Bachelor Thesis

Mathematics

The Riemann-Hurwitz Formula

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Abstract

In this thesis we study the Riemann-Hurwitz formula. For a (possibly ramified) covering $f : S_1 \to S_2$ of two compact Riemann surfaces, this formula relates the genus of S_1, the genus of S_2, the degree of f and the ramification indices of f. We study theory on Riemann surfaces and coverings, after which we discuss and prove the Riemann-Hurwitz formula. For a thorough understanding we include some applications of the formula, including proving Hurwitz's automorphisms theorem. We end this thesis by discussing the equivalence between the category of compact Riemann surfaces and the category of complete, non-singular, irreducible algebraic curves over $\mathbb{C}$.

Contents

1 Introduction

Georg Friedrich Bernhard Riemann (1826 - 1866) was a German mathematician who was of great value to various fields in mathematics in his brief life. He made major contributions to real and complex analysis, differential geometry and number theory. With his contributions to differential geometry he laid the foundations of the mathematics of general relativity. Riemann's paper ""Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" from 1859 on the prime-counting function is considerd to be one of the most influential papers in analytic number theory. This paper also included the original statement of the famous Riemann Hypothesis. In real analysis he is mostly known for the Riemann integral, whereas in complex analysis he is mostly known for the introduction of Riemann surfaces. Bernhard Riemann is considered by many to be one of the greatest mathematicians of all time.

In 1851 Riemann defined (what we now call) Riemann surfaces in his PhD dissertation "Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen komplexen Grösse". Riemann advanced his theory in 1857 in his paper on abelian functions, "Theorie der Abelschen Functionen". Today, the theory of Riemann surfaces plays a central role in modern mathematics and mathematical physics. It can be seen in areas such as algebraic topology, partial differential equations and even string theory [1]. Although Riemann invented Riemann surfaces, he was on the shoulders of some iconic mathematicians, among them Niels Hendrik Abel and Carl Gustav Jacob Jacobi. Various important results were obtained by these mathematicians, 20 years prior to Riemann's PhD dissertation. Other major contributions to the theory were Cauchy's work in complex analysis and Euler's work on elliptic integrals and elliptic functions.

The basic idea of a Riemann surface is that it is a topological space which locally looks like an open set of the complex plane. When considering holomorphic maps of two Riemann surfaces, one of the tools used is a formula called the Riemann-Hurwitz formula, named after Riemann and Adolf Hurwitz. For a given (possibly ramified) covering of two Riemann surfaces, the formula describes the relationship between the genus of both the Riemann surfaces, the degree of the map and the ramification indices. Despite the continuous nature of these holomorphic maps, the formula allows us to deal with integer values and derive strong claims from them. Several results in complex analysis and algebraic topology followed from this formula. One of the applications of this formula is that it can be used to prove Hurwitz's automorphisms theorem. This theorem states that the order of the group of automorphisms of a compact Riemann surface of genus greater than one is bounded above. The Riemann-Hurwitz formula also holds for complete, non-singular, irreducible algebraic curves over $\mathbb{C}$ since there is an equivalence between this category and that of Riemann surfaces.

In this thesis we start by studying some basic theory on Riemann surfaces and (possibly ramified) coverings in Chapter 2, leading up to the Riemann-Hurwitz formula. In Chapter 3 we discuss theory on triangulations of Riemann surfaces, after which we will prove the Riemann-Hurwitz formula. Subsequently in Chapter 4 we discuss and prove Hurwitz's automorphisms theorem by applying the Riemann-Hurwitz formula. The proof in Chapter 4 is based on Hurwitz's article "Ueber algebraische Gebilde mit eindeutigen Transformationen in sich" from 1892. In Chapter 5 we study the equivalence between the category of compact Riemann surfaces and the category of complete, non-singular, irreducible algebraic curves over $\mathbb{C}$.

Before we start Chapter 2, I would like to express my sincere gratitude to Prof. dr. Frans Oort and Prof. dr. Carel Faber for their guidance and encouragement throughout the process of this thesis.

2 Preliminaries

Before getting into the Riemann-Hurwitz formula, we will discuss some basic theory on Riemann surfaces and (possibly ramified) coverings. We will also consider some examples of Riemann surfaces and an example of a branched covering. For a visual representation of a Riemann surface, an unramified topological covering and a branched covering, some figures are included.

2.1 Riemann Surfaces

Riemann surfaces can be thought of as a topological space, which locally looks like an open set of the complex plane. In the interest of letting a topological space T locally look like an open set of the complex plane, we want to have a local complex coordinate at every point of the space. This leads to the definition of a complex chart.

Definition 2.1.1. A *complex chart* for a topological space T is a homeomorphism

$$\varphi : U \to V$$

from an open subset $U \subset T$ to an open subset $V \subset \mathbb{C}$, also denoted as (U, φ).

Definition 2.1.2. Two complex charts

$$\varphi_1 : U_1 \to V_1$$
$$\varphi_2 : U_2 \to V_2$$

are called *compatible* if either $U_1 \cap U_2 = \emptyset$ or the transition function

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic.

Note that this definition is symmetric. The transition functions $\varphi_2 \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi_2^{-1}$ are inverses of one another. Recall that a function is holomorphic if and only if it is analytic. The transition function $\varphi_2 \circ \varphi_1^{-1}$ is injective, we have $\varphi_2 \circ \varphi_1^{-1}(\varphi_1(U_1 \cap U_2)) = \varphi_2(U_1 \cap U_2)$ and we assume it is analytic on the open set $\varphi_1(U_1 \cap U_2)$. Then by [2], p. 82, Theorem 6.4 the transition function $\varphi_1 \circ \varphi_2^{-1}$ is analytic and hence holomorphic.

Definition 2.1.3. An indexed family $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}$ of pairwise compatible complex charts for which $T = \bigcup_\alpha U_\alpha$ is called a *complex atlas* on T.

Definition 2.1.4. We call two complex atlasses $\mathcal{A}_1$ and $\mathcal{A}_2$ *equivalent* if every complex chart of one is compatible with every complex chart of the other.

Definition 2.1.5. A *complex structure* on T is an equivalence class of complex atlasses on T.

We recall that a *Hausdorff space* is a topological space T where for any two distinct points $x, y \in T$, there exists neighbourhoods of x and y which are disjoint. If there is a countable basis for the topology of T we call T second countable.

Definition 2.1.6. A *Riemann surface* S is a (second countable) connected Hausdorff space endowed with a complex structure.

For an intuitive understanding of a Riemann surface we consider Figure (1). The Riemann surface in this figure has local charts (U_x, φ_x) and (U_y, φ_y) to open subsets V_x and V_y of $\mathbb{C}$, where on overlapping charts the transition functions T_{xy} and T_{yx} are holomorphic ([3], p. 4).

