MSc Project : Fast and Stable Representation for the Semi-Graphoid Models

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1 Introduction

Many are the mathematical models consisting of random variables over which joint probability distributions are defined. A successful and well-known class of such models are the probabilistic graphical models [3, 4, 5, 7, 9]. These models represent the conditional independences between a set of variables in a graphical structure and associate with this structure a collection of (conditional) probability distributions. Especially probabilistic networks are included in this class which allow computing probabilities concerning the states of the variables. In order to build a probabilistic network or any other graphical model which obeys a unique joint probability distribution, we need the independence relation between the variables and the values for the conditional probabilities that derive from these relations.

Each independence relation satisfies certain properties. According to these properties, we can distinguish different categories of relations. A well-established basic category is the group of relations satisfying the *semi-graphoid* properties, which are simply called *semi-graphoid* models. Studený [11] has presented computationally efficient solutions for *closure* computation for semi-graphoid models. Closure is the set of all independencies that can be generated by the (semi-)graphoid properties to arrive at a model. Baioletti et al. [2] improved Studený's work by computing the closure for semi-graphoid models through a simple and unique inference rule which requires fewer computations. Additionally, they extended their findings to the models satisfying the *graphoid* properties.

In this research we focus on the notion of closure for semi-graphoid models. We derive various new properties of a closure and introduce a new algorithm for computing in a fast way a set of independences that describes as best as possible the closure and consequently the model. The new algorithm builds upon the concept of *non-symmetric maximal* sets and exploits conditions for the exclusion of independences from the closure computation. The algorithm uses a simpler inference rule than the earlier algorithm by Baioletti et al.

Moreover, we study semi-graphoid models which contain a special type of independences, the *stable* independences. For these models, we also present an algorithm for the closure computation based on *non-symmetric maximal* sets, conditions for the exclusion of independences and a unique inference rule. This algorithm improves the existing algorithm by De Waal and Van der Gaag [6].

2 Preliminaries

Before reviewing the semi-graphoid properties, we introduce some notations. Let $S=\{X_1, \ldots, X_n\}$ be a finite non-empty set of $n \geq 2$ random variables. We define a *triplet*, denoted with $\langle A,B|C\rangle$, to be any statement where A,B,C are disjoint subsets of S with $A, B \neq \emptyset$. The triplet indicates that A, B are independent given C; in terms of a probability distribution P, the triplet thus indicates that $P(A \cap B|C) = P(A|C)P(B|C)$. The definition of semi-graphoid models now is as follows.

Definition 2.1 *: Let $S^{(3)}$ be the set of all triplets $\langle A,B|C\rangle$. A semi-graphoid model is a set $J \subseteq S^{(3)}$ which satisfies the following properties:*

G1 *: if $\langle A,B|C\rangle \in J$, then $\langle B,A|C\rangle \in J$ (Symmetry)*

G2 *: if $\langle A,B|C\rangle \in J$, then $\langle A,B'|C\rangle \in J$ for any non-empty set $B' \subseteq B$ (Decomposition)*

G3 *: if $\langle A,B_1 \cup B_2|C\rangle \in J$ with B_1, B_2 disjoint, then $\langle A,B_1|C \cup B_2\rangle \in J$ (Weak Union)*

G4 *: if $\langle A,B|C \cup D\rangle \in J$ and $\langle A,C|D\rangle \in J$, then $\langle A,B \cup C|D\rangle \in J$ (Contraction)*

In addition, a graphoid model is a semi-graphoid model which satisfies also the property:

G5 *: if $\langle A,B|C \cup D\rangle \in J$ and $\langle A,C|B \cup D\rangle \in J$, then $\langle A,B \cup C|D\rangle \in J$ (Intersection)*

The properties G1-G4 are called the semi-graphoid properties and they are also referred to as Pearl's Axioms [8] for semi-graphoid models. Basically, models under the semi-graphoid properties are conditional independence models for joint probability distributions. The properties G1-G5 are called the graphoid properties and are related to conditional independence models for strictly positive probability distributions. Throughout this thesis, we will refer to the properties G1-G5 as rules. The semi-graphoid models form a more general model family than the graphoid models. In section 4 we elaborate more on semi-graphoid models which include *stable* triplets. Stable triplets derive from the concept of stability of independence. Stability indicates independences that remain to hold as the set of triplets is changing through application of the inference rules.

Based upon the notion of triplet and the (semi-)graphoid properties, we now define the notion of closure. Given a starting set of triplets, its closure is the set of all triplets that can be generated by the (semi-)graphoid properties to arrive at a model. For a semi-graphoid model, given a starting set of triplets $J \subset S^{(3)}$, a triplet $\theta \in S^{(3)}$ being derived from J by means of G1-G4 will be written as $J \vdash^* \theta$. The closure of J is denoted with $\overline{J}$, and is defined as $\overline{J} = \{\theta \in S^{(3)} : J \vdash^* \theta\}$. Note that $\overline{J}$ can be produced by J through finite application of the rules G1-G4. For a graphoid model, the respective closure can be computed through finite applications of the rules G1-G5.

By applying the rules to the triplets of a model we produce new independence statements. Now given a set of triplets J and a triplet θ, a well-known problem is to decide whether $\theta \in \overline{J}$. This is called the "implication problem" and it has been already faced in [12]. It is clear that the "implication problem" is closely related to the computation of the closure. However, the computation of all elements included in the closure of a set J is expensive as the size of a closure $\overline{J}$ can be exponentially larger than that of J. Solving the "implication problem" by generating the full closure therefore is not feasible.

Extensive work on closure computation has been conducted by Studený for models under the semi-graphoid properties. He also provided a closure procedure for models under the graphoid properties [11], which was later improved by Baioletti et al. [2] by a fast closure algorithm based on a unique inference rule. In this thesis, we present our idea of providing also a unique inference rule for models under the semi-graphoid properties in order to improve[1] and simplify the procedure Studený proposed. We further implement the idea of stable triplets, which is already studied in [6], through a unique inference rule as well.

We have mentioned that a closure can be exponentially larger than the initial set of a model. Studený managed to generate a set reduced in size compared to the closure that still contains the same information. To describe his procedure, we introduce the notions of *generalized inclusion*(*g-inclusion*) and *dominance*. G-inclusion is used for identifying single triplets from which other triplets can be generated. More specifically, the definition is as follows.

Definition 2.2: *Let $J \subseteq S^{(3)}$. We say that any triplet $\theta_1 \in J$ which can be generated from $\theta_2 \in J$ by means of G1-G3, is* g-included *in θ_2 and we denote this with $\theta_1 \sqsubseteq \theta_2$. A triplet which is not g-included in any other triplet in J is called* maximal.

Note that the G4 rule is not included in the definition as g-inclusion deals with triplets generated from a single triplet and G4 requires two triplets in order to be applied. The following property [2] now states the necessary conditions for g-inclusion between two triplets.

Lemma 2.3: *Let $J \subseteq S^{(3)}$ and let $\theta_1 = \langle A_1,B_1|C_1\rangle$, $\theta_2 = \langle A_2,B_2|C_2\rangle \in J$. Then, $\theta_1 \sqsubseteq \theta_2$ if and only if the following conditions hold:*

1) $C_2 \subseteq C_1 \subseteq X_2$

[1] in terms of complexity

2) *either* $A_1 \subseteq A_2$ *and* $B_1 \subseteq B_2$,
or $B_1 \subseteq A_2$ *and* $A_1 \subseteq B_2$

where $X_i = A_i \cup B_i \cup C_i$, i=1,2

The conditions stated in the above lemma capture all possible situations in which θ_1 can be generated from $\theta_2 = \langle$A,$B|C\rangle$ through the rules G1-G3. By applying decomposition(G2) and weak union(G3) to θ_2 as many times as possible, we end up with triplets $\theta_1 = \langle$A,$B'|C'\rangle$ with $B' \subseteq B$ and $C \subseteq C'$. Combining these two rules with symmetry (G1), we further find triplets $\theta_1 = \langle A',B'|C'\rangle$ and $\theta_1 = \langle B',A'|C'\rangle$ with $A' \subseteq A$. Hence, in each case θ_1 satisfies the two aforementioned conditions.

Another useful definition is that of *dominance* [11] and it is closely related to g-inclusion.

Definition 2.4 *: Let $J \subseteq S^{(3)}$. We say that any triplet $\theta_1 \in J$ which can be generated from $\theta_2 \in J$ by means of G2, G3, G2s and G3s is* dominated *by θ_2, denoted with $\theta_1 \prec \theta_2$, where G2s and G3s are symmetric versions of G2, G3 and are defined as follows:*

G2s *: if* $\langle A,B|C\rangle \in J$, *then* $\langle A',B|C\rangle \in J$ *for any non-empty* $A' \subseteq A$

G3s *: if* $\langle A_1 \cup A_2,B|C\rangle \in J$, *then* $\langle A_1,B|C \cup A_2\rangle \in J$

A triplet which is not dominated by any other triplet in J is called dominant.

The following property now states the necessary conditions for dominance between two triplets.

Lemma 2.5*: Let $J \subseteq S^{(3)}$ and let $\theta_1 = \langle A_1,B_1|C_1\rangle$, $\theta_2 = \langle A_2,B_2|C_2\rangle \in J$. Then, $\theta_1 \prec \theta_2$ if and only if the following conditions hold:*

1) $C_2 \subseteq C_1 \subseteq X_2$

2) $A_1 \subseteq A_2$ *and* $B_1 \subseteq B_2$

Evidently, there is a close relation between g-inclusion and dominance. Baioletti et al. [2] clarify the relation between these two notions: For any two triplets θ_1 and θ_2,

$$\theta_1 \sqsubseteq \theta_2 \text{ if and only if } \theta_1 \prec \theta_2 \text{ or } \theta_1 \prec \theta_2^T. \tag{1}$$

with θ_2^T the transpose of θ_2. The transpose θ^T is the triplet generated from θ by applying the rule of symmetry. In particular, if $\theta = \langle A,B|C\rangle$, then $\theta^T = \langle B,A|C\rangle$.

The definitions of g-inclusion and dominance show that we can construct triplets in which other triplets are included. Using these definitions we can build a set of triplets which has the same information as the closure. The idea is that the generated set contains all dominant triplets from a closure. We will refer to this set as a *basis* of the closure.

3 Basis of Closure

In this section we extend the properties reviewed in the previous section to sets of triplets in order to find a basis of a closure which is reduced in size. Any reduced basis contains the *dominant* triplets which are extensively discussed in Section 3.1. In Section 3.2 we present an algorithm for closure computation based on a single inference rule by Baioletti et al. [2]. This algorithm requires fewer iterations than the existing procedures and potentially produces a more reduced closure. In Sections 3.3 and 3.4, we propose two further improvements which are based on the concept of *non-symmetric maximal triplet* and the conditions for the exclusion of triplets from the closure computation procedure.

3.1 Dominant Triplets

In the previous section, we reviewed the concepts of g-inclusion and dominance and now we will connect them with the contraction rule(G4) and thereby extend them to sets of triplets. The following lemma, whose proof is stated in [2], shows an interesting relation between them.

Lemma 3.1: *Let $\theta_1, \theta_2, \theta_3, \theta_4 \in S^{(3)}$ be triplets such that $\theta_1 \sqsubseteq \theta_3$, $\theta_2 \sqsubseteq \theta_4$. If $\theta_1, \theta_2 \vdash_{G4} \theta$ and $\theta_3, \theta_4 \vdash_{G4} \theta'$, then $\theta \sqsubseteq \theta'$*

From this lemma we have that the contraction of two g-included triplets θ_1, θ_2 is itself g-included in the contraction of the triplets θ_3, θ_4. Yet, the contraction rule cannot always be applied to two randomly chosen triplets. Suppose that the two triplets θ_3 and θ_4 belong to the initial set J. Even when the contraction rule cannot be applied, we might be able to generate g-included triplets from θ_3 and θ_4, to which contraction can be applied. The following example clarifies the previous statement.

