# Black Holes, The Information Paradox and The Island Formula


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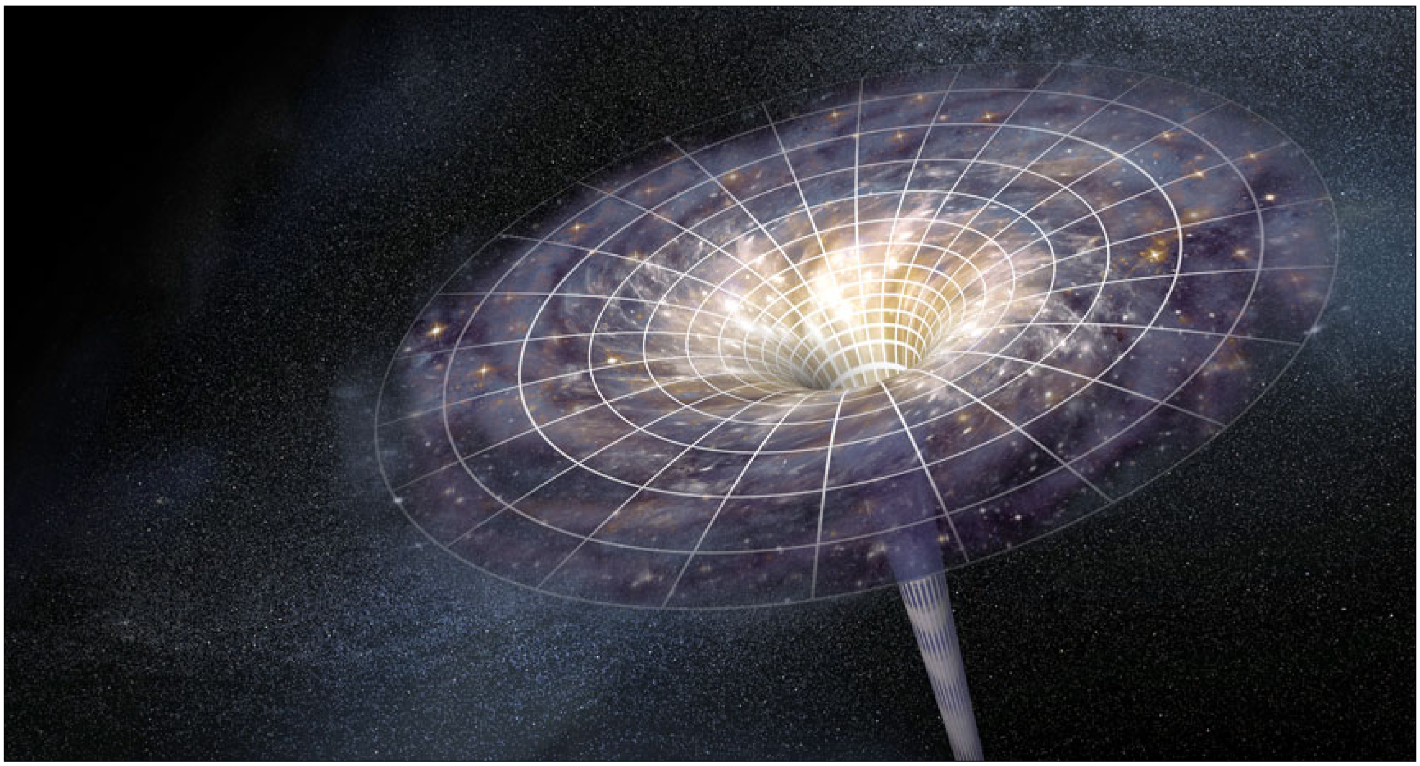

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**Abstract**

Black holes pose fundamental challenges in theoretical physics. It is believed that its resolutions reveal important features of quantum gravity. One such problem involves the black hole information paradox. From the discovery of Hawking radiation on, for years, physicists have tried find out the precise mechanism by which information of matter, that collapses into a black hole during the formation process, can be retrieved from its Hawking radiation at later times. It is essential to understand how unitarity is preserved in semi-classical or quantum gravity, in order to exclude the possibility of information loss.

In this thesis, we dive into black holes and the information paradox. In order to understand black holes and the information paradox, we investigate the fundamental theories of quantum mechanics and general relativity. Hereafter we take a look at important properties of black holes, and the emergence of Hawking radiation. Next, we study the information paradox and especially the Page curve closely, just as proposed solutions to the paradox. We also take a look at the AdS/CFT-duality, from which becomes clear that information is preserved. Finally, we will explore recent advances in the field by Alhmeiri et al. (2019) and Penington et al. (2019). These studies provide a possible explanation to the information problem by a new mechanism for information retrieval, called "islands". We arrive at a unitary Page curve via the island formula, a gravitational fine-grained entropy formula for the Hawking radiation. The island formula is derived from the replica trick by including new saddles: the replica wormholes.

# Contents

# 1 Introduction

Sir Isaac Newton [2] was the first to introduce the formulation of absolute space and time by his law of universal gravitation. Introducing his law in 1687, Newton believed that there exists an attractive gravitational force between two point masses proportional to the product of the masses, and inversely proportional to their squared distance. In 1905, Albert Einstein extended our knowledge of space and time by a theory of special relativity, in which he introduced the fundamental concept of spacetime [3]. In 1915, from inconsistencies in special relativity, Einstein formulated his famous theory of general relativity. In this theory, gravity is considered no longer as a force, but rather as a consequence of the curvature of spacetime.

From the Einstein equations, one instinctively tries to find solutions with maximal radial symmetry. Later on, more general solutions to the equations with less symmetry were found. However, from these solutions, there appeared some physical difficulties such as singularities that were hard to understand. After all, it became clear that once moved inside such region of spacetime, nothing, including light, is able to escape. John Archibald Wheeler [4] was the first to name these curious objects 'black holes'.

At first, just like many other scientists, Einstein believed that the emergence of black holes from his theory were the result of an incomplete physical description. Nonetheless, the current understanding of the black hole solution is much more complete and the existence of black holes is widely accepted. Furthermore, we have gained more knowledge about the formation of black holes. Above all, the existence of black holes is one of the most exciting predictions from Einstein's theory of general relativity .

In 1973, Jacob Bekenstein [5] proposed that a black hole has a finite entropy, proportional to the area of its event horizon. Not much later, in 1974, Stephen Hawking [6] came up with a spectacular discovery: black holes behave as thermal objects and thus emit thermal radiation, which we denote as "Hawking radiation". Hence, black holes are not completely "black". Hawking's discovery of black hole evaporation has lead to deep puzzles in general relativity, quantum mechanics and especially in quantum information theory. By Hawking's argument, black hole evaporation seems to violate a fundamental property in quantum mechanics, called unitarity, which means that quantum information in a system is preserved over time. Unlike other quantum and classical systems, black holes might not hold its information. Indeed, at first, Hawking concluded that information from black holes will eventually be lost.

However, many physicists did not like the idea of information loss, and from Hawking's discovery on, much research has been done in order to find a solution to the paradox. It was soon proposed that a full theory of quantum gravity, that should fit general relativity and quantum mechanics into one complete description, is needed to resolve the problem. Yet, just as Hawking proposed, some scientists speculated that this theory must be non-unitary.

In the years after Hawking's discovery, many ideas about and solutions to the problem have been formulated. At first, these ideas offered some new insights, though they did not fully solved the problem, and researchers were still left with many questions. For example, in 1993, radical insights came from the idea of black hole complementarity [7]. This research offered a new understanding of the paradox, but still, not all scientists were convinced.

In the '90s, results from string theory considering quantum gravity suggested that information indeed must come out. Major progress came from the AdS/CFT-correspondence discovered by Juan Maldacena [8]. In AdS/CFT, one is allowed to compute the entropy of a black hole in $d+2$-dimensional anti-de Sitter (AdS) spacetime, involving a dual $d+1$-dimensional conformal field theory (CFT) on the AdS boundary. Specifically, unitarity on the CFT boundary implies that information is preserved. Hence, Maldacena showed evidence that information can escape the black hole. However, boundary unitarity is not enough to solve the paradox, and a broader understanding is necessary. Furthermore, the question remained whether these results apply to the real universe. Due to the evidence found in AdS/CFT, even Hawking reconsidered his view [9].

After all, by recent research [10][11][12][13], it seems that we have found a definite, more general solution to the information paradox, which is based on findings in AdS/CFT. However, the new understanding is much more general: the results apply to asymptotically flat Minkowski space, and anti-de Sitter spacetime is not required.

The new proposal states that Hawking did not use the right formula for calculating the black hole entropy. The correct formula is the gravitational fine-grained entropy, which was originally studied by Ryu and Takayanagi [14] in AdS/CFT, but has now been both extended and generalized by new research. The new formula results in spatially disconnected regions in the black hole interior, which we call ”islands” [15]. The formula is often named as the ”island formula” [16][10], and can be derived by a mathematical tool called ”the replica trick” [13][11]. The island formula ultimately results in a unitary ”Page curve” [17], which denotes the unitary behaviour of the black hole. However, the fact that unitarity is preserved by the computation of the Page curve, is only a part of the paradox, since one would like to know exactly how information ends up in the outgoing Hawking radiation.

In this thesis, we will take a look at black holes, the information paradox and new research that describes the island formula. In chapter 2 and 3, we discover fundamental theories in quantum mechanics and general relativity. In chapter 4, we take a look at the properties of black holes. In chapter 5, we study Hawking radiation, and we arrive at the information paradox in chapter 6. In chapter 7, we discover proposed solutions to the paradox, which offer important insights in the understanding of the paradox. In chapter 8, we consider the AdS/CFT-correspondence in some detail. Finally, we discover the island formula in chapter 9. Furthermore, we use the conventions that $c = 1$, unless specified otherwise. In general, we will study uncharged, non-rotating black hole, though we will make a note about this in chapter 4.

# 2 Quantum Entanglement

Quantum entanglement is a quantum mechanical property. It states that pairs or groups of particles cannot be described independently of each other. So, a member of the state can only be described relative to all the other states involved. Essentially an entangled system is defined to be one whose quantum state cannot be seen as a product of states of its parts: the system can be described by one wave function which we cannot separate into a wave function for every subsystem involved. So, these parts are individual independent particles, but are inseparable as a total system [18].

Consider two subsystems, $S_1, S_2$ with corresponding Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. We define the states $|n\rangle \in \mathcal{H}_1$ and $|m\rangle \in \mathcal{H}_2$. The space of these two states is spanned by the tensor product of the two Hilbert spaces, $\mathcal{H}_1 \otimes \mathcal{H}_2$. We can form a basis $|n \otimes m\rangle = |n\rangle \otimes |m\rangle$ and can compute a general state. A general state

$$|\Phi\rangle = \sum_{n,m} a_{nm} |n \otimes m\rangle \tag{2.1}$$

is defined to be entangled when the state cannot be written as a product state.

The general state can be written as a tensor product of two states of the different Hilbert spaces, with $|\phi\rangle_1 \in \mathcal{H}_1$ and $|\phi\rangle_2 \in \mathcal{H}_2$ so that $|\phi\rangle_1 = \sum_n c_n |n\rangle$ and $|\phi\rangle_2 = \sum_m d_m |m\rangle$ , only if the coefficients can be written as a product too: $a_{nm} = c_n d_m$. States that can be written as a product state form a subset of the Hilbert space, but not a subspace. This also holds for a more general example. When a state cannot be written in the form of $|\Phi\rangle = \bigotimes_{i=1}^{N} |\phi\rangle_i$ when considering a joint Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$ with $|\phi\rangle_i \in \mathcal{H}_i$, we have an entangled state.

We will give an example for this. Lets define a couple two spin-1/2 particle

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{2.2}$$

where $|\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle$ and $|i\rangle_j \in \mathcal{H}_j$, with $i \in \{\uparrow, \downarrow\}$. This state is entangled. Here we have spin-up and spin-down in different Hilbert spaces. The full state here is in a pure state, but the separate components are not. We cannot define a pure state for one of the systems until we made a measurement on the total state, so as a whole the system will be in one of the two states. Therefore, the complete description of this state is about the parts of the different Hilbert spaces seen relative to each other.

When one makes a measurement on one of the systems, we observe the following. When we measure a spin-up state of system 1, this immediately gives a spin-down state of system 2. Though both systems may be cut apart or distant by a space-like gap, so no information could be exchanged between the two, we will always get this outcome. When we measure a spin-down on system one, this will give a spin-up for the state of system 2. What we experience here, is the non-local character of quantum mechanics. Quantum entanglement is widely discussed by the EPR-paradox. [1][19].

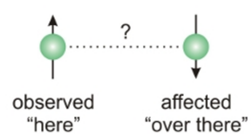



Figure 1: An impression for quantum entanglement: a measurement on one systems, affects the outcome for the other system. [19]

Quantum bits are often named as qubits. In fact these qubits are the quantum mechanical equivalent of the classical bit. The qubit is the most fundamental unit of quantum mechanical information . It can be seen as a two state system, for example a particle with spin-up or spin-down. When we take two possible states, the classical bit is always in the state $|0\rangle$ or $|1\rangle$. For the quantum bit, linear combination are also possible:

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \tag{2.3}$$

Here the qubit is presented by a wave function, where one has a probability of $|\alpha_0|^2$ to find a spin-down state and a probability of $|\alpha_1|^2$ to find a spin-up state. This is classically not possible: here one of the two coefficients must be equal to 1, so the other will be zero. For qubits, entangled states are possible, which is not the case for classical bits.

[1] We will come back to EPR in section 7.4

## 2.1 The density matrix

To study entanglement, we must study the density matrix. The density matrix essentially characterizes the quantum state of the physical system. One can use this matrix to study pure and mixed states. The state of a system is defined to be pure when we only need one wave function or state. A state is mixed when we cannot describe the state as just one wave function. To describe the state, we need an ensemble of wave functions. In this case, we do not have full knowledge about the full state of the system. Mixed states can be described by a density matrix where the eigenvalues of the matrix give the probability to be in the state related to this eigenvalue. In fact, a pure state is a subset of a mixed state. When we have a pure state, we know everything about the system, and so the exact state. The full system will then have a density matrix with every eigenvalue set to zero except for one. We can define the density matrix as

$$\rho = \sum_i p_i \left|\phi_i\right\rangle\left\langle\phi_i\right| \tag{2.4}$$

Here the coefficients are probabilities that have the properties $\sum_i p_i = 1$ and $0 \leqslant p_i \leqslant 1$. Also, the wave functions are orthonormal states. Properties of this density matrix are $\rho = \rho^\dagger$, $\mathrm{Tr}(\rho) = 1$ and $\langle\phi|\,\rho\,|\phi\rangle \geqslant 0$, $\forall\,|\phi\rangle$. An important property is that for every state the trace of the matrix is unity, but only for a pure state we see that $\mathrm{Tr}\big(p^2\big) = 1$ holds. In general, a state is pure if $\mathrm{Tr}(\rho^n) = \mathrm{Tr}(\rho)^n$, for $n \in \mathbb{N}$. For a mixed state, this is not the case: $\mathrm{Tr}\big(p^2\big) < 1$. So, the density matrix can be used to see whether the system is in a pure or mixed state.

To illustrate this principle, we look at an example. We study the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)$ called the Bell state. Computing the density matrix gives, according to the definition , the form

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Here we make use of the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$. When taking the trace of the squared matrix, one can easily show that $\mathrm{Tr}\big(\rho^2\big) = 1$. So, this Bell state is a pure state. Later on, we will see an example for a mixed state.

## 2.2 The reduced density matrix

The entropy of a system is a property that is used to measure the state of disorder. It measures the amount of microstates that creates its macrostate. We define the thermodynamic entropy to be $S = k_B \log \Omega$, with $k_B$ the Boltzmann constant and $\Omega$ the number of microstate that a macrostate can have. Here every microstate has an equal probability to arise. In the theory of quantum information, the entropy is often used.

### 2.2.1 The reduced density matrix: the entanglement entropy

We can construct the von Neumann entropy. It is defined as

$$S = -\,\mathrm{Tr}(\rho \log \rho) \tag{2.5}$$

It is often used to define the entropy of a quantum system. The von Neumann entropy also shows the availability of information and the mixture of the state. For a pure state, the von Neumann entropy equals zero, $S = 0$. When the state is pure, we know all the information about the state. Calculating the von Neumann entropy of our previous example shows indeed that this entropy is zero. On the other side, when a state has a von Neumann entropy $S > 0$, this state will be mixed. A state is said to be maximally mixed when its entropy is given by $S = \log(N)$, with $N$ the dimension of the system. More specifically, this is the case when the density matrix is a diagonal matrix.

We can also define a reduced density matrix. This can be useful, for example when we would like to study a subsystem of a larger total system. Let's take two subsystems, 1 and 2, so the total larger system will be $1 + 2$ . We construct the reduced density matrix to be a partial trace of the reduced density matrix of the full system

$$\rho_1 = \mathrm{Tr}_2(\rho) \tag{2.6}$$

with $\rho_{1+2} = \rho$, or alternatively stated $\rho_1 = \sum_{n,n'} \rho_{1,nn'}\,|n\rangle\langle n'|$. Now we computed the reduced density matrix, we can also compute the entanglement entropy. In fact, the von Neumann entropy of the subsystem is the entanglement entropy of a subsystem $i$

$$S_{EE} = -\,\mathrm{Tr}(\rho_i \log \rho_i) \tag{2.7}$$

When the entanglement entropy is greater than zero, $S_{EE} > 0$, we have entanglement. It is for example easy to see that the entanglement entropy of the previously mentioned Bell state is greater than zero. Here, the density matrix is given by

$$\rho_1 = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Then the reduced density matrix $S_1 = \log(2)$. Via this example, we see that the total system is in a pure state, while the entanglement entropy is nonzero, so the subsystem acts as a mixed state. Thus the entanglement entropy shows the entanglement between system 1 and 2. This can also be interpreted as the amount of missing information of the total system when only considering system 1.

Another example for this is the singlet state for two spin-1/2, for which the state is given by

$$\Phi = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{2.8}$$

When computing the density matrix and hereafter the trace, we arrive at $\mathrm{Tr}(\rho^2) = 1$. Thus, the state is a pure state. By looking at the reduced density matrix,

$$\rho_1 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and the corresponding entanglement entropy, we see that the substate acts again as a mixed state: $S_{EE} = -tr(\rho_1 \log \rho_1) = \log(2)$.

### 2.2.2 The reduced density matrix: coupling to thermodynamics

The formalism of a reduced density matrix can be very useful in many applications as well, for example a system which is coupled to a heat bath, see the figure.[18] Here the reduced density matrix can describe the Hilbert space of the subsystem, including the effect of the coupling to the other subsystem. The density matrix can also be used to describe the time evolution of a system, to arrive at the von Neumann equation of a subsystem

$$i\hbar\frac{\partial\rho(t)}{\partial t} = [H(t), \rho(t)], \tag{2.9}$$

One can compute a similar equation for the reduced density matrix:

$$i\hbar\frac{\partial\rho_1(t)}{\partial t} = [H_1(t), \rho_1(t)] + \int_{-\infty}^{t} dt' \sum(t,t')\rho_1(t') \tag{2.10}$$

Here, the second term is given due to the coupling to $\mathcal{H}_1$. Furthermore, one is able to show that the density matrix for a setup given in the figure below is given by $\rho = \frac{1}{Z}e^{-\beta H}$, where the partition function $Z = \mathrm{Tr}(e^{-\beta H})$ is involved, with $H$ the Hamiltonian for the system. Here we can make a coupling to thermodynamics, specifically to the free energy, entropy and the specific heat:

$$F = -k_B T \log(Z) \tag{2.11}$$

$$S = -\frac{\partial F}{\partial T} \tag{2.12}$$

$$C = T\frac{\partial S}{\partial T} \tag{2.13}$$

In this case, the density matrix is given by maximizing the entropy under the constraint that the average energy is fixed due to the heat bath.

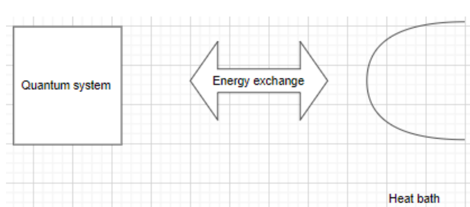



Figure 2: A system with Hilbert space $\mathcal{H}_1$ coupled to a heat bath.

## 2.3 Properties of entanglement entropy

In this section, we discuss some interesting properties of the entanglement entropy. We will look at the statement of strong subadditivity and the Schmidt decomposition.

### 2.3.1 Subadditivity

The entanglement entropy has some interesting properties. First of all, the entanglement entropy of two systems obeys the general triangular law $|S_1 - S_2| \leqslant S_{12}$. Also, for a system for three or more systems involved, with Hilbert space $\mathcal{H}_{123} = \bigotimes_{i=1}^{3} \mathcal{H}_i$, one will find

$$S(\rho_{123}) + S(\rho_2) \leqslant S(\rho_{12}) + S(\rho_{23}) \tag{2.14}$$

for a tripartite system. This is the statement of strong subadditivity.

For a bipartite system, subadditivity states that

$$S(\rho_{12}) \leqslant S(\rho_1) + S(\rho_2) \tag{2.15}$$

For a Hilbert space $\mathcal{H}_{12} = \bigotimes_{i=1}^{2} \mathcal{H}_i$

### 2.3.2 The Schmidt decomposition

The von Neumann entropy for pure states is a well-defined measure for the entanglement entropy, which is a consequence from a theorem called the Schmidt decomposition.[20] It is stated as follows.

Assume we have a pure state which is a composite system $AB$. Then we can find for both subsystems orthonormal states such that the composite wave function can be described as

$$|\Psi\rangle = \sum_n \lambda_n |n\rangle_A \otimes |n\rangle_B \tag{2.16}$$

with $0 \leqslant \lambda_i \leqslant 1$ and $\sum_i \lambda_i^2 = 1$. These coefficients are called the Schmidt coefficients. In fact, in this way it can be seen that each of the states of system A will be correlated to a state in system B. We can proof this theorem.

Suppose we have a subsystem A with basis $|\Phi\rangle_A = \sum_n |n\rangle_A$ and that we have a subsystem B with basis $|\Phi\rangle_B = \sum_j |j\rangle_B$, where $|n\rangle_A$ and $|j\rangle_B$ are orthonormal states. Then we can express the wave function of the total system as $|\Phi\rangle_{AB} = \sum_{nj} a_{nj} |n\rangle_A \otimes |j\rangle_B$. Here $a_{nj}$ are matrix elements of a certain matrix $A$. We can express these elements in some basis $\{l_i, m_i\}$ as

$$a_{nj} = \langle n|_A A |j\rangle_B = \langle n|_A [\sum_i b_i |l_i\rangle\langle m_i|] |j\rangle_B = \sum_i b_i \langle n|_A |l_i\rangle\langle m_i|j\rangle_B \tag{2.17}$$

When we insert this in the expression for the total state.

$$|\Psi\rangle_{AB} = \sum_{nj}[\sum_i b_i \langle n|_A |l_i\rangle\langle m_i|j\rangle_B] |n\rangle_A \otimes |j_B\rangle = \tag{2.18}$$

$$\sum_i b_i[\sum_n \langle n|_A |l_i\rangle |n\rangle_A] \otimes [\sum_j \langle m_i|j\rangle_B |j\rangle_B] = \tag{2.19}$$

$$\sum_i b_i[|l_i\rangle_A \otimes |m_i\rangle_B^*] = \tag{2.20}$$

$$\sum_i |b_i| e^{i\theta_i} [|l_i\rangle_A \otimes |m_i\rangle_B^*] \tag{2.21}$$

here the introduced coefficients is $b_i \in \mathbb{C}$. Now we can use another basis $k_i, v_i$, and define the states $|k_i\rangle_A = e^{i\theta_i} |l_i\rangle_A$ and $|v_i\rangle_B = |m_i^*\rangle_B$. Here, $\lambda_i = |b_i| \in \mathbb{R}$. We just used a trick here, so that $|b_i| \in \mathbb{R}$ instead of $b_i \in \mathbb{C}$. Now, we can express the total state as

$$|\Psi\rangle_{AB} = \sum_i \lambda_i (|k_i\rangle_A \otimes |v_i\rangle_B) \tag{2.22}$$

Since we obtain for both subsystems the same coefficient $\lambda_i$, the reduced density matrices will look the same. They have the same eigenvalues:

$$\rho_A = \sum_i \lambda_i^2 |k_i\rangle_A \langle k_i|_A \tag{2.23}$$

$$\rho_B = \sum_i \lambda_i^2 |v_i\rangle_B \langle v_i|_B \tag{2.24}$$

One can state therefore that the entanglement entropies of the subsystems will be the same:

$$S_A = -\operatorname{Tr}(\rho_A \log \rho_A) = -\lambda_i^2 \log \lambda_i^2 \tag{2.25}$$

$$S_B = -\operatorname{Tr}(\rho_B \log \rho_B) = -\lambda_i^2 \log \lambda_i^2 = S_A \tag{2.26}$$

This is an interesting result which we will interpret later, when studying the Page curve. In general, when there will be more than one coefficient that is non-zero, the state is entangled. The total state is only separable if there is only one coefficient which is either 1 or 0. Hence, this composition tells us something about the separability of the state.

## 2.4 The Rényi entropy

In chapter 9, we will make use of a quantity called the Rényi entropy. It is defined as

$$S_\alpha = \frac{1}{1-\alpha} \log(\operatorname{Tr}(\rho^\alpha)) \tag{2.27}$$

$$\alpha \in (0,1) \cup (1,\infty) \tag{2.28}$$

We arrive at the von Neumann entropy when taking the limit $\alpha \to 1$:

$$\lim_{\alpha\to 1} S_\alpha(\rho) = S_{vN}(\rho) \tag{2.29}$$

Furthermore, we can show the following. We define $\lambda_i$ to be the eigenvalue of $\rho$. Then we can write

$$\log(\operatorname{Tr}\rho^n) = \log\left(\sum_i \lambda_i^n\right) \tag{2.30}$$

Now, we take the derivative of this with respect to $n$:

$$-\frac{\partial \log \operatorname{Tr}(\rho^n)}{\partial n}|_{n=1} = -\frac{\sum_i \lambda_i^n \log(\lambda_i)}{\sum_i \lambda_i^n}|_{n=1} = -\frac{\sum_i \lambda_i \log(\lambda_i)}{\sum_i \lambda_i (2.31)}$$

Now, since $\operatorname{Tr}(\rho) = 1$, we have $\sum_i \lambda_i = 1$. Hence, we arrive at [21]

$$-\frac{\partial \log \operatorname{Tr}(\rho^n)}{\partial n}|_{n=1} = -\sum_i \lambda_i \log \lambda_i = -\operatorname{Tr}(\rho \log \rho) \tag{2.32}$$

## 2.5 Unitary time evolution

Time evolution can be written in terms of an evolution operator. A state is defined as [18]

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle \tag{2.33}$$

by the time evolution operator $U(t, t_0)$, which thus describes the time evolution of a quantum state. Here the operator is unitary. An operator is unitary if it obeys $UU^\dagger = \mathbb{1} = U^\dagger U$. Whenever we make a measurement in time, the sum of all the probabilities of the possible outcomes of the involved state must be equal to one: probabilities are conserved over time.

## 2.6 Fine-grained and coarse-grained entropy

Basically there are two kinds of entropy that we might consider in physics. We already defined the simplest one, the von Neumann entropy. When we have all knowledge of the system, the von Neumann entropy reduces to zero.