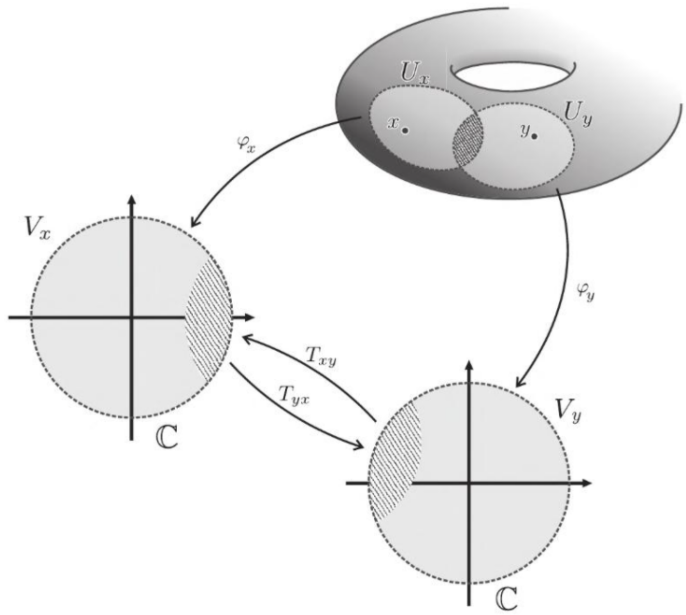

Figure 1: A visual representation of a Riemann surface with complex charts (U_x, φ_x) and (U_y, φ_y) to open subsets V_x and V_y of $\mathbb{C}$ respectively, and holomorphic transition functions T_{xy} and T_{yx} ([4], p. 33, Figure 3.1).

Example 2.1.7. The most basic example of a Riemann surface is the complex plane $\mathbb{C}$. The identity map $f(z) = z$ defines a complex chart for $\mathbb{C}$, and the set $\{f\}$ acts as an atlas for $\mathbb{C}$.

Example 2.1.8. Another example of a Riemann surface is $S = \mathbb{C} \cup \{\infty\}$. Define two subsets of S to be $U_1 = S \backslash \{0\}$ and $U_2 = S \backslash \{\infty\}$, and define two maps $\varphi_1 : U_1 \to \mathbb{C}$ and $\varphi_2 : U_2 \to \mathbb{C}$

$$\varphi_1(z) = \begin{cases} \frac{1}{z} & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \end{cases}$$

$$\varphi_2(z) = id_{\mathbb{C}}$$

Then φ_1 and φ_2 are complex charts. They are compatible complex charts since the transition function $\varphi_2 \circ \varphi_1^{-1} = 1/z$ is holomorphic on $\mathbb{C} \setminus \{0\} = U_1 \cap U_2$. Since φ_1 and φ_2 are compatible complex charts and $S = \bigcup_\alpha U_\alpha$ with $\alpha = 1, 2$, we have that $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}$ with $\alpha = 1, 2$ and $V_\alpha = \mathbb{C}$ is a complex atlas on S. This particular compact Riemann surface is called the *Riemann sphere*. The complex projective line $\mathbb{P}^1(\mathbb{C})$ is isomorphic to S.

Definition 2.1.9. Two Riemann surfaces S_1 and S_2 are *isomorphic* if there are holomorphic maps $f : S_1 \to S_2$ and $g : S_2 \to S_1$ such that $g \circ f = id_{S_1}$ and $f \circ g = id_{S_2}$. The holomorphic maps f and g are called *isomorphisms* ([4], p. 50).

In the next example of a Riemann surface we study the complex torus.

Example 2.1.10. This example is based on the work done in [5], pp. 9-10. Let L be the lattice

$$L = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2 = \{c_1\tau_1 + c_2\tau_2 \mid c_1, c_2 \in \mathbb{Z}\}$$

with $\tau_1, \tau_2 \in \mathbb{C}$ fixed and linearly independent of $\mathbb{R}$. Let $T_\tau = L\backslash\mathbb{C}$ be the quotient group, let $\pi : \mathbb{C} \to T_\tau$ be the projection map and impose the quotient topology on T_τ. In this example we will show that T_τ is a compact Riemann surface.
Note that π is continuous and since connectivity is preserved under continuous mapping ([6]), T_τ is connected since $\mathbb{C}$ is connected. Every open set $U \subset T_\tau$ is the image of an open set $V \subset \mathbb{C}$, since π is continuous, this open set $U = \pi(\pi^{-1}(U))$.
We will show that π is also an open mapping, i.e. any open set in $\mathbb{C}$ is mapped by π to an open set in T_τ. Let V be an open set in $\mathbb{C}$. To show that π is an open mapping we want to check that $\pi(V)$ is open. Since π is continuous and $\pi(V) \subset T_\tau$ we have to show that $\pi^{-1}(\pi(V))$ is open in $\mathbb{C}$. We write $\pi^{-1}(\pi(V))$ as the union of translates of V

$$\pi^{-1}(\pi(V)) = \bigcup_{\tau \in L} (\tau + V)$$

which are open sets in $\mathbb{C}$, and the union of open sets is open.
We define for any $z \in \mathbb{C}$ the closed parallelogram

$$P_z = \{z + \lambda_1\tau_1 + \lambda_2\tau_2 \mid \lambda_1, \lambda_2 \in [0, 1]\}$$

Any point in $\mathbb{C}$ is congruent to a point in P_z modulo L. Hence π maps P_z onto T_τ and since P_z is compact, T_τ is too. We fix an $\varepsilon > 0$ such that $|\tau| > 2\varepsilon$ for $\tau \in L$ with τ nonzero. Such an ε exists since L is a discrete subset of $\mathbb{C}$. We also fix a point $p_0 \in \mathbb{C}$ and consider the open disc $D_\varepsilon(p_0)$. Due to the ε we fixed, no points of this disc can differ by a nonzero element of L.
The map $\pi|_{D_\varepsilon(p_0)} : D_\varepsilon(p_0) \to \pi(D_\varepsilon(p_0))$ is a homeomorphism since it is continuous, open and a bijection. It is continuous, open and onto since π is and because of our choice of ε it is also 1-1.
We fix ε as above and let $D_\varepsilon(p_i)$ for every $p_i \in \mathbb{C}$. Define $\varphi_i : \pi(D_\varepsilon(p_i)) \to D_\varepsilon(p_i)$. These are complex charts for T_τ by Definition 2.1.1 since they are homeomorphisms from an open subset in T_τ to an open subset in $\mathbb{C}$.
To show that T_τ is a Riemann surface we also have to check if these complex charts are pairwise compatible as defined by Definition 2.1.2. For this we fix two points p_1, $p_2 \in \mathbb{C}$, and show that φ_1 and φ_2 are compatible. Let $U_1 = \pi(D_\varepsilon(p_1))$ and $U_2 = \pi(D_\varepsilon(p_2))$. If $U_1 \cap U_2 = \emptyset$ we are done. If this intersection is not empty we check if the transition function $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is holomorphic. We write $\varphi_2 \circ \varphi_1^{-1}(z) = \varphi_2(\varphi_1^{-1}(z)) = \varphi_2(\pi(z))$ for $z \in \varphi_1(U_1 \cap U_2)$. Note that $\pi(\varphi_2(\pi(z))) = \pi(z)$ for all $z \in \varphi_1(U_1 \cap U_2)$. Hence $\varphi_2(\pi(z))$ and z differ by a lattice element, so $\varphi_2(\pi(z)) - z = \tau \in L$ for all $z \in \varphi_1(U_1 \cap U_2)$. The function $\tau : \varphi_1(U_1 \cap U_2) \to L$ is continuous and L is discrete, so τ is locally constant on $\varphi_1(U_1 \cap U_2)$. Hence the transition function is $\varphi_2(\pi(z)) = \tau + z$ for $z \in \varphi_1(U_1 \cap U_2)$ and a fixed $\tau \in L$, which is holomorphic. Hence φ_1 and φ_2 are complex compatible charts and the indexed family $\mathcal{A} = \{\varphi_i : \pi(D_\varepsilon(p_i)) \to D_\varepsilon(p_i)\}$ for $p_i \in \mathbb{C}$ is a complex atlas on T_τ. Hence T_τ is a compact Riemann surface. We call this compact Riemann surface the *complex torus*.