Example 1. Let $J \subset S^{(3)}$ with $S = \{1, 2, 3, 4, 5, 6\}$ and let $\tau = \langle\{1, 2\},\{4, 5\}|\{6\}\rangle$, $\phi = \langle\{1, 3, 6\},\{2\}|\{4, 5\}\rangle$ both be in J. We cannot apply the contraction rule to τ and ϕ. Yet, we can generate the g-included triplet $\tau' = \langle\{1\},\{4, 5\}|\{6\}\rangle$ by applying decomposition to τ and the triplet $\phi' = \langle\{1\},\{2\}|\{4, 5, 6\}\rangle$ by applying decomposition and weak union to ϕ. Now, we can apply the contraction rule to the triplets τ', ϕ' and produce $\langle\{1\},\{2, 4, 5\}|\{6\}\rangle$ which is not g-included in τ or ϕ. Hence, if J consisted only of τ and ϕ, then both of them plus the generated one would be dominant/maximal triplets.

Before further studying application of the contraction rule, it is useful to extend g-inclusion to sets of triplets. The definition is as follows.

Definition 3.2 *: Let $H, J \subset S^{(3)}$. H is g-included in J, denoted with $H \sqsubseteq J$, if and only if $\forall \theta \in H$, $\exists \theta' \in J$ such that $\theta \sqsubseteq \theta'$.*

The associated definition of dominance over sets of triplets is shown below.

Definition 3.3 *: Let $H, J \subset S^{(3)}$. J dominates H, or H is dominated by J, denoted with $H \prec J$, if and only if $\forall \theta \in H$, $\exists \theta' \in J$ such that $\theta \prec \theta'$.*

Taking into account property (1), we now extend Definition 3.2 to involve the concept of dominance.

Lemma 3.4 *: Let $H, J \subset S^{(3)}$. $H \sqsubseteq J$ if and only if $\forall \theta \in H$, $\exists \theta' \in J$ such that $\theta \prec \theta'$ or $\theta \prec \theta'^T$.*

Studený now provided an operator for creating dominant triplets in (semi-)graphoid models by combining the notion of dominance with the contraction rule. In this thesis, we will focus only on his findings concerning semi-graphoid models. Studený introduced the set $H_{G4}(\theta_1, \theta_2) = \{\tau : \exists \theta_1' \prec \theta_1, \exists \theta_2' \prec \theta_2 \text{ such that } \theta_1', \theta_2' \vdash_{G4} \tau\}$ and proved that if it is possible to apply his new operator to the triplets θ_1 and θ_2, then the triplet produced dominates any other triplet in $H_{G4}(\theta_1, \theta_2)$. The conditions for creating the τ triplets, as well as the operator for creating the dominant ones are stated below. These are the conditions Baioletti et al. proposed, which are slightly stronger than those of Studený's .

Lemma 3.5 *Let $J \subseteq S^{(3)}$ and let $\theta_1 = \langle A_1, B_1 | C_1 \rangle, \theta_2 = \langle A_2, B_2 | C_2 \rangle \in J$. Furthermore, consider the conditions*

$$A_1 \cap A_2 \neq \emptyset \tag{2a}$$

$$C_1 \subseteq X_2 \text{ and } C_2 \subseteq X_1 \tag{2b}$$

$$B_2 \backslash C_1 \neq \emptyset \tag{2c}$$

$$B_1 \cap X_2 \neq \emptyset \tag{2d}$$

$$|(B_2 \backslash C_1) \cup (B_1 \cap X_2)| \geq 2 \tag{2e}$$

$$\text{where } X_i = A_i \cup B_i \cup C_i \text{ for } i = 1, 2$$

Then, $gc(\theta_1, \theta_2) = \langle A_1 \cap A_2, (B_2 \backslash C_1) \cup (B_1 \cap X_2) | C_1 \cup (A_1 \cap C_2) \rangle$ and

- *$H_{G4}(\theta_1, \theta_2) = \emptyset$ if and only if at least one of the conditions (2a)-(2e) does not hold.*
- *if $H_{G4}(\theta_1, \theta_2) \neq \emptyset$, then $gc(\theta_1, \theta_2)$ is included in $H_{G4}(\theta_1, \theta_2)$ and dominates any other triplet from $H_{G4}(\theta_1, \theta_2)$*

The five conditions (2a)-(2e) are necessary for creating dominated triplets from θ_1 and θ_2 to which the contraction rule can be applied. The operator for creating the dominant triplets from θ_1 and θ_2 is denoted with $gc(\theta_1, \theta_2)$. Now we will give an informal explanation of how the dominant triplets are generated and why the conditions mentioned above are necessary; a formal proof can be found in [10]

Let $\theta_1 = \langle A_1, B_1 | C_1 \rangle, \theta_2 = \langle A_2, B_2 | C_2 \rangle \in J \subseteq S^{(3)}$ as stated in the lemma. In order to create dominant triplets we should generate, if possible, from θ_1 and θ_2 triplets to which the contraction rule can be applied. The rules G1-G3 are applied to a single triplet only and the generated triplets are either dominated by or g-included in this triplet. By applying the contraction rule to two triplets dominated by θ_1 and θ_2, we cannot be guaranteed that the new triplet is also dominated, as we already saw in Example 1. We recall that in order to apply the contraction rule to two triplets τ and ϕ their form should be $\tau = \langle X, W | Z \rangle$ and $\phi = \langle X, Y | W \cup Z \rangle$. As θ_1 and θ_2 might not have this form, we have to generate appropriate triplets by applying the rules G1-G3. We now first observe that for θ_1 and θ_2 it should hold that $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$ (2b); otherwise there is no possibility for creating the conditioning parts of the triplets τ and ϕ. Now we recall that for any two sets of variables X and Y, it always holds that

$$X = (X \cap Y) \cup (X \backslash Y) \tag{3}$$

If we apply equation (3) to a component of θ_1, for example A_1, and X_2 we have the following result

$$A_1 = (A_1 \cap X_2) \cup (A_1 \backslash X_2) = (A_1 \cap (A_2 \cup B_2 \cup C_2)) \cup (A_1 \backslash X_2) \tag{4}$$

By applying the distributive property to equation (4), we find that

$$A_1 = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (A_1 \cap C_2) \cup (A_1 \backslash X_2) \tag{5}$$

Assuming that condition (2b) holds and taking, we can now create triplets of the form $\langle X, B_1 | C_1 \cup (A_1 \cap C_2) \rangle$ and $\langle X, B_2 | C_2 \cup (A_2 \cap C_1) \cup (A_2 \cap B_1) \rangle$ by applying decomposition and weak union to θ_1 and θ_2 respectively, where $X = A_1 \cap A_2$. Note that X has to be non-empty for both triplets to be defined, from which we have that condition (2a) must hold. The next step is based on the requirement that one of the newly generated triplets should have the form $\tau = \langle X, W | Z \rangle$. We choose the triplet $\langle A_1 \cap A_2, B_1 | C_1 \cup (A_1 \cap C_2) \rangle$ to generate τ. By applying the decomposition rule we transform it to $\langle A_1 \cap A_2, B_1 \cap X_2 | C_1 \cup (A_1 \cap C_2) \rangle$. It is clear that we need $B_1 \cap X_2 \neq \emptyset$, that is, we need condition (2d) to hold, if we are to take the $\langle A_1 \cap A_2, B_1 \cap X_2 | C_1 \cup (A_1 \cap C_2) \rangle$ for $\tau = \langle X, W | Z \rangle$. Note that we made the least possible changes to the triplet θ_1 so as to keep the size of X as large as we can, and the sizes of W and Z as small as we can. The last step now is to generate from the triplet $\langle A_1 \cap A_2, B_2 | C_2 \cup (A_2 \cap C_1) \cup (A_2 \cap B_1) \rangle$ a triplet of the form $\langle X, Y | W \cup Z \rangle$. This can be achieved by applying weak union and generating the triplet $\phi = \langle A_1 \cap A_2, (B_2 \backslash (C_1 \cup B_1) | C_2 \cup (A_2 \cap C_1) \cup (A_2 \cap B_1) \cup (B_2 \cap C_1) \cup (B_2 \cap B_1) \rangle$.

By analysing the generated triplets, we will see why the remaining two conditions are necessary for the contraction rule to be applied. The two generated triplets are

$\tau = \langle A_1 \cap A_2, B_1 \cap X_2 | C_1 \cup (A_1 \cap C_2) \rangle$ and
$\phi = \langle A_1 \cap A_2, (B_2 \backslash (C_1 \cup B_1)) | C_2 \cup (A_2 \cap C_1) \cup (A_2 \cap B_1) \cup (B_2 \cap C_1) \cup (B_2 \cap B_1) \rangle$

Expanding the components of the triplets by taking into account equation (5), we find the following properties:

- $B_1 \cap X_2 = (B_1 \cap A_2) \cup (B_1 \cap B_2) \cup (B_1 \cap C_2) = W$
- $B_2 \backslash (C_1 \cup B_1) = (B_2 \cap A_1) \cup (B_2 \backslash X_1) = Y$
- $C_1 \cup (A_1 \cap C_2) = (C_1 \cap A_2) \cup (C_1 \cap B_2) \cup (C_1 \cap C_2) \cup (A_1 \cap C_2) = Z$
- $C_2 \cup (A_2 \cap C_1) \cup (A_2 \cap B_1) \cup (B_2 \cap C_1) \cup (B_2 \cap B_1) = (C_2 \cap A_1) \cup (C_2 \cap B_1) \cup (C_2 \cap C_1) \cup (A_2 \cap C_1) \cup (A_2 \cap B_1) \cup (B_2 \cap C_1) \cup (B_2 \cap B_1) = W \cup Z$

By applying the contraction rule we get the triplet $\langle X, W \cup Y | Z \rangle = \langle A_1 \cap A_2, (B_1 \cap X_2) \cup (B_2 \backslash (C_1 \cup B_1)) | C_1 \cup (A_1 \cap C_2) \rangle$ and because $(B_2 \cap B_1) \subseteq (B_1 \cap X_2)$, $Y = (B_2 \backslash (C_1 \cup B_1))$ can be written as $(B_2 \backslash C_1)$. We conclude that the last property to be satisfied is $Y \neq \emptyset$. This property prompts condition (2c) as it implies that we should have $B_2 \backslash C_1 \neq \emptyset$. We further need $B_2 \backslash B_1 \neq \emptyset$: in case we had $B_2 \backslash B_1 = \emptyset$, this would imply $B_2 \backslash C_1 = (B_1 \cap B_2)$, which would mean that we cannot apply weak union as we would then find a triplet having the second component equal to the empty set[2]. Hence, we can reformulate the condition $B_2 \backslash C_1 \neq (B_1 \cap B_2)$, by saying that we need $B_1 \cap X_2$ and $B_2 \backslash C_1$ to have at least one disjoint element, which amounts to condition (2e).

Studený proved that with *gc* operator we can construct dominant triplets [10]. He further gave a procedure for generating all the dominant triplets of a semi-graphoid model [11]. The set produced by this procedure is referred to as the *basis* of the closure. His procedure starts by adding to the initial set the symmetric triplets of the existing ones if they are not already included. Then the operator is applied to any pair of triplets where the five aforementioned conditions are satisfied. Finally, the dominated triplets are removed and the procedure is repeated on the produced set till no new triplets can be generated.

The basis of the closure reflects the same information as the closure itself and we will analyse some properties to clarify this. By adding symmetric triplets, Studený's operator covers the extra cases to which G4 can be applied. Based on this observation, Baioletti et al. [2] introduced the following set $GC(\theta_1, \theta_2)$:

$$GC(\theta_1, \theta_2) = \{gc(\theta_1, \theta_2), gc(\theta_1, \theta_2^T), gc(\theta_1^T, \theta_2), gc(\theta_1^T, \theta_2^T)\}$$

Through this set Baioletti et al. proposed a new generalized inference rule which is:

$G4^*$(generalized contraction): from θ_1, θ_2 deduce any triplet $\tau \in GC(\theta_1, \theta_2)$

The advantage of the rule above is that we do not need to add symmetric triplets to a set in order to generate new (dominant) ones. Now by defining the set J^{G4^*} to be the closure of an initial set J by applying a finite number of times the rule $G4^*$, the following properties concerning the closure $\overline{J}$ hold[2].