Another kind of entropy is the coarse-grained entropy. The difference between the Von Neumann entropy and the coarse-grained entropy is that we do not take a measure of all the observables involved. We just measure a subset of so called coarse-grained observables $A_i$. The coarse-grained entropy is defined in the following way. We look at all possible density matrices $\tilde{\rho}$ which produce the same values as the observables we study. So this implies $\operatorname{Tr}(\tilde{\rho}A_i) = \operatorname{Tr}(\rho A_i)$ . From this, we state the already known von Neumann entropy of the studied matrices. As a last step, we maximize over all possible outcomes of the density matrices involved. This entropy is used in thermodynamics. Take for example some observables we would like to study, such as the energy and volume of a system. Then one is able to find the thermodynamic entropy by just maximizing the von Neumann entropy over all these states with the studied energy and volume.

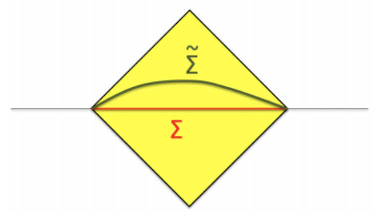



Figure 3: A causal diamond for a spatial slice Σ.Another slice $\tilde{\Sigma}$ has the same causal diamond.[22]

A property of the coarse-grained entropy is that it increases under unitary time evolution. So, it obeys the second law of thermodynamics. In general, the von Neumann entropy - which, as we have seen earlier, has much to do with quantum entanglement – is often named as 'quantum entropy' or 'fine-grained entropy', in comparison to the 'coarse-grained entropy'. Essentially the fine-grained entropy must be smaller than the coarse-grained entropy.

$$S_{VonNeumann} \leqslant S_{Coarse-grained} \tag{2.34}$$

This is a result of the way we defined these entropies. The coarse-grained entropy puts an upper bound to the amount of entanglement of the system: it measures the total amount of degrees of freedom that is accessible to the system.

Furthermore, one can also state the fine-grained entropy on a spatial region. We define this spatial region on a fixed time slice. This spatial part has its own density matrix, and we define the fine-grained entropy of this part as

$$S_{VN}(\Sigma) = S_{VN}(\rho_\Sigma) \tag{2.35}$$

The spatial region Σ is a Cauchy slice. It describes a causal diamond, a region of which we can understand physically if we know the initial properties of the surface. This is not the case of the outside of the slice. The von Neumann entropy of another spatial slice is the same, as long as they have the same causal diamond, see the figure.When we have just a part of the complete slice, the von Neumann entropy on this slice is often time-dependent. So, it can increase or decrease with time if we change its place in time.

This is a manifold from general relativity which can be seen as a collection of pairs of locations and times. Typically, a Cauchy surface gives a notion of direction and time. Especially in a curved spacetime manifold, it is hard to study time evolution of physical quantities by the fact that there is no natural direction of time. A Cauchy slice can be seen as a slice at a fixed time: between these points, there is no time difference. So, from this slice at a fixed time, one may predict the future or past of the physics on this slice.

When we treat gravity in the semiclassical approximation– that is, when we treat gravity classically adding quantum mechanical theory – we can also describe the semiclassical entropy of the spatial slice. Here, the semiclassical entropy is simply the von Neumann entropy of quantum mechanical fields in the semiclassical approximation: it is the fine-grained entropy described by quantum field theory in curved spacetime.

# 3 General Relativity

General relativity (GR) is a theory by Albert Einstein that describes gravity in a geometry of curved spacetime. Specifically, Einstein's theory defines gravitation to be a consequence of the curvature of spacetime. For example, other consequences of general relativity include gravitational time dilation, redshifts, light bending an the presence of horizons.

In general relativity, we work in four-dimensional spacetime: one time dimension and three spatial dimensions. Just as in regular 3D-coordinates, the space is described by points in three dimension. Spacetime is then an extension to this, by adding a time coordinate, so we end up with a four-dimensional set of points $\{t, x, y, z\}$. A main difference between Newtonian physics – Newton also described gravitation – and general relativity is that the first takes a look at time and space separately, while the latter interprets it as one. In practice, when calculating physical quantities in GR, on has always a fourth dimension due to the time coordinate which has to be taken into account.

For his laws of universal gravitation, Newton considered two elements: an equation for the gravitational fields as influenced by matter, and an equation for the response of matter to this field [23]. Newton manifested these elements into his equations:

$$\vec{F} = \frac{GMm}{r^2}\hat{e}(r) \tag{3.1}$$

$$\vec{F} = m\vec{a} \tag{3.2}$$

which describe the forces between particles and the resulting acceleration. Equivalently, one can state these expressions in terms of the gravitational potential $\Psi$ and the mass density $\rho$ as

$$\vec{\nabla}^2\Psi = 4\pi G\rho \tag{3.3}$$

$$\vec{a} = \nabla\Psi \tag{3.4}$$

These equations define Newtonian gravity, and have to be described,or even replaced, by a description which involves the curvature of spacetime.

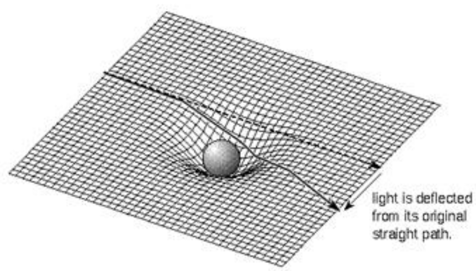



Figure 4: Spacetime curvature in general relativity. In order to describe the curvature, we use manifolds: mathematical complex topological spaces. For example, a consequence of curvature is that light is deflected from its original path.

To describe the curvature of spacetime in GR, we use manifolds. Manifolds are defined as topological spaces which, locally, have comparable properties to $n$-dimensional flat Euclidean space. Though, globally these manifolds may be curved, so they can be seen as a clever mathematical tool, a structure, to describe curvature in spacetime. Manifolds that correspond to a curved space, often have a complicated topology. Interestingly they look locally just like $\mathbb{R}^n$. Besides, spherically symmetric manifolds can be fit together into $n$-spheres. In mathematics, an $n$-sphere is defined as a set of $n+1$ points that are located at a constant value from a certain central point. This sphere is mathematically defined as $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = r\}$, using a standard norm $\|x\|$. For a unit radius, we simply have $r = 1$. A simple example for this is in 3D-Euclidean space, namely a 2-sphere. For higher dimensions, we obtain more complicated manifolds. Furthermore, it is possible to keep spherical symmetry without an origin: an example for this is a wormhole.

## 3.1 The metric

Another basic element in the description of spacetime is the metric tensor $g_{\mu\nu}$. The metric contains the geometry of space. The Riemann curvature tensor can be derived from the metric, a property that is used in the Einstein equations which describes the curvature of manifolds. There are some restrictions on the components of the metric tensor: for example, the determinant of the metric tensor does not vanish, $det(g_{\mu\nu}) = 0$. It also has an symmetric inverse metric $g^{\mu\nu}g_{\nu\sigma} = g_{\lambda\sigma}g^{\lambda\mu} = \delta^{\mu}_{\sigma}$, with $\delta^{\mu}_{\sigma}$ the Kronecker delta. In GR, the metric tensor has some interesting properties. For example, it is used to find the shortest distance between two separate points. Also, by the metric tensor, we can define physical properties, such as the inner product, in four-dimensions instead of three. From the metric tensor, we can define a line element, by using the Einstein convention, as

$$\mathrm{d}s^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3.5}$$

with d$x$ an infinitesimal distance. For example, in three dimensional flat space, the metric tensor is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

with corresponding metric

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3.7}$$

in Cartesian coordinates. In spherical coordinates $\{t, r, \phi, \theta\}$, the line element takes the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin\theta^2 d\phi^2 \tag{3.8}$$

In Special relativity, we often use the Minkowski metric tensor $\eta_{\mu\nu}$, which describes four-dimensional flat spacetime. It is defined as

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.9}$$

This is used to define the line element as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{3.10}$$

$$= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{3.11}$$

In spherical coordinates, this Minkowski metric becomes

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \tag{3.12}$$

By this metric, the time-coordinate is distinguished by a minus sign in comparison to the other coordinates involved. In special and general relativity, this is often the case, while in flat Euclidean space, with only spatial coordinates, the signs will be the same.Therefore, a metric with only positive eigenvalues is called Euclidean, while metrics that contain a single minus sign, are called Lorentzian. For example, a Schwarzschild metric[2] tensor can be defined in spherical coordinaates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$ as

$$g_{\mu\nu} = \begin{pmatrix} (1 - \frac{2GM}{r^2}) & 0 & 0 & 0 \\ 0 & -(1 - \frac{2GM}{r^2})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \tag{3.13}$$

Furthermore, one can define co-vectors and contra-vectors which are related by the metric tensor:

$$x_\mu = g_{\mu\nu} x^\nu \tag{3.14}$$

$$x^\mu = g^{\mu\nu} x_\nu \tag{3.15}$$

In four-dimensional spacetime, the four vectors have the form

$$x^\mu = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} t/c \\ x \\ y \\ z \end{pmatrix} \tag{3.16}$$

The same procedure can be generalized in other dimensions, for a line element

$$ds^2 = g_{ij}(x) dx^i dx^j \tag{3.17}$$

for $g_{ij}$ a symmetric matrix. [24]

[2] we will encounter this metric later on.

## 3.2 The Equivalence Principle

In general relativity, the dynamical field that causes gravitation is the metric tensor , which describes the curvature of spacetime. It is the metric tensor itself that describes curvature in spacetime, rather than some other field moving through spacetime. This significant observation inspired Einstein to state the Principle of Equivalence. The principle is described in many forms, and we start with the Weak Equivalence Principle (WEP). It states that the inertial mass and gravitational mass of an arbitrary object are equal. The inertial mass can be seen in Newton's Second law, and the gravitational mass is the proportionality constant in Newton's laws of gravitation.

$$\vec{F} = m_i \vec{a} \tag{3.18}$$

$$\vec{F}_g = -m_g \nabla\Phi \tag{3.19}$$

$$m_i = m_g \tag{3.20}$$

So, putting it simply, WEP is stated by the last equation, $m_i = m_g$ . A consequence of this statement is that the nature of a free-falling particle is quite general: the acceleration is independent of the mass, $\vec{a} = -\nabla\Phi$, due to gravity. In particular, the WEP-statement implies a universality of gravity. It also states that one cannot distinguish an uniform accelerating object to an object in a gravitational field, when we look at small enough regions of spacetime. In larger regions, we will find inhomogeneities in the fields of gravity, which will lead to tidal forces.

We also have the Einstein Equivalence Principle (EEP). It states that the laws of physics can be seen as the laws of special relativity, if we take the studied region of spacetime to be small enough. This statement comes from the idea that gravity is something universal: it must act on all particles.From this, it makes sense to state that a particle which feels no acceleration, is freely falling. This relates to the often stated idea that gravity is not a force, because a force eventually creates the acceleration. Additionally, the strong equivalence principle defines all the laws of nature to be the same, when in a uniform static gravitational field and in the corresponding reference frame of acceleration. In fact, it says that the laws of gravitation do not rely on the velocity and location involved.

## 3.3 Special relativity

Special relativity (SR) is the theory of spacetime without gravity, and so without curvature. Hence, it is a special form of general relativity. Special relativity describe Newtonian mechanics, which is also about the structure of spacetime. In Newtonian physics, we have different slices of space at different times. Particles move forward in times, along worldlines, at any speed. In this theory, there is a universal agreement on when and where events in space occur. Hence, in this theory, it is clear when two events happen at the same time. In special relativity, there is no well defined slice of space, with all of the space at a single moment of time. However, this spacetime is not completely structureless, since we can define a light cone. A light cone describes a set of trajectories that can be taken by particles and it defines the causal structure of spacetime. From the origin of the light cone, there is no particle that can reach a point of space out of the light cone. Specifically, all future and early paths of a particle at the origin are defined by the light cone. A light cone consists of two parts, the future and the past. Particles move via time-like or light-like worldlines, paths in four-dimensional spacetime. On these worldlines, particles move with less than (time-like) or equal to (light-like) the speed of light. In special relativity, we assume that it is impossible to exceed the speed of light. Space-like regions lay outside the light cone, and we cannot define a causal relation between a body at the origin and a space-like point, since we have $v > c$ here. When different events happen in different light cones, we cannot define which event took place earlier in time. So, the formalism of the light cone replaces the formalism of spacetime that is divided into unique parts of space parametrized by time. [23] Special relativity relies on the theory that every law in nature is invariant under a set of transformations of space and time:

$$x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{3.21}$$

or in simple matrix notation $x' = \Lambda x$. More specifically, a Lorentz group is a general rotation group. It is the rotation group [3] SO(3) which states that a three dimensional length $d\vec{x}$, and the line element $ds^2$, must be invariant. The Lorentz group can be defined as

$$x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu} : \Lambda^{\mu}_{\nu} \in SO(3) \tag{3.22}$$

In special relativity, it is indeed stated that physical properties are invariant under this Lorentz group. [24] In Minkowski space, the spacetime interval between different events, points in spacetime, should be preserved. Hence, if the spacetime interval is not invariant, the Lorentz transformations cannot be used. An ordinary Lorentz transformation is a transformation from one coordinate frame in spacetime to another one, where one of the reference frames moves with a constant velocity relatively to the other. We can distinguish different forms of reference frames: an inertial frame, that

[3] In mathematics, one can define a special orthogonal group, SO(n), which contains all orthogonal matrices with determinant 1. SO(3) describes the rotation around a line.

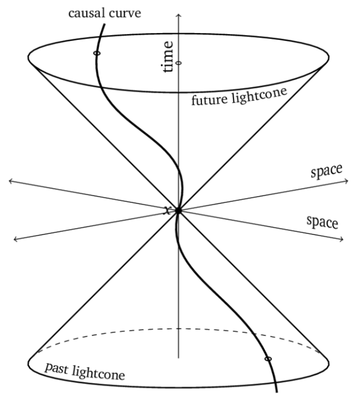



Figure 5: The light cone, which describes the causal structure of spacetime. When different events happen in different light cones, we cannot know which event took place earlier in time.

moves with a constant velocity, and a non-inertial, that has an acceleration or moves along a curved path. By Lorentz transformations, we only consider inertial frames. When these properties stated above hold, we call the matrix $\Lambda^{\mu}_{\nu}$ in (36) a Lorentz transformation, which is Lorentz invariant. An example is given by the most common form, for a velocity in the x direction, given by the matrix

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.23}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. In this way, one finds

$$ct' = \gamma(ct - \beta x) \tag{3.24}$$
$$x' = \gamma(x - \beta ct) \tag{3.25}$$
$$y' = y \tag{3.26}$$
$$z' = z \tag{3.27}$$

with $\beta = \frac{v}{c}$.

in each reference frame, an observer is able to measure physical quantities such as lengths and time intervals. By the Lorentz transformations, the coordinates of the different reference frames are expressed relative to each other.

## 3.4 Properties in spacetime curvature and curved coordinates

Spacetime curvature relies on the metric that describes the geometry of the manifold. The way curvature is expressed depends on some kind of mathematical property called a 'connection', as we will see. For measuring and describing distance and time, Cartesian coordinates are used. Now, we take a look at different coordinates $u = \{u^{\mu}, \mu = 0, .., 3\}$. We already defined the line element, which we now state by using new coordinates

$$\mathrm{d}s^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g'_{\mu\nu}(u) du^{\mu} du^{\nu} \tag{3.28}$$

From this, we can state

$$g'_{\mu\nu}(u) = \frac{\partial x^{\alpha}}{\partial u^{\mu}} \frac{\partial x^{\beta}}{\partial u^{\nu}} g_{\alpha\beta}(x) \tag{3.29}$$

In the original coordinates, a particle will move via a straight line when there is no force: $\frac{\partial^2 x^{\mu}}{\partial \tau^2} = 0$, with $\tau$ the proper time.The new coordinates are curved coordinates: the particles will move on curved lines here, in stead of straight lines. This becomes clear form the following equations:

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}u^{\lambda}} \frac{\mathrm{d}u^{\lambda}}{\mathrm{d}t} \tag{3.30}$$
$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} = \frac{\partial^2 x^{\mu}}{\partial u^{\lambda} \partial u^{\kappa}} \frac{\mathrm{d}u^{\kappa}}{\mathrm{d}\tau} \frac{\mathrm{d}u^{\lambda}}{\mathrm{d}\tau} + \frac{\partial x^{\mu}}{\partial u^{\lambda}} \frac{\mathrm{d}^2 u^{\lambda}}{\mathrm{d}\tau^2} \tag{3.31}$$

For the new coordinates, in stead of $\frac{\partial^2 x^\mu}{\partial\tau^2} = 0$, we arrive at

$$\frac{\mathrm{d}^2 u^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\kappa\lambda}(u)\frac{\mathrm{d}u^\kappa}{\mathrm{d}\tau}\frac{\mathrm{d}u^\lambda}{\mathrm{d}\tau} = 0 \tag{3.32}$$

$$\Gamma^\mu_{\kappa\lambda}(u) = \frac{\partial u^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial u^\kappa u^{\lambda 2}} \tag{3.33}$$

Thus while a particle, with no force acting on it follows a straight line, this no longer holds in terms of the curved coordinates [25]. The property in the above equation is called the connection field, or the Christoffel symbol:

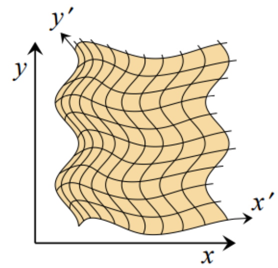



Figure 6: A transformation to curved coordinate frame $\{x', y'\}$[25]

$$\Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\nu}(\partial_\mu g_{\nu\sigma} + \partial_\mu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{3.34}$$

The Christoffel symbol is symmetric under change of subscript indices, $\Gamma^\rho_{\mu\lambda} = \Gamma^\rho_{\lambda\mu}$. Though it looks like a tensor, this is not the case. Moreover, it is a symbol. This connection is used in the geodesic equation.In general relativity, free particles move via geodesics. A geodesic is essentially a curved-space generalization of the description of a straight line in Euclidean space. As we have seen, without spacetime curvature, these particles would move in straight lines, however due to the curvature, they follow a path $x^\mu(\lambda)$ parametrized by the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu(\lambda)}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 \tag{3.35}$$

Still, in a gravitational field, a particle follows a path which comes close to a straight line. It does not feel an acceleration, since the particle is free falling. The nature of the geodesic motion of particles is important in GR: it is relevant in the claim that gravity is not just a force, but a result of the curvature of spacetime.

We can also define the Riemann tensor, which describes the curvature of spacetime:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{3.36}$$

Naturally, the Riemann tensor vanishes when we have a flat metric. It can be shown that this tensor obeys the Bianchi identity $\nabla[_\lambda R_{\rho\sigma}]_{\mu\nu} = 0$, or equivalently,

$$D_\alpha R^\mu_{\kappa\beta\gamma} + D_\beta R^\mu_{\kappa\gamma\alpha} + D_\gamma R^\mu_{\kappa\alpha\beta} = 0 \tag{3.37}$$

$$\text{where } D_\mu A^\kappa(x) = \frac{\partial A^\kappa}{\partial x^\mu} + \Gamma^\kappa_{\mu\nu}(x)A^\kappa(x) \tag{3.38}$$

$$\text{and } D_\mu B_\lambda(x) = \frac{\partial B_\lambda}{\partial x^\mu} - \Gamma^\lambda_{\mu\nu}(x)A_\nu(x) \tag{3.39}$$

$$\tag{3.40}$$

for $A_\alpha(x)$ and $B_\beta(x)$ a vectorial function. This identity appears to be useful in a description of general relativity. One can also find the Ricci tensor $R_{\mu\nu}$ from the Riemann tensor, which is a symmetric tensor. Then, the trace of the Ricci tensor is the Ricci scalar, $R^\mu_\mu = R$. This scalar is used in Einstein's field equations. These equations capture the response of spacetime curvature to the existence of matter and energy. Typically, Einstein's field equations show in which way energy and momentum affect the metric. The equations are quite complicated, since they are a set of non-linear second order differential equations. The solution to these equations are spacetime metric tensors. The equation is stated as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.41}$$

The left-hand side of this equation measures the curvature of spacetime, while the right-hand side measures the energy and momentum of matter: the equation presents a description of energy and momentum in terms of curvature. The energy-momentum tensor $T_{\mu\nu}$ is a generalization of a mass density: in a region of spacetime, it states the division of energy and mass. In vacuum, one can state the equations simply as $R_{\mu\nu} = 0$, since there will be no mass and energy left. The right-hand side of the original equation, $8\pi G T_{\mu\nu}$, is often stated as he Einstein tensor $G_{\mu\nu}$.

## 3.5 The Schwarzschild metric

A spherically symmetric gravitational field is very interesting in a theory of gravity. It is also relevant when considering for example the Earth or the Sun in good approximation, and especially in studying black holes. In general relativity, a unique vacuum solution which involves spherical symmetry is the Schwarzschild metric. The Schwarzschild metric is a non-trivial solution found by Karl Schwarzschild. Just as Minkowski space, it is a very important metric. By using spherical coordinates, one can define the metric as

$$\mathrm{d}s^2 = -(1 - \frac{2GM}{r})\mathrm{d}t^2 + (1 - \frac{2GM}{r})^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{3.42}$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2 \tag{3.43}$$

Here M is the mass of the gravitational object, and the Schwarzschild radius is defined as $r_s = 2GM$. The Schwarzschild equation is a description of a static, spherically symmetric vacuum solution to the equations Einstein postulated. When the mass approaches zero, the metric will behave as normal Minkowski space. This is also the case when the value of the radius reaches infinity: a property that is stated as asymptotic flatness.

## 3.6 Singularties and coordinate transformations

From the given Schwarzschild metric, one can see that some components of the metric diverge to infinity at two points, for $r = 0$, and $r = 2GM$, at the Schwarzschild radius. At these points, we have a singularity. In terms of curvature, a singularity of curvature appears when its curvature, which is measured by the Riemann tensor, grows to infinity. However, the singularity at the Schwarzschild radius can be ruled out by a proper coordinate transformation. What we will see, is that an event horizon can be found when the radius is equal to the Schwarzschild radius. Still, for the breakdown in the metric at $r = 0$, there is a real physical singularity where gravity diverges to infinity.

Also, the Schwarzschild coordinate system is not very suitable for describing near-horizon physics: when approaching the horizon in these coordinates, light cones close up. In this way, it looks like a light ray that arrives at the horizon will never reach it.

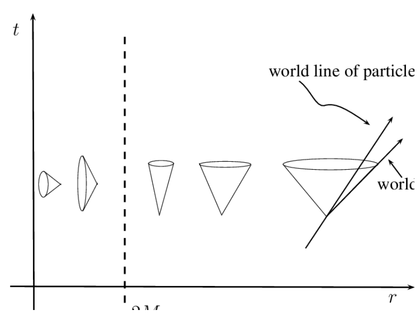



Figure 7: Future light cones in the Schwarzschild metric. Far away,the slope of the cones is $\pm 1$. Near the horizon the slope reaches $\pm\infty$. Behind the horizon, coordinates $t$ and $r$ have changed: they become time-like instead of space-like, and vice-versa.

### 3.6.1 Eddington-Finkelstein coordinates

In order to do avoid the singularity at Schwarzschild radius and the light cones to close up at this point, we look for other coordinate systems that describe the metric of the black hole properly. We start by parameterizing the radial null, light-like curves of the Schwarzschild solution, $\frac{\mathrm{d}t}{\mathrm{d}r} = \pm(1 - \frac{2GM}{r})^{-1}$, by using Tortoise coordinates:

$$t = \pm r^* + C \tag{3.44}$$

$$r^* = r + 2GM \log\Big(\frac{r}{2GM} - 1\Big) \tag{3.45}$$

With $C \in \mathbb{R}$. Next, we define coordinates that fit into the description of null geodesics:

$$v = t + r^* \tag{3.46}$$
$$u = t - r^* \tag{3.47}$$

Here infalling and outfalling null geodesics are described by $v, u \in C_1, C_2 \in \mathbb{R}$.

Now the metric can described in Eddington-Finkelstein coordinates:

$$\mathrm{d}s^2 = -(1 - \frac{2GM}{r})\mathrm{d}v^2 + (\mathrm{d}v\mathrm{d}r + \mathrm{d}r\mathrm{d}v) + r^2\mathrm{d}\Omega^2 \tag{3.48}$$

Via these transformations, the light cones behave normally at the Schwarzschild radius: we can still describe the paths of light-like and time-like particles behind this radius. Here, the Eddington-Finkelstein metric makes clear that the singularity at the Schwarzschild radius is just a coordinate singularity.

### 3.6.2 Kruskal coordinates

We can extend this by going to Kruskal coordinates. We do this by defining the following coordinates

$$U = -e^{-\frac{U}{4GM}} \tag{3.49}$$
$$V = e^{\frac{V}{4GM}} \tag{3.50}$$

Here, constant $U$ and $V$ can be seen as radial null geodesics. These coordinates obey $UV = (1 - \frac{r}{2GM})e^{\frac{r}{2GM}}$, thus a singularity is found for $UV = 1$. The horizon is reached when either $U$ or $V$ equals zero. Up to now, the metric has off-diagonal term, which we can remove by defining a time and space coordinate

$$T = \frac{1}{2}(V + U) \tag{3.51}$$
$$X = \frac{1}{2}(V - U) \tag{3.52}$$

so that the metric becomes

$$\mathrm{d}s^2 = \frac{32G^3m^3}{r}e^{\frac{r}{2GM}}(-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2\mathrm{d}\Omega^2 \tag{3.53}$$

in terms of Kruskal coordinates $(T, X, \theta, \phi)$. As expected, in this coordinate system, there is no singularity to be found at $r = 2GM$.

Furthermore, the metric has some interesting properties. First of all, the radial null (light-like) curves take the same form as in flat space: $T = \pm X + C$ , $C \in \mathbb{R}$. The horizon is defined at $T = \pm X$. Also, constant surfaces, $r =$ constant, are given by hyperbolae: $\frac{T}{X} = \tanh\left(\frac{t}{4GM}\right)$. The region that is covered by the coordinates T and X is given by

$$-\infty < R < +\infty \tag{3.54}$$
$$T^2 < R^2 + 1 \tag{3.55}$$

We can draw a spacetime diagram for the T-X plane, which is the Kruskal diagram, where every point in this diagram is a two-sphere. By this description, we expect to describe the full manifold of the Schwarzschild geometry, see figure 7.

We can divide the diagram into different parts. In the first part, I, our original coordinates are well-defined. The future-directed null rays can be found in the second region II and the past-directed null rays in the third region, III. Spacelike geodesics will be found in the fourth region IV. Here, region two corresponds to a black hole: when something moves from region I to II, it actually will never return. The third region is the time-reverse of the second region, simply a part from where something might escape to us. However, one can never reach this part and it is therefore a white hole. The boundary of this region is the past event horizon, while in the second region, we have a future event horizon.

The fourth region is also an asymptotically flat region in spacetime, just like the first region. It looks like a mirror image compared to our space in region I. Yet, we cannot reach region IV from region I. Still, these parts are connected by a wormhole, a configuration that joins two separate parts in space: an Einstein-Rosen bridge.

Specifically, we can create wormholes by taking slices of space-like surfaces of constant $T$ in the Kruskal diagram. Like this, two asymptotically flat regions from both region I and II are connected by a wormhole for some small time, but will eventually disconnect. It will be impossible for a time-like observer to travel through the wormhole, since it will vanish too quickly to actually pass it. One cannot send a signal from one region to the other, but it seems that both singularities of both regions meet each other.