2.2 Coverings

The Riemann-Hurwitz formula relates several invariants involved when considering a (possibly ramified) covering of Riemann surfaces. This section is dedicated to cover some basic theory on unramified and ramified coverings. All Riemann surfaces considered in this section are compact.

Definition 2.2.1. Let T_1 and T_2 be topological spaces. A continuous surjective map $f : T_1 \to T_2$ is an *unramified topological covering* if for every $p \in T_2$ there is an open set $U_p \subset T_2$ containing p such that $f^{-1}(U_p) = \coprod V_i$ is a union of disjoint open sets V_i in T_1, where $f|_{V_i} : V_i \to U_p$ is a homeomorphism.

Note that that a space covers itself with the identity map being the unramified topological covering. Another example is the following.

Example 2.2.2. The continuous surjective map $p : S^1 \to S^1$, with S^1 the unit circle, given by $p(z) = z^n$ with $n \in \mathbb{N}$ is an unramified topological covering. This map wraps the unit circle around itself n times.

For a visual representation of a holomorphic surjective map of two Riemann surfaces we consider Figure (2). The Riemann surfaces have complex charts (U_x, φ_x) and $(U_{f(x)}, \varphi_{f(x)})$ to open subsets of $\mathbb{C}$. The *local expression* for f in this figure is given by $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$. This figure gives an intuitive understanding of how a holomorphic surjective map f of Riemann surfaces can locally be seen as a holomorphic map of the open subsets V_x and $V_{f(x)}$ of the complex plane.

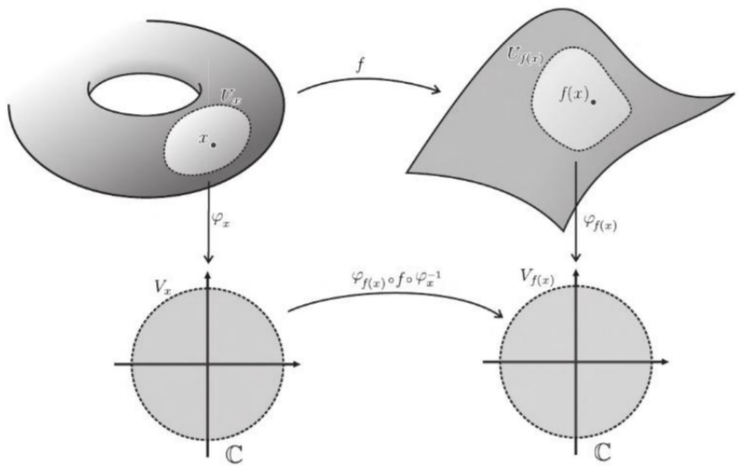

Figure 2: A covering of two Riemann surfaces with complex charts (U_x, φ_x) and $(U_{f(x)}, \varphi_{f(x)})$ to open subsets V_x and $V_{f(x)}$ of $\mathbb{C}$ and local expression $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ for f ([4], p.48, Figure 4.1).

For the study of Riemann surfaces we are interested in possibly ramified coverings since these naturally come up in studying multi-valued functions.

Definition 2.2.3. A holomorphic surjective map $f : S_1 \to S_2$ of compact Riemann surfaces is a *branched covering* if there exists a finite, minimal and non-empty set $W \subset S_2$ such that the induced map

$$f|_{f^{-1}(S_2\backslash W)} : f^{-1}(S_2\backslash W) \to (S_2\backslash W)$$

is an unramified topological covering.

Definition 2.2.4. The subset $W \subset S_2$ as above, where f fails to be an unramified covering, is called the *branch locus* $B(f)$ of f.

In Figure (3) we see a visualization of a branched covering of two Riemann surfaces, both with complex charts ϕ and ψ to open subsets of $\mathbb{C}$ and local expression F.

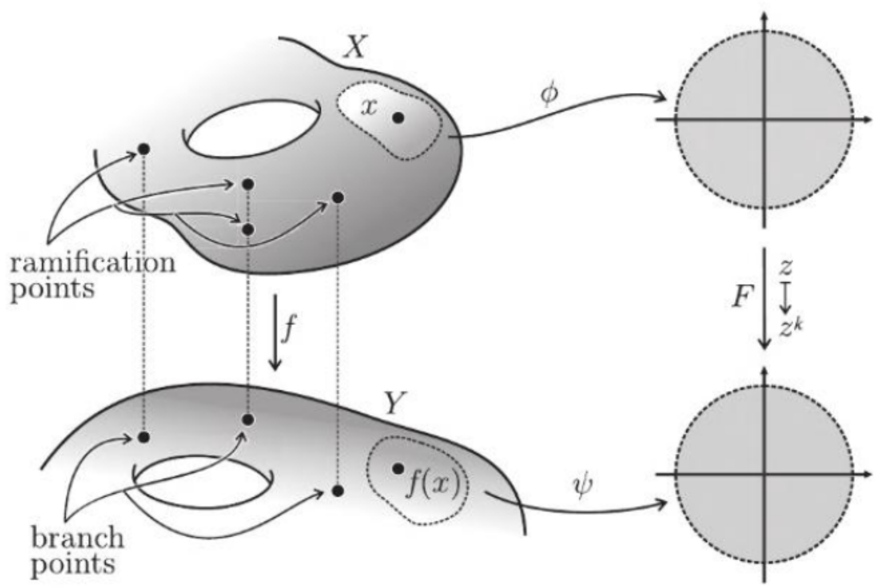

Figure 3: A visual representation of a branched covering of two Riemann surfaces with local charts ϕ and ψ to open subsets of $\mathbb{C}$ and local expression F for f([4], p.53, Figure 4.5).

We will consider an example of branched covering with its non-empty branch locus.

Example 2.2.5. The map $f : \mathbb{C}\backslash 0 \to \mathbb{C}\backslash 0$ defined by $f(z) = z^2$, is an unramified covering, where the cardinality $|f^{-1}(y)| = 2$, independent of the choice of $y \in f(z)$. If we include the point $z = 0$ to this map, f becomes a branched covering, with branch locus $B(f) = \{0\}$. Note that the Riemann surfaces considered in this example are not compact.