[2]informally it is $\langle A_1 \cap A_2, \emptyset | C_2 \cup (A_2 \cap C_1) \cup (A_2 \cap B_1) \cup (B_2 \cap C_1) \cup (B_2 \cap B_1) \rangle$

Lemma 3.6: *Let $J \subseteq S^{(3)}$ and let $\overline{J}$ and J^{G4^*} be as defined above . Then, it holds that*

$$J^{G4^*} \subseteq \overline{J} \tag{6a}$$

$$\overline{J} \sqsubseteq J^{G4^*} \tag{6b}$$

Property (6a) states that the set J^{G4^*} found through application of the rule $G4^*$ is included in the closure of J. Property (6b) implies all that triplets in $\overline{J}$ are g-included in J^{G4^*}. Triplets that they themselves or their symmetries are g-included in others, will be called them *redundant*. It is possible that J^{G4^*} contains some redundant triplets, but the number of triplets produced is bounded with respect to the number of variables each triplet contains. In particular, for a set J with N triplets, the maximum number of generated triplets is $4N^2$. This is because each pair of triplets θ_1, θ_2 produces at most 8 triplets through $G4^*$, as we have to take into account that $GC(\theta_1, \theta_2)$ is different from $GC(\theta_2, \theta_1)$. Therefore, we have 8 times the summation of all possible triplet pairs from a set with size N, which is $N^2/2$.

At this point we introduce the notion of *maximal* triplet set with respect to g-inclusion.

Definition 3.7: *Let $J \subseteq S^{(3)}$. A* maximal *triplet set for J, denoted with $J_{/\sqsubseteq}$, is a set of triplets such that*

$$J_{/\sqsubseteq} = \{\tau \in J : \nexists \overline{\tau} \in J \text{ with } \overline{\tau} \neq \tau, \tau^T \text{ such that } \tau \sqsubseteq \overline{\tau}\}$$

A *maximal* triplet set thus contains only *maximal* triplets. Note that it is not unique. Basically, given a starting set J, then the set $J_{/\sqsubseteq}$ contains a essential information in terms of independence. Using the definition of maximal triplet set, the following property can now be proved [2].

Lemma 3.8 : *Let $J \subseteq S^{(3)}$ and let $\overline{J}_{/\sqsubseteq}$ and $J^{G4^*}_{/\sqsubseteq}$ be as defined above. Then, it holds that*

$$\overline{J}_{/\sqsubseteq} \sqsubseteq J^{G4^*}_{/\sqsubseteq} \text{ and } J^{G4^*}_{/\sqsubseteq} \sqsubseteq \overline{J}_{/\sqsubseteq} \tag{7}$$

In order to explain the usefulness of the above property, we relate triplets and g-inclusion by a property called *weak anti-symmetry* [2].

Lemma 3.9 : *Let $\theta_1, \theta_2 \in S^{(3)}$. Then,*

$$\text{if } \theta_1 \sqsubseteq \theta_2 \text{ and } \theta_2 \sqsubseteq \theta_1 \Rightarrow \text{ either } \theta_1 = \theta_2 \text{ or } \theta_1 = \theta_2^T \tag{8}$$

By using this property over sets of triplets we get the following result

Lemma 3.10 : *Let $J, J' \in S^{(3)}$. Then,*

$$\text{if } J \sqsubseteq J' \text{ and } J' \sqsubseteq J \Rightarrow \forall \theta_1 \in J, \exists \theta_2 \in J' \text{ such that either } \theta_1 = \theta_2 \text{ or } \theta_1 = \theta_2^T$$

$$\text{and } \forall \theta_2 \in J', \exists \theta_1 \in J \text{ such that either } \theta_2 = \theta_1 \text{ or } \theta_2 = \theta_1^T$$

This property implies that the sets $\overline{J}_{/\sqsubseteq}$ and $J^{G4^*}_{/\sqsubseteq}$ share the same information. In Lemma 3.6 we saw that J^{G4^*} contains all the maximal triplets of $\overline{J}$. Lemma 3.10

now implies that the maximal set of the closure $\overline{J}$ coincides with a respective one of J^{G4^*}. Hence, closure computation can be reduced to the computation of the set generated from a finite number of applications of the rule $G4^*$. Furthermore, J^{G4^*} shares the same information as the basis and from now on we will refer to J^{G4^*} as the basis of a closure.

3.2 Closure Algorithm

In this section, we propose a new algorithm for closure computation based on the $G4^*$ rule and the notion of maximal triplet set. In Section 3.2.1 we introduce some necessary conditions for excluding triplets to which Studený's operator can never be applied throughout the closure computation. In Section 3.2.2, we further state that we do not need adding or keeping any symmetric triplet; note that not having to consider symmetric triplets can result in a more reduced size of the basis. In section 3.2.3, we present an inference rule, different from $G4^*$, with which we can simplify the closure procedure and reduce the number of iterations for the computation of the basis. Finally, in Section 3.2.4 we present our new algorithm for closure computation based on the elements of the previous sections.

3.2.1 New Conditions

In Section 3.1, we mentioned the five necessary conditions for application of the $G4^*$ rule as formulated by Baioletti et al. We now show for single triplets the conditions under which the rule cannot be applied throughout the closure procedure.

Lemma 3.11 *Let $\theta_1 = \langle A_1,B_1|C_1\rangle \in J \subseteq S^{(3)}$. Furthermore, consider the following conditions:*

$$\forall\theta_2 = \langle A_2,B_2|C_2\rangle \in J \text{ with } \theta_2 \neq \theta_1, \theta_1^T, C_1 \nsubseteq X_2 \tag{9}$$

$$\forall\theta_2 = \langle A_2,B_2|C_2\rangle \in J \text{ with } \theta_2 \neq \theta_1, \theta_1^T, A_1 \cap (A_2 \cup B_2) = \emptyset \text{ and } B_1 \cap (A_2 \cup B_2) = \emptyset \tag{10}$$

$$\forall\theta_2 = \langle A_2,B_2|C_2\rangle \in J \text{ with } \theta_2 \neq \theta_1, \theta_1^T, (A_1 \cup B_1)\backslash C_2 = \emptyset \text{ and } (A_2 \cup B_2)\backslash C_1 = \emptyset \tag{11}$$

$$\forall\theta_2 = \langle A_2,B_2|C_2\rangle \in J \text{ with } \theta_2 \neq \theta_1, \theta_1^T, (A_1 \cup B_1) \cap X_2 = \emptyset \text{ and } (A_2 \cup B_2) \cap X_1 = \emptyset \tag{12}$$

where $X_i = A_i \cup B_i \cup C_i$ for $i = 1, 2$

Then, the $G4^*$ rule cannot be applied to θ_1 and any other triplet in J if at least one of the conditions (9)-(12) *hold.*

Proof. We begin by stating all possible triplets which can be generated by application of $G4^*$. For $\theta_1 = \langle A_1, B_1 | C_1 \rangle$ and $\theta_2 = \langle A_2, B_2 | C_2 \rangle$ the generated triplets come from the sets $GC(\theta_1, \theta_2)$ and $GC(\theta_2, \theta_1)$, and hence we have the following results:

$$gc(\theta_1, \theta_2) = \langle A_1 \cap A_2, (B_2 \backslash C_1) \cup (B_1 \cap X_2) | C_1 \cup (A_1 \cap C_2) \rangle$$

$$gc(\theta_1, \theta_2^T) = \langle A_1 \cap B_2, (A_2 \backslash C_1) \cup (B_1 \cap X_2) | C_1 \cup (A_1 \cap C_2) \rangle$$

$$gc(\theta_1^T, \theta_2) = \langle B_1 \cap A_2, (B_2 \backslash C_1) \cup (A_1 \cap X_2) | C_1 \cup (B_1 \cap C_2) \rangle$$

$$gc(\theta_1^T, \theta_2^T) = \langle B_1 \cap B_2, (A_2 \backslash C_1) \cup (A_1 \cap X_2) | C_1 \cup (B_1 \cap C_2) \rangle$$

$$gc(\theta_2, \theta_1) = \langle A_1 \cap A_2, (B_1 \backslash C_2) \cup (B_2 \cap X_1) | C_2 \cup (A_2 \cap C_1) \rangle$$

$$gc(\theta_2, \theta_1^T) = \langle B_1 \cap A_2, (A_1 \backslash C_2) \cup (B_2 \cap X_1) | C_2 \cup (A_2 \cap C_1) \rangle$$

$$gc(\theta_2^T, \theta_1) = \langle A_1 \cap B_2, (B_1 \backslash C_2) \cup (A_2 \cap X_1) | C_2 \cup (B_2 \cap C_1) \rangle$$

$$gc(\theta_2^T, \theta_1^T) = \langle B_1 \cap B_2, (A_1 \backslash C_2) \cup (A_2 \cap X_1) | C_2 \cup (B_2 \cap C_1) \rangle$$

If condition (9) holds we cannot apply the operator $G4^*$ to θ_1 and θ_2 because its application requires $C_1 \subseteq X_2$(see condition (2b) from Lemma 3.5). It is trivial to see that when condition (10) holds, none of the above 8 triplets are defined as their first elements would always be equal to the empty set. Similarly, if either condition (11) or (12) holds, the second elements of the 8 triplets above are not defined properly as one of the two components of the union will always be the empty set.

Now we have to show that we cannot apply the $G4^*$ rule to θ_1 and any triplet that may be generated by applying $G4^*$ to any $\theta_2, \theta_3 \in J$. We already showed that we cannot apply $G4^*$ to θ_1 and any triplet belonging to $J_0 = J$ if at least one of the conditions (9)-(12) is satisfied. Now, suppose that after application of $G4^*$ to any two triplets from J_0, we have generated the set J_1 whose form is

$$J_1 = \{\theta | \theta \in J_0 \text{ or } \theta \in GC(\theta_2, \theta_3) \cup GC(\theta_3, \theta_2) \text{ with } \theta_2, \theta_3 \in J_0\}$$

and the recursive algorithmic scheme for closure computation based on $G4^*$ rule is the following.

$$\text{Let } J \subseteq S^{(3)} \text{ with } J_0 = J \text{ and } J_k = J_{k-1} \cup \{\bigcup GC(\theta_1, \theta_2) \text{ with } \theta_1, \theta_2 \in J_{k-1}\} \tag{13}$$

We get the closure of J when $J_k = J_{k-1}$.

We will show that we cannot apply the $G4^*$ rule to θ_1 and any triplet θ from J_1 with $\theta \neq \theta_1$. Let $\theta_2 = \langle A_2, B_2 | C_2 \rangle$ and $\theta_3 = \langle A_3, B_3 | C_3 \rangle$ be triplets in J_0 such that we can apply the rule to θ_2 and θ_3 in all possible ways. The newly generated triplets for J_1 have the form of the 8 aforementioned triplets, where θ_3 is substituted for θ_1. We write $\theta = \langle A,B|C\rangle$ to denote the form of the newly generated triplets. It is easy to see that the following statements hold

$$C \subseteq C_2 \cup C_3 \tag{14}$$

$$A, B \subseteq A_2 \cup A_3 \cup B_2 \cup B_3 \tag{15}$$

If we were to apply $G4^*$ to θ_1 and θ, we would find the following triplets:

$$gc(\theta_1, \theta) = \langle A_1 \cap A,(B\backslash C_1) \cup (B_1 \cap X)|C_1 \cup (A_1 \cap C)\rangle$$

$$gc(\theta_1, \theta^T) = \langle A_1 \cap B,(A\backslash C_1) \cup (B_1 \cap X)|C_1 \cup (A_1 \cap C)\rangle$$

$$gc(\theta_1^T, \theta) = \langle B_1 \cap A,(B\backslash C_1) \cup (A_1 \cap X)|C_1 \cup (B_1 \cap C)\rangle$$

$$gc(\theta_1^T, \theta^T) = \langle B_1 \cap B,(A\backslash C_1) \cup (A_1 \cap X)|C_1 \cup (B_1 \cap C)\rangle$$

$$gc(\theta, \theta_1) = \langle A_1 \cap A,(B_1\backslash C) \cup (B \cap X_1)|C \cup (A \cap C_1)\rangle$$

$$gc(\theta, \theta_1^T) = \langle B_1 \cap A,(A_1\backslash C) \cup (B \cap X_1)|C \cup (A \cap C_1)\rangle$$

$$gc(\theta^T, \theta_1) = \langle A_1 \cap B,(B_1\backslash C) \cup (A \cap X_1)|C \cup (B \cap C_1)\rangle$$

$$gc(\theta^T, \theta_1^T) = \langle B_1 \cap B,(A_1\backslash C) \cup (A \cap X_1)|C \cup (B \cap C_1)\rangle$$