(a) (b)

Figure 8: On the left: the Schwarzschild solution in Kruskal coordinates. Lines of constant $r$ and $t$ are drawn, just as the event horizons at $r = 2GM$. Light cones are all at $\pm 45$ degrees.[23]. On the right: We can divide the diagram into different parts. Asymptotically flat spaces are connected by a wormhole. [23]

### 3.6.3 Penrose diagrams

Kruskal diagrams are very useful for studying black holes, but one might also want to pu describe Schwarzschild solution by a finite region. We will do this by constructing Penrose diagrams. Penrose diagrams are used to describe complex curved spacetime manifolds by their general properties and their causal structure. We often call Penrose diagrams also conformal diagrams. In this diagram, we try to capture a normal metric by a proper coordinate transformation. Specifically, we try to fit infinite coordinate values into finite values. We also want to keep the light cones at 45 degrees everywhere, which correspond to a slope of $\pm 1$ for the light cones: the radial null waves satisfy $\frac{\mathrm{d}T}{\mathrm{d}X} = \pm 1$. So, we would like to find a metric that is conformally related[4] to another metric for which the 45 degrees hold for the light cones.

Starting from the Kruskal metric, we define coordinates to capture infinity in some finite coordinates

$$A = \arctan\left(\frac{V}{\sqrt{2GM}}\right) \tag{3.56}$$

$$B = \arctan\left(\frac{U}{\sqrt{2GM}}\right) \tag{3.57}$$

$$-\frac{\pi}{2} < A < \frac{\pi}{2}, -\frac{\pi}{2} < B < \frac{\pi}{2}, -\frac{\pi}{2} < A + B < \frac{\pi}{2} \tag{3.58}$$

In this coordinate system, the singularities are straight lines which go from one asymptotic part to the other, both in time like infinity. The full diagram for a black hole is given by the figure. In this Penrose diagram, we can now distinguish five different regions, similar to Minkowski spacetime.

$$i^+ = \text{future time-like infinity} \tag{3.59}$$

$$i^0 = \text{spatial infinity} \tag{3.60}$$

$$i^- = \text{past time-like infinity} \tag{3.61}$$

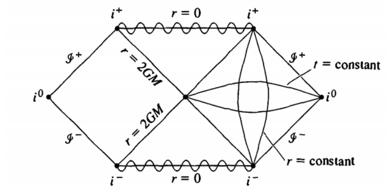
$$\mathbb{J}^+ = \text{future null (light-like) infinity} \tag{3.62}$$

$$\mathbb{J}^- = \text{past null (light-like) infinity} \tag{3.63}$$

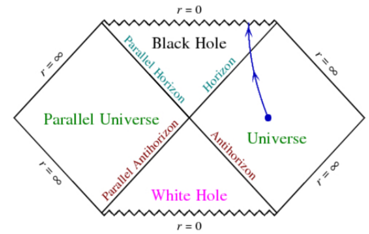
Hence, light-like geodesics move from $\mathbb{J}^-$ to $\mathbb{J}^+$ and time-like particles from $i^-$ to $i^+$. Space-like geodesics begin and start at $i^0$.

Not to mention, in the diagram, time runs upwards. By this construction, the light cone is preserved and so the causal structure. Here, the light cones in the diagram are all again at $\pm 45$ degrees. Also by using Penrose diagrams, we can get an insight in the causal structure of a black hole and its surroundings.

[4]Two metrics which only differ via a multipliciation of a positive scalar function, related by $g'_{\mu\nu} = e^{2\omega(x)} g_{\mu\nu}$ with $\omega(x)$ a smooth real function, are named as conformally equivalent. Here time-like and space-like curves in one metric will be the same in the other metric. [26]




(a) A Penrose diagram for the full Kruskal expansion of Schwarzschild spacetime.[23]




(b) Similarly, the full Penrose diagram for a black hole.

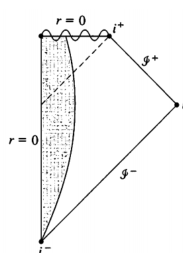
Figure 9

# 4 Black holes

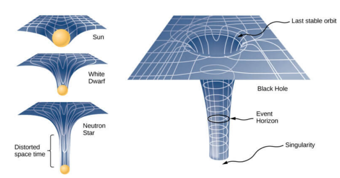
Black holes are one of the most fascinating objects in our universe. Black holes are regions of spacetime where gravity is so strong, that it is impossible for particles, and even for light, to escape the black hole. Black holes have been predicted by Einstein's theory of general relativity. According to this, high mass is able to form spacetime into a black hole. Since light cannot escape the black hole, we cannot see it, and therefore it is called black. Once fallen in, it is impossible to get out. Still, due to quantum fluctuations, radiation seems to come out. By this, black holes have created some troubles in a theory of quantum information. This is stated by the information paradox in section 6. In this section, we will look at some important properties of black holes, which will also be important later on.

## 4.1 Black holes: the collapse of heavy stars

Black holes can be created by collapse of heavy stars. Primordial black holes, which are created short after the big bang, exist too, but we will not study them here. In the full Penrose diagram for a black hole, we have seen the appearance of a white hole and a second asymptotically flat region, which are connected by a wormhole to our universe. For stellar collapse to a black hole, however, we won't see this: the full Schwarzschild solution does not describe such a past of the spacetime here.




(a)




(b)

Figure 10: On the left: A Penrose diagram for a black hole formed out of stellar collapse. The shaded region presents matter. The horizon is given by the dotted line and there is a singularity at $r = 0$. [23]. On the right, an example of stages is stellar evolution.

The non-vacuum region that is shaded can be seen as the stellar evolution. Gravitational collapse can be the final stage of stellar evolution, but it is not a mandatory condition. General relativity states certain requirements for gravitational collapse to a black hole. Essentially, from a certain amount of mass, the collapse will always create a black hole.

For a star to exist, the pressure from heat production of light nuclei into heavy ones by fusion is essential. When this process stops, the temperature will lower and the star will shrink due to gravity. This process might be slowed down by an outward pressure due to the Pauli exclusion principle: squeezing the electrons further together will put them in the same state, which is not possible according to this principle. When a star finds itself in such an end state, it is called a white dwarf. However, when the mass is high enough, the gravitational pull will be stronger than the outward pressure, and the star will collapse further to a smaller radius. Then, neutrons and neutrinos are created by putting protons and neutrons together. Now what's left is a neutron star. It is also possible that the white dwarf collapses to a black hole immediately, mainly if the mass is high enough. For a white dwarf to be stable, there is a limit on its mass, which is the Chandrasekhar limit. This is about 1.4 times the solar mass. Neutron stars have a typical radius of around 10 km. Quite often, these stars are fast spinning objects and consist of strong magnetic fields. Together, this gives rise to pulsars. Pulsars give an acceleration to particles in jets emitted from the magnetic poles.

Eventually, massive neutron stars will collapse even further to a black hole. We estimate the maximum possible neutron-star mass to be around three or four time the solar mass. This limit is called the Oppenheimer-Volkoff limit. It is believed that the collapse to a black hole is the final stage in the stellar evolution.

## 4.2 The event horizon

From the event horizon on, particles are never able to escape to infinity when they passed this horizon. Thus, since nothing is able to escape the event horizon, there is no-one to see what's inside. Therefore, we use the name black holes. Thus, black holes can be seen as a region that separates a part of spacetime, by its event horizon, from outer infinity. Essentially, an event horizon is a hypersurface that separates spacetime points that are connected to infinity by timelike paths that are not. For us it is not of great interest what happens far away, but we define infinity as being "well outside the black hole", and we approximate the spacetime far away from the black hole to be Minkowski space and thus asymptotically flat [23]. We can distinguish two kinds of event horizons: a future and past event horizon. The future event horizon is a surface from which timelike curves are not able to escape to infinity. Similarly, the past event horizon is a surface that one can only pass by curves in the direction of the past. When arrived inside, all matter is forced out, and finally out of the past horizon, one is not able to enter the region behind this horizon again. Looking at the Schwarzschild solution, at the event horizon, light cones are tilt over: at the Schwarzschild radius we have a light-like surface rather than a time-like surface. In general relativity, singularities are hidden behind event horizons [23].

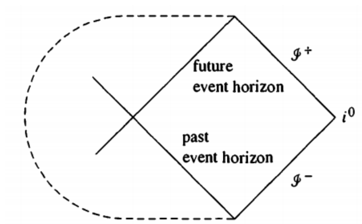



Figure 11: Past and future event horizons.We have asymptotically flat Minkowski spacetime. $\mathbb{J}^{\pm}$ and $i^0$ have the same structure as flat Minkowski space. The space between the dashed lines can be a different form of spacetime.[23]

As we have seen, in a Schwarzschild metric, an event horizon corresponds to the region in spacetime where $r = 2GM$. From the event horizon on, light will move radially inwards approaching the singularity. We can easily see this. Consider the Schwarzschild metric and a light ray that moves radially inward. Then we can express such a light ray via [25]

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{1 - \frac{2MG}{r}} \tag{4.1}$$

$$ds^2 = 0 \text{ and } d\Omega^2 = 0 \tag{4.2}$$

A solution for this expression is

$$t = t_0 \pm [r + 2GM \log(r - 2GM)] \text{ , } t_0 \in \mathbb{R} \tag{4.3}$$

Choosing now the minus sign, so that when $r$ comes very close to $2GM$, we can express it as

$$r(t) \to 2GM + e^{\frac{t_0 - t}{2GM - 1}} \tag{4.4}$$

In a similar way, radially outgoing rays are given by

$$r(t) \to 2GM + e^{\frac{t - t_0'}{2GM - 1}} \text{ , } t_0' \in \mathbb{R} \tag{4.5}$$

One can see that both light rays will never pass through the horizon at $r = 2GM$. Also at other angles, this will indeed not happen.[25] More general, looking at the Kruskal diagram, from region I the geodesics can move to a boundary at infinity at a future direction. In contrast, geodesics starting in region II can never reach to a boundary at infinity. The boundary between those regions is the horizon: a surface composed of light-like geodesics that move radially outwards. According to the equivalence principle, an observer traveling across the event horizon will not feel anything special [27].

Also, the size of a black hole is captured by the area of its event horizon. According to the laws of black hole thermodynamics, black holes never shrink in size too: from this laws, it becomes clear that the area of a future event horizon in asymptotically flat spacetime is never decreasing. For Schwarzschild black holes, this automatically implies that black holes can only become heavier, looking at the increase of mass. However, this is not the case for spinning black holes, since the area then depends not only on the mass, but also on the angular momentum. Can the mass then never decrease? It can, due to Hawking radiation as we will discover in the following chapters.

## 4.3 No-hair theorem

Though black holes are created from massive amount of energy and matter, and above all seem quite complicated objects, only a small number of parameters define the black hole. Asymptotically flat black hole solutions can be fully stated by their mass, magnetic and electric charge and angular momentum. This is a fundament of the no-hair theorem. However, in order to fully describe the black hole, one would think of including all the matter that has fallen into the black hole. It is a remarkable result, since many microscopic systems can be described by more parameters than the three describing the black hole. For accurately describing other cosmological fenomena, such as planets for example, we would often need much more information. Also, the matter falling into the black hole is stated by many parameters, but when arrived at the black hole, it seems that this is not necessary anymore.

Ultimately this will lead to some puzzles in physics. In our physical theories, we want that by the information of a state at an arbitrary moment in time, one is able to predict the state of another moment of time. Consequently, we would like the information in these states to be preserved. However, it now seems that in general relativity that this is not the case. In GR this does not seem a big problem: information could hide itself behind the horizon without being lost. But as we will see, black holes evaporate as discovered by Hawking, and information seems to be lost forever.

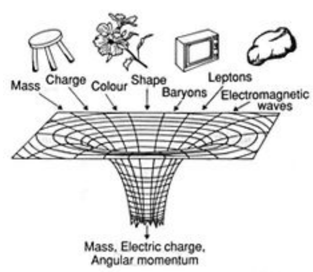



Figure 12: An impression for the no-hair theorem.

## 4.4 Black hole thermodynamics

Black holes can be described by a small number of macroscopic parameters, but "the microscopic degrees of freedom that lead to their thermal behavior have not yet been adequately identified" [28]. Black holes seem to behave as thermal object, and these properties are described by four classical laws of black hole mechanics, with great analogy to the four laws of thermodynamics. Every law of thermodynamics has an equivalent in the laws of black hole mechanics. [28]

The zeroth law of black hole mechanics states that the surface gravity $\kappa$ of a stationary black hole is constant over its event horizon. In thermodynamics, the zeroth law states that the temperature for a system in thermal equilibrium is constant. The first law is an expression for the conservation of energy. It states that the change in the black hole mass $M$ equals a certain change in its area $A$, angular momentum $J$ and electric charge $Q$:

$$\delta M = \frac{k}{8\pi G}\delta A + \Omega dJ + \Phi\delta Q \tag{4.6}$$

analogous to the well-known first law of thermodynamics $dE = TdS - pdV$. One can extend the zeroth law by also stating that the angular velocity $\Omega$ and the electrostatic potential $\Phi$, the other constants that appear in (4.6), are constant values over the black hole event horizon. The second law of black hole mechanics states that the area of the horizon never decreases, analogous to the statement that entropy never decreases. Finally the third law states that the surface gravity cannot become zero in any physical process, analogous to the thermodynamic statement that the temperature of a system cannot become zero. Here, the third law applies to some situations, but is not in fact a truly fundamental property, since there are physical systems that violate it.

As Hawking showed that black holes radiate, we can state $S_{BH} = \frac{A}{4G}$ as the actual entropy for the black hole, called the Bekenstein-Hawking entropy. Furthermore, Bekenstein came up with a generalized second law, which defines the total entropy of matter and the black holes can never decrease:

$$\delta(S + \frac{A}{4G}) \geqslant 0 \tag{4.7}$$

Normally, we think of the entropy of a system as a logarithm of a number of available quantum states. However, according to the no-hair theorem, there is only one possible state for a black hole with a constant charge, spin and mas: here we experience some awkwardness between quantum mechanics and general relativity. The question still remains what to do with the accessible states which are calculated by the ordinary thermodynamic entropy: where are the degrees of freedom of the black hole, and what exactly are they?

Many ideas about this have been proposed. The most common one, and also the simplest one, is that these degrees of freedom are inside the black hole, by matter that has fallen in or by particle-antiparticle production. The question remains how and if information by these degrees of freedom is able to come out. For example, it has been proposed that quantum gravity does not make a distinction between the outside and inside of the black hole, in contrast with the semiclassical approximation. In this way, information that would be hidden in semiclassial physics, would be able to come out in a theory of quantum gravity.

## 4.5 Euclidean gravity

One can express properties in space-time coordinates by complex values of the same coordinates. Here, the original equation still hold. In particular, it is interesting to look at the imaginary time $t = it_E$. Often we call the Euclidean time the imaginary time and the Lorentzian time the real time.[22] [25] Using these coordinates, the Schwarzschild metric becomes

$$ds_E^2 = (1 - \frac{r_s}{r})dt_E^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2 \tag{4.8}$$

with $r_s$ the Schwarzschild radius. Interesting here is that imaginary-time periodicity can be seen as a temperature in such an Euclidean geometry. The partition function in a thermal state is $\mathrm{Tr}\left(e^{-\beta H}\right)$. Such an observable is periodic under certain conditions. In quantum field theory, the trace can be computed via a path integral. In order to do this, we need to look for a geometry that is a spherically symmetric solution of the Euclidean Einstein equation [26]. Via imaginary-time evolution $e^{-\beta H}$, we can compute a path integral on such a Euclidean geometry. In quantum field theory, one can calculate the trace of imaginary-time evolution via a path integral on a Euclidean cylinder with conditions $\theta = \theta + \beta$. Being in Euclidean geometry, a black hole does not have an interior. The geometry of the Euclidean Schwarzschild black hole looks like a sigar, see the figure.

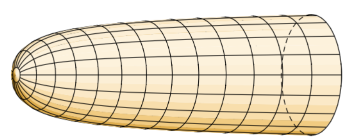


Figure 13: A Schwarzschild black hole in Euclidean geometry. There is no singularity to be found on this geometry. [25]

The tip of sigar corresponds to the horizon. We can then look at the path integral as the partition function with contributions of both gravity and quantum fields:

$$Z(\beta) \sim e^{-I_{classical}} Z_{quantum} \tag{4.9}$$

The gravitational part is found in the Einstein action, which can be found by evaluating it on the Euclidean Schwarzschild geometry , while the quantum part can be found by the partition function of quantum fields in this geometry.

We take a closer look at path integrals. For a quantum system, we can find the ground state by acting on a state $|a\rangle$ wit $e^{-HT}$, wit $T$ a long time. This is stated as [26]

$$\langle\phi|b\rangle = \frac{1}{\langle b|a\rangle} \lim_{T\to\infty} \langle\phi| e^{-HT} |a\rangle \tag{4.10}$$

In the formalism of the Euclidean path integral, this wave function is stated as

$$\langle\phi|b\rangle \sim \int_{\hat{\phi}(T_E=-\infty)=0}^{\hat{\phi}(t_E=0)=\phi} D\phi e^{-I_E} \tag{4.11}$$

$$\text{with } I_E(\hat{\phi}) = \frac{1}{2}\int d^3x dt_E [(\partial_{t_E}\hat{\phi})^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2] \tag{4.12}$$

Here, $D\phi$ defines the integration over all possible paths on the denoted interval and $I_E$ the Euclidean action for a free massive scalar field [26]. Now we evaluate the integral from $-\infty$ to 0. Alternatively, one can evaluate the integral by angular directions.

For the horizon to be smooth at the point corresponding to the black hole horizon in the Schwarzschild metric, we do not have a singularity in this geometry. This is because an observer will, when approached classically, not notice anything special when passing the horizon. Via the computed partition function, one can obtain the generalized entropy via the common thermodynamic formula $S = (1 - \beta\partial_\beta)\log(Z(\beta))$.

By Euclidean path integrals, it seems we have found a nice method for computing thermodynamic quantities of systems in which gravity is involved.[26]

## 4.6 Rotating and charged black holes

In this thesis we consider non-rotating, uncharged black holes, but let's have a small note about them here. Describing the metric for a rotating black holes is slightly more difficult, because here we do not have spherical symmetry. Kerr was the first to find the description for this metric. It is stated by the Kerr metric [23]:

$$ds^2 = -(1 - \frac{2GMr}{\rho^2})dt^2 - \frac{2GMar\sin(\theta)^2}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2(\theta)}{\rho^2}[(r^2 + a^2)^2 - a^2\delta\sin^2\theta d\phi^2] \quad (4.13)$$

$$\text{with } \delta(r) = r^2 - 2GM + a^2 \quad (4.14)$$

$$\text{and } \rho^2(r,\theta) = r^2 + a^2\cos^2(\theta) \quad (4.15)$$

with $a = \frac{J}{M}$, the angular momentum per mass. Rotating black holes are often created in gravitational collapse of a spinning star or in a collision of cosmological objects with a nonzero angular momentum. As far as we know, many stars rotate and realistic collisions do consist of a nonzero angular momentum. Therefore, rotation is a realistic property of black holes.

Black holes with a nonzero charge do exist too. This feels a bit counter-intuitive, since stars do not have a charge and black holes are created by stellar collapse. Still, they do exist. They were discovered soon after the Schwarzschild solution by Reissner and Nordström. Their solution is called the Reissner-Nördstrom metric. [23] It is given by

$$ds^2 = -\Delta t^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2 \quad (4.16)$$

$$\text{where } \Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} \quad (4.17)$$

Here, $M$ is the mass of the hole, $Q$ the total electric charge and $P$ the total magnetic charge. Also, Reissner and Nordström found out that the horizon of a charged black hole is found at the point

$$r_\pm = GM \pm \sqrt{G^2M^2 - G(Q^2 + P^2} \quad (4.18)$$

By this, we have to solutions for the horizon: we will get an inner and outer horizon.

Charged, rotating black hole with nonzero angular momentum are described by the Kerr-Newman metric [29], which gives a general solution for a spacetime region surrounded by a charged, rotating mass. It is a solution to the Einstein-Maxwell equations from general relativity.

Of course, the physics that describe rotating, charged black holes, differ from the description of non-rotating, uncharged black holes. We will simply not discuss this here, though it is nice to mention some properties of such beautiful objects.

# 5 Quantum field theory: the Unruh effect and Hawking radiation

General relativity is a fully classical theory. Still, the world we live in is fundamentally described by quantum mechanics. Ultimately, by combining both as a theory for quantum gravity we might come closer to a full understanding of black holes. By studies using quantum field theory, Hawking [9] found out that black holes do actually radiate.

Quantum field theory (QFT) is a theory that specifically defines fields by infinite sets of harmonic oscillators. QFT describes particles in the form of excited states of their quantum fields. Each of these harmonic oscillators has its own vibration, namely via its eigenfrequency. By using interaction terms in the Langrangian description of these quantum fields, interactions of these particles can be described.

The radiation emitted by black holes is called Hawking radiation and is exactly thermal, without any correlations between the outgoing particles. Eventually, this radiation will lead to the famous information paradox. The basis for the Hawking radiation comes from the Unruh effect, which states that an uniformly accelerating observer, that takes a measure of normal Minkowski vacuum, will see a thermal spectrum of particles, a thermal bath. However, an inertial observer in the Minkowski vacuum would not be able to see this bath.

The Unruh effect and the derivation of the Hawking temperature is described in many papers and by many researchers. We will give a compact derivation here, though for a full derivation one can take a look at [30] or [31].

## 5.1 Rindler space

First of all, we will define Rindler space, which is a space to understand the observation of space by an observer that is uniformly accelerated. The observer follows a path with a constant acceleration that obeys the expression

$$x^2(\tau) = t^2(\tau) + \alpha^2 \tag{5.1}$$

with $\alpha$ a constant acceleration, called the proper acceleration. It is felt by the accelerating observer, called the Rindler observer. The path has the shape of a hyperbola, and thus the motion is hyperbolic. For this motion, the asymptotes are null paths given by $x = \pm t$. By choosing new coordinates in the original two-dimensional Minkowski space, we can create a coordinate system fitted for an accelerated observer:

$$t = \frac{1}{\alpha} e^{\alpha\xi} \sinh(\alpha\eta) \tag{5.2}$$

$$x = \frac{1}{\alpha} e^{\alpha\xi} \cosh(\alpha\eta) \tag{5.3}$$

$$-\infty < \eta, \xi < +\infty \text{ and } x > |t| \tag{5.4}$$

These coordinates have ranges stated above and indeed they cover the region between the asymptotes as seen in the figure. The path for an observer that has a constant acceleration is

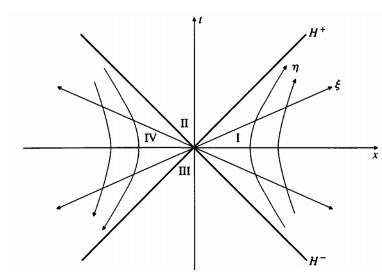



Figure 14: Minkowski spacetime in Rindler coordinates.Region I is the region that is accessible for the Rindler observer, $H^{\pm}$ are the past and future boundaries. Just like region I, region IV can be described by $(\eta, \xi)$.[23]

$$\eta(\tau) = \frac{a}{\alpha}\tau \tag{5.5}$$

$$\xi(\tau) = \frac{1}{a}\log\left(\frac{a}{\alpha}\right) \tag{5.6}$$

$$\text{for } \alpha = a \text{ we have } \eta = \tau, \xi = 0 \tag{5.7}$$

Then,the metric of the Rindler space is given by

$$\mathrm{d}s^2 = e^{2\alpha\xi}(\mathrm{d}\eta^2 - \mathrm{d}\xi^2) \tag{5.8}$$

It can be shown mathematically that this metric is the same as the Schwarzschild metric, if we come close to the event horizon. Physically seen, this is a consequence of the equivalence principle, since uniform acceleration is locally the same as a gravitational field.

Note that the Rindler space is still just a part of Minkowski space. Here straight lines starting at the origin are lines of constant time for the Rindler observer, while for this observer, the hyperbolic lines correspond to a constant position. Furthermore, this observer will experience the presence of an event horizon. One can interpret this horizon in Rindler space just as in the Kruskal coordinates. Consequently, a signal from region IV will never reach region I, when one is constantly accelerated. Thus all observers that move with a constant acceleration will have an event horizon, just like a black hole event horizon: the Rindler wegdes four and one are causally disjoint spacetimes.

## 5.2 The Unruh effect

We will study the Unruh effect. We start with the wave equation for a massless scalar field, obtained from the Klein-Gordon equation $\Box\phi = 0$. The field equation for this solution is

$$\psi(t,x) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} [e^{-i|k|t+ikx} \hat{a_k}^{-} + e^{i|k|t-ikx} \hat{a_k}^{\dagger}] \tag{5.9}$$

using creation and annihilation operators $\hat{a_k}^{-}, \hat{a_k}^{\dagger}$. By using the Rindler metric, we find a similar field equation for the right Rindler wedge

$$\psi_{Rindler}(\tau,\xi) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} [e^{-i|k|\tau+ik\xi} \hat{b_k}^{-} + e^{i|k|\tau-ik\xi} \hat{b_k}^{\dagger}] \tag{5.10}$$

with $\hat{b_k}^{-}, \hat{b_k}^{\dagger}$ specified for Rindler space. We now looked at the right wedge, but for the left wedge, the equation has the same form. We define the zero eigenvector for the vacuum state for all annihilation operators, and in the same way, the vacuum state in the Rindler vacuum is defined by:

$$\hat{a_k}^{-} |0_M\rangle = 0 \ \forall k \tag{5.11}$$

$$\hat{b_k}^{-} |0_R\rangle = 0 \ \forall k \tag{5.12}$$

By introducing light cone coordinates $\tilde{u} = \tau - \xi, \tilde{v} = \tau + \xi$, we can put the equation for the Rindler wedge in another form

$$\psi_R(\tilde{u},\tilde{v}) = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega\tilde{u}} \hat{b_\omega}^{-} + e^{i\omega\tilde{u}} \hat{b_\omega}^{\dagger} + e^{-i\omega\tilde{v}} \hat{b_{-\omega}}^{-} + e^{i\omega\tilde{v}} \hat{b_{-\omega}}^{\dagger}] \tag{5.13}$$

Now the Klein-Gordon solutions to the field equations are $\frac{\partial^2\phi}{\partial\tilde{u}\partial\tilde{v}} = 0$. From this, we can compute two independent solutions for a wave moving from the left and to the right. Now these parts do not interact and we can decompose the field into two parts: right-moving waves and left-moving waves.

$$\psi(\tilde{u},\tilde{v}) = A_-(\tilde{u}) + B_+(\tilde{v}) \tag{5.14}$$

with $A_-(\tilde{u}), B_+(\tilde{v})$ respectively right-moving and left-moving waves. Now, because these parts do not interact, we can for example compose the wave function for all the modes that move left:

$$B_+(\tilde{v}) = \int_0^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} (e^{-i\omega\tilde{v}} \hat{b_{-\omega_r}}^{-} + e^{i\omega\tilde{v}} \hat{b_{-\omega_r}}^{\dagger}) + \int_{-\infty}^{0} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} (e^{-i\omega\tilde{v}} \hat{b_{-\omega_l}}^{-} + e^{i\omega\tilde{v}} \hat{b_{-\omega_l}}^{\dagger}) \tag{5.15}$$

Here we defined the operators for the right ($r$) and left ($l$) Rindler wedge. Both operators for the left and right wedge obey a commutation relation. The operators in Minkowski space obey such an equation:

$$\left[\hat{b_\Omega^-}, \hat{b_\Omega}^{\dagger}\right] = \delta(\Omega - \Omega') \tag{5.16}$$

$$\left[\hat{a_\omega^-}, \hat{a_\omega}^{\dagger}\right] = \delta(\omega - \omega') \tag{5.17}$$

with $\omega, \Omega \in \mathbb{R}$. Now, by the Bogoliubov transformations, we can see the relation between the Rindler and Minkowski operators. These transformations state that the Rindler operators can be written as

$$\hat{b_\Omega}^{-} = \int_0^\infty d\omega[\alpha_{\omega\Omega}\hat{a_\omega}^{-} + \beta_{\omega\Omega}\hat{a_\omega}^{\dagger}] \tag{5.18}$$

$$\hat{b_\Omega}^{\dagger} = \int_0^\infty d\omega[\alpha^*_{\omega\Omega}\hat{a_\omega}^{\dagger} + \beta^*_{\omega\Omega}\hat{a_\omega}^{-}] \text{ where} \tag{5.19}$$

$$\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}}F(\omega, \Omega) \tag{5.20}$$

$$\beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}}F(-\omega, \Omega) \tag{5.21}$$

Here we use some assisting functions $F(\pm\omega, \Omega)$.