Definition 2.2.6. Let $f : S_1 \to S_2$ be a (possibly ramified) covering of compact Riemann surfaces S_1 and S_2. Let $p_1 \in S_1$, and $f(p_1) = p_2$. The *ramification index* of f at p_1 is the positive integer $m_{p_1}(f)$ such that there is an open neighbourhood U of p_1 so that p_2 only has one preimage in U, i.e., $f^{-1}(p_2) \cap U = p_1$, and for all the other points $z \in f(U)$ it holds that $|f^{-1}(z) \cap U| = m_{p_1}(f)$ (see [7]).

Definition 2.2.7. We say that a point $p \in S_1$ is a *ramification point* if it has $m_p(f)>1$. Then $f(p)$ is a *branch point*.

For an intuitive understanding of the definition of a ramification index we will consider a figure. In Figure (4) below we see a domain X and the map f takes points in X to points in Y. Almost all points in Y have 3 preimages in X except for the blue dots which are the branch points of f. The first branch point of f has one preimage in X. The ramification index of f at this point in X is 3, hence it is a ramification point. The second branch point of Y has two preimages in X. At the preimage on top f has ramification index 2, so this point is a ramification point. At the other preimage of this branch point f has ramification index 1 and hence this point is not a ramification point.

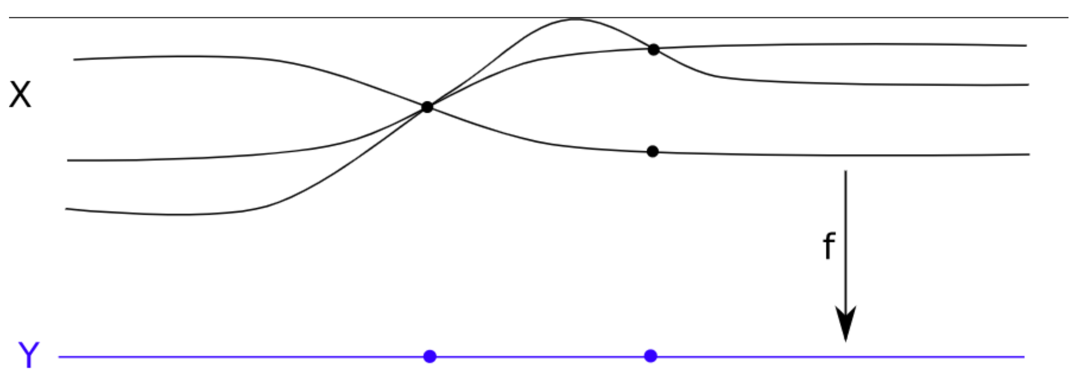

Figure 4: A map f which takes points in X to points in Y. The map has two branch points in Y (the blue dots), and two ramification points in X.

Definition 2.2.8. The ramification locus of $f : S_1 \to S_2$ as defined above is the subset $R(f) \subset S_1$ consisting of the ramification points.

Note that $R(f) \subseteq f^{-1}(B(f))$, but that equality does not need to hold. Also note that an unramified covering is the occurrence of an empty ramification locus $R(f) = \emptyset$.

Definition 2.2.9. Let f be as defined in Definition 2.2.6 . For each point $p_2 \in S_2$, we define the sum of ramification indices of f at the points $p_1 \in S_1$ mapping to p_2 as

$$\mathrm{d}_{\mathrm{p}_2}(f) = \sum_{p_1 \in f^{-1}(p_2)} m_{p_1}(f)$$

.

Proposition 2.2.10. *$d_{p_2}(f)$ is constant, independent of the choice of p_2.*

Proof. The proof can be found in [5], pp. 47-48, proposition 4.8. □

Definition 2.2.11. The *degree* of f, denoted by $\deg(f)$, is the integer $d_{p_2}(f)$.

Note that if a function has degree one it is an isomorphism of Riemann surfaces.

In the following chapter we discuss the Riemann-Hurwitz formula in which we will make use of the above definitions. All Riemann surfaces considered in the following chapters are connected (by Definition 2.1.6) and compact.

3 The Riemann-Hurwitz Formula

In this section we will give a topological proof of the Riemann-Hurwitz formula. This section is based on the work done in [5], pp. 47-53. The idea of the proof is to write the Riemann surfaces as a union of simply connected areas where the branch points and ramification points are on the boundaries, and by connecting the Euler characteristics and the ramification indices. Before we can get to the proof some additional definitions and propositions are required. All Riemann surfaces considered in this section are compact (for further conditions see definition 2.1.6).

3.1 Triangulation and the Euler characteristic

Definition 3.1.1. Let S be a Riemann surface. A *triangulation* of S is a decomposition of S into finitely many closed subsets where each of these closed subsets is homeomorphic to a triangle. Any two of the triangles is either disjoint, share one edge or share one vertex.

One might ask oneself if every compact, connected Riemann surface can be triangulated. There is an important theorem that states that it can. The proof can be found in [8], p. 60.

Definition 3.1.2. Let a triangulation of S be given with v vertices, e edges and t triangles. The *Euler characteristic* is defined by $\chi(S) = v - e + t$.

A *refinement* of a triangulation consists of a sequence of elementary refinements. There are two elementary refinements we can take of a triangulation of a Riemann surface. First we can add a vertex in the interior of a triangle along with three edges connecting the other three vertices. This replaces the first triangle with three triangles. Note that this leaves the Euler characteristic unchanged since $\chi(S) = (v + 1) - (e + 3) + (t + 2) = v - e + t$. A second way is by adding a vertex on an edge shared by two triangles. This creates an extra edge. Then add two new edges to the opposing vertices of the two triangles. This operation bisects both of the triangles, which gives an additional 2 triangles. This also leaves the Euler characteristic unchanged since $\chi(S) = (v+1) - (e+3) + (t+2) = v - e + t$. The elementary refinements leave the Euler characteristic unchanged, hence a refinement leaves it unchanged.

Lemma 3.1.3. *Two triangulations of a Riemann surface S have a common refinement.*

Proof. We start this proof by superimposing two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of S. Let $\mathrm{T}(\mathcal{T}_1)$ and $\mathrm{T}(\mathcal{T}_2)$ be the sets of triangles of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let a triangle $\delta_1 \in T(\mathcal{T}_1)$ and a triangle $\delta_2 \in T(\mathcal{T}_2)$. To prove this lemma it is sufficient to take a triangulation from superimposing δ_1 and δ_2. The intersection of δ_1 and δ_2 consists of finitely many connected components. There are three different cases for such connected components.

Case 1: The connected component is a point. Then the intersection is on the boundary of δ_1 and of δ_2. If the point of intersection is a vertex shared by δ_1 and δ_2 we are done. If not we add a vertex at the point of intersection (in the case that it was not a vertex). In case it was a vertex of one of the triangles we also make it a vertex of the other one. The edge containing the point of intersection is then replaced by two edges. We add an edge connecting this vertex and a vertex of one of the triangles δ_1 and δ_2 such that a triangulation is obtained. This bisects one of the triangles δ_1 and δ_2.

