (Z(n), N(n)) & \text{if } N(n) = 0 \\ (Z(n), N(n), J_1(n), J_2(n), \ldots, J_{N(n)}(n)) & \text{if } N(n) > 0. \end{cases}$$

Note that in virtue of (4.6), $\mathcal{X}$ completely determines the incomplete data set $\mathcal{R}$. In order to find $Q(\rho, \rho_0)$, we first determine the complete log-likelihood $\ln q(\mathcal{R}, \mathcal{S}|\rho) = \ln q(\mathcal{X}|\rho)$:

$$\begin{aligned} \ln q(\mathcal{X}|\rho) = \sum_{i=1}^{T} \ln q(X(i)|\rho) &= -T \ln \sqrt{2\pi} - \frac{T}{2} \ln \tau^2 - \frac{1}{2\tau^2} \sum_{i=1}^{T} (Z(i) - \nu)^2 \\ &- T\lambda_Y + \ln(\lambda_Y) \sum_{i=1}^{T} N(i) - \sum_{i=1}^{T} \ln N(i)! - \ln \sqrt{2\pi} \sum_{i=1}^{T} N(i) \\ &- \frac{\ln \sigma_Y^2}{2} \sum_{i=1}^{T} N(i) - \frac{1}{2\sigma_Y^2} \sum_{i=1}^{T} \sum_{k=1}^{N(i)} (J_k(i) - \mu_Y)^2, \end{aligned}$$

where again the last term is interpreted as zero if $N(i) = 0$. Independence is several times used in the above derivation. Given the observations $\mathcal{R}$, we maximize $Q(\rho|\rho_0)$ with respect to the elements of $\mathcal{P}$, i.e., take the corresponding derivative and set it equal to zero. We tried to solve this system of equations in Mathematica, but unfortunately we did not succeed because of the large amount of parameters; even putting $r = \mu_1 = \mu_2$ does not remedy our problem. To get an idea of the complexity of the computations, the derivative with respect to λ is partially presented in Appendix A.6.

4.3 Inverse Problem

Another way of calibrating our model is the so-called inverse problem: it is the inverse of the (option) pricing problem. The basic idea is to find the model parameters in order to match the market prices of a certain dataset (of options). But it can happen that several parameter sets are equally consistent with the market prices, which makes the problem ill-posed (Hamida and Cont [35]). Since the problem is not well-defined, it does not make sense to try to exactly match the parameters to the market data and so the calibration problem becomes an optimization problem. More precisely: the aim is to minimize the difference between the market prices and the model prices. Now, assume we have observed M option prices, then a possible formulation of the problem is the following:

$$\min_{\mathcal{P}} \sum_{i=1}^{M} w_i (P^{model}(t, T_i, V_t) - P^{market}(t, T_i, V_t))^2, \tag{4.7}$$

where $P^{model}(t, T_i, V_t)$ and $P^{market}(t, T_i, V_t)$ denote the i^{th} put option prices from the model and market, respectively. The strictly positive quantities w_i are weights which we will choose equal to $|\text{ask}_i - \text{bid}_i|^{-1}$, where ask_i (bid_i) corresponds to the ask (bid) price of the i^{th} option. This choice is intuitively clear since if the spread is bigger for a given observation, then the range of the consistent prices for the model is wider which means that we should put less weight on this observation. The objective function in (4.7) could alternatively be defined as $\min_{\mathcal{P}} \sum_{i=1}^{M} \frac{|P^{model}(t,T_i,V_t) - P^{market}(t,T_i,V_t)|}{P^{market}(t,T_i,V_t)}$ or one may prefer absolute values instead of using squares.

The market price in (4.7) is chosen to be the average of the bid and ask price (call it the *mid price*), but we will also accept a parameter set such that

$$\sum_{i=1}^{M} w_i(P^{model}(t,T_i,V_t) - P^{market}(t,T_i,V_t))^2 \leq \sum_{i=1}^{M} w_i(\text{bid}_i - \text{ask}_i)^2.$$

In other words: we do not insist that the model price exactly correspond to the mid price, but, on average, to fall within the bid-ask spread (Moodley [60]). One way to implement this constraint into the objective function is by adding the term

$$\max\left(\sum_{i=1}^{M} w_i(P^{model}(t,T_i,V_t) - P^{market}(t,T_i,V_t))^2 - \sum_{i=1}^{M} w_i(\text{bid}_i - \text{ask}_i)^2, 0\right)$$

to the objective function in (4.7). Such optimization problems are usually done numerically by a gradient-based technique (Andersen and Andreasen [2]). However, in our situation this approach may not succeed since the objective function is not convex and does not have any particular structure. So we need to consider other optimization methods. Unfortunately, a lot of these methods will struggle to find a *global* optimum. Some of the most popular global optimization algorithms are Branch and Bound, Clustering, Evolutionary algorithms (e.g. Genetic algorithm), Simulated annealing and Tabu search (Venkataraman [68]). In this thesis we will use the simulated annealing (SA) algorithm. Its main advantages are: (1) It can be used for arbitrary problems. (2) In general, the algorithm provides a "good" solution. (3) Statistically, SA guarantees an optimal solution. The primary criticism of the algorithm is that it is too slow. One way to improve its speed performance is presented in [40].

Remark 1: In contrast with MLE, this calibration method is executed under the risk-neutral probability measure $\mathbb{Q}$, since in our framework option prices are computed under the risk-neutral measure.

Remark 2: As mentioned before, the inverse problem is ill-posed. (There may be no solution at all or an infinite number of solutions.) Often one tries to remedy this ill-posedness by means of *regularization strategies*. The idea behind it is to add a convex *penalization criterion* to the objective function in order to make the problem well-defined.

In this thesis, further investigation on regularization strategies is out of scope. The interested reader is referred to [28] and Chapter 6.
Another possibility might be to add more market price information. However, we must be aware of the fact that adding too much data can also have a negative impact on the results of the inverse problem (see Engl [27]).

4.4 Computing Probabilities of Default

Once the model is calibrated, we are able to compute the 1-year real-world default probability for a given firm, that is, the probability that the firm will default over exactly one year. It is important to notice that the default probabilities in this context are *real-world* default probabilities. This is because we are not concerned with pricing of derivatives (see Chapter 1). Hence, we are working under the real-world measure $\mathbb{P}$. To compute these probabilities we need to find the conditional probabilities in (4.1). For simplicity we put $f_j(i) := f(\ln E_i|\theta_i = j, \ln E_{i-1}, \mathcal{P})$ for each $i \in \{1, \dots, M\}$ and denote by $f_j^S(i)$ the same expression but only if the switch has occured at time t_{i-1}. Furthermore, for each $i \in \{2, \dots, M\}$ and $k \in \{1, 2\}$, we write $P(\theta_{i-1} = k|\cdots)$ instead of $P(\theta_{i-1} = k|\ln E_{i-1}, \dots, \ln E_0, \mathcal{P})$. Now, according to (4.1-4.3) one gets

$$P(\theta_{i-1} = 1|\cdots) = \frac{P(\theta_{i-2} = 1|\cdots)(1-\lambda dt)f_1(i-1)}{P(\theta_{i-2} = 1|\cdots)(1-\lambda dt)f_1(i-1) + P(\theta_{i-2} = 1|\cdots)\lambda dt f_2^S(i-1) + P(\theta_{i-2} = 2|\cdots)f_2(i-1)}$$

and

$$P(\theta_{i-1} = 2|\cdots) = \frac{P(\theta_{i-2} = 1|\cdots)\lambda dt f_2^S(i-1) + P(\theta_{i-2} = 2|\cdots)f_2(i-1)}{P(\theta_{i-2} = 1|\cdots)(1-\lambda dt)f_1(i-1) + P(\theta_{i-2} = 1|\cdots)\lambda dt f_2^S(i-1) + P(\theta_{i-2} = 2|\cdots)f_2(i-1)},$$

for each $i \in \{2, \dots, M\}$. If $i = 1$ one has $P(\theta_0 = 1|\ln E_0, \mathcal{P}) = 1$ and $P(\theta_0 = 2|\ln E_0, \mathcal{P}) = 0$ by construction. At time t_i, we choose the threshold level K_i to be equal to 80% of the current liability L_i, where we use linear interpolation between data points if needed. Since we are looking for the 1-year default probability we always take $T_i = 1 + t_i$. Observe that these assumptions are very rough; we need more data about the debt structure of the firm in order to choose the parameters more properly. Hence, at time t_i, the real-world probability of default PD_i is given by

$$PD_i = P(\theta_i = 1|\cdots) \cdot \mathbb{P}_{RSJD}(V_{T_i} < K_i|\mathcal{G}_{t_i}) + P(\theta_i = 2|\cdots) \cdot \mathbb{P}_{JDM}(V_{T_i} < K_i|\mathcal{G}_{t_i}),$$

where

$$\mathbb{P}_{JDM}(V_T < K|\mathcal{G}_{t_i}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda_Y T}(\lambda_Y T)^k}{k!} \mathcal{N}\left(\frac{\ln \frac{K}{V_T} - \left(\left(\mu_j - \lambda_Y \kappa - \frac{\sigma_j^2}{2} \right) T + k\mu_Y \right)}{\sqrt{\sigma_j^2 T + k\sigma_Y^2}} \right)$$

if $\theta_i = j$ and $\mathbb{P}_{RSJD}(V_T < K|\mathcal{G}_{t_i})$ as given by Proposition 2 where r is replaced by μ_j since we are working under the real-world measure $\mathbb{P}$. The formula for $\mathbb{P}_{JDM}(V_T < K|\mathcal{G}_{t_i})$ is derived in [72]. We must mention that this kind of methodology is also used by credit rating agencies such as Moody's (Bohn [11]). However, the determination of a credit rating does not only require the real-world PD's; it is a combination of (estimated) PD's and additional economically and statistically relevant factors.