Now the amount of particles in a certain state is measured by the operator $N_\Omega = \hat{b_\Omega}^{\dagger}\hat{b_\Omega}^{-}$. It is interesting to look at the amount of particles that is observed by an accelerating observer in Minkowski space:

$$< N_\Omega >= \langle 0_M | \hat{b_\Omega}^{\dagger}\hat{b_\Omega}^{-} | 0_M \rangle \tag{5.22}$$

$$= \int d\omega |\beta_{\omega\Omega}|^2 \tag{5.23}$$

The mean density with momentum $\Omega$ is found to be

$$< N_\Omega >= [\exp\left(\frac{2\pi\Omega}{a}\right) - 1]^{-1} \tag{5.24}$$

Considering theory from statistical physics, this expression looks like the Bose-Einstein distribution, which is defined as

$$< N_{BE} >= [\exp\left(\frac{E}{T}\right) - 1]^{-1} \tag{5.25}$$

Then we arrive at $\Omega = E, T = \frac{a}{2\pi}$. So, an accelerating observer sees particles with an energy in the Minkowski vacuum. Also, the observer finds a temperature of $T = \frac{a}{2\pi}$, or in SI-units:

$$T = \frac{a\hbar}{2\pi k_B c} \tag{5.26}$$

the Unruh temperature. With this, we will be able to calculate the temperature of the black hole. Furthermore, what Unruh found, is that the we can describe the Minkowski vacuum as an entangled state. This can be stated as [30]

$$|0_M\rangle = \prod_j [N_j \sum_{n_j} e^{\frac{-\pi n_j \Omega_i}{a}} |n_j, r\rangle_R \otimes |n_j, r\rangle_R] \tag{5.27}$$

with $N_j$ from, and $|n_j, r\rangle_R$ is a state with $n_j$ particles in the right ($r$) Rindler wedge. From this, one can see the strong correlations between the left and right region: particles in the right and left Rindler wedge are entangled.

## 5.3 The Hawking temperature

By the equivalence principle we can assume locally that acceleration and gravity are for now the same. The gravitational acceleration of a body near the horizon is $\alpha = \frac{1}{4M}$, seen by an observer far away. For this observer at infinity, the Hawking temperature of the black hole is then $T_H = \frac{1}{8\pi M}$, or in SI-units

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} \tag{5.28}$$

Here, the radiation that is emitted is the same as the radiation that a blackbody of the same temperature would emit. Interestingly according this formula, the higher the mass, the lower the temperature will be. This temperature is a temperature measured by an observer far away. At distances very close to the black hole, the temperature becomes incredibly high.

Still, the discovery that black holes radiate is quite counter-intuitive since general relativity states that it is impossible for particles to escape from behind the black hole event horizon. What is going on, is the creation of particle-antiparticle pairs. Normally, these particles would annihilate, but due to the event horizon the particles will be separated: one

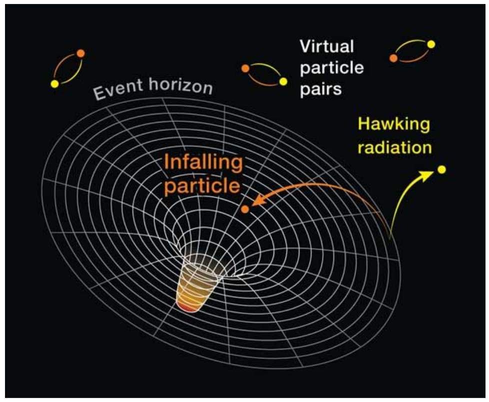



Figure 15: The particle-antiparticle creation discovered by Hawking [32]

particle falls into the black hole, while the other escapes to infinity. The particle that escapes out of the black hole is the particle we denote and observe as Hawking radiation. Since the total energy must be zero, the particle that falls into the black hole will have a negative energy. The outgoing particle has a positive energy. In this way, the black hole will lose mass and will eventually evaporate completely.

Yet, in QFT, we make the assumption that information is defined to be located in a region of space. However, quantum gravity might be different, and information might be available somewhere, non-locally, around the black hole.

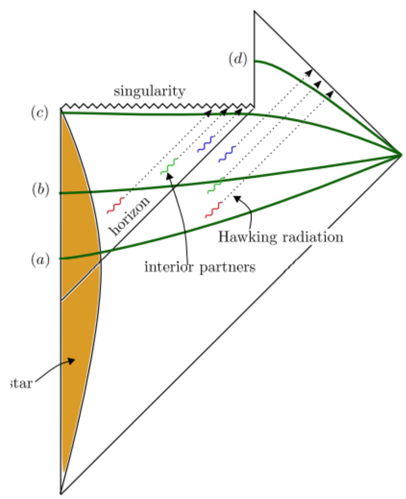



Figure 16: A Penrose diagram for an evaporating black hole that has been fromed by stellar collapse. (a) - (d) correspond to spatial slices, defining stages in black hole evaporation. Eventually, by the Hawking particle-antiparticle creation, the black hole will vanish. The Hawking particles will head to the future $\mathbb{J}^+$, where we have causal Minkowski space. [22]

# 6 The Information paradox

By the fact that the radiation emitted by the black hole is exactly thermal, with no correlations between the outgoing particles, information, which is used to specify the states, seems to be lost. Imagine that we put different states into different black holes with the same mass, charge and spin. Then, after they have completely evaporated, the radiation of these two black holes - due to the effects Hawking described – seems completely the same. Here it looks like that the information, previously thrown into the black hole, has disappeared. This is what we call the "black hole information paradox". Both quantum field theory and general relativity state that information at early times has to be available at later times, by the description of their equations of motion. This is unitary evolution, which thus must be conserved. However, as we have seen, combining these two important theories seem to give the result that unitary is not conserved.

So, it is a problem then if black holes radiate. Stars for example radiate too, however, the radiation that comes from a star depends on what fell into the star: there are correlations between matter that fell down to the star and the outgoing radiation. So, when the star and its outgoing radiation are in a pure state, it will remain in a pure state. Exactly this seems not the case for black holes and their Hawking radiation. The Hawking radiation is fully thermal and the density matrix corresponds to a fully mixed state: this violates the essential principle in quantum mechanics of unitary time evolution.

## 6.1 The S-matrix

Essentially, the question is what happens to a pure quantum state that collapses to a black hole will emit radiation. Hawking believed that the black hole would disappear, and that we would be left with a mixed state: information would be lost in the black hole, and we would not have a certain matrix $S$ to bring an initial pure state to a final pure state. [26]

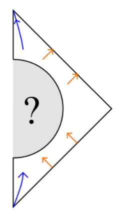



Figure 17: The S-matrix maps an initial state to a final state. Here massless particles are given by orange lines, massive particles are denoted by blue. For a black hole, we would like to have an S-matrix that maps an initial pure state to a final pure state. [26]

Still, if a pure state is brought to a mixed state, this would violate important principles in quantum mechanics. Indeed, it has been believed by many researchers that it should not be the case that information is lost. For a quantum system, the ordinary time evolution is stated by $\phi_a^{final} = U_{ab}\phi_b^{initial}$. Here, a pure state is mapped to another pure state. However, what seems to be the case, is that the black hole can be stated as a matrix \$ that brings the density matrix of a pure state to a mixed state, such that $\rho_{aa'}^{mixed} = \$_{aa',bb'}\rho_{bb'}^{pure}$. Now, if $\$_{aa',bb'} = U_{ab}U^*_{a'b'}$, unitary will be preserved. After all, this is what one would like to see to avoid complicated problems with well-known laws of physics. It is believed, and hoped, that a full theory of quantum gravity is able to solve the problem. [28]

Thermal aspects of Hawking radiation arise because we split the initial vacuum in two parts via the particle-antiparticle production: one of the particles moves to the interior, the other to the outside of the horizon. The two particles are entangled and form together a pure state. Still, looking at one of the particles will give a mixed state. This is not a problem, since quantum systems often radiate (close to) thermal radiation. The problem here is that information, stored at the ingoing particles, cannot influence the outgoing radiation to arrive at a total pure state, which is due to the event horizon. Therefore, according to Hawking, the entanglement of Hawking radiation will increase till the black hole has completely evaporated. This differs from an ordinary quantum system, for example example a piece of coal: the late radiation has encoded information in the early radiation that has been emitted. In this way, the entropy will start decreasing at some point, and finally we arrive at a pure state, just as the initial state.Thus, if we want to end with a pure state, the entropy must decrease at some point.

## 6.2 Information in black hole radiation

One of the first to propose that unitary must be preserved, was Don Page. In many famous articles, he argued his belief that the essential principle of unitarity should not be violated. He argued that there might be other solutions to the problem of information loss. Indeed, according to Page, Hawking's calculations are a problem: thermal emission of the black hole may lead to information loss since the black hole might fully evaporate into a mixed state. By his calculations, he showed that "information might come out initially so slowly, or else be so spread out, that it would never show up in an perturbative analysis" [17].

Page shows the following.[17] We study a black hole subsystem with dimension$n \sim e^{s_h}$ with $s_h = \frac{A}{4}$ the semiclassical entropy of area A. Also, assume that we have a radiation subsystem with dimension $m \sim e^{s_r}$ with $s_r$ the radiation entropy. We propose that the information is not lost in the process of black hole evaporation. Hence we assume that the subsystems form a total system. We describe the system by a Hilbert space with dimensions $mn$, with density matrices $\rho_{rh} = \rho_{rh}^2$. Both subsystems will be correlated if they are in a mixed state, so

$$\rho_r = \mathrm{Tr}_h(\rho_{rh}) \qquad \rho_h = \mathrm{Tr}_r(\rho_{rh}) \tag{6.1}$$

The corresponding entanglement entropies are

$$S_r = -\mathrm{Tr}_r(\rho_r \log \rho_r) = S_h = -\mathrm{Tr}_h(\rho_h \log \rho_h) \tag{6.2}$$

The information, which is defined as the deviation of the entanglement entropy from its maximum, is given by

$$I_r = \log(m) - S_r \simeq s_r - S_R \qquad I_h = \log(n) - S_h \simeq s_h - S_h \tag{6.3}$$

It is useful to know how much information one has in the radiation at different stages of the black hole evaporation. The information $I_r$ lies close to the average information of a subsystem with dimension $m$ when the total system is in a random pure state $mn$. Now, if $m \leqslant n$, then this average information is

$$I_{m,n} = \log(m) + \frac{m-1}{2n} - \sum_{k=n+1}^{mn} \frac{1}{k} \tag{6.4}$$

For $1 << m \leqslant n$ one can show that

$$I_{m,n} \simeq \frac{m}{2n} \simeq e^{s_r - s_h} \tag{6.5}$$

for $m \geqslant n$, the previous equations imply, toghether with the fact that $S_r = S_h$, that

$$\log(m) + \frac{n-1}{2m} - \sum_{k=m+1}^{mn} \frac{1}{k} \sim \log(m) - \log(n) + \frac{n}{2m} \tag{6.6}$$

The average information is shown in the figure, together with the average entanglement entropy $S_{m,n} = \log(m) - I_{m,n}$

It becomes clear from the second equation that there is little information accessible in the radiation when the subsystem is small compared to the black hole. After it has evaporated for a while, much more information becomes available. Still, if one measures a part of the radiation so that it has smaller dimension than the rest of the system, one would still only see a small amount of radiation. After half of its entropy, at the Page time, information leaves the black hole.

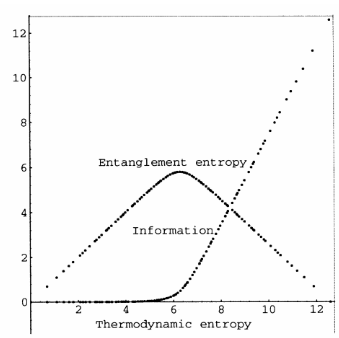



Figure 18: A plot for the entanglement entropy and the information of the radiation, calculated by Page. After the Page time, information stored in the black hole comes out.[28]

## 6.3 The Page curve

Page considered [17] the entanglement entropy for the black hole and its radiation. Since we assume to start at a pure state for the black hole, we finally have to arrive at a pure state again if we assume the conservation of unitary. The entanglement entropy of a pure state is zero, so the entropies of both the begin state and end state should be zero. Hence, the entanglement entropy of the black hole at first should be equal to the entropy of the total radiation at the end. Between these point, there will be entanglement, and the entanglement entropy can only be determined by a reduced density matrix of the radiation that lives outside the black hole.

Page argued that the entropy of a black hole must follow a curve that we call the Page curve, [33] in contrast to Hawking's idea of the curve of the black hole entropy. According to Hawking, the entropy of a black hole keeps increasing until the black hole has completely evaporated. At this moment, all the radiation from a black hole is emitted. From Page's perspective, the maximum entropy is reached at the Page time. This time occurs when around half of the final radiation is emitted. After the Page time, the entropy decreases down to zero.

From the black hole particle-antiparticle production, one is able to calculate the entanglement entropy of the particles that move out by their reduced density matrix. WE consider both the subsystems, the black hole and the radiation, as a bipartite system. Since we start at a pure state, the entanglement entropy of the radiation equals the entanglement entropy of the black hole according to the Schmidt decomposition. This is also why the curve is symmetric up to the Page time, so $S_{blackhole} = S_{rad}$. This is also true, since the radiation and the black hole must produce a pure state. Furthermore, we know have that the fine-grained entanglement entropy must be less than the Bekenstein-Hawking entropy, the coarse grained entropy. So, after all, we must have $S_{blackhole} < S_{Bek.-Haw.}$. This is also why the Page curve should bend down after the Page time. The Page time is exactly at the point $S_{Bek.-Haw.} = S_{radiation}$. [22]

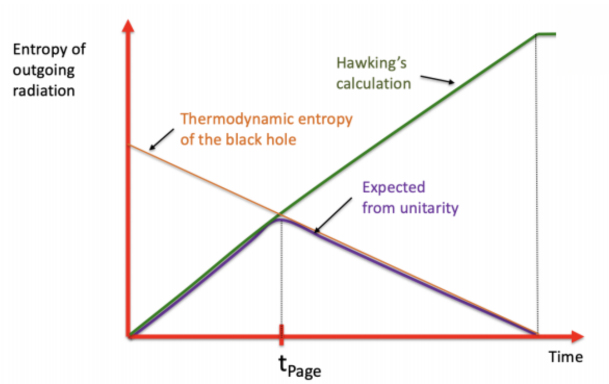



Figure 19: The Page curve, which captures the behaviour of black hole radiation.The green line shows Hawking's result. The thermodynamic, coarse-grained entropy is given by the orange line. For the process to be unitary, it is predicted that the entropy of radiation should be smaller than the thermodynamic entropy. At the Page time, these entropies are exactly equal. [22]

The entanglement entropy of the radiation will increase by emitting more radiation. At the same time, the Bekenstein-Hawking entropy should decrease since the area of the black hole shrinks.From the halfway point on, the Bekenstein-Hawking entropy, the thermodynamic coarse-grained entropy, will be smaller than the entropy of the radiation. If the black hole entropy follows the Hawking curve, we will arrive at a contradiction on this point: since entropy can be seen as an amount of microstates compared to the black hole macrostate, the coarse-grained entropy should always be bigger than the entanglement entropy. By looking at the entropies, we expect that before the Page time, the black hole should follow the curve Hawking predicted. After the Page time, the curve must bend down since the coarse-grained entropy becomes smaller than the entropy for the radiation. This should happen to arrive at a pure state.

Arguments for the Page curve rely on fundamental properties of the fine-grained entropy. Therefore, it is a strong argument. Also, as we will see, it looks like the problem cannot be fixed by small corrections to the Hawking state and this holds for all orders in perturbation theory. Therefore, if there is a solution for the problem, it should be non-perturbative in gravitational coupling $G_N$. [22]

Still, at the Page time, the black hole is a giant hole. A major question in the construction of a solution, is how to bend the curve down in order to preserve unitarity. Sometimes, this is called the real "information puzzle", since it is hard to establish that the entropy indeed follows this curve. Loads of work has been done to show it is the case. After all, as we will see, the Page curve is important in recent studies which claim to solve the paradox.

# 7 Proposed solutions

In order to resolve the paradox, wildly different solutions have been developed. Some of them claim to solve the paradox, while others give some insight towards the solution. Many of them are still under investigation, since the search for a solution has appeared to be difficult. By trying to solve the information paradox, one might have to deal with energies at the Planck scale. Here the known laws of physics seem to fall apart. In order to fully describe the black hole in a physical way, we might need a new theory of quantum gravity, also for a solution for the information paradox. Quantum gravity has to agree on both quantum mechanics and general relativity. Since Hawking found out black holes radiate, a lot of research has been done in order to find and define such a theory. Here, we take a look at some famous proposed solutions involving the information paradox.

## 7.1 Black Hole Complementarity

Black Hole Complementarity (BHC) is a theorem constructed by Leonard Susskind , Lárus Thorlacius and John Uglum, which suggests that black holes have a stretched-horizon. [7] A stretched horizon can be seen as a physical membrane that contains physical characteristics while it also absorbs and emits information that falls into the black hole.The stretched horizon can be found above the real event horizon, around a Planck unit of proper distance, a very small distance. According to BHC, observers outside see the stretched horizon as a place where quantum information, that has moved into the black hole, is stored. A free falling observer that passes this horizon does not notice it. Still, this horizon has a high temperature and we might need a description by quantum gravity to understand its dynamics.

Important for the theory of BHC is the fact that only an outside observer is able to see the stretched-horizon. Furthermore, another aspect of the theory is that it has given up on the principle of locality. BHC can be stated by three main postulates [7]:

- Postulate 1: The process of formation and evaporation of a black hole, as viewed by a distant observer, can be described entirely within the context of standard quantum theory. In particular, there exists a unitary S-matrix which describes the evolution from infalling matter to outgoing Hawking-like radiation.

- Postulate 2: Outside the stretched horizon of a massive black hole, physics can be described to good approximation by a set of semiclassical field equations.

- Postulate 3: To a distant observer, a black hole appears to be a quantum system with discrete energy levels. The dimension of the subspace of states describing a black hole of mass M is the exponential of the Bekenstein entropy $S(M)$.

The first postulate implies that purity is preserved: information is not lost inside the black hole, rather it is emitted by Hawking radiation. The second postulate states that semiclassical gravity is allowed outside the horizon: there is nothing strange about the physical description outside the horizon. The third postulate implies that the description of black hole thermodynamics is valid: from a distance, a black hole can be defined as a quantum mechanical system. Another important note of BHC is that a free falling observer, that crosses the event horizon, sees nothing special in accordance with the equivalence principle.

Still, quite the same postulates are used to describe how the black hole paradox is caused. By looking at the views of different observers and the stretched horizon, BHC claims to offer a solution. The main argument in BHC is that physical laws are not violated when one takes a look at only one view at a time. So, we can look at the interior of the black hole and the exterior, but not at both the same time. Precisely, for an observer outside, the horizon looks like a hot place, with information put into the horizon that comes out by Hawking radiation.

### 7.1.1 BHC and no-cloning

Say we have an outside observer, called Bob. Then, this observer from the outside will see a body, say Alice, that comes closer to the black hole. Due to gravity, Alice appears to move slower in time. Also, light gets more and more redshifted as Alice moves closer to the event horizon. Finally, Alice seems to be standing still above the horizon, and will seem to be more flat by the effects of special relativity of length contraction. Furthermore, Bob cannot see what falls into the black hole, though everything that falls into the black hole should somehow be presented at a membrane surrounding the black hole. Finally, Bob will lose Alice out of sight and he will see this high-temperature membrane. According to his view, Alice has gone being destroyed and has been occupied by the membrane. Following this view of Bob, Alice never passes the event horizon.

Still, the observer moving through the horizon sees nothing special (no drama) and will follow its path to the singularity. This is because for an observer, say Alice, that approaches the event horizon and is in free fall, no other forces than gravity act on her. According the equivalence principle, Alice does not see anything special when falling across the event horizon: she will not observe it. So, both observers see very different things happening.

According to the first postulate, the information that Alice carries is emitted after she has fallen apart at the horizon. Then, if we estimate the information is send out, unitary seems preserved. Thus, following the lines of thoughts of BHC, the information goes both through the black hole and becomes presented at the horizon: the particle going out the black hole and the particle going inside have the same Hilbert space instead of two different Hilbert spaces: both views of the observers seem complementary. Then, if we estimate the information is send out, unitary seems preserved. Yet, it seems like the no-cloning theorem is violated in this way.

The no-cloning theorem says that it is impossible to copy information of a quantum state by linearity of quantum mechanics. Still BHC claims that one cannot think of an observer that is able to study both copies of information of the black hole inside and outside. So, it should be possible that information is available at different places at the same time. Indeed, an observer standing outside the black hole as Bob does, would not observe cloning since he only observes the infalling Alice and the emitted radiation. Bob can never see that Alice passes the event horizon as she observes, simply because he cannot measure anything that passed the horizon. Alice does not observe cloning too: even if her information is cloned and presented at a membrane surrounding the black hole, she will not be able to see the copied information.

One could think of a way to let Bob observe both entangled particles from the particle-antiparticle creation. Let's say that Alice follows the antiparticle going inside the black hole and Bob follows the emitted particle outside. If Bob measures the particle outside, which contains a copy of the information of the infalling particle, and moves into the black hole and measures information of the other particle send by Alice, Bob could see both the information of the particles: a violation of no-cloning. Furthermore, this is a violation of the monogamy of entanglement: the antiparticle is entangled with the Hawking radiation, which is entangled with the particle outside. Still, from the view of BHC, this seems not possible since one would need energies of a Planck scale.

For Bob to get the information from the antiparticle, it will take some time. After the Page time, the black hole will emit radiation which is entangled with early radiation. To get the total information, one has to find the early and late radiation, namely the total system of the entangled particle-antiparticle system. Bob will simply not have enough time to receive both the information from both the particles, also because Alice does not have an infinite lifetime: eventually, she will be destroyed since she approaches a black hole singularity. In order to describe the information in both outside and inside the black hole, we need Planck scale energies: beyond the Planck scale limit, the known laws of physics seem to break down. Thus, according to BHC, when we know something about the inside or outside of the black hole, we do know nothing about the other. Essentially, here we give up the fact that we know where and when something happens.

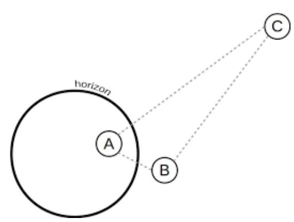



Figure 20: Information from A and B are entangled. If B enters the black hole and measures A, after measuring C, B will violate principles in quantum mechanics. [34]

## 7.2 Hayden-Preskill

In their research, Hayden and Preskill [34] look at an interesting thought experiment in quantum information theory considering the information paradox, which gives insights in the transmission and processing of information in quantum systems. Above all, this might be very useful to broaden the understanding of the questions involved in the paradox. Though they do not propose a solution for the information problem, they present useful information in certain relations between infalling particles and outgoing radiation [34].

In their paper, Hayden and Preskill review what time it takes for a certain observer to decode information, that has been thrown into a black hole, out of Hawking radiation. In their study, the authors assume unitary for the black hole interior. They also assume that black holes proceed quantum information and do not destroy it, just like a normal thermal system. Furhtermore, in their paper, it is believed that the black hole information will be thermalized rapidly. Not to mention, the internal black hole dynamics are defined by a random unitary transformation. Like this, black holes come close to being optimal thermalizers.

The setup used by Hayden and Preskill is shown in the figure. At first, we consider Alice. She falls into the black hole with information $M$, and Bob stays outside, trying to catch some measurements of the black hole radiation. Then we have Charlie, who has a reference system $N$, with a same dimension as the space of Alice, so $|M| = |N|$: thus, these two systems are maximally entangled. Therefore, the joint state of Alice and Charlie can be in a pure state:

$$|\Phi\rangle^{MN} = \frac{1}{\sqrt{|M|}} \sum_{a=1}^{|M|} |a\rangle^M \otimes |a\rangle^N \tag{7.1}$$

Here $N$ gives a purification of the state $M$. One can furthermore divide the black hole into two subsystems, the interior of the black hole and its early radiated system $E$. Here $\log(B)$ is defined to be the Bekenstein-Hawking entropy. Also, we have $\frac{|B|}{|E|} >> 1$ soon after the black hole formation, and one can argue that these systems are maximally entangled. Eventually we will have $\frac{|B|}{|E|} << 1$ . Later on, we have a unitary transformation $V^B$ which operates inside the black hole due to its interior dynamics. From here on, there appears to be another system that emits radiation, $R$. Bob detects radiation, and measures a subsystem $RE$. This subsystem is almost maximally entangled with $N$. Hence, the information that Alice has in possession is now owned by Bob. The rest of the black hole is denoted by $B'$.

In the experiment, we consider information that Alice owns, say a diary, containing $k$ qubits, and an internal state of a black hole with $n-k$ qubits [34]. An observer at the outside, Bob, knows the black hole internal state, but nothing about the information Alice contains. Alice throws a $k$-qubit into the black hole that is entangled with Bob's computer that measures the quantum system. By looking at the collected Hawking radiation and putting this in his measuring system, he is able to decode the information that Alice owned. The question is how long it will take for Bob to get the information. The authors state that Bob needs $k$ qubits or more from the radiation to get to know the information.

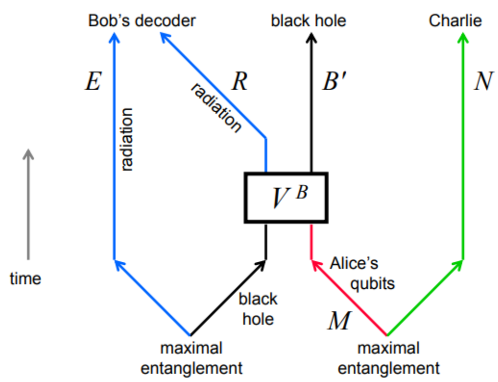



Figure 21: The setup for the experiment described by Hayden and Preskill. From the initial pure state on, the black hole is entangled with the early radiation $E$. Then the information that Alice contains, $M$, is moved inside the black hole. This information is maximally entangled with a reference system by Charlie, $N$. Now, $V^B$ is a strongly mixed unitary transformation, which appears inside the black hole, due to its dynamics. From here on, there appears another system which emits radiation, $R$. Bob detects radiation, and measures a subsystem $RE$. This subsystem is almost maximally entangled with N. Hence, the information that Alice has in possession is now owned by Bob. The rest of the black hole is denoted by $B'$.