Case 2: The intersection of the boundary of δ_1 and δ_2 consists of points and/or intervals. If the intersection is an edge shared by δ_1 and δ_2 we are done. When considering a vertex of one of the triangles in the intersection we also make this a vertex of the other triangle. In case the intersection is an interval we add vertices at the endpoints of the interval (in case these were not vertices). We then add appropriate edges in δ_1 and/or δ_2.

Case 3: Let $\delta_{1,0}$ and $\delta_{2,0}$ be the interior of δ_1 and δ_2, respectively. We consider the case where the connected components are the intersection $\delta_{1,0} \cap \delta_{2,0}$. We let C be such a component. The boundary of C is a polygon. This polygon can consist of three types of intervals. The interval lies only on the boundary of δ_1, only on the boundary of δ_2, or lies on both the boundaries of δ_1 and δ_2. We add vertices where one type of interval meets another type of interval and add appropriate edges. In every connected component of $\delta_{1,0} \cap \delta_{2,0}$ we add a vertex. This is done arbitrarily. Lastly we add edges connecting this vertex and the vertices of the polygon, with the edges not intersecting one another.

Since we can take a triangulation of the superimposition of δ_1 and δ_2 we can take a refinement of the superimposition of two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ which is a refinement of both $\mathcal{T}_1$ and $\mathcal{T}_2$. Hence any two triangulations of a Riemann surface S have a common refinement. □

Note that for case 3 the number of ways to take a common refinement of two triangulations is uncountable, since the number of ways to place the vertex in the interior of the polygon is uncountable.

Proposition 3.1.4. *The Euler characteristic of S is independent of the choice of triangulation.*

Proof. We saw that a refinement of a triangulation leaves the Euler characteristic unchanged. In Lemma 3.1.3 it was proven that any two triangulations of S have a common refinement. Since any two triangulations have a common refinement and a refinement of a triangulation leaves the Euler characteristic unchanged, the Euler characteristic of S is independent of the choice of triangulation. □

A Riemann surface S, as a topological space, is an orientable manifold S_{top} of real dimension two. There is a classification of such topological spaces. By attaching g handles to a sphere, with $g \in \mathbb{Z}_{\geq 0}$, a compact, orientable real surface X can be obtained ([9], p. 10, section 5. Handles and cross-gaps). We call this g the *genus* of X, denoted by $g(X) = g$. Any two surfaces X and Y with the same orientability and same genus are isomorphic as real manifolds ([10], Chapter 17). By this classification, the genus of a Riemann surface S is defined by

$$g(S) = g(S_{\text{top}})$$

Note that for two Riemann surfaces S_1 and S_2, an isomorphism between $S_{1,\text{top}}$ and $S_{2,\text{top}}$ does not imply that S_1 and S_2 are isomorphic as complex manifolds of dimension one.

Another way of constructing a topological surface T_g of genus $g{>}0$ is by taking a $4g$-gon with sides

$$\alpha_1 \cdot \beta_1 \cdot \alpha_1^{-1} \cdot \beta_1^{-1} \cdot \alpha_2 \cdot \beta_2 \cdot \alpha_2^{-1} \cdot \beta_2^{-1} ... \cdot \alpha_g \cdot \beta_g \cdot \alpha_g^{-1} \cdot \beta_g^{-1}$$

and making suitable identifications on the boundary ([10], Chapter 17, p. 239. Also see [9], p. 9, Section 4. Statement of the classification and orientability).

The genus can also be given in terms of the Euler characteristic.

Definition 3.1.5. The *genus* $g(S)$ of a Riemann surface S in terms of the Euler characteristic is given by the relation

$$\chi(S) = 2 - 2g(S)$$

Intuitively the genus can be seen as the number of 'holes' in a closed surface, as can be seen in Figure (5). An often told joke is that topologists cannot tell their coffee cup with handle apart from their doughnut as they both have genus 1.

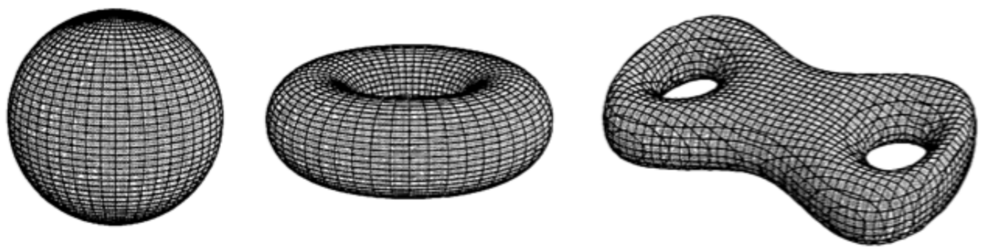

Figure 5: The representation of surfaces of genus 0, 1 and 2.

Example 3.1.6. The Riemann sphere S (see Example 2.1.8) has genus $g(S) = 0$. The Riemann sphere is topologically equivalent to the 2-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^3 = 1\}$ (see [11] for the explicit homeomorphism which establishes the equivalence) which has genus 0. Since the Riemann sphere as a topological space S_{top} has genus $g(S_{\text{top}}) = 0$, the genus of S is $g(S) = g(S_{\text{top}}) = 0$.

Example 3.1.7. The complex torus (Example 2.1.10) T_τ as an orientable manifold $T_{\tau,\text{top}}$ of real dimension 2 is the the torus which has genus $g(T_{\tau,\text{top}}) = 1$. This can be seen by considering $\pi|_{P_0}$ with π and the paralellogram P_0 as defined in Example 2.1.10. The map π restricted to P_0 identifies opposites sides, and no other identifications are made, which gives the structure of a torus ([5], p. 10). Since the complex torus as a topological space has genus $g(T_{\tau,\text{top}}) = 1$, the complex torus T_τ has genus $g(T_\tau) = g(T_{\tau,\text{top}}) = 1$.

We have all the invariants needed to understand and prove the Riemann-Hurwitz formula in the following paragraph.