From the properties (14) and (15) we have

- if condition (10) holds, $A_1 \cap A = A_1 \cap B = B_1 \cap A = B_1 \cap B = \emptyset$ and none of the above cases constitutes a valid triplet;
- if condition (11) holds, $B\backslash C_1 = A\backslash C_1 = B_1\backslash C = A_1\backslash C = \emptyset$ and none of the above cases constitutes a valid triplet;
- if condition (12) holds, $B_1 \cap X = A_1 \cap X = B \cap X_1 = A \cap X_1 = \emptyset$ and none of the above cases constitutes a valid triplet;
- if condition (9) holds, we know that $C_1 \nsubseteq X$. We now distinguish between two cases. In the first case, we assume that C_1 contains at least one variable which does not belong to X_2 nor to X_3. It is trivial to see that $C_1 \nsubseteq X$ holds since $X \subseteq X_2 \cup X_3$. In the second case, we assume that C_1 contains some variables, for example $\{d, e\}$, for which it holds that $d \in X_3, e \in X_2$ and $d \notin X_2, e \notin X_3$. Since we can now apply $G4^*$ to θ_2, θ_3 it holds that $C_2 \subseteq X_3$ and $C_3 \subseteq X_2$(see condition (2b) from Lemma 3.5). Hence, $d \notin C_3$ and $e \notin C_2$. We thus have that $d \in A_3$ or $d \in B_3$ and $e \in A_2$ or $e \in B_2$. Looking again at the 8 triplets generated by applying $G4^*$, we observe that none of them can include both variables d and e, because any of their first elements is always an intersection between elements of θ_2 and $\theta_3(A_2 \cap A_3, A_2 \cap B_3, B_2 \cap A_3, B_2 \cap B_3)$ and the same holds for the second component of the unions in their second elements. Therefore, it still holds that $C_1 \nsubseteq X$ and none of the triplets can be defined.

Using similar arguments it now follows by induction that we are not able to apply the $G4^*$ rule to θ_1 and any triplet in a set J_n which is generated through the procedure of closure computation (13). $\square$

It is useful to mention that for a single triplet θ_1 the two conditions (11) and (12) cannot be satisfied simultaneously. For example, if for $\theta_1 = \langle A_1,B_1|C_1\rangle$ condition (11) is satisfied then for any $\theta_2 = \langle A_2,B_2|C_2\rangle$ in J, it holds that $(A_1 \cup B_1)\backslash C_2 = \emptyset$. This means that $(A_1 \cup B_1) \subseteq C_2$ and also $(A_1 \cup B_1) \cap C_2 \neq \emptyset$. Hence, condition (12) cannot be satisfied. Similarly, if we have condition (12) satisfied we cannot have also condition (11) satisfied.

Through the conditions described above, we have that a starting might include triplets to which Studený's rule can never be applied. We can exclude these triplets from the basis computation, thereby effectively reducing the number of applications of the $G4^*$ rule. The procedure for excluding such triplets requires at most $N^2/2$ iterations, given a starting set with N triplets.

3.2.2 Symmetric Triplets and G-Inclusion

Upon computing a basis for a starting set of triplets, the number of applications of the $G4^*$ rule can be further reduced by not adding symmetric triplets to this starting set. We recall that Studený's algorithm adds all symmetric triplets to cover all possible applications of the rule. Basically, these applications are covered by the sets $GC(\theta_1, \theta_2)$ and $GC(\theta_2, \theta_1)$ for any pair of triplets θ_1, θ_2. In the proof of Lemma 3.11 we showed that we can derive all possible dominant triplets generated from any pair θ_1, θ_2 without explicitly including in the basis their respective transposes. We now show that a basis without symmetric triplets, denoted with $J^{G4^*}_{-}$, shares the same information as the full closure $\overline{J}$ of a starting set J. We recall that Studený's algorithm removes dominated triplets from the basis under construction after each iteration. We further recall that the algorithms by Baioletti et al. are based on g-inclusion instead of dominance. By using g-inclusion instead of dominance for the removal of triplets, we do not need to take into consideration any symmetric triplet. We recall that for the basis J^{G4^*} and the full closure $\overline{J}$ the following property holds(see lemma 3.8):

$$\overline{J}_{/\sqsubseteq} \sqsubseteq J^{G4^*}_{/\sqsubseteq} \text{ and } J^{G4^*}_{/\sqsubseteq} \sqsubseteq \overline{J}_{/\sqsubseteq}$$

From the definition of maximal triplet set, we now have that such a set may contain both a triplet θ and its transpose θ^T. For a triplet θ such that $\theta \in J^{G4^*}, \overline{J}$ and $\theta^T \in \overline{J}$, Lemma 3.8 still holds because $\theta^T \sqsubseteq \theta$. Since it also holds that $\theta \sqsubseteq \theta^T$, we could further have $\theta^T \in J^{G4^*}, \overline{J}$ and $\theta \in \overline{J}$. Hence, it suffices to have either θ or θ^T in J^{G4^*}. Let now $J'^{G4^*} \subseteq S^{(3)}$ be an intermediate basis in the computation from which we are about to remove all dominated triplets. Suppose that the triplets θ',θ and θ^T belong to J'^{G4^*} and that θ' dominates θ ($\theta \prec \theta'$). Since $\theta \prec \theta'$ holds, it also holds that $\theta^T \not\prec \theta'$ and $\theta^T \prec \theta'^T$ from the definition of dominance. In terms of g-inclusion however, we have $\theta, \theta^T \sqsubseteq \theta'$. Therefore, by defining the $J'^{G4^*}_{/\sqsubseteq}$ which shares the same information as J'^{G4^*}, the triplets θ and θ^T are removed without adding or generating θ'^T.

We conclude that during the computation of the basis we can generate all possible dominant triplets from a subject pair of triplets without the need of considering transposes. Moreover, by using maximal triplet sets we basically remove all g-included triplets, which means that if a triplet θ is to be removed, then θ^T will be removed as well if it is included in the generated set of triplets. It is useful to mention that seeking for g-included triplets is almost as demanding as seeking for dominated ones. With respect to dominance, given a triplet $\theta_1 = \langle A_1,B_1|C_1\rangle$, we can remove any triplet $\theta_2 = \langle A_2,B_2|C_2\rangle$ such that $C_1 \subseteq C_2 \subseteq X_1$ and $A_2 \subseteq A_1$, $B_2 \subseteq B_1$; in case of g-inclusion, the condition $A_2 \subseteq A_1$, $B_2 \subseteq B_1$ to be verified is simply replaced by $A_2 \subseteq A_1$, $B_2 \subseteq B_1$ or $A_2 \subseteq B_1$, $B_2 \subseteq A_1$.

From the above consideration, it is now clear that a symmetric triplet does not give us any additional information. Nevertheless, through Studený's operator, it is possible that dominant triplets and some of their associated symmetric ones are generated. It is trivial to see that if a triplet is maximal, then its transpose is also maximal. As we do not need to keep any symmetric triplet during the procedure of basis computation, we propose a new type of maximal triplet set. We will refer to such a set as a*maximal non-symmetric* triplet set(with respect to g-inclusion): basically, a *maximal non-symmetric* triplet set, denoted with $J_{/\sqsubseteq n}$, is a maximal triplet set such that for every triplet $\tau \in J_{/\sqsubseteq n}$ we have that $\tau^T \notin J_{/\sqsubseteq n}$. It is easy to see that $J_{/\sqsubseteq n} \sqsubseteq J_{/\sqsubseteq}$ and $J_{/\sqsubseteq} \sqsubseteq J_{/\sqsubseteq n}$ hold.

3.2.3 Unique Inference Rule

We now introduce a new inference rule for computing dominant triplets from any pair of triplets. Together with the concept of maximal non-symmetric triplet set, this inference rule lies at the core of our new algorithm for computing a reduced basis of the closure which will be detailed in the next subsection.

First of all, we introduce a new way of representing triplets.

Definition 3.12 *Any pair of triplets θ_1 and θ_2 is in* general form *if*
$\theta_1 = \langle A_A \cup A_B \cup A_C \cup A_D, B_A \cup B_B \cup B_C \cup B_D | C_A \cup C_B \cup C_C \cup C_D\rangle$

$\theta_2 = \langle A_A \cup B_A \cup C_A \cup A_{D'}, A_B \cup B_B \cup C_B \cup B_{D'} | A_C \cup B_C \cup C_C \cup C_{D'}\rangle$

where some of the sets can be empty.

Let $\theta_1,\theta_2 \in S^{(3)}$*. Then,* θ_1,θ_2 *are in* almost general form *if they are in general form with* $C_D = C_{D'} = \emptyset$*, that is, if*

$\theta_1 = \langle A_A \cup A_B \cup A_C \cup A_D, B_A \cup B_B \cup B_C \cup B_D | C_A \cup C_B \cup C_C \rangle$

$\theta_2 = \langle A_A \cup B_A \cup C_A \cup A_{D'}, A_B \cup B_B \cup C_B \cup B_{D'} | A_C \cup B_C \cup C_C \rangle$

Note that the concept of almost general form for triplets is actually a representation of a pair of triplets $\theta_1 = \langle A_1,B_1|C_1\rangle$ and $\theta_2 = \langle A_2,B_2|C_2\rangle$ which satisfy the condition (2b) of Lemma 3.5; that is, for which $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$.

Assuming that the remaining conditions (2a),(2c),(2d),(2e) from Lemma 3.5 hold for two triplets in almost general form, we can generate at most 8 triplets; these 8 triplets are actually those produced by $G4^*(\theta_1, \theta_2)$ and $G4^*(\theta_2, \theta_1)$.

Now we introduce the set $J^U(\theta_1, \theta_2) = \{\theta_1, \theta_2, \theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f, \theta_g, \theta_h\}$ with

$\theta_a = \langle A_A, A_B \cup B_A \cup B_B \cup B_C \cup B_{D'} | A_C \cup C_A \cup C_B \cup C_C \rangle$

$\theta_b = \langle A_B, A_A \cup B_A \cup B_B \cup B_C \cup A_{D'} | A_C \cup C_A \cup C_B \cup C_C \rangle$

$\theta_c = \langle B_A, A_A \cup A_B \cup A_C \cup B_B \cup B_{D'} | B_C \cup C_A \cup C_B \cup C_C \rangle$

$\theta_d = \langle B_B, A_A \cup A_B \cup A_C \cup B_A \cup A_{D'} | B_C \cup C_A \cup C_B \cup C_C \rangle$

$\theta_e = \langle A_A, A_B \cup B_A \cup B_B \cup B_D \cup C_B | A_C \cup B_C \cup C_A \cup C_C \rangle$

$\theta_f = \langle B_A, A_A \cup A_B \cup A_D \cup B_B \cup C_B | A_C \cup B_C \cup C_A \cup C_C \rangle$

$\theta_g = \langle A_B, A_A \cup B_A \cup B_B \cup B_D \cup C_A | A_C \cup B_C \cup C_B \cup C_C \rangle$

$\theta_h = \langle B_B, A_A \cup A_B \cup A_D \cup B_A \cup C_A | A_C \cup B_C \cup C_B \cup C_C \rangle$

It is trivial to see that $J^U(\theta_1, \theta_2) = J^U(\theta_2, \theta_1) = GC(\theta_1, \theta_2) \cup GC(\theta_2, \theta_1) \cup \{\theta_1, \theta_2\}$, and hence $J^U_{/\sqsubseteq}(\theta_1, \theta_2) \sqsubseteq J^{G4^*}_{/\sqsubseteq}(\theta_1, \theta_2)$ and $J^{G4^*}_{/\sqsubseteq}(\theta_1, \theta_2) \sqsubseteq J^U_{/\sqsubseteq}(\theta_1, \theta_2)$, where $J^{G4^*}(\theta_1, \theta_2)$ is the set produced from a single application of $G4^*$ to θ_1 and θ_2. We can now introduce our new inference rule, which states:

U_{sem}: from θ_1, θ_2 deduce any triplet $\tau \in J^U(\theta_1, \theta_2)$;

We define the set J^U to be the closure after a finite number of applications of the U_{sem} rule. A useful property of this set is the following:

Lemma 3.13 *: Let* $J \subseteq S^{(3)}$*. Then,* $J^U_{/\sqsubseteq n} \sqsubseteq \overline{J}_{/\sqsubseteq n}$ *and* $\overline{J}_{/\sqsubseteq n} \sqsubseteq J^U_{/\sqsubseteq n}$

Proof. We already know that $\overline{J}_{/\sqsubseteq} \sqsubseteq J^{G4^*}_{/\sqsubseteq}$ and $J^{G4^*}_{/\sqsubseteq} \sqsubseteq \overline{J}_{/\sqsubseteq}$. Therefore, it follows that

$J^U_{/\sqsubseteq n} \sqsubseteq J^U_{/\sqsubseteq} \sqsubseteq J^{G4^*}_{/\sqsubseteq} \sqsubseteq \overline{J}_{/\sqsubseteq} \sqsubseteq \overline{J}_{/\sqsubseteq n}$ and also $\overline{J}_{/\sqsubseteq n} \sqsubseteq \overline{J}_{/\sqsubseteq} \sqsubseteq J^{G4^*}_{/\sqsubseteq} \sqsubseteq J^U_{/\sqsubseteq} \sqsubseteq J^U_{/\sqsubseteq n} \square$

From this lemma, we conclude that in order to compute the basis of the closure without including any redundant triplet, we only need to compute the set J^U and then find a maximal non-symmetric set fromit.