Furthermore, it looks like the black hole can be seen as a quantum information mirror rather than a real black object, since the information that is thrown in is getting out almost immediately: if Alice throws her diary in, it will simply come back. Here, the time it takes, comes from the scrambling time and the radiation time to get the qubits out.

Thus, if Alice would like to hide her secrets that she wrote down in her diary, it would not be smart to throw it into a black hole. When Alice throws her diary in after the Page time, Bob will immediately recover the information. If Alice does the same but now before the Page time, Bob has to wait until half of the black hole qubits have come out, when the black hole has become maximally entangled with the outside. However, if Hawking radiation comes out slowly, it will take some time before the information will move out. Also, Bob must have knowledge of the initial state of the black hole. If more than $k$ bits have been radiated, these secrets will become fully clear to Bob. What's interesting here is that the authors originally constructed an experiment in which the black hole might hide information. Thus, the results are rather the opposite.

When the black hole is at the Page time, where half of the entropy has been radiated, it turns out that the Hawking radiation can be seen quite fast, much more quickly than we have expected at first. This occurs when the internal degrees of freedom of the black hole are almost maximally entangled with the early Hawking radiation, which is expected for a black hole after the Page time. Information stays in the black hole till the Page time, and from the Page time on, it is appears rapidly: $k$ qubits of information, originally put into the black hole, can be found again by Bob after $k$ qubits have come out via Hawking radiation.

Furthermore, states in the black hole interior become mixed in a Schwarzschild time order of $r_s \log\left(\frac{r_s}{l_p}\right)$ with $l_p$ the Planck length and $r_s$ the Schwarzschild radius. After the Page time, k-qubits of information absorbed by the black hole will move out by a Schwarzschild time $O(r_s \log(r_s))$ or $O(kr_s^3)$, depending on which one is most substantial. The information that has fallen in, is rapidly thermalized in the black hole by a "Schwarzschild time" scale $r_s$ locally, when considering non-rotating uncharged black holes. Thus, if one gives a kick to a classical black hole, after this time the balck hole forgets about the kick. However, globally, the time it takes for the information of Alice to become mixed with the black hole interior degrees of freedom, is of an order $O(r_s log(r_s))$.

According to the research, if information is dropped into the black hole from a distance of order $r_s$, it will take a Schwarzschild time of order $r_s \log(r_s)$ to reach the stretched horizon. Interestingly it takes an same amount of time for information from the stretched horizon to move to the observer outside. This indeed shows how a black hole seems to behave as a mirror: it takes approximately the same amount of time for a particle to fall in the black hole as to arrive at the observer outside from the stretched horizon. However, for this to happen, we need a global thermalization time of $O(r_s \log(r_s))$. Yet, the fact that quantum information comes out rapidly after the Page time, holds even if the thermalization time[5] is longer. For the latter, the thermalization must be faster than the evaporation, which happens at a time order of $r_s^3$.

The question rises if black holes are quantum cloners. If Bob is able to receive the information from Alice which is available behind the event horizon, her information is cloned by the outgoing radiation. However, cloning violates the principle of linearity of quantum mechanics. It is proposed that a theory of quantum gravity could solve this. BHC believes that both descriptions of the information are suitable, as we have seen. Still, we would like to have a deeper understanding of this, hopefully via quantum gravity.

After all, it seems that Alice is able to send a message to Bob using super-Planckian frequencies if Bob moves into the event horizon in order to get the information of Alice. This is possible if she has less than a Planck time to communicate with Bob. Yet it is not possible to understand such signals. If Alice's proper time would be much larger than a Planck time, Bob could see the cloning happening analyzed by semiclassical approximations.For cloning not happen, the information in the black hole must be stored until a Schwarzschild time of $\Omega(r_s \log(r_s))$ with $\Omega$ a constant to set the lower bound for asymptotically large $r_s$. This result is consistent with the thermalization time stated earlier, yet it is a close call.

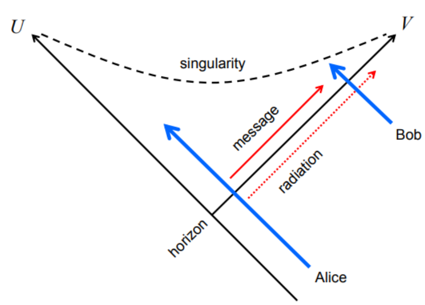



Figure 22: The idea of cloning quantum information. The information that Alice contains, falls into the black hole. Bob is able to receive the information via Hawking radiation. Then Bob moves inside the black hole too. If he is able to measure the information put inside the black hole by Alice, the information will be cloned. According to Hayden and Preskill [34], for cloning not to happen, the information that Alice carries must be stored in the black hole until a Schwarzschild time of $\Omega(r_s \log(r_s))$. [34]

The findings by Hayden and Preskill [34] seem to be hardly consistent with BHC, though they are. From the experiment it becomes clear that some information is destroyed by the formation of black holes after gravitational collapse. Nonetheless, information in a small subsystem can almost completely survive. So, at the end, according to Hayden and Preskill, there might be partial information loss.

To conclude, Hayden and Preskill have shown that information from a small diary, thrown early into the back hole, can be found from radiation at the Page time. However, when the diary is thrown into the black hole after the Page time, one needs to wait a scrambling time to decode the information. This scrambling time is often stated by $\frac{\beta}{2\pi}{}_{BH}$, or alternatively $\frac{1}{T_H} \log S_{BH}$.

[5]The thermalization time is defined to be the time a black hole needs to reach thermal equilibrium.

Why is this true? As we will see, at later times, the diary is not yet encoded in the entanglement wedge of the radiation in the black hole interior.

## 7.3 Firewalls

According by studies in 2013 from Alhmeiri, Marolf, Polchinski and Sully (AMPS) [35], considering the weak equivalence principle from general relativity and the conservation of unitarity from quantum mechanics, would ultimately result in a firewall. This solution is often stated as the AMPS solution by the names of the authors. The AMPS-argument claims that there are inconsistencies in BHC, and their solution is largely based on the idea that BHC is not finalized. The solution is described by the argument that a high energetic surface arises around the black hole, a firewall, which destroys anything that passes the black hole horizon.

Consider the particle-antiparticle creation and Hawking radiation that is emitted at an early stage. We assume the particle and the antiparticle to be entangled. Also, we consider three subsystems here, the antiparticle (A), the particle (B), and the early Hawking radiation (C), which can be seen as separable subsystems. Now, A and B are entangled, but B is also part of the late radiation. Moreover, B is entangled to the early Hawking radiation C. Yet it is impossible for B to be entangled with both A and C according to the principle of the monogamy of entanglement. Precisely, AMPS states that it is possible to observe the three particles which results in the violation of the monogamy of entanglement. According to the authors, one is able to measure C and then move to the inside of the black hole to measure A. While doing this, it will encounter B so all three systems are measured.

If we assume that the monogamy of entanglement holds, then particles cannot be entangled to both the early radiation and the black hole. Here, the entanglement to the letter is important for the preservation of unitarity. In order to fix this problem, the firewall proposal suggests that the monogamy of entanglement must hold: if B and C are entangled, then B and A cannot be entangled. Thus, if A and B are not entangled, the observed state is not the vacuum state but rather an excited state: a firewall of high energy particles is found which destroys anything that falls in. The firewall consists of a high energy to break the entanglement between A and B. Especially, the entanglement between the outgoing and ingoing particles is necessary to keep the vacuum smooth, so there will be no large gradients in quantum fields available in normal Minkowski space.

When we disentangle points close to the event horizon, we will get a large gradient at the horizon of the black hole, which will generate high energies. When considering the Hamiltonian of a scalar field, we see that we arrive at high energies when the gradient of the field becomes large, when breaking the entanglement between the particles:

$$H = \int d^3x(\frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi(x))^2 + \frac{m^2}{2}\phi(x)^2) \tag{7.2}$$

Here $\pi(x)$ is the canonical momentum of the quantum field $\phi(x)$. This equation shows the realization of a high-energetic firewall.

Nevertheless, the idea of a firewall seems to violate the equivalence principle, since a free-falling observer does not measure Minkowski space, but rather an excited state. This has appeared to be a big problem for many researchers. Still, the AMPS-authors argue that the breakdown of the equivalence principle from general relativity is less problematic than the violation unitary from quantum mechanics. Alternatively, if this would be the case, the AMPS-authors emphasize that effective field theory would be violated, and this would be a much more radical proposal.

Though the stretched horizon in BHC and the firewall by AMPS sound like the same, they are clearly not. The stretched horizon appears to have high energies for an accelerated observer near the horizon but still outside, while for all observers, the firewall is found behind the horizon. The presence of the firewall does not depend on the observer, while the stretched horizon does. Still, one can pass the stretched horizon, but not the firewall.

Since the AMPS-authors propose that the firewall is located behind the event horizon, an observer outside the black hole will not see the firewall. Also, it is suggested that the firewall forms right after the Page time.

## 7.4 ER = EPR

Another theory that might give some new insight in the information problem is the ER = EPR proposal.[36][27] This principle combines two ideas from Einstein, ER and EPR. For the first, Einstein described EPR as a 'spooky action at distance', which is his note about the entanglement between quantum particles. The EPR-statement is named by its authors: Einstein, Podolsky and Rosen. The EPR-statement describes how the physical definition of quantum entanglement must be incomplete, which is partly solved by giving up locality. For the second, the idea of ER, which finds its origin in general relativity, describes that black holes can be connected via a wormhole, called Einstein-Rosen (ER) bridges. Such a solutions can be seen as black holes that are maximally entangled, thus forming an EPR pair. Moreover, it is suggested by Susskind and Maldacena [36] that such ER-bridges can be used for other entangled states.


Figure 23: An impression for an Einstein-Rosen bridge between two black holes. Entanglement can be stated by geometry as the ER-bridge. The entanglement is presented by coupling the different horizons via space-time interior regions beyond the horizon of the black holes. On the right we have the Penrose diagram for an entangled black holes in a extended Schwarzschild spacetime.Here, the asymptotic regions are connected by an ER-bridge. The regions $\epsilon_L$ and $\epsilon_R$ are connected by an interface $I$. [37]

The duality between ER-bridges and quantum entanglement is famously called "ER = EPR". Interestingly, by studying entangled black holes in this way, we might solve the AMPS firewall paradox.

In physics, the locality of spacetime is well-understood. By this, we state that it is not possible for systems or particles to send information to each other faster than the speed of light. Still, it could be possible that locality is violated by general relativity via ER-bridges and by quantum mechanics via EPR-correlations. In general, it is believed that these do not violate locality for real. This is because, for EPR-particles, one is not able to transfer information faster than the speed of light. It can be shown quite easily that non-local operations on a particle that is entangled, do influence the other before information has been send to the other. Also, ER-bridges cannot bring information from one asymptotic region to the other when certain energy conditions are embraced. In quantum theory, these conditions may not hold since they appear for the classical theory, but still this would not be enough to make wormholes traversable. Thus, it is assumed [36] that these wormholes are un-traversable in quantum theories. Maybe, via ER = EPR, one will be able to describe how one might combine gravity and quantum mechanics to a full description of quantum gravity.

According to Maldacena and Susskind, it is stated that an ER-bridge is made out of EPR-correlations from two different black hole microstates. So, for the ER-bridge, "there exists an EPR-correlation for which its quantum system has an Einstein gravity description." [36] Also by them, it is suggested that all EPR-systems consist of some kind of ER-bridge, though these bridges can be difficult quantum objects. It is believed [36] that, for example, a singlet state of two spins might be connected by such a quantum mechanical bridge.

Hence, by a theory of quantum gravity, ER-bridges might be the same as EPR-correlations. There could be a Planckian bridge for all entangled particles, however this might be a quantum bridge which we cannot describe classically. We know that Hawking radiation is highly entangled with the black hole, and by the proposal of ER=EPR, it is expected that this entanglement creates the geometry of the black hole interior, see the figure. Essentially, it has become clear that black holes are entangled if they have an ER-bridge, however it is more difficult to show that entanglement or two different systems implies an ER-bridge. Following the theorem of ER=EPR, to Susskind, it seems that "geometry and quantum mechanics are so inseparably joined that each may not make sense without the other". [27] The main question he proposes is whether "the identification of entanglement and ER-bridges are consistent with the standard rules of quantum mechanics."

Black hole that have one side, which are the normal black holes that come from example stellar collapse, can get two sides at the Page time. This can happen due to outgoing Hawking radiation, that has at least the amount of degrees of freedom that the remaining black hole has after this time. From this time, the radiation is maximally entangled with the black hole. The early radiation, from before the Page time, can be seen as the second black hole in the previous examples.

In studying ER=EPR, black holes are described as if they are in a specific entangled state. [36] Interestingly, the creation of black hole pairs in magnetic or electric fields are exactly in this state. A geometry with a constant magnetic field is also defined by Euclidean geometry. Also, it has become clear that extremal black holes have a fixed charge to mass ratio, and that the acceleration of the black holes is fully defined by the magnetic field, and independent of the black hole mass. The charged black holes in the Euclidean geometry go around in circles by Euclidean time, and by their acceleration, which defines the circle, one can obtain its Rindler temperature. Now, these black holes have this temperature, and they are in thermal equilibrium with outgoing radiation. The rate at which this happens has some factor $e^S$, with $S$ the black hole entropy. What's interesting here, is that this is exactly expected if these two black holes would be entangled. [36]

Two black holes that are maximally entangled but do not interact, can be described by the extended Schwarzschild

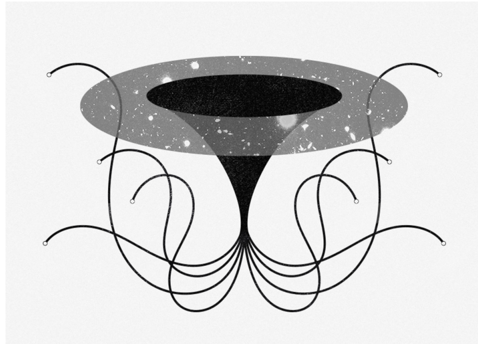


Figure 24: Entanglement between Hawking radiation and the black hole. It is expected that the entanglement creates the geometry of the black hole interior. ER = EPR proposes that particles from Hawking radiation are connected by particles from the black hole interior via wormholes. From these wormholes, the black hole looks like an octopus, where the arms represent the wormholes.

spacetime Penrose diagram. The entanglement is drawn by the horizons that touch each other in the middle. In the figure, two asymptotic regions are connected by an ER-bridge. Although these regions are connected, it is not possible for information to get through the bridge, This is constistent with the view that entanglement cannot create information signals that are not local. [36]

### 7.4.1 ER=EPR and firewalls

Maldacena and Susskind believe [36] that ER=EPR might be important for the question whether black hole horizons are smooth. Yet, there has been no answer to this. However, they state that the firewall argument of AMPS does not have to be true. Essentially, there is a main difference between AMPS and ER=EPR. AMPS believes that parts inside and outside the horizon are independent, while ER=EPR believes that particles on the both sides are connected by a wormhole.

Black holes, with a large distance between each other but connected via an ER-bridge, show that a black hole can be maximally entangled with another system, while it still has a smooth horizon. However, the AMPS-authors believe that a smooth black hole interior will be removed when this black hole will be entangled by the outgoing radiation. Indeed, By AMPS, firewalls seem to solve the information paradox. Essentially, its solution comes from the fact that AMPS does not considers important properties from quantum mechanics.

Though entangled black holes often do not have firewalls beyond the horizon, this can be the case. If we consider Alice to sit far away from the left black hole, she is able to send information in the black hole by doing some manipulations on the boundary. Bob is on the right side, and he will not be able to see information send by Alice if he does not pass the right black hole horizon. The information can be seen if Alice will send the information early enough. Ultimately, according to the authors of ER=EPR, the question whether Bob's black hole has a firewall depends on the actions of Alice.

One can consider two black holes on the same slice of space. If Bob is settled close to one of the black holes, while Alice is placed far away from the other black hole, their communication can only occur via the outer space, and it is impossible to do this via the ER-bridge. Still, when some conditions are preserved, they can meet at their singularities. It turns out that it is now possible for Alice to make a firewall occur at Bob's black hole by sending information, say shock waves, though she must do this early enough.

After all, it has not become completely clear how ER=EPR solves the firewall problem, and if it does.In fact, to let it work, we need specific kinds of space and entanglement, and so we need a specific kind of wormhole. ER=EPR seems to work in specific situations, however the complete information problem seems broader than that. Still, ER=EPR has been an interesting idea. In some way, it has been realized in the description of black holes, and even in solutions to the paradox, by the Ads/CFT-correspondence and the Island proposal. We will encounter both later on.

## 7.5 Quantum error corrections

In the search for a solution to the black hole information paradox, it is sometimes believed that Hawking did not formulate his statements well. Some physicists, for example Raju [38], believe that looking at quantum corrections to the leading order Hawking state will eventually leave the radiation to be unitary. Specifically, Raju states in his recent article that "exponentially small corrections in the radiation emitted by a black hole are sufficient to resolve the original paradox put forward by Hawking".[38]. However, Mathur [39] stated earlier in his paper that this cannot be true, by showing that "small corrections to the leading order Hawking computation cannot remove the entanglement between the radiation and the hole" [39]. Mathur shows that, by using the known laws of physics at the horizon and the assumption that locality holds, we will arrive at remnants or mixed states. Furthermore, he arrives at the conclusion that black hole interior must have a 'fuzzball' structure.

In our universe, it is a common belief that we can do experiments with our current knowledge of the traditional physics without taking details of quantum gravity into account. This is because we believe that there exists a limit from whereon quantum gravity effects become significantly unimportant. Locally, we can use well defined approximate evolution equations.[39] If we find ourselves in this limit, we deal with so called "solar system physics". Here we can deal with normal physics, because the spacetime curvatures are of the same order as the ones that can be found in our solar system.

The state of created pairs of particles can be described by the Hawking state

$$|\Phi\rangle_{Hawking} = |0\rangle_b |0\rangle_c + |1\rangle_b |1\rangle_c \tag{7.3}$$

for b and c ingoing and outgoing particles discovered by Hawking. This immediately shows the entanglement between particles b and c. Hawking's conclusion is right if this is indeed the state of the particle pairs. Still it is believed that there are small corrections available that make black hole radiation the same as normal radiation from a hot body, so that the information problem is solved.

According to Mathur, such corrections to the leading order state will not work out, unless "we make an order unity modification to the leading order result". [39] Though this is not possible since then "there will be a breakdown of the solar system limit even if we are given the niceness condition N". [39]

If we look at the entropy of particles that already have been emitted from the black hole, it is interesting to see whether one can get rid of the entanglement between the emitted radiation and the black hole. Mathur [39] wonders whether corrections lead to a decrease in the entanglement entropy. Eventually, he shows that it not possible to happen. Indeed, by corrections to the leading order state, we cannot find a solution to the paradox in the form of purity: the entanglement will only rise and we still have a mixed state if we assume that we cannot show that "corrections to evolution are order unity instead of order $\epsilon << 1$".[39]

Furthermore, Mathur [39] believes, just as in black hole complementarity, that the horizon is not smooth. He also points out [39] that if one likes to hold a smooth interior of the black hole, this will create a problem with the monogamy of entanglement, since then the black hole interior will be entangled with degrees of freedom from the outside of the horizon, and also with degrees of freedom further away. As we have seen, this problem is well described by the authors of the firewall proposal.

### 7.5.1 Arguments for considering quantum error corrections

The question rises: why do we come up with quantum error corrections? We already defined a state for an entangled pair of Hawking radiation. The complete state of the entangled pair is

$$|\Phi\rangle = [|0\rangle_{b_1} |0\rangle_{c_1} + |1\rangle_{b_1} |1\rangle_{c_1}] \otimes [|0\rangle_{b_2} |0\rangle_{c_2} + |1\rangle_{b_2} |1\rangle_{c_2}] \otimes [|0\rangle_{b_3} |0\rangle_{c_3} + |1\rangle_{b_3} |1\rangle_{c_3}] \otimes .... \tag{7.4}$$

since every pair production is independent of the others. If we now add small corrections to create correlations between the pairs, we will arrive at a state for the first pair to be

$$|\phi\rangle = \frac{1}{\sqrt{2}}[(1+\epsilon_1) |0\rangle_{b_1} |0\rangle_{c_1} + (1-\epsilon_1) |1\rangle_{b_1} |1\rangle_{c_1}] \tag{7.5}$$

with a small correction parameter $|\epsilon_1| << 1$. By this parameter, the emitted quantum is able to carry information about the hole. For the second pair, the corrections are dependent of the first pair and the initial black hole. Thus we can write the second pair as

$$|\phi\rangle = \frac{1}{\sqrt{2}}[(1+\epsilon_2) |0\rangle_{b_2} |0\rangle_{c_2} + (1-\epsilon_2) |1\rangle_{b_2} |1\rangle_{c_2}] \tag{7.6}$$

if the state of the first pair was $|0\rangle_{b_1} |0\rangle_{c_1}$. Here the parameter obeys again $|\epsilon_2| << 1$. For the other state, we can write a similar-looking equation. In this way, the early created particles influence the late created particles. The total state can be computed, which looks quite complicated. What is important here, is that the number of correction parameters rises with the number of particles that are emitted. For $N$ emitted particle pairs, we will have $2^{N-1}$ correction parameters. This number may be written down as an exponential $2^N = e^{N \log 2} \sim e^{\alpha(\frac{M}{m_p})^2}$. Here $\alpha$ is of order unity, $m_p$ the particle mass and $M$ the black hole mass. If $N$ becomes extremely large, we can have an exponentially number of correction parameter, so we will have an exponentially large number of correction terms to the leading order state.[40] Then each correction parameter can become exponentially small so that

$$|\epsilon_i| < \epsilon \text{ where } \epsilon \sim e^{\alpha(\frac{M}{m_p})^2} \tag{7.7}$$

Such small corrections can come from quantum gravitational fluctuations of the black hole. Though such effects need a description of quantum gravity, we can estimate that the black hole geometry is expected to have exponentially small corrections from its quantum state. Hence, we do not end up with a paradox. By these corrections, the entanglement between the particles $b_i$ and $c_1$ will be removed in the leading order state. Furthermore, the information of the initial black hole will be described in the outgoing radiation.

Now, the proof of Mathur shows that corrections cannot destroy the entanglement between the outgoing particles and the black hole. We let $N$ quanta be emitted in by $1, ..., N$ steps, $\{b_1, b_2, ..., b_N\}$. After all, one is able to find the relation

$$S_{N+1} > S_N + \log(2) - (\epsilon_1 + \epsilon_2). \tag{7.8}$$

This implies that the radiation state will not become pure. Thus, by corrections to the leading order state, we cannot find a solution to the paradox in the form of purity: the entanglement will only rise and we are still left with a mixed state. [39].

The proof by Mathur [39] uses a tool from quantum mechanics: the strong subadditivity of quantum entanglement entropy, which is quite general. By this and other general assumptions, such as the locality of interactions, the proof states that the entanglement will not stop growing by small corrections in general. Due to this proof, Mathur and other gained more confidence in the description of black holes as fuzzballs. As we will see later, the research in islands agrees in some way to the theories of Mathur. Namely, an island contribution can be seen as a correction of order $O(\frac{1}{4G_N})$, which is non-pertubative and small in order to make it work. [22]

### 7.5.2 Raju: quantum corrections restore unitarity

Still, by some researchers it is believed that small corrections restore unitarity. For example, this is stated by Raju in a recent article [38]. Specifically, he states that “Hawking’s argument is not precise enough that it will lead to a paradox. This is because small corrections to Hawking’s calculation, which are exponentially suppressed by the black hole entropy, are sufficient to ensure that information about the initial state is preserved”. Raju wonders “how different density matrices and pure states are” and if there are calculations “precise enough to distinguish between the two”. Therefore, Raju looks at the exact difference between a mixed state and a pure state. According to Raju, his results[6] imply that “pure states are exponentially close to mixed states in a system with a large number of degrees of freedom”. From his view, this is a general result that follows from kinematic considerations. Hence, the Page curve will be unitary and will bend down.

## 7.6 String theory

Research in string theory provides some evidence for the argument that information should be preserved.[41] In string theory, one-dimensional strings correspond to point-like particles described in particle physics, while ’branes’ generalize point-particles into higher dimensions. This theory is often defined in 10 or 11 spacetime dimensions and it does describe the higher dimensional ’branes’. In string theory, an important aspect is the supersymmetry between bosons and fermions. In a theory of black holes, the black hole geometry can be described in many dimensions by supersymmetric configurations of string and branes.

In string theory, we have a quantity that regulates the strength of forces such as gravity, called the string coupling. The string coupling is found by the expectation value of a field that we call dilaton. If we decrease the string coupling, at a configuration that describes black holes at a value of this parameter, then the Schwarzschild radius will become smaller and smaller, till it has a smaller size than the configuration itself. Then it turns into a number of strings and branes that are weakly coupled. By the high degrees of the supersymmetry, the essence of the state stays the same when we alter the parameter. Specifically, in this theory it is predicted that the number of degrees of freedom do not change. Still, looking in this weakly-coupled region we do not have a black hole. Here, we only have a gas, with the same degrees of freedom, from which we can calculate the entropy.

By considering five-dimensional supersymmetric black holes with different charges, researchers found out that the number of degrees of freedom of the system in this regime is precisely equal to the degrees of freedom of black holes computed by their entropy in the strong-coupling regime. Similar research has been done at different kinds of black holes, and remarkably the same result was found. Therefore, according to string theory, one may believe that the degrees

[6]See the appendix for an insight in the derivation of this result

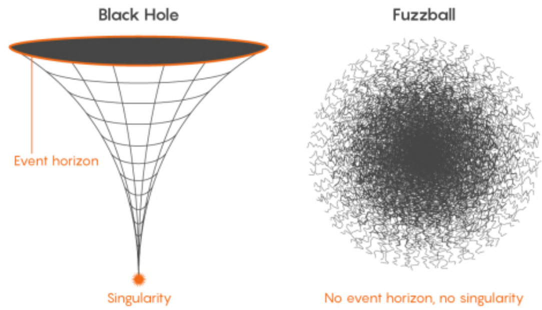



Figure 25: An impression for a black hole fuzzball. In this description, black holes do not have an event horizon, nor a singularity.

of freedom provided by radiation from the black hole are representative, which is some evidence against information loss [23].

Still, by counting states in string theory, we don't know much about how the information of the black hole is eventually coded in the Hawking radiation. Furthermore, we like to get an answer to the way that information from the black hole is put into the Hawking radiation. Yet, evidence from string theory shows that information can be encoded in the black hole's internal degrees of freedom and eventually transferred to the outgoing radiation. [34]

#### 7.6.1 Black hole fuzzballs

To avoid problems with singularities and event horizons, researchers in string theory have proposed to look at black holes as if they are 'fuzzballs'.

Mathur [41] has been one of the main researchers to argue that a black hole can be described as a ball of strings, energy units that vibrate and, by this, describe spacetime. According to Mathur's view, just as in BHC, the black hole horizon is not smooth. From his view, the black hole interior microstates must have a 'fuzzball' structure, a ball containing strings, with no horizon and singularity. He states that "The nontrivial [fuzzball] structure of microstates resolves the information paradox, and gives a qualitative picture of how classical intuition can break down in black hole physics". [39] This is because, by the picture of a fuzzball, the black hole radiates like a normal thermal body, so consequently there is no problem such as entanglement or information loss. Hence, in this view, the information problem will be an illusion.