From a mathematical point of view we would expect that the risk-neutral probabilities are obtained by substituting r for μ_j in the above expressions. But in reality, the relation between these two world is more complicated (Berg [5]). Risk-neutral default probabilities are used for the valuation of credit derivatives such as CDSs (Hull [39]).

4.5 Implementation and Results

In this section we focus on the implementation of the calibration methodologies from the previous sections and the computation of the PDs. The MLE technique will be applied to real data. After that we will test the performance of the inverse problem using option prices that are generated according to a known parameter set. As a final result, using the parameter values obtained by the MLE, we will compute the probabilities of default.

4.5.1 MLE

First, we will apply the MLE to the real firm Netflix, Inc. that has been in financial distress recently. This is easily seen from Figure 4.1 below (source: Yahoo! Finance). We choose the interest rate r to be 5%, the length $\Delta t = \frac{1}{252}$ ($\approx$ 1 trading day) and the maturity $T = 1$ year. The equity value at time i is equal to the stock price at time i multiplied by the number of shares outstanding at time i. We also need the threshold level $K = 0.8L$. For the liability L we choose the "total liability" of the firm. Both, the number of shares outstanding and the liability, are quarterly observed as shown in Table 4.1. For the missing values we use linear interpolation. We have found the following values for the parameters using the MLE:

$$\mu_1 = -0.0642, \quad \mu_2 = -0.0115, \quad \sigma_1 = 0.0001, \quad \sigma_2 = 0.3507,$$
$$\lambda = 9.0760, \quad \mu_Y = -0.0226, \quad \sigma_Y = 0.0487, \quad \lambda_Y = 54.0118.$$

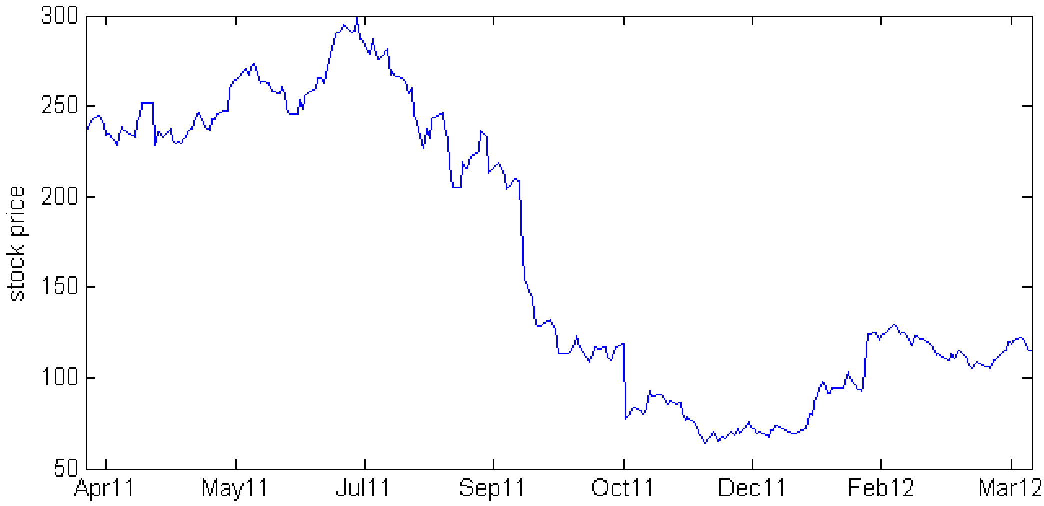

Figure 4.1: The daily observed stock prices of Netflix, Inc. for the period 31-03-11 till 31-03-12.

date	number of shares outstanding	total liability
31 March 2011	52.5200e06	814.5e06
30 June 2011	52.5400e06	1236.7e06
30 September 2011	52.5000e06	1573.2e06
31 December 2011	55.4000e06	2426.4e06
31 March 2012	55.5200e06	2817.1e06

Table 4.1: The total liability and the number of shares outstanding as observed from Bloomberg. The currency is US Dollar.

4.5.2 Inverse Problem

Now we will test the performance of the inverse problem using option prices that are generated according to a known parameter set. In virtue of Proposition 1 and equation (3.3), the price of an European put option is known. Hence, if we assign values to the unknown parameters in the formula, we can generate put prices from it. And then we can optimize the objective function in (4.7) by choosing all weights equal to 1 and identifying the market prices to the generated prices. We generated 11 prices with strikes ranging from 95 to 105 and maturities varying from 1 day to 1 year.

As mentioned in Section 4.3 we will apply the SA algorithm. Therefore we use the function `simulannealbnd` in MATLAB and as hybrid function we choose `fmincon` which will refine the solution found by SA. The obtained results are presented in Table 4.2-4.3.

We see that the algorithm performs better when the Poisson process is absent, which seems reasonable since in this case there is less randomness in the model. The presence of a Poisson process does disturb the situation a bit, but its impact is not enormous since

parameter	σ_1	σ_2	λ	μ_Y	σ_Y	λ_Y
actual value	0.1	0.3	1	-	-	0
estimated value	0.1	0.2926	1.0047	1.4103	1	0.0005

Table 4.2: Parameter estimation according to SA for λ_Y=0. The corresponding minimum of (4.7) equals 1.3e-05.

parameter	σ_1	σ_2	λ	μ_Y	σ_Y	λ_Y
actual value	0.1	0.3	1	0.1	0.2	0.25
estimated value	0.0820	0.3201	1.0475	0.0222	0.0611	2.1834

Table 4.3: Parameter estimation according to SA for λ_Y=0.25. The estimated minimum of the objective function is equal to 1.1e-04.

the estimated minimum is equal to 1.1e-04, which is still quite low. However, finding the exact parameters values seems to be more difficult. Adding more data[2] and/or a penalization criterion will probably improve the results.

Observe that the drift parameters (μ_1 and μ_2) do not appear in the option pricing formula and hence we cannot calibrate them using the inverse problem. These terms are usually estimated statistically (see McDonald and Sandal [55] and Choi [15]).

4.5.3 1-Year Real-World PDs

Using the parameter values found in Section 4.5.1, we are able to compute the real-world probabilities of default. We are using the assumptions and expressions from Section 4.4 to calculate the PDs over the same time period as in Section 4.5.1, that is, 31-03-2011 till 31-03-2012. The result is presented in Figure 4.2 below. One observes that this picture is consistent with the data from Figure 4.1 and Table 4.1 since the probability of default becomes significantly larger at the point where the stock price declines and the total liability increases. At the highest point, this probability approximately equals 4%. We should notice that allowing the default to occur at any time before maturity (hence, applying the approach by Black and Cox from Chapter 3, Section 3.1) will (probably) increase the PD.

To summarize: In this chapter we have considered various calibration techniques. The MLE and inverse problem have been adapted to our setting and have been implemented. The inverse problem has been tested using option prices that are generated according to a known parameter set. The results were sufficient. Furthermore, we have applied the MLE technique to Netflix, Inc., a firm that has been in financial distress recently. Using the parameter values proposed by the MLE, the 1-year real-world probabilities of default have been computed. We saw that at the highest point, the probability that

[2]But of course, not *too much* data, see Remark, Section 4.3.

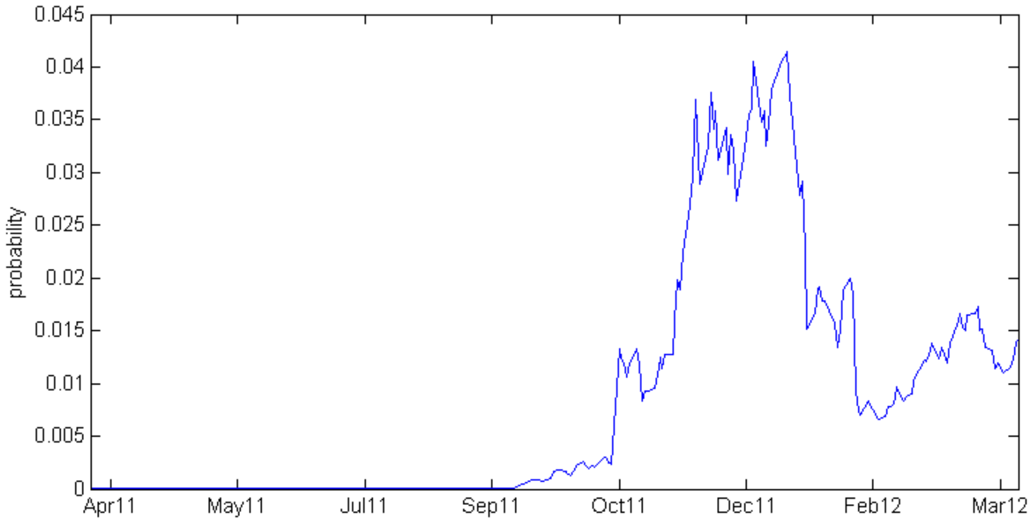

Figure 4.2: 1-year real-world probabilities of default for Netflix, Inc..

Netflix, Inc. will default over exactly one year is approximately 4%, which is significant. Next we will discuss the second main application of the model: option pricing.

Chapter 5

Option Pricing

In this chapter we will focus on the second part of our motivation from Chapter 1: option pricing. For the RSJD model, we have already derived an analytical expression for the price of an European-style put option in the proof of Proposition 1 in Chapter 3. First we will use this formula to obtain volatility smiles for different models. Beside the analytical method for option pricing, we will present a second option pricing technique that is often used for more complex models (e.g. stochastic volatility models) when direct computations are not possible.[1] This technique, known as fast Fourier transform (FFT), is based on the so-called Fourier transforms (FTs). Some background information on these transforms is provided in Appendix A.7. We discuss the implementation of the FFT and at the end of this chapter we compare the two option pricing methodologies. This chapter is about option pricing and hence we will solely work under the risk-neutral probability measure $\mathbb{Q}$.