3.2 The Riemann-Hurwitz Formula

Theorem 3.2.1. (Riemann-Hurwitz Formula) *Let $f\colon S_1 \to S_2$ be a (possibly ramified) covering of compact Riemann surfaces. Then*

$$2g(S_1) - 2 = \deg(f)(2g(S_2) - 2) + \sum_{p \in S_1} (m_p(f) - 1) \tag{1}$$

Proof. Note that the set of ramification points is finite since S_1 is compact, so $\sum_{p \in S_1}(m_p(f) - 1)$ is finite.
We take a triangulation $\mathcal{T}_2$ of S_2 such that each branch point of f is a vertex, and assume that there are v vertices, e edges and t triangles. We pull back this triangulation to S_1 via f. We assume furthermore that the triangulation $\mathcal{T}_1$ of S_1 has v' vertices, e' edges and t' triangles. Every ramification point of f is a vertex on $\mathcal{T}_1$.
Points of $\mathcal{T}_1$ that are not vertices, do not include ramification points. Hence each triangle t in $\mathcal{T}_2$ pulls back to $t' = \deg(f)t$ triangles. The same argument holds for the edges $e' = \deg(f)e$. Let the set of vertices of $\mathcal{T}_2$ be $V(\mathcal{T}_2)$ and the set of vertices of $\mathcal{T}_1$ be $V(\mathcal{T}_1)$. Let $q \in V(\mathcal{T}_2)$. Recall that the definition of the degree of f which was given by $\deg(f) = \sum_{p \in f^{-1}(y)} m_p(f)$, which is independent of the choice of y as proven in Proposition 2.2.8. We can express the number of preimages of this vertex by

$$\begin{aligned}
|f^{-1}(q)| &= \sum_{p \in f^{-1}(q)} 1 \\
&= \deg(f) - \sum_{p \in f^{-1}(q)} m_p(f) + \sum_{p \in f^{-1}(q)} 1 \\
&= \deg(f) + \sum_{p \in f^{-1}(q)} (1 - m_p(f))
\end{aligned}$$

The total number of preimages of vertices of $\mathcal{T}_2$ equals the total number of vertices of $\mathcal{T}_1$. This number is given by

$$\begin{aligned}
v' &= \sum_{q \in V(\mathcal{T}_2)} \big(\deg(f) + \sum_{p \in f^{-1}(q)} (1 - m_p(f))\big) \\
&= \deg(f)v + \sum_{q \in V(\mathcal{T}_2)} \sum_{p \in f^{-1}(q)} (1 - m_p(f)) \\
&= \deg(f)v - \sum_{p \in V(\mathcal{T}_1)} (m_p(f) - 1)
\end{aligned}$$

By the two expressions for the Euler characteristic we can write:

$$\begin{aligned}
2g(S_1) - 2 &= -v' + e' - t' \\
&= -\big(\deg(f)v - \sum_{p \in V(\mathcal{T}_1)} (m_p(f) - 1)\big) + \deg(f)e - \deg(f)t \\
&= \deg(f)(-\chi(S_2)) + \sum_{p \in V(\mathcal{T}_1)} (m_p(f) - 1)) \\
&= \deg(f)(2g(S_2) - 2) + \sum_{p \in S_1} (m_p(f) - 1)
\end{aligned}$$

The last equality holds since every ramification point of f is a vertex of the triangulation of S_1.

$\square$

In the following we will consider three examples of the Riemann-Hurwitz formula.

Example 3.2.2. Let $f : S_1 \to S_2$ be a branched covering of $\deg(f) = 2$ of two Riemann surfaces S_1 and $S_2 = \mathbb{P}^1(\mathbb{C})$, with $g(S_1) = g$ and $g(S_2) = 0$. With the Riemann-Hurwitz formula the number of branch points can be computed. Since the degree of the map is 2, any point with ramification index greater than 1 will have exactly ramification index 2 . From formula (1) follows

$$\begin{aligned}
2g(S_1) - 2 &= \deg(f)(2g(S_2) - 2) + \sum_{p \in S_1} (m_p(f) - 1) \\
2g - 2 &= 2(0 - 2) + \sum_{p \in R(f)} (2 - 1) \\
2g - 2 &= -4 + |R(f)| \\
2g + 2 &= |R(f)|
\end{aligned}$$

Thus, the number of ramification points equals $2g + 2$. Since the degree of f is 2, all the ramification points map to distinct points of S_2. Hence the map has $2g + 2$ branch points.

Example 3.2.3. Let $S_2 = \mathbb{P}^1(\mathbb{C})$ and $Q_1, Q_2, ..., Q_b \in S_2$. Does there exist (S_1, f) with S_1 a Riemann surface and f a branched covering of degree 2 with $B(f) = \{Q_1, ..., Q_b\}$ if b is odd? The answer is negative and can be shown by the Riemann-Hurwitz formula as follows

$$\begin{aligned}
2g(S_1) - 2 &= \deg(f)(2g(S_2) - 2) + \sum_{p \in S_1} (m_p(f) - 1) \\
2g(S_1) - 2 &= -4 + |B(f)| \\
2g(S_1) &= -2 + b \\
g(S_1) &= -1 + \frac{b}{2}
\end{aligned}$$

Hence, for odd b, this (S_1, f) does not exist.

Example 3.2.4. ([4], pp. 57-58, Example 4.4.2) Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a branched covering of degree $\deg(f)$. We let two points $p_1, p_2 \in \mathbb{P}^1(\mathbb{C})$ have ramification indices $m_{p_1}(f) = m_{p_2}(f) = \deg(f)$. We show with the Riemann-Hurwitz formula that the only ramification points of f are the points p_1 and p_2. It is known that the genus of $\mathbb{P}^1(\mathbb{C}) = 0$. By formula (1)

$$
\begin{aligned}
2 \cdot 0 - 2 &= \deg(f)(2 \cdot 0 - 2) + \sum_{p \in \mathbb{P}^1(\mathbb{C})} (m_p(f) - 1) \\
-2 &= -2\deg(f) + \sum_{p \in \mathbb{P}^1(\mathbb{C})} (m_p(f) - 1) \\
2\deg(f) &= 2 + (m_{p_1}(f) - 1) + (m_{p_2}(f) - 1) + \sum_{p \in \mathbb{P}^1(\mathbb{C}) \setminus \{p_1, p_2\}} (m_p(f) - 1) \\
2\deg(f) &= 2\deg(f) + \sum_{p \in \mathbb{P}^1(\mathbb{C}) \setminus \{p_1, p_2\}} (m_p(f) - 1)
\end{aligned}
$$

Hence we see that

$$\sum_{p \in \mathbb{P}^1(\mathbb{C}) \setminus \{p_1, p_2\}} (m_p(f) - 1) = 0$$

and the only ramification points of f are the points p_1 and p_2.

Some other interesting consequences can be drawn from the Riemann-Hurwitz formula. We can see that a Riemann surface S_1 cannot map to a Riemann surface S_2 of higher genus. Also the sum $\sum_{p \in S_1} (m_p(f) - 1)$ is even, and if it is equal to zero then $g(S_1) = \deg(f)g(S_2) - \deg(f) + 1$.

In the next chapter another application will be shown, namely proving Hurwitz's automorphisms theorem via the Riemann-Hurwitz formula.

4 Hurwitz's Automorphisms Theorem

Hurwitz's automorphisms theorem states that the order of the group of automorphisms of a compact Riemann surface of genus $g \geq 2$ is bounded above by the number $84(g-1)$. In this section we will prove this theorem by applying the Riemann-Hurwitz formula. Before proving the theorem we will discuss automorphisms on Riemann surfaces and prove a lemma that we will use in the proof of Hurwitz's automorphisms theorem. Subsequently we prove the theorem and discuss some groups for which the bound is achieved. Recall that Riemann surfaces are connected (for further conditions see definition 2.1.6). All Riemann surfaces considered in this chapter are compact.