An advantage of the fuzzball idea is that the black hole interior structure can be described by the strings and branes, which can be stated in various ways. By putting the strings and branes in the right way, they could generate an entropy for the black hole fuzzball that equals the entropy given by the Bekenstein-Hawking law. By this, it is possible to count the black hole microstates. Furthermore, fuzzballs seem to behave like real black holes: for example, they emit the same Hawking radiation. In contrast with black holes, fuzzballs are also unique, while black holes can be described by the same properties, according to the no-hair theorem. Hence, we might get to know the initial state of the fuzzball from the end state. Also, according to some research, a firewall can be seen as nothing more then a hot fuzzball.

Still, mostly people working with general relativity find it difficult to accept a theory that relies on structures that contain strings, branes and higher dimensions constructed by string theory. Furthermore, many physicist do not like the idea since one must sacrifice many well-known principles as the event horizon and the singularity. However, researchers such as Mathur, believe that a fuzzball must be the right quantum mechanical description for a black hole, since they seem to resolve many difficulties black holes create. In contrast, by general relativity, the black hole will evaporate and eventually the information might be stored in a remnant of a Planck mass. The construction of remnants gives some problems. Remnant will have a large number of possible states in a very small region. In string theory, it is believed that these remnants do not exist.

After all, there have been many proposals for a solution to the paradox. Still, from these solutions, a solution to the paradox has not fully been understood. In the next chapters, we will discover some more interesting research in the black hole information paradox, that finally might fully solve the problem.

# 8 Principles from AdS/CFT

In the previous section, we discussed various possible solutions that are involved in information paradox. They all seem interesting in some way, yet they do not fully solve the problem. By holography and specifically the AdS/CFT-corresondence , we have obtained some more insight in a solution towards black holes and the information problem. Also, a major finding in AdS/CFT is the Ryu-Takayanagi surface. [14]

## 8.1 De Sitter and anti-de Sitter space

Both de Sitter and anti-de Sitter space are used to describe the universe. For example, de Sitter space is used for describing an accelerating universe in a simple model. De Sitter space has been found to be a solution to Einsein's field equations with a positive cosmological constant. Moveover, evidence has been found for the argument that the universe can be described as asymptotically de Sitter space. On the other hand, anti-de Sitter space would describe a universe with negative cosmological constant, so this universe would be negatively curved. It is another way to describe the metric of spacetime, which of course comes from general relativity.[42]

### 8.1.1 De Sitter (dS) space

n-dimensional de Sitter spacetime, $dS^n$, is defined to be a set of points $(x_0, x_1, ..., x_n)$ in Minkowski space $\mathbb{M}^{n,1}$ of $n+1$-dimensions. Hence, it is a subspace of normal $\mathbb{M}^{n,1}$ Minkowski space. Here, the metric for Minkowski space is $\mathbb{R}^{1,n}$ is

$$ds^2 = -dx_0^2 + \sum_{i=1}^{n} dx_i^2 \tag{8.1}$$

and it satisfies the equation

$$-x_0^2 + \sum_{i}^{n} x_i^2 = L_1^2 \tag{8.2}$$

Here, the $L_1$ is the de-Sitter length.

The metric looks almost the same for an $(n+1)$ dimensional sphere[7], if we change $x_{n+1}$ into $x_0$. By this change, we defined a version of the sphere, applied to Minkowski space. The de Sitter metric can be seen as a specific version of the metric of a sphere in a certain dimension.

The line element of de Sitter space appears to be the most symmetrical solution of Einstein's field equations. Einstein introduced a cosmological constant $\Lambda$ which satisfies $L_1 = \sqrt{\frac{3}{\Lambda}}$. Furthermore, de Sitter space is related to Euclidean geometry. By a transformation $x_0 \to ix_0$, Euclidean the Sitter spacetime is defined as a sphere, and defined by an Euclidean the Sitter rotation group SO(5). [42].

In cosmology, the de Sitter metric is of great importance. This is because, in cosmology, one makes a natural choice of cosmic time, which makes the universe "homogenous and isotropic at large scales" [42]. By maximal symmetry in de Sitter spacetime and its topology, one can describe three kinds of spacetime $(\mathbb{S}^3, \mathbb{H}^3, \mathbb{R}^3)$ on a de Sitter manifold, when using the right coordinates. [42]. In black hole physics, one often uses static coordinates in de Sitter space. In this system, nothing explicitly depends on time.

[7] Spheres and hyperboloids are discussed in the appendix.


(a)

(b)

Figure 26: On the left, we see a five-dimensional de Sitter space in yellow. In this picture, the blue parts are lightcones, asymptotic to the de Sitter hyperboloid. As before, timelike geodesics are given by the intersection of hyperboloid and two-spheres that pass the centre. The geodesics can be parametrized by $\xi$ and $\eta$. On the right, we see a representation of Euclidean the Sitter space. [42]

### 8.1.2 Anti-de Sitter (AdS) space

In a similar way, we can define $n$-dimensional anti-de Sitter spacetime as a space with two time-like coordinates, for a set $(x_0, x_1, ..., x_n)$ in $n+1$-dimensional Minkowski space $\mathbb{M}^{n-1,2}$, with a metric

$$ds^2 = -(dt_0)^2 - (dt_1)^2 + \sum_{i=2}^{n} dx_i^2 \tag{8.3}$$

Similarly, it satisfies the equation

$$-t_0^2 - t_1^2 + \sum_{i=2}^{n} = L_2^2. \tag{8.4}$$

Here, $L_2$ is the anti-de Sitter length. [42] So, for example, five-dimensional anti-de Sitter space has two time-like, and three spacelike coordinates. An important property of AdS is that it has positive curvature. For AdS, timelike geodesics are ellipses.

Also, in Euclidean geometry, anti-de Sitter space can be expressed by an imaginary transformation $x_4 \to ix_4$, which is a copy of $\mathbb{H}^4$ [42]. Yet, AdS is not a hyperbolic spacetime. Normally, in hyperbolic manifold, which are non global, it is not enough for the full knowledge of time evolution to have the equations of motion and its initial data. In AdS, there is 'global hyperbolicity' [42], since we have a boundary at spacelike infinity. From here, information is able to flow in. This gives some difficulties with the quantization of fields on AdS-manifold, but it also brings major opportunities: the proposal of AdS/CFT.

Hence, for a nonzero vacuum, normal Minkowski space is not a solution to the Einstein equations. For a negative vacuum energy, anti-de Sitter space ($AdS_{d+1}$) is the solution. The metric can also be stated in polar-like coordinates, such that the metric takes the form [26]

$$ds^2 = -(1 + \frac{r}{r_{AdS}}^2)dt^2 + \frac{dr^2}{1 + (\frac{r}{r_{AdS}})^2} + r^2 d\Omega_{d-1}^2 \tag{8.5}$$

$$\text{where } \frac{1}{r_{AdS}^2} = -\frac{16\pi G\rho_0}{d(d-1)} \tag{8.6}$$

with $t \in (-\infty, \infty)$, r $\in [0, \infty)$. For $r << 1$, the metric looks like Minkowski space. We can also describe the metric by a Penrose diagram. For this, we use a coordinate transformation $r = \tan(\rho)$. We arrive at

$$ds^2 = \frac{1}{\cos^2\rho}[-dt^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2] \tag{8.7}$$

where $\rho \in [0, \frac{\pi}{2})$. By including the boundary $\rho = \frac{\pi}{2}$, we can draw a diagram shown in the figure below. From this, it becomes clear that AdS can be seen as a box. Massive and massless particles move from the center to the boundary and come back from the perspective of a central observer, in a time of order 1 in AdS-units. By this, we can state that the boundary is time-like. In general, roughly speaking, a spacetime is asymptotically AdS if its boundary is time-like, and its boundary approaches the geometry of AdS at $r \to \infty$. Considering quantum gravity, by particles that reach the

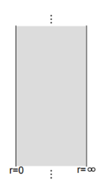


Figure 27: A Penrose diagram for AdS space. Signals can move to the boundary and return to the center in a finite amount of proper time. In this view, the space is not compact since the future and past continue to exist ininitely far away. By another coordinate transformation, this can be undone.[26]

boundary and return to the bulk within a finite amount of time, the geometry can be seen as 'gravity in a box'. [26].

## 8.2 Conformal field theory (CFT)

A conformal field theory (CFT) is of great importance in the study of the AdS/CFT-duality. A CFT is a " relativistic quantum field theory which is invariant under a large set of spacetime transformations, generated by Poincare

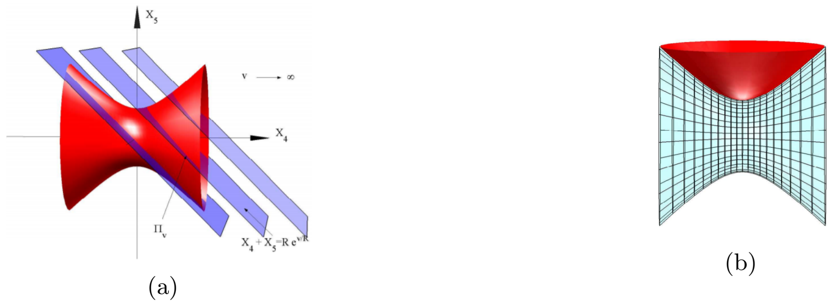



Figure 28: On the left: anti-de Sitter space. Here, timelike geodesics are found by the intersection of two-planes that pass through the center and the AdS-space. On the right: Euclidean anti-de Sitter space. [42]

transformations." [26]. Specifically, this is done by a coordinate transformation

$$x'^{\mu} = \lambda x^{\mu} \tag{8.8}$$

$$x'^{\mu} = \frac{x^{\mu} + a^{\mu} x^2}{1 + 2x_{\nu} a^{\nu} + a^2 x^2} \tag{8.9}$$

by this transformation, in the conformal group, angles are preserved. The conformal group is isomorphic [8] to the group $SO(d, 2)$, which already is a hint to a connection to $AdS_{d+1}$.

An example for this is a CFT of a free massless scalar field in 3 + 1 dimensional Minkowski space. This scalar field is invariant under dilatation transformation [26]

$$x^{\mu} = \lambda x^{\mu} \tag{8.10}$$

$$\phi'(x') = \lambda^{-1} \phi(x) \tag{8.11}$$

CFT's have many interesting properties, yet some are most important. First, for all CFT's, one is able to find a set of primary operators that transform under conformal transformations

$$O'(x') = \lambda^{-\Delta} O(x) \tag{8.12}$$

with $\Delta$ a conformal dimension of the primary operator $O$. If this quantity is real and positive, and for $O$ a scalar field, it obeys $\Delta \geqslant \frac{d-2}{2}$. Such primary operators often do not have complicated correlation functions. For example, for a CFT, a scalar primary $O$ with dimension $\Delta$ has a correlation function

$$\langle \Omega | \, TO(x,t) O(0,0) \, | \Omega \rangle = \frac{1}{(|x|^2 - t^2 + i\epsilon)^{\Delta}} \tag{8.13}$$

Also, CFT's are often studied on a cylinder ($\mathbb{R} \times \mathbb{S}^{d-1}$). One can find a basis of eigenstates to generate the time translation in this metric. For CFT's, there is a bijection between local operators with dimension $\Delta$ and these eigenstate energies. [26]

The main question in studying CFT's is how the CFT leads to a proper semiclassical definition of a theory of quantum gravity.

## 8.3 The holographic principle

The holographic principle considers non-local physics. The principle suggests that degrees of freedom in a region of space are proportional to the area of its boundary, rather than proportional to its volume. At first, Susskind and 'T Hooft [43][44] came up with ideas about holography. Ideas of this principle also came from studies about the black hole entropy. The black hole entropy is proportional to the area of the horizon. Hence, if this entropy describes the amount of possible states, the holographic principle would explain why we only have to consider the area of the black hole. Still, one could become worried about very small boundaries for large regions of space, such as an enclosed universe. This is solved via replacing the enclosed universe by 'light -sheets' that lay beyond the boundary on.[23] Also by holography, it

[8] An isomorphism is mathematically defined to be a mapping between two objects, respecting in some sense the structure of the objects. For two objects, $A$ and $B$, a isomorphism is a bijection $\phi : A \rightarrow B$. For example, scalar multiplication can be respected.

has been thought that the interior of a universe could be projected on its boundary in smaller dimensions as a hologram. [23]

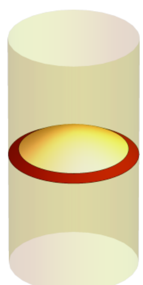
We can describe the holography of information as follows: "In a theory of quantum gravity, a copy of all the information available on a Cauchy slice is also available near the boundary of the Cauchy slice. This redundancy in description is already visible in the low-energy theory." [38]. According to the holographic principle, all the information in the 'bulk space' is is encoded on the boundary of this space. So, in a quantum theory of gravity, it should be possible to describe its properties in a boundary theory. [44] Of course, from the principle, the question rose in which way the exact information in the bulk is encoded in its boundary. [45].

Most importantly, the AdS/CFT-correspondence is a major example of the holographic principle. In this theory, Ads/CFT is stated as "a large limit $N$ of a Conformal Field Theory (CFT) in d-dimensional Minkowski space which can be described by string theory (or M-theory) on $AdS_{d+1} \times K$ with $K$ a suitable compact space. " [45]. The exact relation between the theories can be stated as [46]

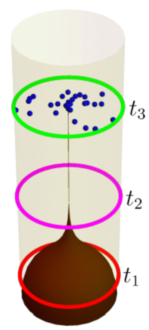
$$Z_{AdS}[\phi_0] = \int_{\phi_0} D\phi \exp(-I[\phi]) = Z_{CFT}[\phi_0] =< \exp\left(\int_{\partial\Omega} d^d x O \phi_0\right) > \tag{8.14}$$

here $\phi_0$ is the value of $\phi$ at the boundary. On the right, $\phi_0$ is an external current that is coupled an operator $O$ in the CFT-boundary. In this way, the correlation function of the boundary CFT is described by the partition function in $AdS_{d+1}$.

The AdS/CFT involves a lot of mathematics, which is quite difficult. We will not dive in to this here, though it would have been very interesting and I would have loved to. However, loads of research has been done in black holes and the information paradox by AdS/CFT. It has been of great importance for finding a solution to the information paradox and yet, it is still of great importance for describing black holes and its properties.



(a)




(b)

Figure 29: On the left: in AdS/CFT, a specific version of holography, all information from the 'bulk' $AdS_{d+2}$ is available on the boundary $CFT_{d+1}$. The figure shows a cauchy slice and its boundary. On the right: a black hole that forms in AdS space. At time $t_1$, the black hole has yet not been formed. This happens at time $t_2$. At time $t_3$, it has completely evaporated into Hawking radiation. [38]

## 8.4 The AdS/CFT-duality

Juan Maldacena was the first to discover this AdS/CFT correspondence, which comes from holographic considerations in string theory. [8]. In the AdS/CFT-correspondence, physical properties of quantum gravity in anti-de Sitter (AdS) space, the bulk that has $d+2$ dimensions ($AdS_{d+2}$), has corresponding properties on the boundary of AdS, a conformal field theory with $d+1$ dimensions ($CFT_{d+1}$). Specifically, there is a relativistic conformal field theory $CFT_d$ on $\mathbb{R} \times \mathbb{S}^{d-1}$, that can be seen as a quantum gravity description in asymptotically Anti-de Sitter ($AdS_{d+1} \times M$) spacetime. $M$ is some manifold here. [37][26]. So, all physical properties we measure in AdS have corresponding properties on a CFT-boundary,which is non-perturbative and well-defined. The latter is an assumption that we make at almost every gauge-gravity dualities. More important, the CFT-boundary is a holographic projection in a lower dimension for properties in the 'bulk'-AdS/CFT, which is also non-local.Thus, holography is realized in AdS/CFT, and specifically, the entanglement entropy studied in QFT is a non-local property. A non-perturbative construction such as CFT is very useful, since it can describe gravitational theories over long distances properly. For example, in this way, the interior of a universe might be projected on its boundary in smaller dimensions as a hologram.

An important question for a solution to the problem, is whether we can find an independent, non-perturbative setup of a theory about gravitation, especially for gravity on long distances. Also, the question is how we can define theories of gravity in other cosmological spaces. What is important, is that the AdS/CFT duality tells us that the information is preserved, but it does not explain how this happens: the results by AdS/CFT towards the paradox lead to a belief among many researchers that information is not lost, including Hawking himself. [26][9] In a quantum theory such as CFT, it is assumed that the Page curve holds, and so we can exclude information loss, and the possibility of remnants. [47].

Still, the AdS/CFT-duality considers negatively curved anti-de Sitter space, while our own universe looks rather like positively curved de-Sitter-space, looking like an expanding sphere.

in AdS/CFT, the Hilbert space of the bulk is identical to the Hilbert space of the CFT. Also, specific symmetry generators in the CFT corerspond to bulk symmetry generators in asymptotically AdS space. [26]. At both AdS and CFT, the Hamiltonian is the same too.

In AdS, we can define boundary limits on local bulk fields. In the description of AdS/CFT, a CFT operator, that is a scalar primary $O$, has an equivalent bulk scalar field $\phi$ such that

$$\lim_{x\to\infty} r^{\Delta}\phi(t,r,\Omega) = O(t,\Omega) \tag{8.15}$$

Thus, if we connect a bulk field to a boundary, we arrive at a value that is exactly described in CFT by an operator with dimension $\Delta$.[26]

Furthermore, every CFT has an exclusive energy momentum tensor $T_{\mu\nu}$, that is, a $d$-dimensional spin two primary operator of dimension $d$. One can find an equivalent bulk property, which is the metric tensor, stated in gravitational theories. [26]

### 8.4.1 AdS/CFT and black holes

In AdS, an important property is that for a black hole, its Hawking radiation is reflected in a finite amount of time by the boundary. For small black holes, this is not relevant since the black hole can completely evaporate before the radiation has reached the boundary. However, for larger black holes, in AdS, it could be that radiation reflects back into the black hole as soon as it is emitted.[26] Small black holes in AdS seem to behave as black holes in asymptotically flat space. [38]. One can define a crossover point between a stable and unstable point for the black hole in AdS, which is when the black hole is of order $\frac{r_{AdS}}{l_p}$ smaller than the AdS radius. For a radius larger than the AdS radius, by increasing the energy, the entropy and Schwarzschild radius of the black hole both grow. Here, the black hole gains more entropy by the fact that there is less space available for its radiation.Ultimately, for a radius larger than the AdS at a sufficient large energy, "nearly all states in the CFT consist of a bulk description as a single gigantic black hole", which shows a clear understanding of the statement by Bekenstein that entropy of a black hole stand for the amount of microstates contained.[26]

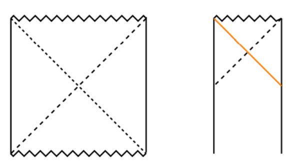


Figure 30: On the left: A Penrose diagram for a two-sided AdS-Schwarzschild wormhole. On the right: a big AdS-black hole created by collapse. [26]

Essentially, by applying AdS/CFT and holography to black holes, these two concepts tell us that the exterior of the black hole will always own a full copy of the information in the bulk. By means of AdS/CFT , the information paradox seems to be resolved: information is preserved. Even if the black hole interior would be completely evaporated, its copy will continue to exist, [38]

By studying black holes and the information paradox in AdS/CFT, we hope to learn more about the information paradox in a theory of quantum gravity.

### 8.4.2 AdS/CFT and ER=EPR

Similar to the symimptotically flat space, in the full AdS-Schwarzschild geometry, two asymptotic parts in space are connected by a wormhole.[48]. Here, the exterior parts are asymptotically AdS regions. To see this more cleary, we define a specific AdS-form of Kruskal coordinates:

$$U = -e^{\frac{r_* - t}{2}} f'(r_s) \tag{8.16}$$

$$V = e^{\frac{r_* + t}{2}} f'(r_s) \tag{8.17}$$

Here, $f'(r_s)$ is a function necessary to make sure both coordinates $U$ and $V$ are real under analytic continuation. [26].The main difference between the normal Kruskal coordinates, is that there are two boundaries to be found at $UV = -1$. To describe the metric, we define $U = T - X$ and $V = T + X$:

$$ds^2 = 4\frac{f(r)}{f'(r_s)} e^{-r_* f'(r_s)}(-dT^2 + dX^2) + r^2 d\Omega_{d-1}^2 \tag{8.18}$$

At the horizon $UV = 0$, the geometry is smooth. The metric can be described by a AdS-Penrose diagram. For the AdS-wormhole that connects the two asymptotic regions, one can choose a proper ground state, which is the Hartle-Hawking state. By the fact that the geometry consists of two asymptotically AdS boundaries, it is suggested that this wormhole is described by a state in the Hilbert space of two CFT copies. [49][26] We can define this state as

$$|\psi_{HH}\rangle = \frac{1}{Z}\sum_i e^{-\beta\frac{H}{2}} |i^*\rangle_L |i\rangle_R \tag{8.19}$$

Here $|i\rangle_R$ is an energy eigenstate of one of the CFT copies, and $|i^*\rangle_L = \Theta |i\rangle_R$ is created by the antiunitary operatur $\Theta$ that relates both CFT's. Also, by this operator, the time direction of each CFT. One is able to compute the state by an Euclidean path integral. This state is frequently named as the thermofield double state. [37]. The CFT definition of such a wormhole is quite stunning, and has some interesting properties. The Hamiltonian of the total system is the sum of both CFT Hamiltonians, without any connection between the two CFT's. More special, the two systems might have a description in which a connecting geometry appears in which two different observers could meet each other in the center of the wormhole. By this, the theory of ER=EPR has been proposed, relating geometry to entanglement.

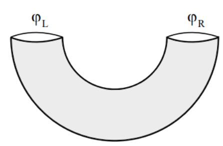



Figure 31: The Hartle-Hawking state in CFT. Here, the CFT is defined on the boundary of the geometry: fields are described by CFT field descriptions at $\mathbb{S}^{d-1}$ at both ends. [26]

## 8.5 The Ryu-Takayanagi formula

From the AdS/CFT correspondence, it is proposed that one can obtain a holographic version of the entanglement entropy in quantum CFT's. Indeed, according to Ryu and Takayanagi [14], the entanglement entropy in $d + 1$ dimensional CFT can be found by looking at a $d$-dimensional minimal surface area in $AdS_{d+2}$. This has some analogies between the computation of the black hole entropy as proposed by Bekenstein and Hawking. In accordance with this, the entropy computed via the method in [14] gives the right entropy in two-dimensional CFT for a 'bulk' $AdS_3$.

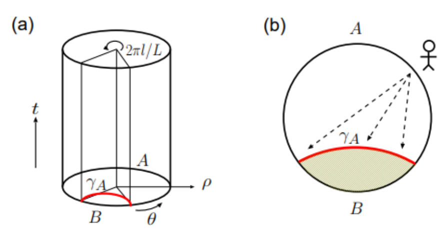



Figure 32: On the right side, in figure (a), the $AdS_3$-space with a boundary $CFT_2$.Here, $\gamma_A$ is a holographic screen as seen by an observer from subsystem A. [14]

We know that the gravitational Bekenstein-Hawking entropy of a black hole is given by its entropy

$$S_{BH} = \frac{\text{Area of the horizon}}{4G_n} \tag{8.20}$$

with $G_n$ the Newtonian gravitational constant. This expression seems to present a connection between "gravitational entropy and the degeneracy of quantum field theory as its microscopic description."[14]. Hence, this formula can be seen as a specific example of a more general form in AdS/CFT.

We can define an entanglement entropy $S_{EE}$ in a certain CFT on $\mathbb{R}^{1,d}$ or $\mathbb{R} \times S^d$, for a subsystem 1 with a boundary $\partial A \in \mathbb{R}^d$ (or $\mathbb{S}^d$) that has $d-1$ dimensions. Then, in AdS/CFT, the 'area law' is given by

$$S_{EE,1} = \frac{Area(\gamma_A)}{4G_N^{(d+2)}} \tag{8.21}$$

here $\gamma_A$ is a $d$ dimensional minimal surface in $AdS_{d+2}$. Its boundary is defined by $\partial A$. Also, $4G_N^{(d+2)}$ is the $d+2$ dimensional Newtonian gravitational constant. Here, it looks like the minimal surface acts like a holographic screen for an observer from the studied subsystem $S_1$. For example, in $AdS_3$, the minimal surface $\gamma_A$ is given by a geodesic line. Furhtermore, basic properties such as subadditivity, $S_1 + S_2 \geqslant S_{1\cup 2}$ and $S_1 = S_2$ are satisfied. In higher dimensions, for example in $AdS_5 \times \mathbb{S}^5$, the formulation of the Ryu-Takayanagi area law also holds.

By the Ryu-Takayanagi, we can compute the fine-grained or von-Neumann entropy by a extremal surface in the bulk. This extremal surface is a codimenisonal two extremal-area, which means that the area has two dimensions less than the full spacetime. Also, the surface must obey $\partial\gamma_A = \partial A$. For more than two of such surfaces, one should pick the smallest.

The Ryu-Takayanagi proposal has been proven to be important in different kinds of research, for example in studying two-sided AdS Schwarzschild geometries. [50]

# 9 The island formula

As we have seen, the information paradox is a fundamental problem in quantum gravity. By the fact that Hawking radiation behaves as thermal radiation, the entanglement entropy outside the black hole naively seems to keep on increasing. A major question in the information problem is how one can derive the Page curve from the entanglement entropy of the Hawking radiation.

From last years, new research has proposed a definite solution, without the usage of firewalls, fuzzballs or other complicated objects. By the understanding of and research on important principles and tools from AdS/CFT and holography, it has been found that the entropy of a black hole, and ultimately its Hawking radiation, can be computed from the usage of Quantum Extremal Surfaces (QES), and interestingly, this description only considers theory from general relativity, quantum mechanics and quantum field theory. [51][15]

Essentially, the description of the QES is a generalized version of the Ryu-Takayanagi definition, which, as we have seen, considers the von Neumann entropy of a subsystem in holographic quantum field theory [14] [51]. When computing the gravitational fine-grained entropy, one will encounter two different QES: a new QES, a Ryu-Takayanagi surface, that Hawking did not include in his calculations and an already known QES. Thus, compared to the calculations Hawking did, a major finding is the existence of the new QES which is found close to the shrinking black hole horizon. Also, by the new QES, the computed entropy will behave similar to the Bekenstein-Hawking entropy and the area of the evaporating black hole. So, the entropy reaches a final vale of zero after the black hole has been completely evaporated, and so the final state for the Hawking radiation will be pure. Furthermore, the QES relates entanglement to area, a geometrical property, which might give an idea about the properties of a theory of quantum gravity. By the new research, it has been proposed that the Page curve can be computed from the "island" formalism, which studies a region inside the black hole defined by the QES. The island formula can be derived from a mathematical tool called the replica trick. We will take a look at these recent studies [12][10].

## 9.1 Gravitational fine-grained entropy

In the last couple of years, researchers have developed a more complete understanding of a gravitational version of the von Neumann entropy. The description is based on an area of a surface, which is not the horizon. In contrast with the Ryu-Takayanagi formula [14], the new description of the von Neumann entropy in gravitational systems is more general. In this description, there is no need for anti-de Sitter space or holography. It is a general formula, resulting in a fine-grained entropy formula for a quantum system connected to gravity [22].