3.2.4 Basic Algorithm

Based upon the inference rule defined above, we will present an algorithm for the closure computation. The algorithm builds upon one more result concerning g-inclusion.

Lemma 3.14 : *Let* $\theta_1 = \langle A_1,B_1|C_1\rangle$, $\theta_2 = \langle A_2,B_2|C_2\rangle \in J \subseteq S^{(3)}$, *such that* $C_1 \subseteq X_2$ *and* $C_2 \subseteq X_1$.

If at least one of the conditions (2a),(2c),(2d),(2e) *from Lemma 3.5 does not hold, the triplet* $gc(\theta_1, \theta_2) = \langle A_1 \cap A_2, (B_2 \backslash C_1) \cup (B_1 \cap X_2)|C_1 \cup (A_1 \cap C_2)\rangle$ *is either not defined or it is g-included in* θ_1 *or* θ_2.

Proof. First of all, we recall that the conditions (2a),(2c),(2d),(2e) are

$A_1 \cap A_2 \neq \emptyset$

$B_2 \backslash C_1 \neq \emptyset$

$B_1 \cap X_2 \neq \emptyset$

$|(B_2 \backslash C_1) \cup (B_1 \cap X_2)| \geq 2$

- if $A_1 \cap A_2 = \emptyset$, it is trivial to see that $gc(\theta_1, \theta_2)$ cannot be defined as the first component would be the empty set.
- if $B_2 \backslash C_1 = \emptyset$, we find that $gc(\theta_1, \theta_2) = \langle A_1 \cap A_2, B_1 \cap X_2|C_1 \cup (C_2 \cap A_1)\rangle$. It is clear that $A_1 \cap A_2 \subseteq A_1$, $B_1 \cap X_2 \subseteq B_1$, $C_1 \subseteq C_1 \cup (C_2 \cap A_1)$ and hence the generated triplet is dominated by θ_1.
- if $B_1 \cap X_2 = \emptyset$, we have that $gc(\theta_1, \theta_2) = \langle A_1 \cap A_2, B_2 \backslash C_1|C_1 \cup (C_2 \cap A_1)\rangle$. Since $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$, we further have that $C_1 = (C_1 \cap A_2) \cup (C_1 \cap B_2) \cup (C_1 \cap C_2)$ and $C_2 = (C_2 \cap A_1) \cup (C_2 \cap B_1) \cup (C_2 \cap C_1)$. Because of $B_1 \cap X_2 = \emptyset$, we can conclude that $C_2 = (C_2 \cap A_1) \cup (C_2 \cap C_1) \subseteq C_1 \cup (C_2 \cap A_1)$. Moreover, $A_1 \cap A_2 \subseteq A_2$ and $B_2 \backslash C_1 \subseteq B_2$, hence the generated triplet is g-included in θ_2.

- if $|(B_2 \backslash C_1) \cup (B_1 \cap X_2)| < 2$, we have that either $|(B_2 \backslash C_1) \cup (B_1 \cap X_2)| = 0$ or $|(B_2 \backslash C_1) \cup (B_1 \cap X_2)| = 1$. In the first case, the triplet $gc(\theta_1, \theta_2) = \langle A_1 \cap A_2, \emptyset | C_1 \cup (C_2 \cap A_1)\rangle$ results which is not defined. The second case implies that $((B_2 \backslash C_1) \cup (B_1 \cap X_2)) = B_1 \cap B_2$ and the generated triplet is $\langle A_1 \cap A_2, B_1 \cap B_2 | C_1 \cup (C_2 \cap A_1)\rangle$ which is clearly g-included in θ_1. $\square$

At first glance, the aforementioned lemma appears contradictory to Lemma 3.5. From this lemma, we have that if one of the conditions (2a),(2c),(2d),(2e) does not hold, then $gc(\theta_1, \theta_2)$ is not defined. The above lemma shows, by using Studený's rule for dominant triplets, that even if at least one of the conditions does not hold for a pair of triplets in almost general form, then the generated triplet still is not defined or it is g-included. This means that we can simply apply the U_{sem} rule without checking the conditions from Lemma 3.14 as the generated redundant or invalid triplets will be removed through the computation of maximal non-symmetric triplet set.

Algorithm 1 computes the $J^U_{/\sqsubseteq n}$ where *NonApplicable* is the function for identifying triplets that satisfy at least one of the conditions (9), (10), (11), (12) from Lemma 3.11, and *FindNonSymmetricMaximal* computes the set $J_{/\sqsubseteq n}$ from a given set $J \subseteq S^{(3)}$.

Algorithm 1 Fast Closure by U_{sem}

function FCSem(J)
  $A \leftarrow NonApplicable(J)$
  $J_0 \leftarrow J \backslash A$
  $N_0 \leftarrow J \backslash A$
  $k \leftarrow 0$
  repeat
    $k \leftarrow k + 1$
    $N_k := \bigcup_{\theta_1 \in J_{k-1}, \theta_2 \in N_{k-1}} J^U(\theta_1, \theta_2)$
    $J_k \leftarrow FindNonSymmetricMaximal(J_{k-1} \cup N_k)$
  until $J_k = J_{k-1}$
  return $J_k \cup A$
end function

Firstly, we add in the set A the triplets to which we cannot apply U_{sem} rule according to Lemma 3.11. Then, we subtract the set A from the initial set J.The set N_k is used in order to avoid applying U_{sem} to a pair of triplets to which we have already applied it. Now we show the completeness and correctness of the algorithm through the following lemma.

Lemma 3.15 : *For any* $J \subseteq S^{(3)}$

$$FCSem(J) \sqsubseteq J^{G4^*}_{/\sqsubseteq} \tag{16}$$

$$J^{G4^*}_{/\sqsubseteq} \sqsubseteq FCSem(J) \tag{17}$$

Proof. We prove equation (17), by first proving that $J^{G4^*} \sqsubseteq$ FCSem(J). For the computation of J^{G4^*} we use the recursive scheme (13) as follows

$J'_0 = J \backslash A$
$J'_k = J'_{k-1} \cup \{\bigcup GC(\theta_1, \theta_2)$ with $\theta_1, \theta_2 \in J'_{k-1}\}$, and
$N'_0 = J \backslash A$, $N'_k = J'_k \backslash J'_{k-1}$

In order to prove that $J^{G4^*} \sqsubseteq$ FCSem(J), we now have to prove that for any $h \in \mathbb{N}$, $N'_h \sqsubseteq N_h$ and $J'_h \sqsubseteq J_h$. It is trivial to see that these properties hold for $h = 0$. Supposing that it holds for $h = k-1$, we prove that it holds also for $h = k$. Let $\tau' = gc(\theta'_1, \theta'_2)$ with $\theta'_1 \in J'_{k-1}$, $\theta'_2 \in N'_{k-1}$. By inductive hypothesis on h we know that $\exists \theta_1 \in J_{k-1}$ and $\exists \theta_2 \in N_{k-1}$ such that $\theta'_1 \sqsubseteq \theta_1$ and $\theta'_2 \sqsubseteq \theta_2$. Let $\tau = gc(\theta_1, \theta_2)$, by using Lemma 3.1 we have $\tau' \sqsubseteq \tau$. Therefore, $GC(\theta'_1, \theta'_2) \sqsubseteq GC(\theta_1, \theta_2)$ which implies that $N'_h \sqsubseteq N_h$.

From FCSem(J) we have that J_k is a maximal non-symmetric set derived from $J_{k-1} \cup N_k$. We further know that $J'_k = J'_{k-1} \cup N'_k$ and hence we have $J_{k-1} \sqsubseteq J_k$. Since J is finite, $\exists n \in \mathbb{N}$ such that $J'_n = J'_{n-1} = J^{G4^*}$ and $J_n = J_{n-1}$ =FCSem(J) $\Rightarrow J^{G4^*} \sqsubseteq$FCSem($J$). From $J^{G4^*}_{/\sqsubseteq} \subseteq J^{G4^*}$, we can now conclude that equation (17) holds.

To complete the proof, we observe that equation (16) derives from the following

$$\text{FCSem}(J) \subset \overline{J} \sqsubseteq J^{G4^*} \sqsubseteq J^{G4^*}_{/\sqsubseteq} \qquad \square.$$

In Algorithm 1 the elements of the set A are added to final generated set, to which we will refer as J_N. The following lemma shows that none of the triplets from A are g-included in a triplet from J_N nor are any triplets from J_N g-included in triplets from A.

Lemma 3.16: *Let* $J \subseteq S^{(3)}$. *Let* FCSem(J)= $J_N \cup A$ *be the function value returned by Algorithm 1. Then, it holds that*

$\forall \theta_1 \in A, \nexists \theta_2 \in J_N$ *such that* $\theta_1 \sqsubseteq \theta_2$ *and*

$\forall \theta_2 \in J_N, \nexists \theta_1 \in A$ *such that* $\theta_2 \sqsubseteq \theta_1$

Proof. In Lemma 3.11 we presented the conditions under which we cannot apply the U_{sem} inference rule to θ_1 and any other (generated)triplet in J. Let $\theta_1 = \langle A_1, B_1 | C_1 \rangle$ and $\theta_2 = \langle A_2, B_2 | C_2 \rangle$ and assume that the conditions (9), (10), (11) and (12) from Lemma 3.11 hold. We will prove that under each of these conditions at least one condition for g-inclusion(see Lemma 2.3) is not satisfied:

- if condition (9) holds, then we have that $C_1 \not\subseteq X_2$. $C_1 \not\subseteq X_2$ implies that the condition $C_2 \subseteq C_1 \subseteq X_2$ for g-inclusion cannot be satisfied.

- if condition (10) holds, then we have that $A_1 \cap (A_2 \cup B_2) = B_1 \cap (A_2 \cup B_2) = \emptyset$. From $A_1 \cap (A_2 \cup B_2) = \emptyset$, we find that $A_1 \not\subseteq A_2$. From $B_1 \cap (A_2 \cup B_2) = \emptyset$, we further find that $B_1 \not\subseteq A_2$. We conclude that the second necessary condition for g-inclusion is not satisfied.

- if condition (11) holds, then we have that $(A_1 \cup B_1)\backslash C_2 = (A_2 \cup B_2)\backslash C_1 = \emptyset$. From the definition of triplet we know that A_1, B_1, C_1 are disjoint. From $(A_2 \cup B_2)\backslash C_1 = \emptyset$, it follows that $A_2 \cup B_2 \subseteq C_1$ and hence $A_1 \not\subseteq A_2$. Similarly, $B_1 \not\subseteq A_2$ and we see that the second necessary condition for g-inclusion is not satisfied.