4.1 Automorphisms of Riemann surfaces

Definition 4.1.1. An *automorphism* of a Riemann surface S is an isomorphism $f : S \to S$.

The collection of automorphisms of S forms a group under composition, with the identity automorphism as the identity element of the group ([12], p. 18). We refer to this collection of automorphisms of S as the automorphism group of S.

Theorem 4.1.2. (H.A. Schwarz, 1878) *The automorphism group* $\mathrm{Aut}(S)$ *of a Riemann surface* S *with genus* $g \geq 2$ *is finite.*

Proof. The proof can be found in [13], p. 21, Theorem 3.1.2. □

Remark 4.1.3. The automorphism group of S with $g < 2$ is infinite.

Lemma 4.1.4. *Let* $q : S \to Y$*, with* $Y = \mathrm{Aut}(S)\backslash S$*, be the quotient map of a Riemann surface* S*. We let* $|\mathrm{Aut}(S)| = r$*,* $g(S) = p$ *and* $g(Y) = \pi$*. If* $q(x') = q(x)$*, then* $m_{x'}(q) = m_x(q)$*. Further, assume that* $b_1, ..., b_w \in Y$ *are the branch points and* k_i *denotes the ramification index* $m_x(f)$ *of* f *at* x *with* $f(x) = b_i$*. Then we have*

$$\frac{2p-2}{r} = 2\pi - 2 + \sum_{i=1}^{w} \left(1 - \frac{1}{k_i}\right)$$

Proof. This proof is based on the proof done in [12], p. 19, Lemma 6.7. For proving that if $q(x') = q(x)$, then $m_{x'}(q) = m_x(q)$ I refer the reader to [12], p. 29, Proposition B.1. By definition we have for the degree of q that

$$\deg(q) = \sum_{x \in q^{-1}(b_i)} k_i$$

Since for $q(x') = q(x)$ it holds that $m_x(q) = m_{x'}(q)$ we have

$$\deg(q) = |q^{-1}(b_i)| \cdot k_i$$

Furthermore the degree of q is equal r, and the quotient Y is a Riemann surface since $\mathrm{Aut}(S)$ acts holomorphically and effectively on S ([5], pp. 78-79, Theorem 3.4). Hence

$$|q^{-1}(b_i)| \cdot k_i = r$$
$$|q^{-1}(b_i)| = \frac{r}{k_i}$$

We apply the Riemann-Hurwitz formula (formula (1) in Chapter 3). The sum over the ramification points in this formula can be written as

$$\begin{aligned}
\sum_{x\in S}(k_i-1) &= \sum_{b_i\in Y}\sum_{x\in|q^{-1}(b_i)|}(k_i-1)\\
&= \sum_{b_i\in Y}|q^{-1}(b_i)|\cdot(k_i-1)\\
&= \sum_{b_i\in Y}\frac{r}{k_i}(k_i-1)\\
&= r\cdot\sum_{i=1}^{w}(1-\frac{1}{k_i})
\end{aligned}$$

in which the last sum is over the w branch points. It follows from the Riemann-Hurwitz formula that

$$\begin{aligned}
2p-2 &= \deg(q)(2\pi-2)+\sum_{x\in S}(k_i-1)\\
2p-2 &= r\cdot(2\pi-2)+r\cdot\sum_{i=1}^{w}\left(1-\frac{1}{k_i}\right)\\
\frac{2p-2}{r} &= 2\pi-2+\sum_{i=1}^{w}\left(1-\frac{1}{k_i}\right)
\end{aligned}$$

□

The result of this lemma is useful in proving Hurwitz's automorphisms theorem in the next section.

4.2 Hurwitz's Automorphisms Theorem

The proof in this section of Hurwitz's automorphisms theorem is based on the proof in Hurwitz's article "Ueber algebraische Gebilde mit eindeutigen Transformationen in sich" from 1892 ([14], pp. 421-424*)*.

Theorem 4.2.1. (Hurwitz's Automorphisms Theorem) *Let S be a Riemann surface of genus $p \geq 2$, $\mathrm{Aut}(S)$ its automorphism group and $r = |\mathrm{Aut}(S)|$, then*

$$r \leq 84(p-1)$$

Proof. We have seen in Theorem 4.1.2 that $\mathrm{Aut}(S)$ is finite.
We consider the quotient map $q : S \to Y$ as defined in Lemma 4.1.4. Then by Lemma 4.1.4 we have

$$\frac{2p-2}{r} = 2\pi-2+\sum_{i=1}^{w}\left(1-\frac{1}{k_i}\right) \tag{1}$$

which is equivalent to formula (1) in Hurwitz's article [14], p.421.
Under the assumption of $p \geq 2$ we take a closer look at equation (1). We distinguish three cases for the genus of Y; $\pi \geq 2$, $\pi = 1$ and $\pi = 0$.

Case 1: $\pi \geq 2$. Equation (1) tells us that $\frac{2p-2}{r} \geq 2$ and hence

$$r \leq p - 1 \tag{2}$$

Case 2: $\pi = 1$. In this case w cannot be 0, because then equation (1) would result in $p = 1$ which violates the assumption of $p \geq 2$. It is therefore that

$$\frac{2p-2}{r} \geq \left(1 - \frac{1}{k_1}\right) \geq \frac{1}{2}$$

where k_i is at least 2. It follows that

$$r \leq 4(p-1) \tag{3}$$

Case 3: $\pi = 0$. In this case we have

$$\begin{aligned}\frac{2p-2}{r} &= -2 + \sum_{i=1}^{w}\left(1 - \frac{1}{k_i}\right) \\ &= w - 2 - \sum_{i=1}^{w}\frac{1}{k_i}\end{aligned} \tag{1'}$$

in which w is at least three. We distinguish three cases for w; $w \geq 5$, $w = 4$ and $w = 3$.

∘ Case 1: $w \geq 5$. In this case it follows from equation (1') that

$$\frac{2p-2}{r} \geq \frac{w}{2} - 2 \geq \frac{5}{2} - 2 = \frac{1}{2}$$

and

$$r \leq 4(p-1) \tag{4}$$

∘ Case 2: $w = 4$. From equation (1') it follows that

$$\frac{2p-2}{r} = 2 - \sum_{i=1}^{4}\frac{1}{k_i}$$

Assuming that $k_1 \leq k_2 \leq k_3 \leq k_4$, we have the following possibilities arranged in table (5) ([14], p. 423, Table (5)).

(5)

k_1	k_2	k_3	k_4	$\frac{2p-2}{r}$	
> 2	> 2	> 2	> 2	$\geqq \frac{2}{3}$	$r \leqq 3(p-1)$
$= 2$	> 2	> 2	> 2	$\geqq \frac{1}{2}$	$r \leqq 4(p-1)$
$= 2$	$= 2$	> 2	> 2	$\geqq \frac{1}{3}$	$r \leqq 6(p-1)$
$= 2$	$= 2$	$= 2$	> 2	$\geqq \frac{1}{6}$	$r \leqq 12(p-1)$

○ Case 3: $w = 3$. In this last case, from equation (1') follows

$$\frac{2p-2}{r} = 1 - \sum_{i=1}^{3} \frac{1}{k_i}$$

By assuming $k_1 \leq k_2 \leq k_3$, we have the following possibilities arranged in table (6) ([14], p. 423, table 6). Note that there has been made a correction in table (6) at the bottom row. In that row k_3 has to be greater than 6 in order to meet the assumption of $p \geq 2$.