The gravitational version of the von Neumann entropy consists of a generalized entropy, which describe the black hole area and the entropy of fields outside the black hole. In this formula, we choose a surface which minimalizes the generalized entropy. In this way, the fine-grained entropy can be stated as [22]

$$S \sim min[\frac{A}{4G_N} + S_{outside}] \tag{9.1}$$

Still, the complete formula is somehow more complicated. For a gravitational definition of the entropy, we want a surface that minimizes the equation above in the spatial direction. Yet, it should maximize the equation in the time direction. By moving the surfaces in space and time, we look for specific extremal surfaces. In the case for more than one extremal surface, we should take the global minimum. From this, one arrives at a specific formula for the entropy [22]:

$$S = min_X\{ext_X[\frac{Area(X)}{4G_N} + S_{semi-cl}(\Sigma_X)]\} \tag{9.2}$$

Here, $\Sigma_X$ is a region bounded by $X$ and a cutoff surface. $S_{semi-cl}(\Sigma_X)$ is the von Neumann entropy of the quantum fields in $\Sigma_X$, which comes from the semiclassical description. $X$ is a codimension-2-surface, which is a surface that has two dimensions less that the full available spacetime. Here, the part in the brackets is the generalized entropy. This entropy obeys the second law of thermodynamics, $\Delta S_{gen} \geqslant 0$, and it increases as a result of Hawking radiation. In order to compute the entropy, we start at a surface outside the black hole, and one moves it into the horizon, looking for its minimum. Hence, the entropy relies on the geometry of the black hole interior. So, black holes with different interiors will have different von Neumann entropies. The surface that extremizes the generalized entropy is called the Quantum Extremal Surface (QES), which is a classical geometric surface in spacetime. However, it is named as a quantum surface, since the generalized entropy contains the entropy of quantum fields. It is found at an $O(G_N)$ radial distance from the horizon [10].

## 9.2 The fine-grained entropy for an evaporating black hole

We will take a look at the fine-grained entropy, applied to all stages of the black hole evaporation.

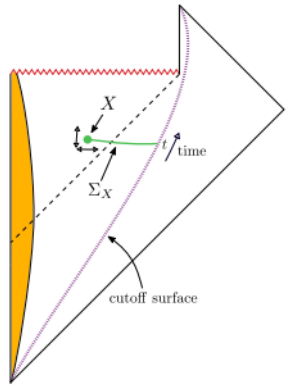



Figure 33: From the cutoff surface on, one dives into the black hole, looking for an extremal surface. $X$ represents a surface, and $\Sigma_X$ is a region between $X$ and the cutoff surface. By QES, we are able to describe a general formula for the fine-grained entropy of a quantum system connected to gravity. [22][52]

Right after the black hole has formed, when yet no Hawking radiation has escaped the black hole, one will not find an extremal surface in the black hole interior. So, when we move our surface $X$ inside the black hole, the minimal surface becomes zero. We call this the vanishing surface. Now, the area term becomes zero, and the fine-grained entropy is just the entropy of the part in the black hole from the cutoff surface on. So, at the initial stage, this means that the entropy will reduce to zero if we assume to start at a pure state. [22]

When the black hole starts to evaporate, the von Neumann entropy will increase, exactly similar to the entropy growth of the outgoing Hawking radiation. Naively seen, from here, it looks like the black hole will reach a larger entropy value than $\frac{A}{4G_N}$, which would be a problem (see chapter 6). However, this will not be the case. There appears another non-vanishing extremal surface, soon after Hawking radiation starts to escape the black hole. The location of the surface relies on the amount of radiation that has escaped, and depends on what time at the cutoff surface, one computes the entropy. [22]

One can find this surface close to the event horizon, where an ingoing light ray, shot from a time order $r_s \log S_{BH}$ back from the cutoff surface, intersects the horizon. Hence, the total generalized entropy has an area term plus a von Neumann entropy term for the quantum fields.[9]

However, the entropy for the quantum field will be small since it does not describe many outgoing particles. Therefore, approximately the area term will only be important for the generalized entropy, and this entropy takes the same path as the coarse-grained thermodynamic entropy of the black hole. So, the entropy is approximately

$$S_{gen} \approx \frac{\text{Horizon Area (t)}}{4G_N} \tag{9.3}$$

Thus, by the fact that the area decreases as Hawking radiation starts to escape the black hole, such an extremal surface causes the entropy to be lowered [22].

For the gravitational fine-grained entropy formula, we should consider the minimum of all available extremal surfaces. Specifically, we have two kinds of such surfaces: a vanishing surfaces which causes the entropy to grow, and a non-vanishing surface, which in turn causes the entropy to become lower. At the start of black hole formation , the vanishing surface is available. From here, the entropy keeps on growing until the black hole has been completely evaporated. Closely after the black hole has formed and starts to evaporate, a non-vanishing surface appears. At its starts, this surface has a large area given by the available black hole area. From the evaporation, the black hole shrinks and so does this surface [22].

From a certain point on, the change in the area of the non-vanishing extremal surface perfectly balances the change of the entropy for the quantum fields. Starting at the cutoff surface, from the horizon on, the von Neumann entropy $S_{semi-cl}$ will be lowered, since the included Hawking antiparticles have been purified by particles which moved to the region outside, and are thus already included in the region, seen from the cutoff surface on. However, after all outgoing Hawking particles have been purified by particles in the black hole interior, moving further inside the black hole would

[9]Note that the entropy contribution for the outside quantum fields, $S_{semi-cl}$, denotes the quantum entropy, which is the entanglement entropy. This is the same as the von Neumann entropy, or alternatively the fine-grained entropy. So, we use all these terms but essentially they are often the same. Still, as we will see in the definition of the entropy for the radiation, one should be careful with these definitions.

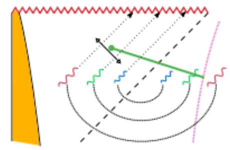


Figure 34: When moving to the inside of the black hole from the cutoff surface on, the entropy $S_{semi-cl}$ will decrease. This happens since from here, Hawking modes are included that purify the Hawking particles outside the horizon, but inside the region covered from the cutoff surface. However, $S_{semi-cl}$ will increase when moving further, at the time when all the inside particles have been purified, because extra interior particles will be included. These extra particles purify particles outside the cutoff surface, which results in an increase in the fine-grained entropy on this slice. The green line represents the area spanned between the cutoff surface and the extremal surface. [22]

result in the inclusion of even more particles inside entangled with particles outside the cutoff surface. By this, the entropy $S_{semi-cl}$ will be increased. Specifically, in the regime of this extremal surface, changes in the entropy exactly compensate for the change in area.[22]

The vanishing surface is representative for the fine-grained entropy of the black hole, until the non-vanishing surface generates a lower entropy. From this point, this surface gives the right representation for the fine-grained black hole entropy. By this, the gravitational entropy of the black hole follows the Page curve, from which one can expect unitary black hole evaporation [22].

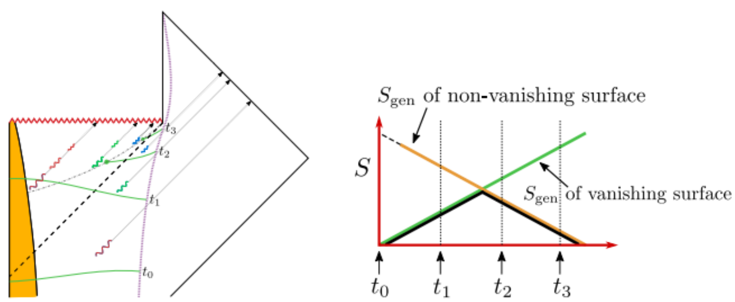



Figure 35: On the left: A Penrose diagram for a black hole from stellar collapse, showing entangled Hawking particles inside and outside the black hole horizon. Again, the green lines show the region spanned between vanishing surfaces and the cutoff surface at different times. On the right: the Page curve for the black hole fine-grained entropy, computed by the effects of a vanishing and non-vanishing surface. Up to the Page time, the vanishing surface generates the black hole fine-grained entropy. After the Page time, the non-vanishing surfaces causes the curve to bend down. Hence, by the contribution of the non-vanishing surface, unitarity is preserved. [22]

## 9.3 The fine-grained entropy for Hawking radiation

In order to solve the information paradox, one should take a look at the entropy of Hawking radiation. Though it is important that the black hole entropy follows the Page curve, the radiation entropy has been a problem since it seems to grow till the end of the black hole evaporation. Outside the cutoff surface, the entanglement entropy $S_{semi-cl}(\Sigma_{Rad})$ keeps on growing. Here, $\Sigma_{Rad}$ is the region where the outgoing radiation lives, outside the cutoff surface. In this spacetime region, gravitational effects are very small: as stated earlier, we approximate this as flat space.

For the computation of the entropy for the radiation, we need a gravitational description since we made use of gravity to determine the radiation state. In the previous section, we studied regions inside the black hole. Now, we will move our view to parts of space outside the black hole, and so outside the cutoff surface. It turns out that the gravitational entropy formula can be applied to the radiation.

By an extra area term, the semiclassical entropy contribution of the radiation can be lowered by including the black hole interior. This extra area in the black hole interior is called an "island". At late times, the island causes the entropy to decrease, since the disconnected island region is included inside the black hole. [22][15]

The fine-grained entropy for the radiation is expressed by

$$S_{Rad} = min_X\{ext_X[\frac{Area(X)}{4G_N} + S_{semi-cl}[\Sigma_{Rad} \cup \Sigma_{Island}]]\} \tag{9.4}$$

Here, the area is the boundary of the boundary. The min/ext operates on the location and appearance of the island. On the left, we have the full entropy for the radiation. The term on the right hand side $[\Sigma_{Rad} \cup \Sigma_{Island}]$ represents the von Neumann entropy of the complete radiation and island state in the semiclassical description, which state differs from the exact quantum state for the radiation. The latter is given on the left, which is the total entropy of the radiation by the gravitational fine grained formula.

For using the gravitational fine-grained radiation entropy formula, one does not need full knowledge of the exact quantum state of the radiation: the formula does not give a full quantum description of the state. Rather, it computes the entropy of it. Essentially, the formula for computing the entropy of the radiation, called the 'island formula', is just a more general form of the gravitational entropy for black holes. Therefore, it considers the same principles. Though there is no black hole involved in the formula, we have used gravity to arrive at the radiation state, so the gravitational fine-grained entropy formula can be applied in this case. To derive the formula, we will take a look at the replica trick later on.

A great advantage of the formula is the fact, as stated earlier on, that we do not need any complicated theories from holographic AdS/CFT or higher-dimensional AdS spacetime. [15] Though we do not need it for the formula here, many ideas for the principles come from AdS/CFT.

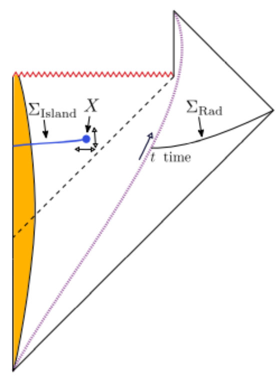



Figure 36: A Penrose diagram for the fine-grained entropy formula for the radiation. The $\Sigma_{Rad}$ and $\Sigma_{Island}$ together contribute to this entropy. The latter is the island region. The area $X$, the boundary of the island, and the union of the entropy of the two regions are important for the formula.[22]

By this formula, we want to find the entropy for the radiation that has escaped from the black hole. Hence, we study the region from the cutoff surface to infinity, which is the region $\Sigma_{Rad}$. The region $\Sigma_{Island}$ contains regions on the other side from the cutoff surface on, inside the black hole. For computing the entropy, we extremize the right part of the equation, which depends on the position of $X$. Hereafter, one minimizes the equation with respect to every available extremal position and possible island.

### 9.3.1 Island an non-island contribution

For computing the entropy, it is possible to have multiple islands. Still, the simplest possibility is having zero islands. If this is the case, we arrive at $S_{Rad} = S_{semi-cl}(\Sigma_{Rad})$, since the area of $X$ vanishes, just as the island part. In this case, by evaporating of the black hole, the entropy will keep on increasing. However, just after the black hole has formed, around a time order of $r_s \log S_{BH}$ a non-vanishing island appears. The island can be found around the origin and its boundary lies close to the event horizon. At later times, it moves closer to the cutoff surface. In the island formula, the von Neumann entropy term, defined for the outgoing radiation and the island together, is always small. This is because the island consists of all interior particles that form a pure state together with the outgoing particles in the region outside the cutoff surface, also defined in this term. Hence, what is left is the area term, defined by the black hole area. Here, for early times, the entropy has a large value by the area of the horizon. At late times, the entropy will decrease and it will reach a value of zero.

By taking the minimum of these two contributions, we obtain the fine-grained entropy for the radiation. By this, we obtain the Page curve. Here, the first part comes from the no-island contribution. From around the Page time on, the

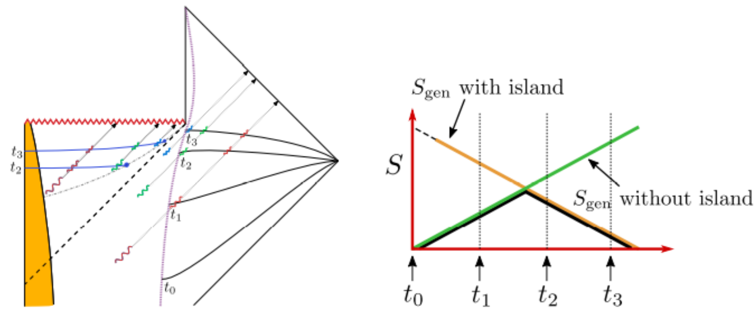



Figure 37: The curve for the entropy of Hawking radiation, computed by two the non-island and island contribution. At every moment of time, we take the minimum of these contributions. The result is the Page curve. At early times, yet there has not appeared an island. At around the Page time, the island appears and bend the entropy curve down.[22]

island-contribution bends the curve down to zero. If we start at a pure state for the black hole, it is believed that the the black hole entropy and the radiation entropy should have the same value. This is the case, since for both we have the same surface $X$. When in a pure state on the total Cauchy slice, we have $S_{semi-classical}(\Sigma_X) = S_{semi-cl}(\Sigma_{rad} \cup \Sigma_{island})$.Hence, if we now minimize and extremize both, we obtain the same function. Therefore, the entropy for the black hole and the radiation follow the same curve.

### 9.3.2 A skeptic's view

Yet, a skeptic's view would be the following: one just include the interior, and we arrive at a pure state. So, by this view, what we have done is just a simple trick to restore unitarity. However, we did not include the interior on purpose. This is because the fine-grained entropy formalism is obtained via the gravitational path integral, similar to the method of computing the entropy of a black hole by path integrals. By this, it is just gravity instructing one to take the interior into account in the equations. By this, gravity states that unitarity must be preserved, though it does not tell anything about the state of the outgoing particles.

## 9.4 Calculations on the island formula

We dive in some deeper in the island formula. In this section, we study eternal Schwarzschild black holes, which are time independent and will not disappear. Looking at a full Schwarzschild Penrose diagram, the island formula can be stated as [15]

$$S(R) = min_I\{ext[\frac{Area(\partial I)}{4G_N} + S_{matter}(R \cup I)]\} \tag{9.5}$$

Here, the region of $R$ is the region outside the black hole that contains Hawking radiation. It contains two regions $R_+$ and $R_-$. We can find the density matrix of $R$ by taking the partial trace over its complementary region $\bar{R}$. $I$ denotes an island region ($I \subset \bar{R}$). Hence, the entanglement entropy of the radiation is $S_{matter}(R \cup I)$. In the formula, we have the gravitational generalized entropy, and the matter entanglement entropy for the specified regions. The first is proportional to the total area of the island boundaries $\partial I$.

In four dimensions, in contrast to two dimensions, the matter entropy is

$$S_{matter}(R \cup I) = \frac{A(\partial I)}{\epsilon^2} + S_{finitematter}(R \cup I) \tag{9.6}$$

with $\epsilon$ a cut-off scale at a short distance. By this, we can state another version of the Newton gravitational constant

$$\frac{1}{G_N^{(r)}} = \frac{1}{4G_N} + \frac{1}{\epsilon^2} \tag{9.7}$$

We call this quantity the renormalized Newton constant. Now, for higher dimensions, the island formula becomes

$$S(R) = min\{ext[\frac{A(\partial I)}{4G_N^{(r)}} + S_{finitematter}(R \cup I)]\} \tag{9.8}$$

At late times, the distance grows between $R_+$ and $R_-$. Eventually, it will be very large. Hence, it is expected that the island contribution will be dominant.

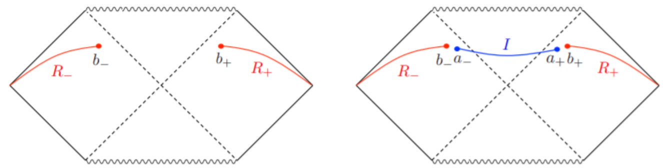



Figure 38: On the right, we have the full Penrose diagram for a Schwarzschild black hole, including an island. On the left, we have a full Penrose diagram without an island. The region $R$ contains the Hawking radiation and has two different parts $R_{\pm}$. The boundaries of these parts are defined by $b_{\pm}$.The island is found between these parts, denoted by $I$. Its boundaries are given by $a_+$ and $a_-$. [15]

In the case of no islands, we are left with the matter entanglement on $R_+$ and $R_-$. This is the minus of the mutual information

$$S_{finitematter}(R) = -I(R_+; R_-) = -[S(R_+) + S(R_-) - S(R_+ \cup R_-)] \tag{9.9}$$

At later times, we will have an island. In turns out that correlations between the two wedges are negligible, since its boundaries act as having opposite charges. [15]. Both wedges give similar results, so we look at one of the wedges. In this case, the matter entanglement entropy is

$$S_{finitematter}(R \cup I) = -2I(R_+; I) = -2[S(R_+ \cup I) - S(R_+) - S(I)] \tag{9.10}$$

Assuming that the contributions of both wedges is the same. In curved spacetime, the mutual information $I(R_+; I)$ is not known. Therefore we should make some assumptions. For two boundary surfaces $A$ and $B$, for a large distance between them, the mutual information is

$$I(A; B) = -\frac{c}{3} \log d(x, y) \tag{9.11}$$

Here, $c$ is a central charge and $d(x, y)$ the distance between $x$ and $y$ that denote the distance between the boundaries. [15] For a small distance between the surfaces, the mutual information is

$$I(A; B) = \kappa c \frac{Area}{L^2} \tag{9.12}$$

with $\kappa$ a constant, $c$ the number of free massless matter fields and $L$ the length between the two boundaries, placed parallel to each other. [15]

### 9.4.1 Case 1: no island

When no island appears, we have two points that form the boundary of the regions $R_+$ and $R_-$. Here, for late times, we have

$$S_{matter} = \frac{c}{3} \log d(b_+, b_-) \tag{9.13}$$

By some calculations, especially involving a conformal map and Kruskal coordinates, one arrives at an entropy

$$S = \frac{c}{6}[\frac{16r_h^2(b - r_h)}{b} \cosh^2 \frac{t_b}{2r_h}] \tag{9.14}$$

Here $r_h$ is the horizon radius. For late times ($t_b >> b$), one can approximate this by

$$S \approx \frac{c}{6} \frac{t_b}{r_h} \tag{9.15}$$

$t_b$ is the time coordinate for the boundary of $R$. Now, at late times, the entropy is much larger than the von Neumann entropy, which results in a contradiction. We know that the island formalism fixes this problem, and we will see this.

We have used the following here. We used coordinates $(t, r) = (t_b, b)$ for $b_+$ and $(t, r) = (-t_b + i\beta, b)$ for $b_-$. The imaginary part denotes that one is in the left wedge.[10] Furthermore, we used Kruskal coordinates and its metric:

[10] By this, the Kruskal coordinates will have an additional minus sign.

$$ds^2 = -\frac{dUdV}{W^2} + r^2\Omega^2 \tag{9.16}$$

$$r_* = r - r_h + r_h \log\frac{r - r_h}{r_h} \tag{9.17}$$

$$U = -e^{-\frac{t-r_*}{2r_h}} = -\sqrt{\frac{r-r_h}{r_h}}e^{-\frac{t-(r-r_h)}{2r_h}}, \qquad V = e^{\frac{t+r_*}{2r_h}} = \sqrt{\frac{r-r_h}{r_h}}e^{\frac{t+(r-r_h)}{2r_h}} \tag{9.18}$$

$$W = \sqrt{\frac{r}{4r_h}\frac{UV}{r-r_h}} = \sqrt{\frac{r}{4r^3}}e^{\frac{r-r_h}{2r_h}} \tag{9.19}$$

The latter is a "conformal factor" of the Schwarzschild black hole geometry [15].

### 9.4.2 Case 2: a close look

In order to look at all degrees of freedom for the radiation, it is useful to look closely to the black hole event horizon. We look at the case where the entanglement region $R$ lies close to the horizon ($b - r_h << r_h$) Hence, formulas we will use are valid ($L << a$). The exact boundaries for the island are given by $(t, r) = (t_a, a)$ for $a_+$ and $(t, r) = (-t_a + i\beta, a)$ for $a_-$. It is assumed that we can treat both wedges separately, since these are separated by a volume that grows in time. [15]

The total entropy will be

$$S \approx \frac{2\pi a^2}{G_N} - 2\kappa c\frac{4\pi b^2}{L^2} \tag{9.20}$$

The distance $L$ is defined as the geodesic distance

$$L = \int_a^b \frac{dr}{\sqrt{1-\frac{r_h}{r}}} \tag{9.21}$$

From here, the entropy must be extremized regarding the island boundary location. The enropy is extremized by a "harmonic gravitational potential" and an "attractive potential", $\frac{2\pi a^2}{G_N}$ and $-2\kappa c\frac{4\pi b^2}{L^2}$. The latter brings particles near the point $r = b$. The entropy is extremized via these contributions by considering it as a potential energy for particles found at $r = a$.

From this, the geodesic distance becomes

$$L \approx 2\sqrt{r_h}(\sqrt{b - r_h} - \sqrt{a - r_h}) \tag{9.22}$$

With respect to $a$, we minimize the entropy. We do this by a variable change $x = \sqrt{a - r_h}$. Also, we look at $\frac{\partial S}{\partial x} = 0$. This is the same as, when we approximate $x << 1$ and $b \approx r_h$, to state that

$$x\left(\sqrt{\frac{b-r_h}{r_h}} - x\right)^3 = \frac{\kappa c G_N}{2r_h^2} \tag{9.23}$$

Ultimately, by the fact that the minimization happens at small values for $x$, namely $x << \sqrt{\frac{b-r_h}{r_h}}$. Also, the right hand side of the equation above is small. By this notions, the island can be found at

$$a = r_h + \frac{(\kappa c G_n)^2}{4(b - r_h)^3} \tag{9.24}$$

Indeed, this point is very close to the black hole horizon. Putting the whole inside the entropy formula stated in this section, we arrive at

$$S = \frac{2_h^2}{G_N} - 2\pi\kappa c\frac{r_h}{b - r_h} \tag{9.25}$$

Hence, we arrive at a constant value for the entropy. In comparison to the contribution without an island, the entropy value stops to grow. Also, the term $\frac{2_h^2}{G_N}$ is two times the Bekenstein-Hawking entropy. The term $2\pi\kappa c\frac{r_h}{b-r_h}$ presents the effects of quantum matter. [15].

### 9.4.3 Case 3: a distant view

Now, we take a look at a distant view: the boundary $r = b$ is far away from the horizon: $b >> r_h$. The entropy is given by

$$S_{matter} = \frac{c}{3}\log\left(\frac{d(a_+, a_-)d(b_+, b_-)d(a_+, b_-)d(a_-, b_-)}{d(a_+, b_-)d(a_-, a_+)}\right) \tag{9.26}$$

By the same Kruskal coordinates as before, the entanglement entropy is

$$S = \frac{2\pi a^2}{G_N} + \frac{c}{6}\left[\frac{2^8 r_h^4 (a - r_h)(b - r_h)}{ab}\cosh^2\frac{t_a}{2r_h}\cosh^2\frac{t_b}{2r_h}\right] + \frac{c}{3}\log\left[\frac{\cosh\left(\frac{r_*(a)-r_*(b)}{r_h}\right) - \cosh\left(\frac{t_a-t_b}{2r_h}\right)}{\cosh\left(\frac{r_*(a)-r_*(b)}{r_h}\right) + \cosh\left(\frac{t_a+t_b}{2r_h}\right)}\right] \tag{9.27}$$

$$\text{where } \cosh\frac{r_*(a) - r_*(b)}{2r_h} = \frac{1}{2}\left[\sqrt{\frac{a - r_h}{b - r_h}}e^{\frac{a-b}{2r_h}} + \sqrt{\frac{b - r_h}{a - r_h}}e^{\frac{b-a}{2r_h}}\right] \tag{9.28}$$

From this equation, one can do a few approximations. First, we assume that the island will be near the black hole horizon, thus $a \sim r_h$. From this, the first term of the equation above can be ignored. Also, we look at a late time approximation, where the distance between the wedges is very large. Also, we take another approximation. These are stated as:

$$\frac{1}{2}\sqrt{\frac{b - r_h}{a - r_h}}e^{\frac{b-a}{2r_h}} << \cosh\frac{t_a + t_b}{2r - h} \tag{9.29}$$

$$\cosh\frac{t_a - t_b}{2r_h} << \frac{1}{2}\sqrt{\frac{b - r_h}{a - r_h}}e^{\frac{b-a}{2r_h}} \tag{9.30}$$

From here, the entropy can be written in an approximate form, which is quite complicated. It is stated by

$$S = \frac{2\pi a^2}{G_N} + \frac{c}{6}\log\left[\frac{16r_h^4(b - r_h)^2}{ab}e^{\frac{b-h}{r_h}}\right] - \frac{2c}{3}\sqrt{\frac{a - r_h}{b - r_h}}e^{\frac{a-b}{2r_h}}\cosh\frac{t_a - t_b}{2r_h} \tag{9.31}$$

This approximation involves also the fact that, at late times, the entanglement entropy is approximated by

$$S_{matter} = \frac{c}{3}\log[d(a_+, b_+)d(a_-, b_-)]$$

. What's important, from this equation, we can extract a local minimum

$$a \approx r_h + \frac{(cG_N)^2}{144\pi^2 r_h^2(b - r_h)}e^{\frac{r_h-b}{r_h}}\cosh^2\frac{t_a - t_b}{2r_h} \tag{9.32}$$

From this value of $a$, the entanglement entropy will be

$$S = \frac{2\pi r_h^2}{G_N} + \frac{c}{6}\log\frac{16r_h^3(b - r_h)^2}{b}e^{\frac{b-r_h}{r_h}} - \frac{c^2 G_N}{36\pi r_h(b - r_h)}e^{\frac{r_h-b}{r_h}}\cosh^2\frac{t_a - t_b}{2r_h} \tag{9.33}$$

The entropy is extremized at $t_a = t_b$. Now, inserting this in the entropy formula, we arrive at

$$S = \frac{2\pi r_h^2}{G_N} + \frac{c}{6}\left[\log\left(\frac{16r_h^3(b - r_h)^2}{b}\right) + \frac{b - r_h}{r_h}\right] \tag{9.34}$$

Here we ignore higher order terms of $G_N$. Also, this entropy does not grow when time becomes larger. Thus, just as in the previous section, the first term looks like the Bekenstein-Hawking term. Hence, by the island contribution, the entropy grow vanishes. Also, the island gives us a new form of the Bekenstein-Hawking entropy for Schwarzschild black holes.