- if condition (12) holds, then we have that $(A_1 \cup B_1) \cap X_2 = (A_2 \cup B_2) \cap X_1 = \emptyset$. This case is similar to this of condition (10) and we can conclude that the second necessary condition for g-inclusion does not hold.

From Lemma 3.15 we have that Algorithm 1 computes a maximal basis of a closure which shares the same information with this by using the $G4^*$ rule. From Lemma 3.16 we see that the triplets which satisfy at least one of the conditions (9), (10), (11), (12) from Lemma 3.11 are added in the final triplet set generated by Algorithm 1 and they are maximal in this set.

4 Stable Triplets

In the previous section we discussed dominant triplets and showed how these can be exploited to easily compute a closure which is reduced in size. In this section we elaborate further on semi-graphoid models by introducing an additional type of triplets, the so-called *stable* triplets. De Waal and Van der Gaag [6] have done significant work concerning stable triplets and we review this concept in Section 4.1. In Section 4.2 we extend the concept of dominance to stable triplets, in a similar way as De Waal and Van der Gaag did. In Section 4.3 we present a unique inference rule for s-dominant triplets. In Section 4.4 we extend the concept of g-inclusion to stable triplets. Finally, in Section 4.5 we present a new algorithm for computing a closure through the inference rule of Section 4.3. The proposed algorithm is combined with the algorithm of the previous section and we provide additional necessary conditions for the exclusion of stable triplets.

4.1 Preliminaries over Stable Triplets

Stable triplets capture concept of stability of independence. As described in [6], stability pertains to independences that remain to hold as its conditioning element is extended. Taking into consideration the concept of stability, we need to distinguish the stable triplets from the other triplets included in the starting set of a model. The definition for stable triplets is as follows.

Definition 4.1 : *Let $J \subseteq S^{(3)}$. Then,*

- *a triplet $\langle A,B|C\rangle$ is called stable in J if $\langle A,B|C'\rangle$ in J for all sets C' with $C \subseteq C'$; if $A \cup B \cup C = S$, then $\langle A,B|C\rangle$ is called trivially stable;*
- *a triplet $\langle A,B|C\rangle$ is called unstable in J if it is not stable in J*

The set of all triplets that are stable in J is called the stable part of J and it is denoted with J_S. The triplets that are unstable in J form the unstable part of the independence model which is denoted by J_U.

For the stable part J_S of an independence model J the four semi-graphoid properties hold plus an additional one. This is the *strong union* axiom [8] and it is as follows.

Lemma 4.2 : *Let $J \subseteq S^{(3)}$ and let J_S be the stable part of J. Then*

S5: if $\langle A,B|C\rangle \in J_S$ then $\langle A,B|C \cup D\rangle \in J_S$, for all (mutually disjoint) sets $A,B,C,D \subset S$

This property implies that if a triplet is stable, its first and second unconditioning parts remain independent given any superset of the conditioning part. Note that stable triplets remain in the basis of a model as their conditioning parts grow. If we allow dominance to further capture the $S5$ rule, it is trivial to see that it always holds that $\langle A,B|C \cup D\rangle \prec \langle A,B|C\rangle$. Hence $\langle A,B|C\rangle$ should remain unless it is dominated by or g-included to a potential generated triplet during the closure computation.

Based on Lemma 4.2, the following property shows another interesting result over stable triplets.

Lemma 4.3 : *Let $J \subseteq S^{(3)}$ and let J_S be the stable part of J. Then,*

if $\langle A,B|D\rangle \in J_S$ and $\langle A,C|D\rangle \in J_S$, then $\langle A,B \cup C|D\rangle \in J_S$ for all sets $A,B,C,D \subset S$

The proof is straightforward by applying the strong union rule to $\langle A,C|D\rangle$ and then apply contraction.

4.2 Stable Dominant Triplets

In the previous section we saw that the stable part of a model satisfies all semi-graphoid properties plus strong union. It is usefule therefore, to identify potential dominant triplets for the stable part of a model and to thereby reduce the closure in size. We introduce now the notion of dominance with respect to stable triplets.

Definition 4.4 : *Let $J \subseteq S^{(3)}$ and let $\theta_1 = \langle A_1,B_1|C_1\rangle,\theta_2 = \langle A_2,B_2|C_2\rangle \in J$. Then,*

$\langle A_1,B_1|C_1\rangle$ *s-dominates* $\langle A_2,B_2|C_2\rangle$, denoted with $\langle A_2,B_2|C_2\rangle \prec_s \langle A_1,B_1|C_1\rangle$ *if* $A_2 \subseteq A_1$, $B_2 \subseteq B_1$ *and* $C_1 \subseteq C_2$

A triplet that is not s-dominated by any other triplet in J is called maximally s-dominant in J.

S-dominance derives directly from the standard concept of dominance which we reviewed in Definition 2.4. In order to distinguish between the two concepts, we will refer to the ordinary dominance as *o-dominance* and the generated triplets will be called *o-dominant*. Note that the difference between the two dominance concepts lies in the fact that we do not need $C_2 \subseteq X_1$ to hold for stable triplets, as a result of Lemma 4.3: through the S5 rule, we can construct triplets having any super-set in the conditioning part. Therefore, if $C_2 \subseteq X_1$ holds before applying S5 to $\langle A_2,B_2|C_2\rangle$, we can find a triplet $\langle A_2,B_2|C'\rangle$ with $C_2 \subseteq C' \not\subseteq X_1$, yet the s-dominance between $\langle A_1,B_1|C_1\rangle$ and $\langle A_2,B_2|C'\rangle$ remains to hold.

Similarly to Studený's algorithm, De Waal and Van der Gaag proposed an algorithm for creating s-dominant triplets. We will review this algorithm and show under which conditions we can apply it to a pair of stable triplets. The conditions we present are slightly different from those proposed in [6]; also our inference rule is different for easier computation.

Definition 4.5.1 : *Let $J \subseteq S^{(3)}$ and let $\theta_1 = \langle A_1,B_1|C_1\rangle$, $\theta_2 = \langle A_2,B_2|C_2\rangle \in J_S$. Then,*

if $A_1 \cap A_2 \neq \emptyset$ and $B_2 \backslash (B_1 \cup C_1) \neq \emptyset$, $gc_s(\theta_1, \theta_2)$ is defined as

$$gc_s(\theta_1, \theta_2) = \langle A_1 \cap A_2,(B_2 \backslash C_1) \cup B_1|C_1 \cup (C_2 \backslash B_1)\rangle$$

else, $gc_s(\theta_1, \theta_2)$ is undefined.

The following lemma shows how the above computational rule derives from the stable semi-graphoid properties.

Lemma 4.6 : *Let $J \subseteq S^{(3)}$ and let $\theta_1 = \langle A_1,B_1|C_1\rangle$, $\theta_2 = \langle A_2,B_2|C_2\rangle \in J_S$. Then,*

If $gc_s(\theta_1, \theta_2)$ is defined, then $gc_s(\theta_1, \theta_2) \in \overline{J}$

Proof. From the triplet $\langle A_1,B_1|C_1\rangle$ we can derive $\langle A_1 \cap A_2,B_1|C_1\rangle$ by applying decomposition. Note that since $gc_s(\theta_1, \theta_2)$ is defined, we have that $A_1 \cap A_2 \neq \emptyset$. Now by applying the strong union axiom to this triplet we get $\langle A_1 \cap A_2, B_1 | C_1 \cup (C_2 \backslash B_1)\rangle$. By applying decomposition to the triplet $\langle A_2, B_2 | C_2\rangle$ we find the triplet $\langle A_2 \cap A_1, B_2 | C_2\rangle$. Through weak union and strong union, we derive from the aforementioned triplet the triplet $\langle A_2 \cap A_1, B_2 \backslash (B_1 \cup C_1) | C_2 \cup B_1 \cup C_1\rangle$. It is obvious that $C_2 \cup B_1 = C_2 \backslash B_1 \cup B_1$. Therefore, the triplet can be written as $\langle A_2 \cap A_1, B_2 \backslash (B_1 \cup C_1) | (C_2 \backslash B_1) \cup B_1 \cup C_1\rangle$. It is easy to see that the triplets we generated have the following form

$$\langle A_1 \cap A_2, B_1 | C_1 \cup (C_2 \backslash B_1)\rangle = \langle X, Y | Z\rangle \text{ and}$$
$$\langle A_2 \cap A_1, B_2 \backslash (B_1 \cup C_1) | (C_2 \backslash B_1) \cup B_1 \cup C_1\rangle = \langle X, W | Y \cup Z\rangle$$

Hence we can apply the contraction rule which results in the following triplet

$$\langle A_1 \cap A_2, (B_2 \backslash (B_1 \cup C_1)) \cup B_1 | C_1 \cup (C_2 \backslash B_1)\rangle$$

Since B_1 contains $B_1 \cap B_2$, we can re-write $(B_2 \backslash B_1 \cup C_1)$ as $(B_2 \backslash C_1)$, and the final generated triplet will have the form

$$\langle A_1 \cap A_2, (B_2 \backslash C_1) \cup B_1 | C_1 \cup (C_2 \backslash B_1)\rangle \qquad \square$$

We now review the inference rule and necessary conditions as formulated by De Waal and Van der Gaag.

Definition 4.5.2 : *Let $J \subseteq S^{(3)}$ and let $\theta_1 = \langle A_1, B_1 | C_1\rangle$, $\theta_2 = \langle A_2, B_2 | C_2\rangle \in J_S$. Then,*

if $A_1 \cap A_2 \neq \emptyset$ and $(B_2 \backslash C_1) \cup (B_1 \backslash B_2) \neq \emptyset$, $gc_s(\theta_1, \theta_2)$ is defined as

$$gc_s(\theta_1, \theta_2) = \langle A_1 \cap A_2, (B_2 \backslash C_1) \cup (B_1 \backslash B_2) | C_1 \cup (C_2 \backslash B_1)\rangle$$

else, $gc_s(\theta_1, \theta_2)$ is undefined.

It is trivial to see that both operators from Definition 4.5.1 and 4.5.2 construct the same triplet. Moreover, the operator presented in Lemma 4.5.2 is proven in [6] to create s-dominant triplets. Additionally in [6], it is stated that the operator has only meaning when applied to triplets from the stable part. Nevertheless, Studený's operator for o-dominant triplets can be applied to any pair of triplets.

4.3 Unique Inference Rule for Stable Triplets

In the previous section we introduced the operator gc_s for constructing s-dominant triplets. For a pair of triplets θ_1 and θ_2, the triplet $gc_s(\theta_1, \theta_2)$ is different from

$gc_s(\theta_2, \theta_1)$. In the work of de Waal and van der Gaag the symmetric triplets are added in order to cover the extra cases of the gc_s application. In this subsection we propose a new rule for creating all possible s-dominant triplets for any pair of stable triplets.

For the application of the rule for o-dominant triplets is necessary that the triplets involved can be represented in an almost general form. This happens because of the condition (2b) which is $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$, for triplets $\theta_1 = \langle A_1{,}B_1|C_1\rangle$ and $\theta_2 = \langle A_2{,}B_2|C_2\rangle$. In the case of s-dominant triplets, any pair of stable triplets can be represented in general form as we do not require such a condition.

We recall that any pair of triplets θ_1,θ_2 can be depicted in a general form which is the following

$$\theta_1 = \langle A_A \cup A_B \cup A_C \cup A_D{,}B_A \cup B_B \cup B_C \cup B_D | C_A \cup C_B \cup C_C \cup C_D\rangle$$

$$\theta_2 = \langle A_A \cup B_A \cup C_A \cup A_{D'}{,}A_B \cup B_B \cup C_B \cup B_{D'} | A_C \cup B_C \cup C_C \cup C_{D'}\rangle$$

Assuming that the necessary conditions for the application of gc_s hold, we can generate at maximum 8 triplets from any pair of stable triplets.