5.1 Volatility Smile

In this section we present volatility smiles produced by several models for short and long maturities. Recall from Chapter 1 that the volatility smile is a plot of the implied volatility of an option as a function of its strike price. And the implied volatility is the volatility that, when used in a given pricing model, provides a theoretical option price equal to the current market price of the option.

Hence, in order to compute the implied volatility, we need to know how to price options. We will consider call options only. According to equation (3.3) the price of a put option at time t with maturity T can be obtained by

$$P(t,T,L) = e^{-r(T-t)}L - D(t,T),$$

[1]Another option pricing technique is discussed in Chapter 6. We choose the FFT because of its speed advantage and the fact that it has become a standard method in this area.

where $D(t,T)$ is given by Proposition 1 in Chapter 3 and L represents the strike price. Once we know the put price, we easily derive the call price by means of the put-call parity that is given by

$$C(t,T,L) - P(t,T,L) = V_t - L \cdot B(t,T),$$

where $B(t,T)$ is the value of the risk-free bond at time t with maturity T. Since we assumed the interest rate to be constant, we have $B(t,T) = e^{-r(T-t)}$. Therefore

$$C(t,T,L) = V_t - D(t,T), \tag{5.1}$$

which we can compute. Notice that since the JDM and RSM model are special cases of the RSJD model, we are now also able to calculate call option prices for these two models. Once we have generated a call option price, the corresponding implied volatility is computed by finding the volatility for which the Black-Scholes formula[2] provides the same option price. In Figure 5.1 the implied volatility has been computed according to the RSM, RSJD and JDM model for a short maturity (1 month). The same is done

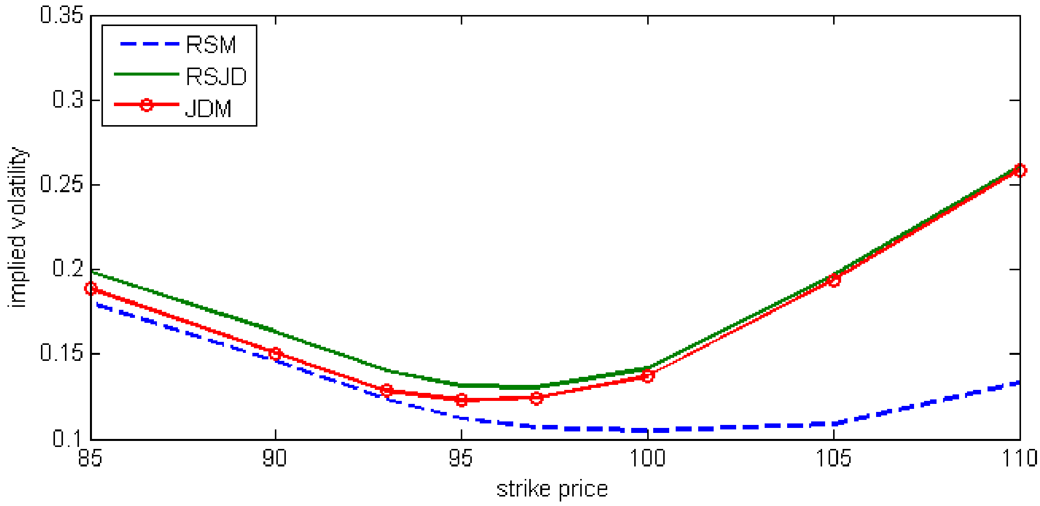

Figure 5.1: Volatility smiles produced by the RSM, RSJD and JDM model for a short maturity (1 month). The remaining parameters are: $V_0 = 100$, $r = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\lambda = 0.5$, $\mu_Y = 0.1$, $\sigma_Y = 0.1$, $\lambda_Y = 1$.

for an 1-year maturity in Figure 5.2. We see that for a short maturity, the JDM and RSJD model are both able to capture the smile by producing a steep volatility smile. The volatility smile produced by the RSM model is more flat. For a long maturity, the RSM and RSJD model perform better than the JDM model (Figure 5.2). In this case, the smile produced by the RSM model is steeper than the one according to the RSJD model. However, since the RSM model is a special case of the RSJD model, our

[2] See Chapter 3, Section 3.1.

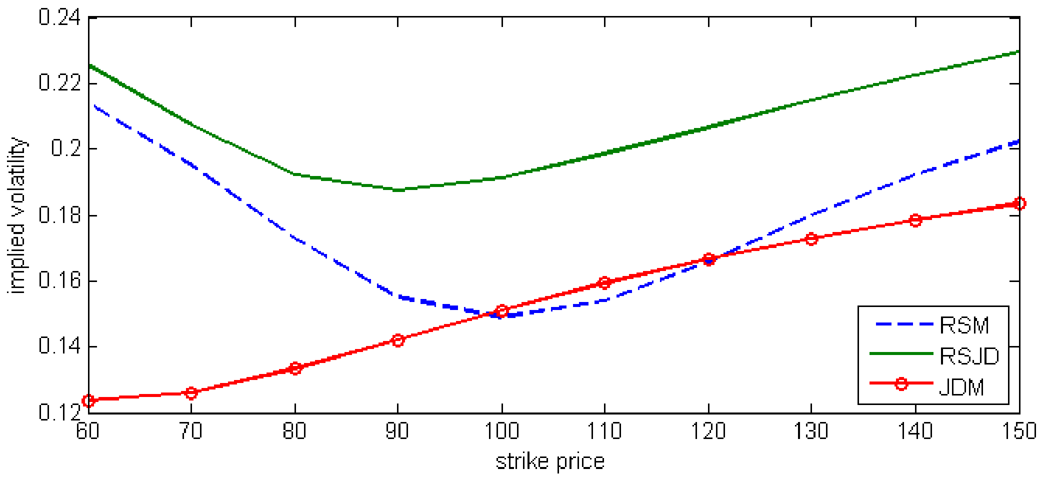

Figure 5.2: Volatility smiles produced by the RSM, RSJD and JDM model for a long maturity (1 year). The remaining parameters are: $V_0 = 100$, $r = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\lambda = 0.5$, $\mu_Y = 0.1$, $\sigma_Y = 0.1$, $\lambda_Y = 1$.

model is also capable to capture such a steep smile. Furthermore from Figure 5.2 we can see that the JDM model is not able to capture the famous U-shape. Instead, the JDM model produces an upward sloping smile which is much less common in reality than the U-shaped volatility smile. Next we will consider another way of option pricing. We recommend the reader to read Appendix A.7 before further proceeding.

Remark: By definition, the BS model produces a flat volatility smile. We know that the BS model is a special case of the RSM, RSJD and JDM model. Hence, the smile according to one of these three models can be made flatter by choosing the parameters properly.

5.2 The Fourier Transform of an Option Price

Several complex pricing models do not have closed-form log-return densities which makes the derivation of the (option) pricing formula, provided in the proof of Proposition 1 in Chapter 3, impossible. One way to get around this is by using FTs. These transforms have the nice feature that for some classes of models (for instance, general exponential Lévy models) the characteristic functions (CFs) are known in closed-form. Albeit our model is not a pure exponential Lévy model due to the Markov process $\{\theta_t\}$ (see Chapter 2), it turns out that computing the CF in this setting is still possible.

Throughout this section the price of a call option at time t, with strike L and maturity T, will be denoted by $C(t, T, L)$. Observe that for notational convenience we suppress the dependence on the firm's value V_t. Without loss of generality we may assume $t = 0$

and we will write $C(T, L)$ instead of $C(0, T, L)$. Now, if $l = \ln(L)$, we have

$$C(T, l) = e^{-rT}\mathbb{E}[(V_T - L)^+] = e^{-rT}\mathbb{E}\left[\left(e^{Z_T} - e^l\right)^+\right],$$

where $Z_T := \ln V_T$.[3] One can observe that $C(T, l)$ does not tend to 0 as l goes to $-\infty$. Therefore the pricing function is not square integrable and hence we cannot directly take the Fourier transform of it.[4] Carr and Madan [14] modified the call option price in order to make it square integrable in l over $\mathbb{R}$. They defined

$$\hat{C}(T, l) := e^{\alpha l} C(T, l),$$

where $\alpha > 0$ represents a damping factor. The choice of α is discussed in Appendix A.8. Once we have a FT for the modified pricing formula, say $\psi_T(u)$, the call option price is easily obtained using an inverse FT of $\psi_T(u)$:

$$\hat{C}(T, l) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iul}\psi_T(u)du$$

whence the price follows, i.e.,

$$C(T, l) = \frac{e^{-\alpha l}}{2\pi}\int_{-\infty}^{\infty} e^{-iul}\psi_T(u)du = \frac{e^{-\alpha l}}{\pi}\int_0^{\infty} \mathrm{Re}\left(e^{-iul}\psi_T(u)\right)du, \tag{5.2}$$

where the second equality follows from the fact that $C(T, l)$ must be a real number, which implies that the imaginary part of the function $\psi_T(u)$ is odd and its real part is even.

In the next proposition we provide a FT for the price of a call option. Recall from Chapter 3 that τ denotes the first jump time of the Markov process $\{\theta_t\}$.

Proposition 3. *On the set $\{\tau > 0\}$, the price of a call option with strike L and maturity T is given by*

$$\begin{aligned} C(T, l) = \frac{e^{-(\alpha l + rT)}}{\pi}\int_0^{\infty} \frac{e^{-iul + U_T(u)}}{\alpha^2 + \alpha - u^2 + i(1+2\alpha)u}&\Bigg[\int_0^T \exp\Bigg\{i(u - i(1+\alpha)) \\ \times\left(Z_0 + (r - \lambda_Y\kappa)T - \frac{1}{2}I(T, s)\right) &- \frac{1}{2}(u - i(1+\alpha))^2 I(T, s)\Bigg\}\lambda e^{-\lambda s}ds \\ + \exp\left\{i(u - i(1+\alpha))\left(Z_0 + (r - \lambda_Y\kappa)T - \frac{1}{2}I(T, T)\right) - \frac{1}{2}(u - i(1+\alpha))^2 I(T, T)\right\}& \\ \times e^{-\lambda T}&\Bigg]du, \end{aligned}$$

[3] Notice the difference with (2.3) where $Z_t := \ln(V_t/V_0)$.
[4] See Chapter 8, Section 8.2 of [54].

where for any u, $U_T(u) = T\lambda_Y \left(\exp\left\{i(u-i(1+\alpha))\mu_Y - \frac{\sigma_Y^2(u-i(1+\alpha))^2}{2}\right\} - 1\right)$,

for every $s \in (0,T]$, $I(T,s) = \sigma_1^2 s + \sigma_2^2(T-s)$ *and* $Z_0 := \ln V_0$.