(6)

k_1	k_2	k_3	$\frac{2p-2}{r}$	
> 3	> 3	> 3	$\geqq \frac{1}{4}$	$r \leqq 8(p-1)$
$= 3$	> 3	> 3	$\geqq \frac{1}{6}$	$r \leqq 12(p-1)$
$= 3$	3	> 3	$\geqq \frac{1}{12}$	$r \leqq 24(p-1)$
$= 2$	> 4	> 4	$\geqq \frac{1}{10}$	$r \leqq 20(p-1)$
$= 2$	$= 4$	> 4	$\geqq \frac{1}{20}$	$r \leqq 40(p-1)$
$= 2$	$= 3$	> 6	$\geqq \frac{1}{42}$	$r \leqq 84(p-1)$.

As in Hurwitz's paper ([14], pp. 423-424):

> **Ein Blick auf die Ungleichungen (2), (3), (4), (5), (6) zeigt, dass *r* die Zahl 84(*p* — 1) nicht übersteigen kann. Mit anderen Worten, es besteht der Satz:**
>
> ***„Die Anzahl der eindeutigen Transformationen in sich, welche eine Riemann'sche Fläche vom Geschlecht p > 1 besitzen kann, beträgt im Maximum* 84(*p* — 1).“**

where the "Ungleichungen (2), (3), (4), (5), (6)" in the paper are equivalent to the inequalities (2), (3), (4), (5) and (6) above.

□

Definition 4.2.2. A *Hurwitz group* is the group of automorphisms of a Riemann surface for which the order is $r = 84(g-1)$.

Definition 4.2.3. A Riemann surface of genus g with $84(g-1)$ automorphisms is called a *Hurwitz surface.*

Example 4.2.4. (The Klein quartic, F. Klein, 1878) Define $\mathcal{C} \subset \mathbb{P}^2$ to be the set of zeros of the polynomial $X^3Y + Y^3Z + Z^3X$; this curve has genus $g(\mathcal{C}) = 3$. Then $\mathcal{C}(\mathbb{C}) = S$ is a compact Riemann surface of genus $g(S) = 3$ (see Chapter 5). This Riemann surface has an automorphism group of order $|\mathrm{Aut}(\mathcal{C})| = 168 = 84(g-1)$, hence S is a Hurwitz surface.
One can easily see that there is an automorphism of order 3 for $\mathcal{C}$, namely by cyclic permutation of the variables.
It is less trivial to see that there is an automorphism of order 7. Let the seventh root of unity be $\zeta := e^{2\pi i/7}$. With the following transformation

$$X \to X$$

$$Y \to \zeta Y$$

$$Z \to \zeta^5 Z$$

we find

$$X^3Y + Y^3Z + Z^3X \to \zeta(X^3Y + Y^3Z + Z^3X)$$

If $(X : Y : Z)$ is a zero of $\mathcal{C}$, then $(X : \zeta Y : \zeta^5 Z)$ is also a zero of $\mathcal{C}$. Hence we see that this transformation is an automorphism of order 7 for $\mathcal{C}$.

5 Riemann surfaces and algebraic curves

In this section we will briefly discuss the equivalence of the category of compact Riemann surfaces and the category of complete, non-singular, irreducible algebraic curves over $\mathbb{C}$. The main result of this equivalence is that over $\mathbb{C}$ as base field, results for Riemann surfaces can be phrased in an equivalent form for the related algebraic curves, and conversely. This is also the case for the Riemann-Hurwitz formula.

Definition 5.0.1. An *algebraic curve* C is an algebraic variety of dimension one defined over κ, in which κ is an arbitrary base field.

Equivalence 5.0.2. Over $\kappa = \mathbb{C}$ as base field, consider the following categories:

(i) Compact, connected Riemann surfaces; as morphisms consider finite holomorphic maps.
(ii) Complete, non-singular, irreducible algebraic curves over $\mathbb{C}$; as morphisms consider finite morphisms.

These categories are equivalent. To understand this result I refer the reader to [15], pp. 1-42. Every compact Riemann surface can be represented as an irreducible, nonsingular, algebraic curve over $\kappa = \mathbb{C}$.

Definition 5.0.3. Let C be an algebraic curve over $\mathbb{C}$, with $S = C(\mathbb{C})$ the related Riemann surface. The definition of its genus is given by Definition 3.1.5.

It is due to equivalence 5.0.2 that for $\kappa = \mathbb{C}$ as base field many results for Riemann surfaces can be stated in their equivalent form for the related algebraic curves and vice versa. This is also the case for the Riemann-Hurwitz formula in Chapter 3. Under this equivalence, the proof of the Riemann-Hurwitz formula in Chapter 3 amounts to the same as proving it for complete, non-singular irreducible algebraic curves over $\mathbb{C}$ as base field.
Also Hurwitz's automorphisms theorem can be rephrased for an algebraic curve.

Theorem 5.0.4. *Let $\mathcal{C}$ be an algebraic curve over $\mathbb{C}$ of genus $g \geq 2$. Then its automorphism* $\mathrm{Aut}(\mathcal{C})$ *has order*

$$|Aut(\mathcal{C})| \leq 84(g-1)$$

Also the proof of Hurwitz's automorphism theorem in Chapter 4 amounts to the same as proving it for complete, non-singular irreducible algebraic curves over $\mathbb{C}$ as base field.

6 Summary

We started this thesis by studying some basic theory on compact Riemann surfaces and (possibly ramified) coverings in Chapter 2. We considered some examples of Riemann surfaces, including the Riemann sphere. Subsequently we discussed theory on triangulations of Riemann surfaces in Chapter 3. We then used this theory to prove the Riemann-Hurwitz formula. For a given (possibly ramified) covering $f : S_1 \to S_2$ of two compact and connected Riemann surfaces, this formula relates the genus of S_1, the genus of S_2, the degree of f and the ramification indices. One of the applications of the Riemann-Hurwitz formula is that it can be used to prove Hurwitz's automorphisms theorem. The latter states that the order of the group of automorphisms of a compact Riemann surface of $g \geq 2$ is bounded above by $84(g-1)$. We studied and proved this in Chapter 4, with use of Hurwitz's proof from 1892. Thereafter we shortly discussed the equivalence between the category of compact Riemann surfaces and the category of complete, non-singular, irreducible algebraic curves over $\mathbb{C}$ in Chapter 5. By this equivalence we saw that proving the Riemann-Hurwitz formula and Hurwitz's automorphisms theorem for compact Riemann surfaces, is equivalent to proving this formula for complete, non-singular irreducible algebraic curves over $\mathbb{C}$.

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