Now we introduce the set $J^S(\theta_1, \theta_2) = \{\theta_1, \theta_2, \theta_{a_s}, \theta_{b_s}, \theta_{c_s}, \theta_{d_s}, \theta_{e_s}, \theta_{f_s}, \theta_{g_s}, \theta_{h_s}\}$ with

$$\theta_{a_s} = \langle A_A{,}A_B \cup B_A \cup B_B \cup B_C \cup B_D \cup B_{D'} | A_C \cup C_A \cup C_B \cup C_C \cup C_D \cup C_{D'}\rangle$$

$$\theta_{b_s} = \langle A_B{,}A_A \cup B_A \cup B_B \cup B_C \cup B_D \cup A_{D'} | A_C \cup C_A \cup C_B \cup C_C \cup C_D \cup C_{D'}\rangle$$

$$\theta_{c_s} = \langle B_A{,}A_A \cup A_B \cup A_C \cup A_D \cup B_B \cup B_{D'} | B_C \cup C_A \cup C_B \cup C_C \cup C_D \cup C_{D'}\rangle$$

$$\theta_{d_s} = \langle B_B{,}A_A \cup A_B \cup A_C \cup A_D \cup B_A \cup A_{D'} | B_C \cup C_A \cup C_B \cup C_C \cup C_D \cup C_{D'}\rangle$$

$$\theta_{e_s} = \langle A_A{,}A_B \cup B_A \cup B_B \cup B_D \cup C_B \cup B_{D'} | A_C \cup B_C \cup C_A \cup C_C \cup C_D \cup C_{D'}\rangle$$

$$\theta_{f_s} = \langle B_A{,}A_A \cup A_B \cup A_D \cup B_B \cup C_B \cup B_{D'} | A_C \cup B_C \cup C_A \cup C_C \cup C_D \cup C_{D'}\rangle$$

$$\theta_{g_s} = \langle A_B{,}A_A \cup B_A \cup B_B \cup B_D \cup C_A \cup A_{D'} | A_C \cup B_C \cup C_B \cup C_C \cup C_D \cup C_{D'}\rangle$$

$$\theta_{h_s} = \langle B_B{,}A_A \cup A_B \cup A_D \cup B_A \cup C_A \cup A_{D'} | A_C \cup B_C \cup C_B \cup C_C \cup C_D \cup C_{D'}\rangle$$

We conclude that we just need to transform any pair of stable triplets into a general form and generate the 8 triplets. These 8 triplets are actually the triplets $gc_s(\theta_1, \theta_2)$, $gc_s(\theta_1, \theta_2^T)$, $gc_s(\theta_1^T, \theta_2)$, $gc_s(\theta_1^T, \theta_2^T)$, $gc_s(\theta_2, \theta_1)$, $gc_s(\theta_2, \theta_1^T)$, $gc_s(\theta_2^T, \theta_1)$ and $gc_s(\theta_2^T, \theta_1^T)$. We do not need to add any symmetric triplet in order to generate all the potential s-dominant triplets and we can therefore introduce a new inference rule

S_{sem}: from θ_1, θ_2 deduce any triplet $\tau \in J^S(\theta_1, \theta_2)$;

As in the case of o-dominant triplets, also in this of s-dominant triplets we do not need to take into account the necessary conditions for the application gc_s. It is trivial to see that for a pair of triplets $\theta_1 = \langle A_1,B_1|C_1\rangle$ and $\theta_2 = \langle A_2,B_2|C_2\rangle$ if $A_1 \cap A_2 = \emptyset$ then the triplet $gc_s(\theta_1, \theta_2)$ cannot be defined. Regarding the second condition, if we are about to to apply the rule and $B_2\backslash(B_1 \cup C_1) = \emptyset$ holds, then the generated triplet will have the form $\langle A_1 \cap A_2,B_1|C_1 \cup (C_2\backslash B_1)\rangle$. This is a valid triplet as it can be produced form the initial triplet $\langle A_1,B_1|C_1\rangle$ by applying decomposition and strong union. It is worth mentioning that the triplet $\langle A_1 \cap A_2,B_1|C_1 \cup (C_2\backslash B_1)\rangle \prec_s \langle A_1,B_1|C_1\rangle$. As it is also stated in the algorithm for closure computation of de Waal and van der Gaag, the s-dominated triplets are removed in each step. Although, through the rule S_{sem} there might be created triplets which are not s-dominated. For example, given that the second condition does not hold in the scheme presented above, we have $\theta_{c_s} \prec_s \theta_1^T$, with θ_1^T not included in the set J^S. In order to deal with this, in the next section we present how we can implement the concept of g-inclusion over stable triplets.

4.4 Stable G-Inclusion

In the previous section we discussed how g-inclusion can be used to include in the closure set only the triplets that have essential information. G-inclusion is a concept that identifies triplets from which other triplets can be constructed by means of G1,G2 and G3. Now we will extend the definition of g-inclusion and combine it with stable triplets, by including the property of strong union. We will refer to this type of g-inclusion as *stable g-inclusion*.

Definition 4.7 *: Let $J \subseteq S^{(3)}$ and let J_S be the stable part of J. Let also $\theta_1 = \langle A_1, B_1|C_1\rangle$,$\theta_2 = \langle A_2, B_2|C_2\rangle \in J_S$. Then,*

θ_1 is stably g-included *to θ_2 (denoted with $\theta_1 \sqsubseteq_s \theta_2$) if and only if the following conditions hold*
1) $C_2 \subseteq C_1$
2) either $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$
or $B_1 \subseteq A_2$ and $A_1 \subseteq B_2$

A triplet that is not stably g-included to any other triplet is called *stably maximal.*

From the definition above we understand that if $\theta_1 \sqsubseteq_s \theta_2$ holds, θ_1 is produced from θ_2 by means of G1, G2, G3 and S5. This is the reason why we have $C_2 \subseteq C_1$ instead of $C_2 \subseteq C_1 \subseteq X_2$(see Lemma 2.3, condition 1). The relation between s-dominance and stable g-inclusion is straightforward from the respective definitions and hence it holds that

$\theta_1 \sqsubseteq_s \theta_2$ if and only if either $\theta_1 \prec_s \theta_2$ or $\theta_1^T \prec_s \theta_2$

We will refer to sets that contain only stable triples as *stable sets*. We now extend stable g-inclusion to sets of stable triplets. The definition is as follows.

Definition 4.8 *: Let $H, J \subset S^{(3)}$ be two stable sets. H is stably g-included in J, denoted with $H \sqsubseteq_s J$, if and only if $\forall \theta \in H$, $\exists \theta' \in J$ such that $\theta \sqsubseteq_s \theta'$.*

It is also useful to introduce the notion of *stably maximal* triplet set with respect to stable g-inclusion. A *stable maximal* triplet set for a stable set J, denoted with $J_{/\sqsubseteq_s}$, is a stable set such that

$$J_{/\sqsubseteq_s} = \{\tau \in J : \nexists \overline{\tau} \in J \text{ with } \overline{\tau} \neq \tau, \tau^T \text{ such that } \tau \sqsubseteq_s \overline{\tau}\}$$

A *stable maximal* triplet set contains only *stable maximal* triplets. Basically, the set $J_{S/\sqsubseteq_s}$ contains essential information in terms of stable independence statements. Alternatively, $J_{/\sqsubseteq_s}$ is free of all stable redundant triplets.

Let now $J^\diamond$ be the closure of the stable part J_S of a set J by adding all the symmetric triplets to J_S and applying gc_s a finite number of times to all pairs. Let also J^S be the closure of the stable part through the application of S_{sem}. We can get the following two properties whose proofs are trivial as they derive from the respective definitions.

Lemma 4.9 *: Let $J \subseteq S^{(3)}$ and let J_S be the stable part of J. Then, $J_S \sqsubseteq_s J_{S/\sqsubseteq_s}$.*

Lemma 4.10 *: Let $J \subseteq S^{(3)}$ and let J_S be the stable part of J. Then, $J^S_{/\sqsubseteq_s} \subseteq J^\diamond$ and $J^\diamond \sqsubseteq_s J^S_{/\sqsubseteq_s}$.*

It is useful to see now that the following property holds

Lemma 4.11 *: Let $J \subseteq S^{(3)}$ and let J_S be the stable part of J. Then, $J^S_{/\sqsubseteq_s} \sqsubseteq_s J^\diamond_{/\sqsubseteq_s}$ and $J^\diamond_{/\sqsubseteq_s} \sqsubseteq_s J^S_{/\sqsubseteq_s}$.*

Proof. *By lemma 4.9 and 4.10 it follows that*

$J^\diamond_{/\sqsubseteq_s} \subseteq J^\diamond \sqsubseteq_s J^S_{/\sqsubseteq_s}$ and also $J^S_{/\sqsubseteq_s} \subseteq J^S \subseteq J^\diamond \sqsubseteq_s J^\diamond_{/\sqsubseteq_s}$ $\Box$.

We proved that the set $J^S_{/\sqsubseteq_s}$ shares the same information as the maximal stable closure set. In order to reduce even more the size of the stable closure and also the number of S_{sem} applications, we can use the concept of maximal non-symmetric set. A *stable maximal non-symmetric set*, denoted with $J^S_{/\sqsubseteq_{sn}}$, is actually a maximal non-symmetric set of $J^S_{/\sqsubseteq_s}$. Therefore, it is easy to see the following property

Lemma 4.12 *: Let $J \subseteq S^{(3)}$ and let J_S be the stable part of J. Then, $J^S_{/\sqsubseteq_{sn}} \sqsubseteq_s J^\diamond_{/\sqsubseteq_{sn}}$ and $J^\diamond_{/\sqsubseteq_{sn}} \sqsubseteq_s J^S_{/\sqsubseteq_{sn}}$*

The proof goes along the same line of Lemma 3.13 proof, by using $J^S_{/\sqsubseteq s}$ and $J^{\diamond}_{/\sqsubseteq s}$ instead of $J^U_{/\sqsubseteq}$ and $\overline{J}_{/\sqsubseteq}$ and taking into account stable g-inclusion. In the next section, we will present the algorithm for the closure computation based on $J^U_{/\sqsubseteq n}$ and $J^S_{/\sqsubseteq sn}$.

4.5 Main Algorithm

Before presenting the main algorithm of closure computation when stable triplets are included, it is useful to see that there can exist conditions under which S_{sem} cannot be applied.

Lemma 4.12 *: Let $J \subseteq S^{(3)}$ and J_S its stable part with $\theta_1 = \langle A_1{,}B_1|C_1\rangle \in J_S$. Furthermore, consider the following conditions*

$$\forall\theta_2 = \langle A_2{,}B_2|C_2\rangle \in J_S \text{ with } \theta_2 \neq \theta_1, \theta_1^T, A_1 \cap (A_2 \cup B_2) = \emptyset \text{ and } B_1 \cap (A_2 \cup B_2) = \emptyset \tag{18}$$

$$\forall\theta_2 = \langle A_2{,}B_2|C_2\rangle \in J_S \text{ with } \theta_2 \neq \theta_1, \theta_1^T, (A_1 \cup B_1)\backslash(C_2) = \emptyset \text{ and } A_2 \cup B_2\backslash(C_1) = \emptyset \tag{19}$$

Then, the S_{sem} rule cannot be applied to θ_1 and any other triplet in J_S if at least one of the conditions (18) *and* (19) *holds*

Proof. This proof follows the same steps of the respective one for Lemma 3.11. Firstly, we show all the possible generated triplets from the application of S_{sem}. For $\theta_1 = \langle A_1{,}B_1|C_1\rangle$ and $\theta_2 = \langle A_2{,}B_2|C_2\rangle$ the potential generated triplets come from the set $J^S(\theta_1, \theta_2)$ and hence we have the following results:

$$gc_s(\theta_1, \theta_2) = \langle A_1 \cap A_2{,}(B_2\backslash C_1) \cup B_1|C_1 \cup (C_2\backslash B_1)\rangle$$

$$gc_s(\theta_1, \theta_2^T) = \langle A_1 \cap B_2{,}(A_2\backslash C_1) \cup B_1|C_1 \cup (C_2\backslash B_1)\rangle$$

$$gc_s(\theta_1^T, \theta_2) = \langle B_1 \cap A_2{,}(B_2\backslash C_1) \cup A_1|C_1 \cup (C_2\backslash A_1)\rangle$$

$$gc_s(\theta_1^T, \theta_2^T) = \langle B_1 \cap B_2{,}(A_2\backslash C_1) \cup A_1|C_1 \cup (C_2\backslash A_1)\rangle$$

$$gc_s(\theta_2, \theta_1) = \langle A_1 \cap A_2{,}(B_1\backslash C_2) \cup B_2|C_2 \cup (C_1\backslash B_2)\rangle$$

$$gc_s(\theta_2, \theta_1^T) = \langle B_1 \cap A_2{,}(A_1\backslash C_2) \cup B_2|C_2 \cup (C_1\backslash B_2)\rangle$$

$$gc_s(\theta_2^T, \theta_1) = \langle A_1 \cap B_2{,}(B_1\backslash C_2) \cup A_2|C_2 \cup (C_1\backslash A_2)\rangle$$

$gc_s(\theta_2^T, \theta_1^T) = \langle B_1 \cap B_2, (A_1 \backslash C_2) \cup A_2 | C_2 \cup (C_1 \backslash A_2) \rangle$

If condition (18) holds, it is trivial to see that none of the triplets above can be defined. If condition (19) holds, we see that $B_2 \backslash C_1 = A_2 \backslash C_1 = B_1 \backslash C_2 = A_1 \backslash C_2 = \emptyset$ and hence we cannot apply the rule as we violate a necessary condition(see Lemma 4.5). It is useful to mention that $B_2 \backslash C_1$ in the triplet $gc_s(\theta_1, \theta_2)$ implies $B_2 \backslash (C_1 \cup B_1)$. Still it holds also that $B_2 \backslash (C_1 \cup B_1) = \emptyset$.