Proof. In order to derive this formula, we first compute $\psi_T(u)$ in (5.2). Suppose $f_{\mathcal{F}_T^\theta}$ is the conditional risk-neutral density of Z_T given $\mathcal{F}_T^\theta$, then:

$$\begin{aligned}
\psi_T(u) &= \int_{-\infty}^{\infty} e^{iul}\hat{C}(T,l)dl \\
&= \int_{-\infty}^{\infty} e^{iul}e^{\alpha l}e^{-rT}\mathbb{E}\left[\left(e^{Z_T}-e^l\right)^+\right]dl \\
&= e^{-rT}\mathbb{E}\left[\int_{-\infty}^{\infty} e^{iul}e^{\alpha l}\mathbb{E}\left[\left(e^{Z_T}-e^l\right)^+\middle|\mathcal{F}_T^\theta\right]dl\right] \\
&= e^{-rT}\mathbb{E}\left[\int_{-\infty}^{\infty} e^{iul}e^{\alpha l}\int_l^{\infty}\left(e^x-e^l\right)f_{\mathcal{F}_T^\theta}(x)dxdl\right] \\
&= e^{-rT}\mathbb{E}\left[\int_{-\infty}^{\infty} f_{\mathcal{F}_T^\theta}(x)\int_{-\infty}^{x}(e^x e^{(\alpha+iu)l} - e^{(1+\alpha+iu)l})dldx\right] \\
&= e^{-rT}\mathbb{E}\left[\int_{-\infty}^{\infty} f_{\mathcal{F}_T^\theta}(x)\left(\frac{e^{(1+\alpha+iu)x}}{\alpha+iu} - \frac{e^{(1+\alpha+iu)x}}{1+\alpha+iu}\right)dx\right] \\
&= e^{-rT}\mathbb{E}\left[\frac{\phi_{\mathcal{F}_T^\theta}(u-i(1+\alpha))}{\alpha+iu} - \frac{\phi_{\mathcal{F}_T^\theta}(u-i(1+\alpha))}{1+\alpha+iu}\right] \\
&= \frac{e^{-rT}\mathbb{E}\left[\phi_{\mathcal{F}_T^\theta}(u-i(1+\alpha))\right]}{\alpha^2+\alpha-u^2+i(1+2\alpha)u},
\end{aligned} \tag{5.3}$$

where

$$\phi_{\mathcal{F}_T^\theta}(u) = \mathbb{E}\left[e^{iuZ_T}\middle|\mathcal{F}_T^\theta\right] = \int_{-\infty}^{\infty} e^{iux}f_{\mathcal{F}_T^\theta}(x)dx$$

denotes the characteristic function of Z_T conditioned on $\mathcal{F}_T^\theta$. Determining $\phi_{\mathcal{F}_T^\theta}$ can be achieved in at least two different way: by involving the Lévy-Khinchine representation or by direct computation. Here we present the second approach.
Decompose Z_T as follows:

$$Z_T = Z_T^{(1)} + Z_T^{(2)},$$

$$\text{where } Z_T^{(1)} = \ln V_0 + \int_0^T\left(r - \lambda_Y\kappa - \frac{1}{2}\sigma_s^2\right)ds + \int_0^T \sigma_s dW_s \quad \text{and } Z_T^{(2)} = \sum_{i=1}^{N_T}\ln Y_i.$$

(This decomposition easily follows from the fact that $Z_T := \ln V_T$ and equations (2.3-2.4).) Observe that $Z_T^{(1)}$ and $Z_T^{(2)}$ are independent by construction. Hence,

$$\phi_{\mathcal{F}_T^\theta}(u) = \mathbb{E}\left[e^{iuZ_T}\middle|\mathcal{F}_T^\theta\right] = \mathbb{E}\left[e^{iuZ_T^{(1)}}\middle|\mathcal{F}_T^\theta\right]\cdot\mathbb{E}\left[e^{iuZ_T^{(2)}}\right].$$

The last factor represents the characteristic function of a compound Poisson process, which is well-known:[5]

$$\mathbb{E}\left[e^{iuZ_T^{(2)}}\right] = \exp\left\{T\lambda_Y \int_{-\infty}^{\infty} (e^{iux} - 1) f(dx)\right\},$$

where f denotes the jump size distribution. In our case f is the normal density, since $\ln(Y_t) \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ for each t, i.e.,

$$\mathbb{E}\left[e^{iuZ_T^{(2)}}\right] = \exp\left\{T\lambda_Y \left(e^{iu\mu_Y - \frac{\sigma_Y^2 u^2}{2}} - 1\right)\right\}. \tag{5.4}$$

Note that given $\mathcal{F}_T^\theta$, $Z_T^{(1)}$ is normally distributed with mean $\ln V_0 + (r - \lambda_Y \kappa)T - \frac{1}{2}\int_0^T \sigma_s^2 ds$ and variance $\int_0^T \sigma_s^2 ds$. It follows

$$\mathbb{E}\left[e^{iuZ_T^{(2)}} \middle| \mathcal{F}_T^\theta\right] = \exp\left\{iu\left(Z_0 + (r - \lambda_Y\kappa)T - \frac{1}{2}\int_0^T \sigma_s^2 ds\right) - \frac{1}{2}u^2 \int_0^T \sigma_s^2 ds\right\}, \tag{5.5}$$

where the characteristic function of the normal distribution is used and $Z_0 := \ln V_0$. Combining (5.4) and (5.5) gives us $\phi_{\mathcal{F}_T^\theta}(u)$. In order to obtain the FT $\psi_T(u)$, the only thing left to do is to take the expectation over the CF $\phi_{\mathcal{F}_T^\theta}(u)$ in (5.3). This is done by conditioning on the occupation time $\eta_1(0,T)$ that has a known density derived in [48], namely $p_{\eta_1(0,T)}(s) = e^{-\lambda s}\left(\lambda \mathbf{1}_{[0,T)}(s) + \delta(T - s)\right)$, where δ stands for the Dirac delta function. Finally, an application of formula (5.2) yields the desired option price. $\square$

5.3 The Fourier Transform of Out-Of-The-Money Option Prices

In the previous section we multiplied the call option value by an exponential function in order to make the FT possible. However, for very short maturities the call value tends to the non-differentiable terminal option payoff causing high oscillations in the Fourier inversion. This implies that the numerical integration becomes very hard.[6] The numerical errors are significant and discussed in Carr and Madan [14]. They also proposed the following alternative approach. The basic idea is the same: the call option value must be modified. The only difference is that in this case the modification involves a hyperbolic sine function instead of an exponential function.

Without loss of generality we may assume $V_0 = 1$ and define by

$$CP(T,l) = e^{-rT}\mathbb{E}\left[\left(e^l - e^{Z_T}\right)^+ \mathbf{1}_{l<0} + \left(e^{Z_T} - e^l\right)^+ \mathbf{1}_{l>0}\right] \tag{5.6}$$

[5] See Proposition 3.4 of [19].
[6] The "numerical integration" is discussed in Section 5.4.

the value of an out-of-the-money option, i.e., if $l < 0$ we have the T maturity put price for $CP(T, l)$ and for $l > 0$ we have the T maturity call price.

Let $\zeta_T(u)$ denote the Fourier transform of $CP(T, l)$, i.e.,

$$\zeta_T(u) = \int_{-\infty}^{\infty} e^{iul} CP(T, l) dl.$$

Inverting this transform gives us the price of the out-of-the-money option, that is,

$$CP(T, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iul} \zeta_T(u) du.$$

Considering the price function $CP_T(l)$, we see that it has a maximum at $l = \ln V_0 = \ln 1 = 0$ and declines in both directions as l goes to $\pm\infty$. Furthermore, as Carr and Madan had shown, the value at $l = 0$ can get quite steep as T tends to 0. Therefore it is useful to consider the transform of $\sinh(\alpha l) CP(T, l)$ since this function vanishes at $l = 0$. Call the FT $\gamma_T(u)$ and observe

$$\begin{aligned} \gamma_T(u) &= \int_{-\infty}^{\infty} e^{iul} \sinh(\alpha l) CP(T, l) dl = \int_{-\infty}^{\infty} e^{iul} \frac{e^{\alpha l} - e^{-\alpha l}}{2} CP(T, l) dl \\ &= \frac{\zeta_T(v - i\alpha) - \zeta_T(v + i\alpha)}{2}. \end{aligned}$$

The value of an option is now obtained by

$$CP(T, l) = \frac{1}{\pi \sinh(\alpha l)} \int_{0}^{\infty} e^{-iul} \gamma_T(u) du.$$

Notice that the parameter α serves as a control parameter for the steepness of the integrand around zero. The only thing left to do is to compute ζ_T. This calculation is provided in Appendix A.9.

5.4 FFT Implementation

In this section we provide an implementation of the previously found FTs by means of a fast Fourier transform (FFT). This implementation is also based on Carr and Madan [14]. The FFT is an efficient algorithm for computing summations of the form

$$\sum_{j=1}^{M} e^{-\frac{2\pi i}{M}(j-1)(k-1)} x(j), \quad \text{for} \quad k = 1, \ldots, M, \tag{5.7}$$

where M is the length of the vector x. This algorithm is significantly faster than direct computations since it reduces the number of arithmetic operations from $\mathcal{O}(M^2)$ to $\mathcal{O}(M \log M)$ if M is chosen to be some power of 2 (Kwok et al.[49]).

We apply the FFT to the FT from Section 5.2. So we need to rewrite the integral in (5.2) as a sum similar to (5.7). First, employ the composite trapezoidal rule to obtain a discrete approximation for this integral, i.e.,

$$
\begin{aligned}
C(T,l) &\approx \frac{e^{-\alpha l}}{\pi}\frac{\eta}{2}\left(\psi_T(0) + 2\sum_{j=2}^{M-1} e^{-iu_j l}\psi_T(u_j) + e^{-iu_M l}\psi_T(u_M)\right) \\
&\approx \frac{e^{-\alpha l}}{\pi}\sum_{j=1}^{M} e^{-iu_j l}\psi_T(u_j)\eta, \qquad (5.8)
\end{aligned}
$$

where $u_j = \eta(j-1)$ for some positive grid spacing η. From $\frac{M\eta - 0}{M} = \eta$ one concludes that the semi-infinite integration domain $[0,\infty)$ is approximated by a finite integration domain where $M\eta$ is its upper limit.

In this context, we would like to compute near-the-money call option prices with l taking discrete values ranging from $-a$ to a for some real number a, i.e.,

$$l_v = -a + h(v-1), \quad \text{for } v = 1, \ldots, M,$$

where h is the regular spacing size and a satisfies the equation $a = \frac{Mh}{2}$. Substituting l_v for l in (5.8) yields

$$
\begin{aligned}
C(T,l_v) &\approx \frac{e^{-\alpha l_v}}{\pi}\sum_{j=1}^{M} e^{-iu_j(-a+h(v-1))}\psi_T(u_j)\eta \\
&= \frac{e^{-\alpha l_v}}{\pi}\sum_{j=1}^{M} e^{-ih\eta(j-1)(v-1)}e^{ibu_j}\psi_T(u_j)\eta, \quad \text{for } v = 1, \ldots, M.
\end{aligned}
$$

When comparing this formula and the one in (5.7), one sees that applying the FFT algorithm requires

$$h\eta = \frac{2\pi}{M} \qquad (5.9)$$

to be true. Hence we take this as a definition. Due to this restriction, it is obvious that we are able to choose only two out of the three parameters (h, η and M) freely, since the third one will be determined by (5.9). The choice of the parameters is provided in Appendix A.8. For accurate integral approximation it is necessary to properly select η and M. This might lead to a not sufficiently dense grid spacing for the log strikes l_v, with option prices that are computed for log strikes that are too large or too small, i.e., it is possible that only few strikes will fall within the preferred area near V_0. To make the integration more precise even for large values of η we incorporate the composite

Simpson's rule into the summation to finally obtain

$$C(T, l_v) = \frac{e^{-\alpha l_v}}{\pi} \sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(v-1)} e^{ibu_j} \psi_T(u_j) \frac{\eta}{3} (3 + (-1)^j - \delta_{j-1} - \delta_{N-j}), \qquad (5.10)$$

where the Kronecker delta function δ_n is defined to be unity for $n = 0$ and zero otherwise. The FFT implementation of an out-of-the-money option is similar to the one in (5.10) where the factor $e^{-\alpha l_v}$ is replaced by $(\sinh(\alpha l))^{-1}$ and the function $\psi_T(u)$ is replaced by $\gamma_T(u)$.

Remark: A generalized version of the FFT algorithm, the FRFT algorithm, has been developed in Chourdakis [16]. Its main advantage is that the two grid spacings η and h can be chosen independently which in some situations improves the running time of the FFT.

5.5 Results

Now we compare the option prices obtained by the analytical expression and the FFT.

Figure 5.3 shows the call option prices provided by the two techniques for different strike prices. For $\lambda_Y = 2$ the biggest percent deviation is approximately 0.79% and the average

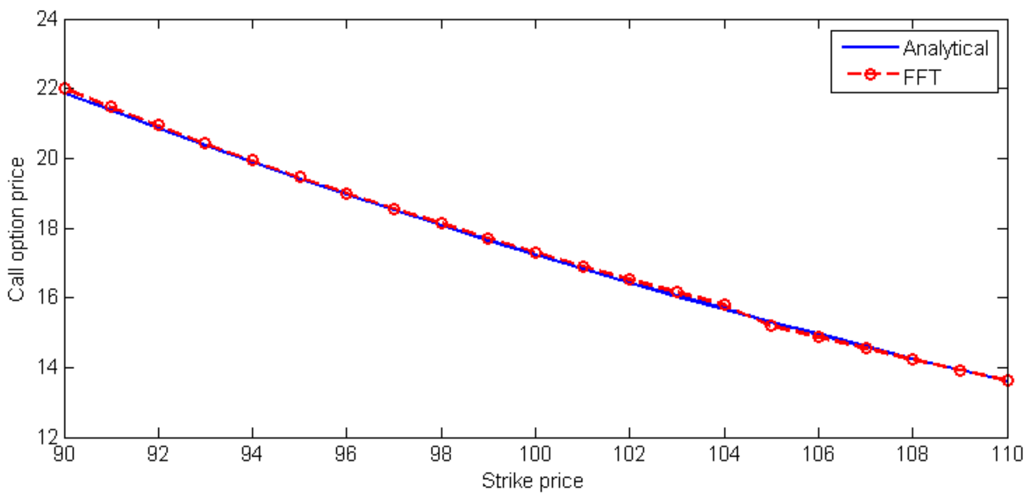

Figure 5.3: Call option prices for various strike prices. The maturity is chosen to be equal to 1 year and the remaining parameters are: $V_0 = 100$, $r = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\lambda = 1$, $\mu_Y = 0.1$, $\sigma_Y = 0.2$, $\lambda_Y = 2$.

deviation equals 0.39%. If we set $\lambda_Y = 1$, we get 0.40% and 0.21%, respectively. Hence, the accuracy depends on the value of λ_Y. The MATLAB code in provided in Appendix A.10. The performance of the FFT for out-of-the-money options from Section 5.3 (FFT adjusted) is given in Table 5.1-5.2. We see that for very short maturities, i.e., less than 1 day the FFT performs very badly while the FFT adjusted provides a good approximation of the option price. Also in this case, the quality of the approximation depends on λ_Y.

To summarize: We have provided some volatility smiles for various (pricing) models. The volatility smiles produced by the RSJD for short maturities as well as for long maturities were satisfactory. After that we have considered an option pricing technique called FFT. The FFT algorithm was successfully implemented and we saw that this algorithm provides a good approximation for the price of an European option. Then the FFT has been adjusted in order to perform well for even very short maturities. In the context of pricing European options, this result seems merely theoretical. Nevertheless, this solution technique can be applied to other problems and is not limited to options and our model. A final, but very important, remark is the following. After implementing the FFT algorithm, this method can be used to price more sophisticated options (e.g. exotic options). We recommend that the interested reader looks at, for instance, [12], [51] and [61].

Maturity	Analytical	FFT	FFT adjusted
1	0.1682	0.1684	0.1769
0.1	0.0313	0.0312	0.0328
0.01	0.0037	0.0036	0.0038
0.001	3.1326e-04	3.1904e-04	3.1404e-04
0.0001	3.1275e-05	5.1980e-05	3.2480e-05
0.00001	3.1279e-06	2.4941e-05	3.2851e-06
0.000001	3.1279e-07	2.2217e-05	3.2892e-07

Table 5.1: Call option prices for different maturities. The maturities are given in years and the other parameter values are: $V_0 = 1$, $L = 1.01$, $r = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\lambda = 1$, $\mu_Y = 0.1$, $\sigma_Y = 0.3$ and $\lambda_Y = 2$.

Maturity	Analytical	FFT	FFT adjusted
1	0.1350	0.1350	0.1419
0.1	0.0213	0.0212	0.0223
0.01	0.0023	0.0022	0.0023
0.001	1.5726e-04	1.6371e-04	1.4983e-04
0.0001	1.5939e-05	3.6840e-05	1.6034e-05
0.00001	1.5939e-06	2.3436e-05	1.6403e-06
0.000001	1.5939e-07	2.2067e-05	1.6444e-07

Table 5.2: Call option prices for various maturities. The maturities are given in years and the remaining parameter values are: $V_0 = 1$, $L = 1.01$, $r = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\lambda = 1$, $\mu_Y = 0.1$, $\sigma_Y = 0.3$ and $\lambda_Y = 1$.

Chapter 6

Concluding Remarks and Implications for Future Research

We started this thesis with the intention to improve the empirical performance of the BS model. The motivation for the new model (RSJD) came from credit risk and option pricing. After the RSJD model was formulated in mathematical terms, we focussed on credit risk, and in particular on pricing risky debt. The approach initiated by Merton [57] was adapted to our model and we derived an analytical expression for the firm's debt value. Then we looked at credit spreads. We saw that the credit spreads produced by our model are realistic, that is, our model is able to fit historical credit spreads. Moreover, due to the fact that calibration is feasible, the RSJD model can be used for practical purposes. We discussed three calibration techniques: MLE, EM and inverse problem. Using the MLE, our model was calibrated to a real firm called Netflix, Inc. that has been in financial distress recently. The obtained parameter values from this procedure have been used to compute the 1-year real-world probabilities of default for this firm. It turned out that during the distress, the probability that the firm will default over exactly one year was indeed significantly high. As a second part of this thesis we considered option pricing. The volatility smiles produced by the RSJD model for short maturities as well as for long maturities were adequate. Beside an analytical expression for European options, a more sophisticated pricing method, FFT, has been discussed. We successfully implemented the FFT algorithm for European options. The observed difference between these two pricing methodologies was very low (below 1%). Unfortunately, the FFT does not perform well on very short maturities (less than 1 day). As a final part of this thesis we adjusted the FFT technique in order to improve its performance on these short maturities. This adjusted algorithm achieved much better results.