As in the proof of Lemma 3.11, we just have to show that we cannot apply the respective rule to θ_1 and any generated triplet. Let $\theta_3 = \langle A_3, B_3 | C_3 \rangle$ be a triplet in J_S such that we can apply S_{sem} to θ_2 and θ_3. The new generated triplets have the form of the 8 triplets shown above by allowing the same substitutions between θ_1 and θ_3 as in Lemma 3.11 .

We assume that each of the generated triplets in the set $J^S(\theta_3, \theta_2)$ have the form $\theta = \langle A, B | C \rangle$. It is easy to see that $A, B \subseteq A_2 \cup A_3 \cup B_2 \cup B_3$ and $C \subseteq C_2 \cup C_3$. If we apply the operator to θ_1 and θ we have the following cases

$gc(\theta_1, \theta) = \langle A_1 \cap A, (B \backslash C_1) \cup B_1 | C_1 \cup (C \backslash B_1) \rangle$

$gc(\theta_1, \theta^T) = \langle A_1 \cap B, (A \backslash C_1) \cup B_1 | C_1 \cup (C \backslash B_1) \rangle$

$gc(\theta_1^T, \theta) = \langle B_1 \cap B, (B \backslash C_1) \cup A_1 | C_1 \cup (C \backslash A_1) \rangle$

$gc(\theta_1^T, \theta^T) = \langle B_1 \cap B, (A \backslash C_1) \cup A_1 | C_1 \cup (C \backslash A_1) \rangle$

$gc(\theta, \theta_1) = \langle A_1 \cap A, (B_1 \backslash C) \cup B | C \cup (C_1 \backslash B) \rangle$

$gc(\theta, \theta_1^T) = \langle B_1 \cap A, (A_1 \backslash C) \cup B | C \cup (C_1 \backslash B \rangle$

$gc(\theta^T, \theta_1) = \langle A_1 \cap B, (B_1 \backslash C) \cup A | C \cup (C_1 \backslash A \rangle$

$gc(\theta^T, \theta_1^T) = \langle B_1 \cap B, (A_1 \backslash C) \cup A | C \cup (C_1 \backslash A \rangle$

Since $A, B \subseteq A_2 \cup A_3 \cup B_2 \cup B_3$ and $C \subseteq C_2 \cup C_3$ hold, thenwe have

- if condition (18) holds, $A_1 \cap A = A_1 \cap B = B_1 \cap A = B_1 \cap B = \emptyset$ and and none of the above cases constitutes a valid triplet;
- if condition (19) holds, $B \backslash C_1 = A \backslash C_1 = B_1 \backslash C = A_1 \backslash C = \emptyset$ and none of the above cases constitutes a valid triplet;

As in Lemma 3.11, by induction we understand that we are not able to apply the operator between θ_1 and any triplet generated through the procedure of stable closure computation. $\square$

Now we will present the algorithm for computing the closure when we have stable triplets. The initial set J consists of the stable J_S and the unstable part J_U, meaning that $J = J_S \cup J_U$. Algorithm 2 computes the $J^U_{/\sqsubseteq n} \cup J^S_{/\sqsubseteq sn}$.

Algorithm 2 Fast Stable Closure by U_{sem} and S_{sem}

function FSCsem($J_S \cup J_U$)
  $A_S \leftarrow$*S-NonApplicable*(J_S)
  $A_U \leftarrow$*NonApplicable*$(J_U \cup J_S)$
  $A_U \leftarrow A_U \backslash J_S$
  $J_{U_0} \leftarrow J_U \backslash A_U$
  $J_{S_0} \leftarrow J_S \backslash A_S$
  $N_{U_0} \leftarrow J_U \backslash A_U$
  $N_{S_0} \leftarrow J_S \backslash A_S$
  $k \leftarrow 0$
  repeat
    $k \leftarrow k + 1$
    $N_{U_k} := \bigcup_{\theta_1 \in J_{U_{k-1}}, \theta_2 \in N_{U_{k-1}}} J^U(\theta_1, \theta_2)$
    $N_{S_k} := \bigcup_{\theta_1 \in J_{S_{k-1}}, \theta_2 \in N_{S_{k-1}}} J^S(\theta_1, \theta_2)$
    $J_{U_k} \leftarrow$ *FindNonSymmetricMaximal*$(J_{U_{k-1}} \cup N_{U_k})$
    $J_{S_k} \leftarrow$ *S-FindNonSymmetricMaximal*$(J_{S_{k-1}} \cup N_{S_k})$
    $J_{U_k} := J_{U_k} \bigcup_{\forall \theta_1' \prec_s \theta_1 : \theta_1 \in J_{S_k}, \theta_2 \in J_{U_k}} J^U(\theta_1', \theta_2)$
    Check for new Stably Maximal triplets in J_{S_k}
    $J_{U_k} \leftarrow J_{U_k} \backslash$*G-Inclusion*$(J_{U_k}, J_{U_k} \cup J_{S_k})$
    $J_{U_k} \leftarrow J_{U_k} \backslash$*StableG-Inclusion*$(J_{U_k}, J_{S_k})$
    $J_{S_k} \leftarrow$*S-FindMaximal*(J_{S_k})
  until $J_{U_k} \cup J_{S_k} = J_{U_{k-1}} \cup J_{S_{k-1}}$
  return $J_{U_k} \cup A_U \cup J_{S_k} \cup A_S$
end function

In Table 1 we show the sets produced by the application of the various functions in the main algorithm

Function	**Output Set**
S-NonApplicable(J)	$M = \{\tau \in J\colon \tau$ satisfies (18) or (19) $\}$
NonApplicable(J)	$M = \{\tau \in J\colon \tau$ satisfies (9) or (10) or (11) or (12)$\}$
FindNonSymmetricMaximal(J)	$J_{/\sqsubseteq n}$
S-FindNonSymmetricMaximal(J)	$J_{/\sqsubseteq sn}$
G-Inclusion(J, J')	$M = \{\tau \in J : \exists \tau' \in J'$ such that $\tau \sqsubseteq \tau'\}$
StableG-Inclusion(J, J')	$M = \{\tau \in J : \exists \tau' \in J'$ such that $\tau \sqsubseteq_s \tau'\}$
S-FindMaximal(J)	$J_{/\sqsubseteq s}$

Table 1.

Below we will elaborate more on certain steps of the algorithm

- **Step 3 & 4**: We have mentioned that stable triplets satisfy the semi-graphoid properties and thus it makes sense to apply the U_{sem} to a stable triplet and an unstable one. In order to remove the unstable triplets to which U_{sem} cannot be applied, we have to check if an unstable triplet satisfies the conditions (9)-(12) for any other triplet in the initial set(step 3). However, there can be stable triplets to which we cannot apply the U_{sem} and these should not be excluded(step 4).

- **Step 16**: We understand from the algorithm that J_{U_k} and J_{S_k} contain o-dominant and s-dominant triplets respectively. Since we are able to apply U_{sem} to a stable triplet and an unstable one, we apply it, if possible, to every triplet in J_{U_k} and every s-dominated triplet for the s-dominant triplets in J_{S_k}. It is proven that such an application can produce new o-dominant triplets [6], which in our algorithm, as in this of De Waal and Van der Gaag, are added in the unstable part.

- **Step 17**: We can result in a case where a number of s-dominant triplets in J_{S_k} can lead to an s-dominant triplet which is not included in J_{S_k}. De Waal and van der Gaag have already faced that in [6]. In particular,

  Let $J \subseteq S^{(3)}$ and S is a union of disjoint set $S = T \cup U \cup W \cup X \cup Y \cup Z$ and we have the following three stable triplets in J_{S_k}

  $$\langle X,Y|W \cup Z\rangle,\ \ \langle X,Y|U\rangle,\ \langle X,Y|T \cup Z\rangle \tag{20}$$

  By applying the $S5$ axiom to the second triplet we get $\langle X,Y|U \cup Z\rangle$ and (20) is now equal with $\langle X,Y|U\rangle$, $\langle X,Y|Z\rangle$. In order to check for such s-dominant triplets we use the following scheme

  Let $\langle A,B|C\rangle \in J_S$ with $C \neq \emptyset$ and $\{d\}$ be a singleton such that $C' = C\backslash\{d\}$, then $\langle A,B|C'\rangle \in J_S$ if and only if

  $$\forall_{e \in S\backslash ABC'} \exists w \in J_S :\ \langle A,B|C' \cup \{e\}\rangle \prec_s w \tag{21}$$

  In Algorithm 2, we do not have any symmetric triplets in the stable part and thus s-dominance between two triplets might not always be obvious. Hence, we implement the concept of stable g-inclusion and the scheme (21) has as follows

  $$\forall_{e \in S\backslash ABC'} \exists w \in J_S :\ \langle A,B|C' \cup \{e\}\rangle \sqsubseteq_s w$$

- **Step 18, 19 & 20**: Basically, in these two steps we remove the remaining redundant triplets. In steps 13 and 14, the removal of redundant triplets was held to the unstable part separately from the stable one and vice versa. As new triplets can be produced through the steps 15 and 16 we need to remove properly the new possible redundant ones. It is important to note that an unstable triplet can be g-included from a triplet belonging either in the unstable part or in the stable one, whereas it can be stably g-included to a triplet in the stable part only. A stable triplet can be g-included, stably or not, only to another stable triplet. Hence, at step 18 we remove the unstable triplets which are g-included to other triplets in the stable or unstable part. In step 19 we remove the unstable triplets that are stably g-included to stable triplets. Finally, in step 20 we find the maximal set of the stable part so as to remove the redundant stable triplets.

- **Step 22**: In Lemma 3.16 we proved that the triplets included in A_U are not g-included to any triplet in the final set. The case is similar with stable g-inclusion and stable triplets.

5 Conclusions

In this thesis, we studied properties of semi-graphoid models in order to compute efficiently the closure of such models. We improved existing algorithms by introducing conditions for exclusion of triplets from the closure computation. Moreover, we introduced a new type of triplet sets, the maximal non-symmetric triplet sets, with which we can remove more triplets and thus reduce the number of the inference rule application. We further included in our research the concept of stability and provided a unique inference rule for creating s-dominant triplets. In future research we would like experimentally study the amount of reduction in the closure size and in the total time of closure computation achieved by our new conditions and concepts. Furthermore, we plan to use the closure set for representing graphically the independence statements for (semi-)graphoid models. This issue has been partially addressed in [1] with respect to graphoid models but has not been studied as yet for semi-graphoid models. Another topic for further research is whether we can provide conditions for the exclusion of triplets from the closure computation and provide similar types of sets, like the maximal non-symmetric sets, for other types of conditional independence models.

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