Here follow some implications for future research. It should be noticed that the RSJD model is not widely used, and therefore it is hard to say which methods/applications are most suitable for this model. Nevertheless, we made the following choice:

- The two main topics we have discussed, credit risk and option pricing, are obviously only a selection of a wide gamut of possible research areas. We address two other subjects of interest: hedging and pricing (other) derivatives. A hedge is a position that reduces the risk on another investment. In the current uncertain times, hedging has become a must. Options are certainly not the only specimen in the world of derivatives. There is a lot more, e.g., forwards, futures, warrants, swaps, etc.

- Within the credit risk there is much more possible. The original approach by Merton was later extended by, among others, Black and Cox [9] and Geske [34]. Black and Cox assumed that default may occur at any time and not only at maturity, which is, of course, more realistic. Geske developed an extension of Merton's approach by allowing the simultaneous existence of multiple debt issues that can differ in maturity, coupon size and seniority. This is very useful since in reality there is a distinction between, for instance, long-term and short-term liabilities. A combination of these two extensions, the one by Black and Cox and by Geske, is worth further investigation as well.

- Beside the three calibration techniques discussed in Chapter 4, there exist also other methods that are often used for the calibration of jump-diffusion models. Two examples are differential evolution (DE) (Ardia, Ospina and Gomez [4]) and Kalman filter (Bhar, Colwell and Xiao [6]).

- Calibration by means of option prices (inverse problem) can be further refined by using regularization strategies as mentioned in Chapter 4, Section 4.3. A regularization method based on relative entropy for jump-diffusion models, is presented by Cont and Tankov [18]. Another regularization technique are provided in [46] and [47].

- After implementing the FFT, this technique can be extended to value more complex and even exotic options (see, for instance, [12], [51] and [61]).

- FFT is not the only one known generic option pricing technique. Depending on the model, option prices of various options often safisty a so-called partial integro-differential equation (PIDE) (or just PDE in the diffusion case). For general regime switching Lévy processes such a PIDE has been derived in [41]. This equation is solved using a new Fourier Space Time-stepping (FST) algorithm. Another way of solving PIDEs is by using the notion of *viscosity solution*. For exponential Lévy models, this has been done by Cont and Voltchkova [21]. The same authors presented a different approach in [20]. In this article, the PIDE is solved using a finite difference scheme.

- Calculation of the market price of risk for a particular market.
- Comparison different calibration techniques.
- Comparison option prices obtained by different risk-neutral measures.
- Examination of the penalty function proposed in Appendix A.5.

Appendix A

A.1 Volatility Smiles

As mentioned in Chapter 1 there are also other “types” of volatility smiles than the one presented in Figure 1.3. The U-shape is typical for currency markets. In other markets, e.g., equity options, the graph is downward sloping. In this case one often uses the term *volatility skew* instead of *volatility smile*.

The interesting thing about the volatility skew for equities is that it has existed only since the stock market crash of October 1987 (Hull [38]). This phenomenon is clearly visible in Figure A.1. There are several reasons suggested for the equity volatility skew.

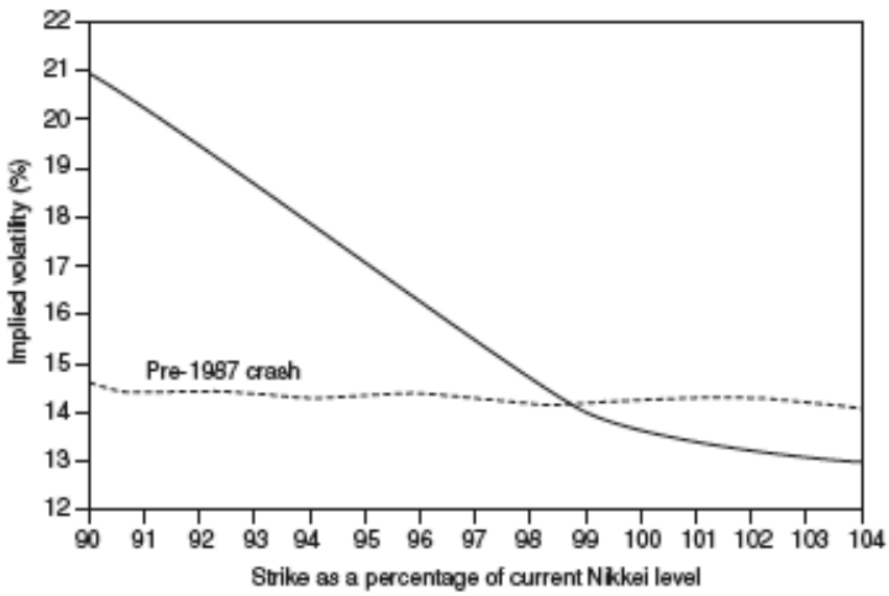

Figure A.1: Volatility skew (or smile) for equities before and after the crash of 1987.

One of these reasons originates from Mark Rubinstein who proposed that traders are concerned about the possibility of another crash similar to October 1987, and they price options accordingly (Hull [38]). This is also known as “crashophobia”.

A.2 Complete and Incomplete Markets

Economists distinguish between two "types" of financial markets: complete and incomplete markets. Since financial modeling is about building mathematical models in order to represent the performance of some investment in a certain market, these terms appear in mathematics as well. We say that a market is complete if for every derivative there exists a perfect hedge. Recall that by a derivative we mean an agreement between two parties that has a value determined by the price of something else. A perfect hedge is a position that entirely eliminates the risk on another investment. Depending on the particular choice, a model describes a complete or an incomplete market. The question is now what type of market should a proper model describe. An incomplete market is from a practical point of view the better choice, since in reality not every derivative can be perfectly hedged. On the other hand, from mathematical perspective we may prefer complete markets because in this case there exists an unique risk-neutral measure. As mentioned in Chapter 1, the RSJD model contains two jump processes. The jumps produced by these processes cannot be perfectly hedged, which implies that we are dealing with an incomplete market. So we need to choose an equivalent risk-neutral measure because in incomplete markets, there are in fact infinitely many candidates. However, there is a problem when one tries to choose an equivalent measure. The reason for this problem lies in the fact that equivalence does not capture the stochastic properties of a process as we will see in the example below. This means that two stochastic processes can define equivalent measures on scenarios while having different statistical properties. The example is the following: consider, for instance, two Poisson processes with the same jump size but different intensities. Then every trajectory of the one process can also be a trajectory of the other and vice versa. In fact, the two measures these processes define are equivalent! (see Chapter 9 of [19]) Hence, incompleteness either does not seem to be a desirable property. But it is probably the more realistic one since vital market risks (e.g. gamma and vega risk) are simply absent in complete markets (see again Cont and Tankov [19]).

A.3 Derivation of Formulae (2.3-2.4)

Denote by X a jump-diffusion process of the form $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta X_i$, where the processes $\{b_t\}$ and $\{\sigma_t\}$ correspond to the drift diffusion rate and the volatility, respectively. The term $\sum_{i=1}^{N_t} \Delta X_i$ represents a compound Poisson process.

Given $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, the Itô's formula for the above process looks like[1][2]

$$df(t, X_t) = \frac{\partial f(t, X_t)}{\partial t}dt + b_t\frac{\partial f(t, X_t)}{\partial x}dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f(t, X_t)}{\partial x^2}dt + \sigma_t\frac{\partial f(t, X_t)}{\partial x}dW_t + [f(X_{t_-} + \Delta X_t) - f(X_{t_-})].$$

Put $V_t = f(t, X_t)$, where f is given by $f(t, x) = \ln x$. An application of the previous formula yields

$$\begin{aligned} d\ln V_t &= (\mu_t - \lambda_Y \kappa)V_t\frac{1}{V_t}dt + \frac{\sigma_t^2 V_t^2}{2}\left(-\frac{1}{V_t^2}\right)dt + \sigma_t V_t\frac{1}{V_t}dW_t + [\ln V_t Y_t - \ln V_t] \\ &= \left(\mu_t - \lambda_Y \kappa - \frac{\sigma_t^2}{2}\right)dt + \sigma_t dW_t + \ln Y_t. \end{aligned}$$

Integrating on both sides gives

$$\ln V_t - \ln V_0 = \int_0^t \left(\mu_s - \lambda_Y \kappa - \frac{1}{2}\sigma_s^2\right)ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \ln Y_i,$$

whence the result follows.

A.4 How to Choose a Risk-Neutral Probability Measure?

Recall from Appendix A.2 that we are dealing with an incomplete market and hence there exist infinitely many risk-neutral probability measures. In order to choose one, there are a lot of well-known theoretical principles. Examples are minimal martingale measure (MMM) ([30]), minimal entropy martingale measure (MEMM) ([59]), variance-optimal martingale measure (VMM) ([63]), utility martingale measure (UMM) ([31]) and q-optimal martingale measure ([45]). In this thesis we will use the Esscher transform to choose a risk-neutral measure. The resulting measure is often called the Esscher martingale measure (ESSMM). The Esscher transform was introduced by Gerber and Shiu [33]. They provided the following economic justification of the use of the Esscher transform: Gerber and Shiu showed that the option price given by the Esscher transform can be justified by maximizing the expected power utility. Unfortunately, theoretical arguments only do not suffice to choose an adequate risk-neutral measure; we need to involve practical aspects as well.

In complete markets derivatives are priced using a no-arbitrage argument. Recall from Chapter 1 that an arbitrage is the practice of taking advantage of a price difference between two or more markets. However, in incomplete markets pricing using arbitrage

[1] See Chapter 8, Proposition 8.14 in Cont and Tankov [19].

[2] t_- denotes a time instance just before time t.

is impossible since different risk-neutral measures may yield different derivative prices. Albeit we are not able to determine a unique price for a specific derivative, prices of derivatives must satisfy an internal consistency requirement (Björk [8]). Namely, all derivatives must have the same *market price of risk*. The market price of risk can be seen as “risk premium per unit of volatility”. In an incomplete market, one cannot construct a portfolio that will completely eliminate risk (Appendix A.2) and so an agent will require a premium to balance his diminished utility function resulting from taking that risk.

It turns out that choosing a risk-neutral measure is equivalent to choosing a market price of risk (see again Björk [8]). But the market price of risk is determined on the market! So if we want to obtain information about the market price of risk for some market, we must go to that particular market and gather relevant information (e.g. price data) using empirical techniques ([36] and [29]).

A.5 Discussion on the MLE

The MLE has a lot of desirable properties, e.g., consistent, asymptotically normal and unbiased, has the lowest asymptotic variance of any consistent estimator. However, it remains a “large sample” technique and so in many cases, when we do not have a large sample, all these nice properties can actually fail. In addition, like for our model, computing the likelihood function can be very complicated and computationally demanding. As proposed by Serlin [64], for any jump-diffusion model the global optimum probably does not even exist. Furhtermore, for some observed equity data the likelihood function can contain singularities (Serlin [64] and Wong and Pi [69]). Wong and Pi suggested an alternative approach called the maximum penalized likelihood estimation (MPLE). The idea is to maximize the likelihood function plus some additional term instead of maximizing solely the likelihood function. This penalty function equals $-2b\log\sigma - a/\sigma^2$ for $a > 0$, $b > 1$ and σ representing the volatility.

A.6 Derivative with Respect to λ

Given the observations $\mathcal{R} = (R_1, R_2, \dots, R_T)$, one gets[3]

$$
\begin{aligned}
&\frac{\partial Q(\rho,\rho_0)}{\partial \lambda} \\
&= \frac{\partial}{\partial \lambda}\mathbb{E}_0\left[-\frac{T}{2}\ln \tau^2 - \frac{1}{2\tau^2}\sum_{i=1}^{T}(Z(i)-\nu)^2 \middle| \mathcal{R}\right] \\
&= \frac{\partial}{\partial \lambda}\left[-\frac{T}{2}\ln \tau^2 - \frac{1}{2\tau^2}\sum_{i=1}^{T}\mathbb{E}_0\left[Z(i)^2 \middle| R(i)\right] + \frac{\nu}{\tau^2}\sum_{i=1}^{T}\mathbb{E}_0[Z(i)|R(i)] - \frac{T\nu^2}{2\tau^2}\right].
\end{aligned}
\tag{A.1}
$$

For instance, working out the derivative of the third term yields

$$
\begin{aligned}
&\frac{\partial}{\partial \lambda}\frac{\nu}{\tau^2}\sum_{i=1}^{T}\mathbb{E}_0[Z(i)|R(i)] \\
&= \left[\frac{T\left(\mu_2 - \mu_1 + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)\right)}{\sigma_1^2 T^2 + \lambda^2\left(\sigma_1^2 + \sigma_2^2\right)} \right. \\
&\left. - \frac{2T^3\left(\left(\mu_1 - \lambda_Y\kappa - \frac{\sigma_1^2}{2}\right) + \frac{\lambda}{T}\left(\mu_2 - \mu_1 + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)\right)\right)}{\left(\sigma_1^2 T^2 + \lambda^2\left(\sigma_1^2 + \sigma_2^2\right)\right)^{3/2}} \right] \sum_{i=1}^{T}\mathbb{E}_0[Z(i)|R(i)].
\end{aligned}
$$

Even simplifications such as $r = \mu_1 = \mu_2$ do not reduce this expression to a much simpler one. Furthermore, the derivatives of the second and the last term are also not that easy to work with. If we set the derivative in (A.1) equal to zero and try to solve this equation for λ in Mathematica, we do not get any solution probably due to the complexity of the formula.

A.7 Background Information on Fourier Transforms

A Fourier transform (FT) is basically a mathematical device to express a function of time as a function of frequency. In general, the FT $F(\omega)$ of a function $f(t)$ can be defined using two arbitrary constants a and b as

$$
F(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}}\int_{-\infty}^{\infty} f(t)e^{ibwt}dt.
$$

[3]Notice that in this notation we dropped the conditioning on ρ in the expectation.

The inverse of this transform is called the inverse Fourier transform and it is given by

$$f(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} F(\omega) e^{-ibwt} dw.$$

Now, put $(a, b) = (1, 1)$ to get

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{iwt} dt, \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-iwt} dw, \end{aligned} \tag{A.2}$$

and observe that, if $f(t)$ is the probability density function of a random variable X then its Fourier transform $F(\omega)$ is, by definition, the characteristic function (CF) of X. In virtue of equation A.2 this transform is invertible and hence there exists an one-to-one correspondence between CFs and probability density functions.

Remark: Every FT in Chapter 5 is one with parameters $(1, 1)$!

A.8 The Choice of α, h and M

First, observe the following: if the call option function was not modified, that is, if $\alpha = 0$, then the denominator in (5.3) will vanish when $u = 0$. This will cause a singularity at $u = 0$ in the integrand of (5.2). Hence, modification is required. The modified call option fuction must be square integrable and so $\psi_T(0)$ must be finite. From (5.3), it follows that this condition is equivalent to $\phi_{\mathcal{F}_T^\theta}(-i(1+\alpha))$ being finite, which implies, by definition, that

$$\phi_{\mathcal{F}_T^\theta}(-i(1+\alpha)) = \mathbb{E}\left[e^{(1+\alpha)Z_T} \middle| \mathcal{F}_T^\theta\right] = \mathbb{E}\left[V_T^{1+\alpha} \middle| \mathcal{F}_T^\theta\right] < \infty$$

must hold true. It turns out that for the most models $\alpha = 4$ is a good choice for α. Hence, we put $\alpha = 4$. The log strike spacing h is chosen to be equal to $\frac{\pi}{300}$ and for the quadrature we use $M = 4096$. The implementation seems to be very accurate since altering these numbers did not significantly improve the results.

A.9 Computing ζ_T

We have

$$\zeta_T(u) = \int_{-\infty}^{\infty} e^{iul} CP(T, l) dl,$$

where $CP(T, l)$ is as given in (5.6). Recall that $f_{\mathcal{F}_T^\theta}$ denotes the risk-neutral density of Z_T conditioned on $\mathcal{F}_T^\theta$. Now substituting the expression for $CP(T, l)$ from (5.6) and conditioning on $\mathcal{F}_T^\theta$ yields

$$
\begin{aligned}
\zeta_T(u) &= \int_{-\infty}^{\infty} e^{iul} e^{-rT} \mathbb{E}\left[\left(e^l - e^{Z_T}\right)^+ \mathbf{1}_{l<0} + \left(e^{Z_T} - e^l\right)^+ \mathbf{1}_{l>0}\right] dl \\
&= e^{-rT} \int_{-\infty}^{\infty} e^{iul} \mathbb{E}\left[\mathbb{E}\left[\left(e^l - e^{Z_T}\right)^+ \middle| \mathcal{F}_T^\theta\right] \mathbf{1}_{l<0} + \mathbb{E}\left[\left(e^{Z_T} - e^l\right)^+ \middle| \mathcal{F}_T^\theta\right] \mathbf{1}_{l<0}\right] dl \\
&= e^{-rT} \int_{-\infty}^{\infty} e^{iul} \mathbb{E}\left[\int_{-\infty}^{l} \left(e^l - e^x\right) \mathbf{1}_{l<0} f_{\mathcal{F}_T^\theta}(x) dx + \int_{l}^{\infty} \left(e^x - e^l\right) \mathbf{1}_{l>0} f_{\mathcal{F}_T^\theta}(x) dx\right] dl \\
&= e^{-rT} \int_{-\infty}^{0} e^{iul} \mathbb{E}\left[\int_{-\infty}^{l} \left(e^l - e^x\right) f_{\mathcal{F}_T^\theta}(x) dx\right] dl \\
&+ e^{-rT} \int_{0}^{\infty} e^{iul} \mathbb{E}\left[\int_{l}^{\infty} \left(e^x - e^l\right) f_{\mathcal{F}_T^\theta}(x) dx\right] dl \\
&= e^{-rT} \mathbb{E}\left[\int_{-\infty}^{0} f_{\mathcal{F}_T^\theta}(x) \int_{x}^{\infty} \left(e^{(1+iu)l} - e^x e^{iul}\right) dl dx\right] \\
&+ e^{-rT} \mathbb{E}\left[\int_{0}^{\infty} f_{\mathcal{F}_T^\theta}(x) \int_{0}^{x} \left(e^x e^{iul} - e^{(1+iu)l}\right) dl dx\right] \\
&= e^{-rT} \mathbb{E}\left[\int_{-\infty}^{0} f_{\mathcal{F}_T^\theta}(x) \left[\frac{e^{(1+iu)l}}{1+iu} - \frac{e^x e^{iul}}{iu}\right]_{l=x}^{l=0} dx\right] \quad \text{(since } l<0\text{)} \\
&+ e^{-rT} \mathbb{E}\left[\int_{0}^{\infty} f_{\mathcal{F}_T^\theta}(x) \left[\frac{e^x e^{iul}}{iu} - \frac{e^{(1+iu)l}}{1+iu}\right]_{l=0}^{l=x} dx\right] \\
&= e^{-rT} \mathbb{E}\left[\int_{-\infty}^{0} f_{\mathcal{F}_T^\theta}(x) \left[\frac{1}{1+iu} - \frac{e^x}{iu} - \frac{e^{(1+iu)x}}{1+iu} + \frac{e^x e^{iux}}{iu}\right] dx\right] \\
&+ e^{-rT} \mathbb{E}\left[\int_{0}^{\infty} f_{\mathcal{F}_T^\theta}(x) \left[\frac{e^x e^{iux}}{iu} - \frac{e^{(1+iu)x}}{1+iu} - \frac{e^x}{iu} + \frac{1}{1+iu}\right] dx\right] \\
&= e^{-rT} \mathbb{E}\left[\int_{-\infty}^{\infty} f_{\mathcal{F}_T^\theta}(x) \left[\frac{1}{1+iu} - \frac{e^x}{iu} - \frac{e^{(1+iu)x}}{u^2 - iu}\right] dx\right] \\
&= e^{-rT} \left[\frac{1}{1+iu} - \frac{1}{iu} \mathbb{E}\left[\int_{-\infty}^{\infty} f_{\mathcal{F}_T^\theta}(x) e^x dx\right] - \frac{1}{u^2 - iu} \mathbb{E}\left[\int_{-\infty}^{\infty} f_{\mathcal{F}_T^\theta}(x) e^{(1+iu)x} dx\right]\right] \\
&= e^{-rT} \left[\frac{1}{1+iu} - \frac{e^{rT}}{iu} - \frac{\mathbb{E}\left[\phi_{\mathcal{F}_T^\theta}(u-i)\right]}{u^2 - iu}\right],
\end{aligned}
$$

where $\phi_{\mathcal{F}_T^\theta}$ denotes the characteristic function of Z_T conditioned on $\mathcal{F}_T^\theta$ and

$$
\int_{-\infty}^{\infty} f_{\mathcal{F}_T^\theta}(x) e^x dx = \mathbb{E}\left[e^{Z_T} \middle| \mathcal{F}_T^\theta\right] = \mathbb{E}\left[V_T \middle| \mathcal{F}_T^\theta\right] = V_0 e^{rT} = e^{rT}
$$

follows from the martingale property and the assumption $V_0 = 1$. Finally, observe that the expectation of the characteristic function $\phi_{\mathcal{F}_T^\theta}$ is obtained by conditioning on the occupation time in the same way as in the proof of Proposition 3.

A.10 Matlab Code Option Prices

Analytical option prices

```
function E = Analytical(V0,T,t,sigma1,sigma2,strike,r,lambda,a,b,lambda2)
L=strike;                                                    %a=mu_Y
T=T-t;                                                       %b=sigma_Y
St=V0;                                                  %lambda2=lambda_Y
E=St-DPF(St,T,sigma1.^2,sigma2.^2,L,r,lambda,a,b.^2,lambda2);
%see equation 5.1 in Chapter 5
end

function D = DPF(St,T,sigma1,sigma2,L,r,lambda,a,b,lambda2)
%derived in Chapter 3, Proposition 1
N=max(40,ceil(20.*lambda2)); %maximum number of jumps
D=L.*exp(-r.*T);
for n=0:N
    std=U(sigma1,sigma2,b,T,T,n);
    mean=(r-(exp(a+b./2)-1).*lambda2).*T+a.*n;
    substract=exp(-lambda.*T).*(L.*exp(-r.*T).*normcdf(myfunz1(St,L,std...
        ,mean,b,n))-St.*exp(-(exp(a+b./2)-1).*lambda2.*T+n.*a+n.*b./2).*...
        normcdf(myfunz2(St,L,std,mean,b,n)));
    substract=substract+quad(@(s) myfun(St,T,s,sigma1,sigma2,L,r,mean,lambda...
        ,a,b,n,lambda2),0,T,1e-3);
    D=D-substract.*exp(-lambda2.*T).*(lambda2.*T).^n./factorial(n);
end
end

function y=U(sigma1,sigma2,b,s,T,n)
    y=sqrt(sigma1.*s+sigma2.*(T-s)+b.*n);
end

function y = myfunz1(St,L,std,mean,b,n)
    y=(1./std).*(log(L./St)-mean+(1/2).*(std.^2-b.*n));
end

function y = myfunz2(St,L,std,mean,b,n)
    y=myfunz1(St,L,std,mean,b,n)-std;
end

function y = myfun(St,T,s,sigma1,sigma2,L,r,mean,lambda,a,b,n,lambda2)
    std=U(sigma1,sigma2,b,s,T,n);
    y =(L.*exp(-r.*T).*normcdf(myfunz1(St,L,std,mean,b,n))-St.*exp(-(exp(a...
        +b./2)-1).*lambda2.*T+n.*a+n.*b./2).*normcdf(myfunz2(St,L,std,mean,...
        b,n))).*lambda.*exp(-lambda.*s);
 end
```

FFT option prices

```
function y=FFT(V0,strike,r,a,b,sigma1,sigma2,lambda,lambda2,T) %a=mu_Y
kappa=exp(a+0.5.*b.^2)-1;                                        %b=sigma_Y
alpha=4;                                                     %lambda2=lambda_Y
M=4096;
c=600;
eta=c/M;
a1=pi/eta;
u=[0:M-1].*eta;
h=2*a1/M;
position=(log(strike)+a1)/h + 1; %position of call
v=u-(alpha+1)*1i;
vect=zeros(1,M);
for i=1:M
    vect(i)=Sum(V0,v(i),r,a,b.^2,sigma1.^2,sigma2.^2,lambda,lambda2,kappa,T);
end
vect_mod=vect.*exp(-r.*T)./(alpha.^2+alpha-u.^2+1i.*(1+2.*alpha).*u);
Simpson=1/3.*(3+(-1).^[1:M]-[1, zeros(1,M-2), 1]);
FftFunc=exp(1i.*a1.*u).*vect_mod.*eta.*Simpson;
payoff=real(fft(FftFunc));
CallValueM=exp(-log(strike).*alpha).*payoff./pi;
format short;
y=CallValueM(round(position));
end

function y=Sum(V0,v,r,a,b,sigma1,sigma2,lambda,lambda2,kappa,T)
y=exp(1i.*v.*log(V0)).*exp(UT(v,a,b,lambda2,T)).*(Inte(v,r,sigma1,sigma2,...
    lambda,lambda2,kappa,T)+exp(1i.*v.*((r-lambda2.*kappa).*T-0.5.*sigma1...
    .*T)-0.5.*v.^2.*sigma1.*T).*exp(-lambda.*T));
end

function y=Inte(v,r,sigma1,sigma2,lambda,lambda2,kappa,T)
y=quad(@(s) exp(expo(v,r,sigma1,sigma2,lambda2,kappa,s,T)).*lambda...
    .*exp(-lambda.*s),0,T);
end

function y=expo(v,r,sigma1,sigma2,lambda2,kappa,s,T)
y=1i.*v.*((r-lambda2.*kappa).*T-0.5.*I(sigma1,sigma2,s,T))-0.5.*v.^2 ...
.*I(sigma1,sigma2,s,T);
end

function y=I(sigma1,sigma2,s,T)
y=sigma1.*s+sigma2.*(T-s);
end

function y=UT(v,a,b,lambda2,T)
y=T.*lambda2.*(exp(1i.*v.*a-0.5.*b*v.^2)-1);
```

```
end
```

FFT adjusted option prices

```
function y=FFT_adjusted(V0,strike,r,a,b,sigma1,sigma2,lambda,lambda2,T)
kappa=exp(a+0.5.*b.^2)-1;                                            %a=mu_Y
alpha=4;                                                             %b=sigma_Y
M=4096;                                                          %lambda2=lambda_Y
c=600;
eta=c/M;
a1=pi/eta;
u=[0:M-1].*eta;
h=2*a1/M;
position=(log(strike)+a1)/h + 1; %position of call
w1=u-1i.*alpha;
w2=u+1i.*alpha;
v1=u-1i.*alpha-1i;
v2=u+1i.*alpha-1i;
vect1=zeros(1,M);
vect2=zeros(1,M);
for i=1:M
    vect1(i)=Sum(V0,v1(i),r,a,b.^2,sigma1.^2,sigma2.^2,lambda,lambda2,kappa,T);
    vect2(i)=Sum(V0,v2(i),r,a,b.^2,sigma1.^2,sigma2.^2,lambda,lambda2,kappa,T);
end
vect_mod1=exp(-r.*T).*(1./(1+1i.*w1)-exp(r.*T)./(1i.*w1)-vect1./(w1.^2-1i.*w1));
vect_mod2=exp(-r.*T).*(1./(1+1i.*w2)-exp(r.*T)./(1i.*w2)-vect2./(w2.^2-1i.*w2));
vect_mod_combi=(vect_mod1-vect_mod2)./2;
Simpson=1/3.*(3+(-1).^[1:M]-[1, zeros(1,M-2), 1]);
FftFunc=exp(1i.*a1.*u).*vect_mod_combi*eta.*Simpson;
payoff=real(fft(FftFunc));
CallValueM=payoff./pi./sinh(alpha.*log(strike));
format short;
y=CallValueM(round(position));
end

function y=Sum(V0,v,r,a,b,sigma1,sigma2,lambda,lambda2,kappa,T)
y=exp(1i.*v.*log(V0)).*exp(UT(v,a,b,lambda2,T)).*(Inte(v,r,sigma1,sigma2,...
    lambda,lambda2,kappa,T)+exp(1i.*v.*((r-lambda2.*kappa).*T-0.5.*sigma1.*T)...
    -0.5.*v.^2.*sigma1.*T).*exp(-lambda.*T));
end

function y=Inte(v,r,sigma1,sigma2,lambda,lambda2,kappa,T)
y=quad(@(s) exp(expo(v,r,sigma1,sigma2,lambda2,kappa,s,T)).*lambda.*exp(...
    -lambda.*s),0,T);
end

function y=expo(v,r,sigma1,sigma2,lambda2,kappa,s,T)
y=1i.*v.*((r-lambda2.*kappa).*T-0.5.*I(sigma1,sigma2,s,T))-0.5.*v.^2.*I(...
```

```
    sigma1,sigma2,s,T);
end

function y=I(sigma1,sigma2,s,T)
y=sigma1.*s+sigma2.*(T-s);
end

function y=UT(v,a,b,lambda2,T)
y=T.*lambda2.*(exp(1i.*v.*a-0.5.*b*v.^2)-1);
end
```

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