### 9.4.4 implications

Above all, the same result hold in higher dimensional space, for $D \geqslant 4$. [16] We will not show this here, though it is cleverly shown in both [15] and [16]. In summary, the no-island contribution entanglement entropy grows linearly in time for $D \geqslant 4$ dimensions:

$$S = \frac{c}{6}(D - 3)\frac{t}{r_h} \tag{9.35}$$

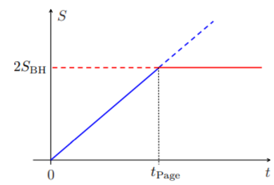



Figure 39: The Page curve for a Schwarzschild black hole. One can ignore higher order terms $\frac{cG_N}{r_h^{D-2}}$, since these are small compared to $S_{BH}$ and $t_{Page}$. Notice that, since we study eternal Schwarzschild black holes here, the Page curve will take a certain value. However, the Page curve for real black holes does bend down. [15]

For the case where an island appears at late times, by saddle point analysis[11] [11][13][15], this island can be found at the point

$$a = r_h + O(\frac{(cG_N)^2)}{r_h^{2D-5}}) \tag{9.36}$$

and its entanglement entropy will be

$$S = 2S_{BH} + O(c) \tag{9.37}$$

with $S_{BH}$ The Bekenstein-Hawking entropy, and $O(c)$ comes from quantum effects of the matter. [15]

As we have seen in the previous sections, the contribution that is dominant will be the one with minimal entropy, and at early times, the non-island contribution will be dominant. After the emergence of an island, the island contribution will take over. We find the Page time by equating 9.36 to 9.37 and from this, the Page time for a Schwarzschild black hole becomes

$$t_{Page} = 3\frac{\Omega_{D-2}}{D-3}\frac{r_h^{D-1}}{cG_N} + O(r_h) \tag{9.38}$$

Higher order corrections will depend on $b$, but the first, leading term is universal [15]. Using the Hawking temperature $T_H = \frac{D-3}{4\pi r_h}$, we arrive at

$$t_{Page} = \frac{3}{\pi}\frac{S_{BH}}{cT_H} \tag{9.39}$$

The entropy is equal to $2S_{BH}$ after the Page time, which is the case in both wedges of the full Schwarzschild Penrose diagram. So, looking at one of the wedges, it is approximately equal to $S_{BH}$, which one would expect.

Furthermore, which we will not derive here but is shown in [15], the estimate scrambling time is

$$t_{scr} \approx 2r_h \log\left(\frac{r_h^2}{G_N}\right) \approx \frac{1}{2\pi T_H}\log S_{BH} \tag{9.40}$$

which is indeed proportional to the predicted scrambling time $\frac{1}{T_H}\log S_{BH}$ [53][34].

## 9.5 The entanglement wedge

Another important question is whether the degrees of freedom, that describe the black hole from the outside, are sufficient to describe the black hole interior. Essentially, we have three options here: the degrees of freedom do or do not describe the interior, or just a part of the interior.

The formula for the fine-grained entropy for the black hole should present us the entropy, obtained from a density matrix which defines the black hole seen from the region outside. What is important, it that the entropy depends on the geometry from the cutoff surface up to the extremal surface. Hence, if we put a certain property with a random state between the extremal surface and the cutoff surface, the fine-grained entropy will change. So, if the entropy describes the exact state of the radiation, operators acting in the radiation entanglement wedge will influence the exact state.

By this observation, it has been predicted that the black hole degrees of freedom describe a part of the the black hole interior. This part is exactly the part between the cutoff surface till the minimal surface. From this, we can define a causal diamond for the region, which is named as the entanglement wedge. This wedge can be seen as the fine-grained entropy region.[10][12]

As we have seen, before the Page time, we have a vanishing surface. This surface is found at the origin, so the entanglement wedge will cover a huge part of the black hole interior, see the figure. After the Page time, we have

[11]This subject will be treated in the section 9.6.

different entanglement wedges. One describes the degrees of freedom of the black hole, the other for the radiation. Interestingly, the degrees of freedom for the black hole only define a small part of the black hole interior. Another significant part of the interior is defined by a wedge that belongs to the radiation. The latter follows from the definition of the fine-grained entropy. Since a serious part of the black hole interior belongs to the radiation at late times, the radiation entropy will change if properties in the region are changed.After the black hole has been completely evaporated, the entanglement wedge of the radiation will include the full black hole interior. The interior has become flat space.

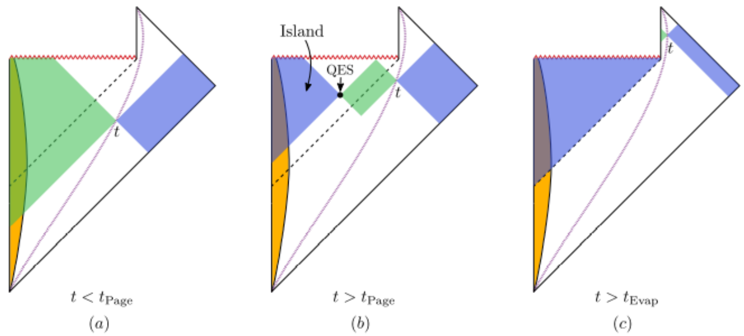



Figure 40: In green, we have the entanglement wedge of the black hole. In blue, we have the entanglement wedge of the Island. At later times, after the Page time, the black hole interior is largely described by the entanglement wedge of the island. Here, only a small portion of the interior is described by the black hole degrees of freedom. After complete evaporation, the black hole interior is fully described by the entanglement wedge for radiation. In order to describe the white parts, we need information of the two different entanglement wedges.[22]

After all, a huge part of the black hole interior will be encoded in the radiation for times after the Page time. So, most of the black hole interior will not be described by its degrees of freedom. It will just describe a small part, which can be seen in the figure.

Inside the entanglement wedge, it is proposed that one can read off the state of qubits inside the wedge by doing the right operations on the degrees of freedom. [12].

### 9.5.1 Hayden-Preskill, ER=EPR and BHC

Now, from the fact that early radiation is encoded in the entanglement wedge of the black hole, we can understand why, after the Page time, we need a scrambling time of $\frac{\beta}{2\pi}\log(S_{BH})$ to recover the information. This is shown in the figure below. The figure shows a black hole in AdS-spacetime, with a boundary CFT. Furthermore, the fact that early

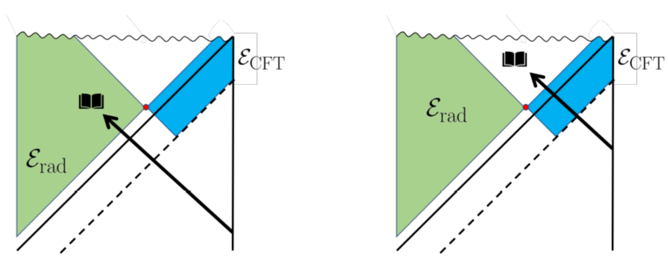



Figure 41: For a diary that was thrown into the black hole after before the Page time, its information will be found in the entanglement wedge of the radiation inside the black hole. If the diary is thrown into the black hole after the Page time, we need to wait for a certain time before the diary can be found in the entanglement wedge. [10]

radiation is encoded inside the black hole, can be seen as a realisation of ER=EPR. It can also be seen as a realisation of BHC, since information is both stated inside and outside the black hole: BHC states that information is both reflected into and outside the horizon.

[12]These degrees of freedom are not the same as the black hole degrees of freedom mentioned earlier on. Black hole degrees of freedom are used to describe the black hole from the outside. Here, the degrees of freedom define the quantum field that live in the black hole interior. Hence, degrees of freedom that lie outside the quantum extremal in the black hole entanglement wedge surface describe the black hole, while in the radiation wedge, these degrees of freedom describe the quantum fields for the radiation

### 9.5.2 Bags of gold

According to Wheeler [54], it could be possible that certain geometries have a larger entropy than the area of their horizon would suggest. Such geometries may look like a black hole from the outside, and are referred to as "Bags of gold". Such geometries are found in general relativity as a classical solution to the Einstein equations, but conflict with the holographic principle. [55]. The formalism of the entanglement wedge solves the problem: when the interior entropy exceeds the value, a value which one would predict from the area of the neck, the entanglement wedge for the degrees of freedom for the black hole will only define a certain part of the interior. By this, the large entropy value will not be taken into account. This is similar to what happens at black holes after the Page time. Specifically, the entropy for the bag exterior is just proportional to the area of the neck.

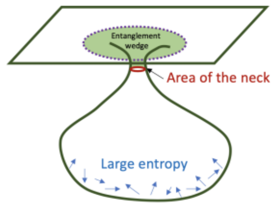



Figure 42: The geometry for a "bag of gold", stated by Wheeler [54]. Its entropy value is much highter than one would ordinarily expect from the area of its neck. In this case, the entanglement wedge of the black hole will only include a small part of the bag.[16]

## 9.6 Derivation of the island formula: the replica trick

The fine-grained entropy for the radiation can be derived by a mathematical tool called the replica trick. The trick is used to compute the von Neumann entropy in the case we do not have full knowledge of the density matrix elements $\rho_{ij}$. Essentially, replica wormholes explain how the black hole interior must be included in the definition of the fine-grained entropy for radiation. The replica wormholes come from new saddles, which connect different spacetime copies. Unitarity is recovered by including these new saddles in the gravitational path integral. [13] By the fact that the replica wormholes come from gravitational theory, and therefore it is believed that one can apply the island formalism to every black hole.[15] [13][11] At late times, the partition function of the replica geometry is dominant, and it generates the minimal entanglement entropy. By using the replica trick, one arrives at the same statements as the description for QES. [15]

Specifically, for the replica trick, we use a certain manifold $M_n$ from which we compute the $n$-th Renyi entropy. In gravitational regions, we can use all kinds of manifolds that obey the boundary conditions. [13] Then, the entropy is found by taking the path integral on this manifold. Hence, eventually, we can calculate the von Neumann entropy by "analitically continuing the Renyi entropies to $n = 1$". [11]

Recall that the total state of an evaporating black hole is given by

$$|\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\phi_i\rangle_B \, |i\rangle_R \tag{9.41}$$

Here, $|\phi_i\rangle_B$ is the black hole state, and $|i\rangle_R$ is the state of the radiation.

In gravity, we should sum over all possible topologies, in order to calculate the purity using the Euclidean path integral. By this, we must sum over various possible ways to connect the interior regions. The same is used in theories involving AdS/CFT, where the sum is needed in order to couple properties from CFT.

For example, this should be done to calculate the purity

$$\mathrm{Tr}\big(\rho_R^2\big) = \frac{1}{k^2} \sum_{i,j=1}^{k} |\langle\phi_i|\phi_j\rangle|^2 \tag{9.42}$$

$$\rho_R = \frac{1}{k} \sum_{i,j=1}^{K} |j\rangle\langle i|_R \, \langle\phi_i|\phi_j\rangle_B \tag{9.43}$$

The latter is the density matrix of a system R, for example the radiation. $k$ is the amount of possible states. Specifically, for $k << e^{S_{BH}}$, the entanglement wedge covers the complete black hole spacetime, while at $k >> e^{S_{BH}}$, an island appears. The matrix elements of the density matrix are gravity amplitudes $\langle\phi_i|\phi_j\rangle$. [11]. One can find these amplitudes

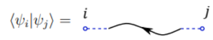



Figure 43: The boundary condition for the gravity amplitudes $\langle\phi_i|\phi_j\rangle$. [11]

by gravitational calculations with certain boundary conditions. These are given in the figure. The arrow shows the direction of time evolution, starting at the ket, moving to the bra.

A leading configuration in gravity which satisfies the boundary conditions for $\langle\phi_i|\phi_j\rangle$ is the classical Hawking solution. However, the purity $\mathrm{Tr}(\rho_R^2)$ looks like the Renyi 2-entropy. The boundary conditions for $|\langle\phi_i|\phi_j\rangle|^2$ are shown in the

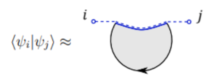



Figure 44: A classical Hawking solution that satisfies the boundary conditions.[11]

figure below. Here, we obtain two different geometries that satisfy the boundary conditions. Option one is a disconnected geometry with a topology of two disks, and the other a connected Euclidean wormhole[13] such that it is one disk. Specifically, the purity is found by summing over $i$ and $j$, connecting the dashed lines. The disconnected geometry has a

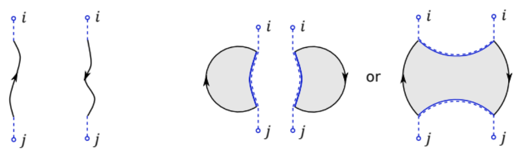



Figure 45: On the left we have the boundary conditions for $|\langle\phi_i|\phi_j\rangle|^2$. On the right, two possible geometries that satisfy the boundary conditions. [11]

so called one "k index loop".Here, two copies of the geometry define $Z_1$, while the connected geometry has two "k-index loops" $Z_2$. Here $Z_n = Z_n(\beta), n \in \mathbb{N}$ denotes the partition function via a gravitational path integral on such a disk topology.[11] , so $Z_n = Z_n[M_n]$. We can state this as

$$\mathrm{Tr}(\rho_R^2) = \frac{kZ_1^2 + k^2 Z_2}{(kZ_1)^2} = \frac{1}{k} + \frac{Z_2}{Z_1^2} \tag{9.44}$$

Here, the term in the denominator normalizes the density matrix.

For $k$ very small, the disconnected geometries are dominant. However, for $k$ very large, the connected geometry will be dominant, and by this, the entropy will stop growing at a certain point.[11]

The von Neumann entropy is calculated via the replica method in the following way:

$$S_R = -\mathrm{Tr}(\rho_R \log \rho_R) = -\lim_{n\to 1} \frac{1}{n-1} \log \mathrm{Tr}(\rho_R^n) = (1 - n\partial_n) \log \mathrm{Tr}(\rho_R^n)|_{n=1} \tag{9.45}$$

We call the right side the Renyi entropy.[14] [11] When $n$ becomes large, many different geometries satisfy the boundary conditions for the computation of $\mathrm{Tr}(\rho_n)$. In a simplified situation, we can consider two extreme limits. For $k << e^{S_{BH}}$, the disconnected geometry with $n$ disks are dominant. By a single k-index loop, this gives the result

$$\mathrm{Tr}(\rho_R^n) \supset \frac{kZ_1^n}{k^n Z_1^n} = \frac{1}{k^{n-1}} \tag{9.46}$$

In the situation $k >> e^{S_{BH}}$, the connected geometry is dominant. Now we have $n$ k-index loops. We arrive at

$$\mathrm{Tr}(\rho_R^n) \supset \frac{k^n Z_n}{k^n Z_1^n} = \frac{Z_n}{Z_1^n} \tag{9.47}$$

In order to evaluate the von Neumann entropy, we should look at the region near $n = 1$. This can be done by the fact that the $Z_n$ geometry has a "$\mathbb{Z}_n$ replica symmetry". [11]Ultimately, as the replica number moves to one, $n \to 1$, this

[13]Euclidean wormholes are spacetime wormholes. These differ from Einstein-Rosen bridges, since these are spatial wormholes

[14]In appendix F, we show that the von Neumann entropy can be calculated from the Renyi entropy.

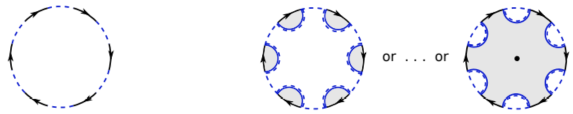



Figure 46: On the left, the boundary conditions are shown for $\mathrm{Tr}(\rho_R^6)$, $n = 6$. On the right, two geometries that satisfy the boundary conditions are shown: one disconnected, and one connected topology. The connected region has a $\mathbb{Z}_n$ symmetry, and by this, there is a fixed point which rotates the replicas. [11]

leads to the appearance of the island formula.[13].Hence, the island formalism comes from a Renyi entropy which makes use of replica wormholes. We will not show this explicitely here, though it is showed in [11]. After using the trick and taking the limit $n \to 1$, the latter computation of the von Neumann entropy reduces to the thermodynamic entropy of the black hole. In general, there are much topologies that may contribute to $\mathrm{Tr}(\rho_R^n)$. However, for $k$ small, two topologies dominate, the disconnected (Hawking) topology and the connected (replica wormhole) topology. Yet, fore $n \geqslant 2$, an analytic replica wormhole geometry has not been found.

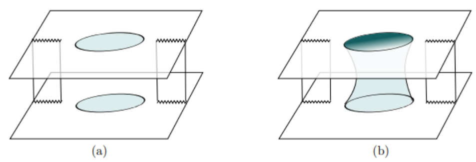



Figure 47: Another representation of the two different saddle points that satisfy the boundary conditions. By these replica's, we can compute $\mathrm{Tr}(\rho_R^2)$ . On the right, the replica is called the replica wormhole. At late times, this saddle is dominant and restores unitarity. [13]

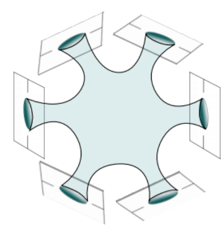


Figure 48: Another representation of the topology of a replica wormhole with $n = 6$.[13]

### 9.6.1 Computing the entropy by the replica trick: a completely evaporated black hole

Essentially, the procedure for using the replica trick can be summarized as follows. Starting at a state $|\Psi\rangle$, which for example might be a star that has collapsed. Eventually, we will arrive at a final state for an evaporating black hole. We do this by the gravitational path integral on a semiclassical geometry. Essentially, its density matrix can be obtained from a gravitational path integral. As we have seen, for the density matrix $\rho = |\Psi\rangle\langle\Psi|$, its matrix elements are

$$\rho_{ij} = \langle i|\Psi\rangle\langle\Psi|j\rangle \tag{9.48}$$

The gravitational path integral is given in the figure below. It illustrates how the trace of the density matrix and the matrix elements can be found. We use the gravitational path integral to calculate $\mathrm{Tr}(\rho_R)^n$. The path integral is dictated by a saddle point, and the obvious saddle is the Hawking saddle. Here, we we have a geometry that consists of $n$ copies of the original black hole. By this, for the von Neumann entropy, we arrive at the result Hawking obtained. Yet, another kind of saddles is given by the replica wormholes, that connects replicas of the black hole. This saddle will dominate the gravitational path integral and will restore unitarity. [13][11]

We look at the purity of the end state to see whether its entropy will reach a value of zero. For large entropies, we expect $\mathrm{Tr}(\rho^2) << [\mathrm{Tr}(\rho)]^2$. Now, $\mathrm{Tr}(\rho^2)$ is found by a path integral that connects two exterior regions.

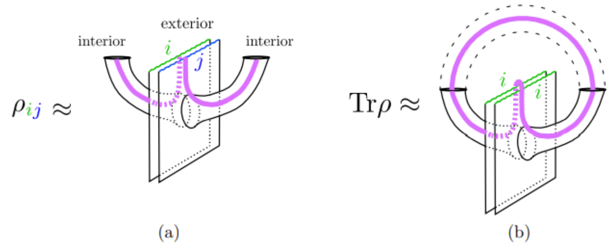



Figure 49: The density matrix elements can be computed by a path integral, just as the trace of the density matrix. Here, the purple lines show entanglement. We choose a topology that we use in order to compute the entropy. This topology is shown and used in the figure.[22]

We will take a look at the situation where the black hole has completely evaporated. Different examples for connecting the interior are shown in the figure below. For a Hawking saddle, we arrive at a large entropy. So we have for the Hawking saddle

$$\mathrm{Tr}\big(\rho_R^2\big)_{\text{Hawking saddle}} << [\mathrm{Tr}(\rho_R)]^2 \tag{9.49}$$

For a replica wormhole, we have

$$\mathrm{Tr}\big(\rho_R^2\big)_{\text{Wormhole saddle}} = [\mathrm{Tr}(\rho_R)]^2 \tag{9.50}$$

We arrive at zero entropy at late times: the replica wormhole contribution will overshadow the Hawking saddle contribution. Here, the replica wormhole saddle computation is leading and implies unitarity.

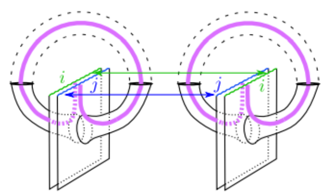


Figure 50: The Hawking saddle for a computation of $\mathrm{Tr}\big(\rho_R^2\big)$.[22]

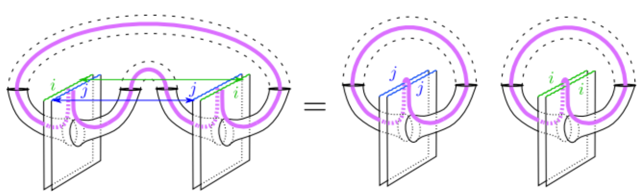


Figure 51: Replica wormhole saddle of $\mathrm{Tr}\big(\rho_R^2\big)$. Here, the black holes are connected via their interior. The right figure shows that $\mathrm{Tr}\big(\rho_R^2\big) = [\mathrm{Tr}(\rho_R)]^2$.[22]

### 9.6.2 Euclidean geometry

The Hawking saddle and replica wormhole saddle can be computed in Euclidean geometry. To compute a Hawking saddle point, we need two copies of the sigar geometry. For the replica wormhole, the black holes are connected trough its interior, which is also shown in the figure. Yet, the saddle point geometry for a replica wormhole for an evaporating black hole is rather complex, so these illustrations are simplified.

Also for calculating the von Neumann entropy, we make use of the replica method. We take a look at $n$ copies of the structure which describes the system. From this, we consider $\mathrm{Tr}(\rho_R^n)$. Here, $\rho_R$ is the density matrix for the black hole radiation, though it can also be used for the black hole itself. The entropy is given by

$$S = (1 - n\partial_n)\log \mathrm{Tr}(\rho_R^n)|_{n=1} \tag{9.51}$$

For $n \neq 1$, for $n$ copies, we can connect the black hole interior in different ways. If these interiors are completely disconnected, the result will be the entropy computed by Hawking. When fully connected, the entropy is described by the non-vanishing quantum extremal surface. Here, at the end, taking the minimum of both contributions result in the Page curve.

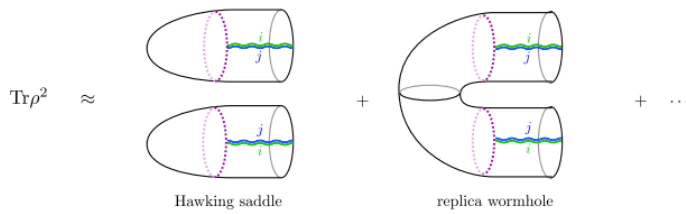



Figure 52: The Euclidean replica trick, studying the purity of radiation outside the cutoff surface. Here, the dotted lines show different kinds of topologies that might be used. In general, such topologies are not dominant.[22]

After all, it is a fun fact that it was originally believed that wormholes would destroy information. After all, exactly the opposite seems to be true. Yet, by the Euclidean path integral, we arrive at the computation of the entropy, but we did not obtain a picture for the specific Hilbert space, nor a definition for the microstates involved.

## 9.7 Firewalls revisited

In the firewall paradox, it is stated [35] that it is impossible for a particle, after the Page time, to be entangled with the interior antiparticle and the early radiation. Hence, entanglement of the particles must be broken, which creates a firewall beyond the horizon. In the AMPS statement [35], two things have been assumed. First of all, we assume that the black hole can be described as a quantum system. Here, its degrees of freedom are proportional to its horizon area, seen from the outside. Also, the time evolution for the system is unitary.

The second assumption is important. It states that the degrees of freedom which describe the black hole from the outside, are also sufficient to define the black hole interior. However, by the island formalism, we have seen that only a certain part of the black hole interior can be described by its degrees of freedom, specifically its entanglement wedge.

Hence, the solution to AMPS [35] is that the antiparticle from the interior belong to early radiation. The particle outside is entangled to this early radiation. In contrast with the solution to AMPS [35], we will have a smooth horizon.

# 10 Discussion

As we have seen, by new research, it has been possible to reproduce the Page curve. From this, unitarity is preserved and the paradox seems to be solved. In a theory of gravity, we have seen that the interior of the black hole is included in the final state of the radiation. Hence, the radiation entanglement wedge lies inside the black hole interior. From this, it looks like wormholes in the radiation reach into the black hole to get the information out. Still, important questions remain. For example, we cannot derive an expression for the exact matrix elements from the density matrix for the radiation. Still, we can compute the right entropy by the trace of the density matrix, or a function involving the density matrix. In the same way, we do not know how to compute the exact elements of the black hole S-matrix, which describe the specific evolution for each microstate. A major question in the information paradox remains how to describe the elements of both the density matrix and the S-matrix from a theory of gravity, without the usage of AdS/CFT and holography. Furthermore, we have seen that one is able to find the exact entropy of the Hawking radiation. Still, we do not know how to describe the exact state of the radiation. This is a part of the black hole information paradox that has not become completely clear yet.

Focusing on the most recent research on islands, a point of discussion is the fact that the derivation for the island formula depends on the Euclidean path integral. Yet, in a theory of gravity, it is not fully clear how this quantity is defined. The question arises which saddle points we should consider and which integration form we should use. Hence, it might be possible that unitarity is preserved in some gravitational theories, while in other theories this would not be the case.

Also, to derive the island formula, we made use of a cutoff surface. A full understanding of the island formula would allow gravity to be everywhere, also beyond the cutoff surface. However, there are important arguments that seem to validate the use of the cutoff surface. For example, in studying the black hole in AdS/CFT, this cutoff surface takes the form of the boundary CFT.

Moreover, in the fine-grained entropy formula for the radiation, we considered two different quantum states: the semiclassical state and the exact quantum state of the black hole. To arrive at the latter, we need to sum over all geometries. However, in more complicated gravity theories involving quantum fields of Hawking radiation, it is also not sure if this trick works out correctly.

It is important to mention that we have discussed results that involve only gravity. However, string theory and holography have been very important in making sure these results are correct, and are used in famous articles that form the basis for the island argument. For example, the important paper [10] describes black holes completely in AdS/CFT.

Not to mention, the result of the island formula has come from very recent research. Though the researchers have tried to present a complete resolution, it is not completely sure if the results hold to be true. In the years from Hawking's discovery on, many solutions have been proposed, and after all it remains unclear whether this will be the final answer. Still, looking at other proposed solution, we see the following. The island formalism gives a solution to the firewall proposal, and as we have seen, it somehow realises the ideas of BHC and ER=EPR. It also explains a great deal of the ideas given by Hayden and Preskill. Furthermore, the island formula seems to support the ideas of Mathur, since the island contribution is a correction of the order $O(\frac{1}{G_N})$. However, this correction is not pertubative and small, which should be the case to restore unitarity, indeed what Mathur argues. Still, the island argument differs from the idea that nothing special happens at the horizon, in contrast to Mathur's view of black holes as fuzzballs. Many scientists believe that the fuzzball view is not right. Alternatively, we have encountered Raju's argument, which states the opposite of Mathur's view. This view does not receive much support, though he might be right. Yet, what is sure, is the fact that the black hole information problem is a very poorly understood problem. Though the new research is exciting, both time and further research must show what will be a definite, complete solution to the problem.

Above all, an important motivation for studying the information paradox is to find a complete theory of quantum gravity, which is essentially the search for a better understanding in the fundaments of spacetime. Another major motivation for studying quantum gravity is the desire to gain more insights in the early days of our universe, where it is believed that quantum gravity play a crucial role. Specifically, research on the island formula presents great insights in how fundamental quantum degrees of freedom may construct the geometry of spacetime. Therefore, it has been spectacular research towards a better understanding of quantum gravity.

# A The expectation value and the projection operator

Here, we define some useful quantum mechanical properties we encounter in this thesis. We define the expectation value for an observable $A$ and a state $|\phi\rangle$ to be

$$< A >= \langle\phi| A |\phi\rangle \tag{A.1}$$

$$= \sum_{m,n} \langle\phi|m\rangle\langle m| A |n\rangle\langle n|\phi\rangle \tag{A.2}$$

by inserting a basis $\mathbb{1} = \sum_m |m\rangle\langle m|$. We can write this then as

$$< A >= \sum_{m,n} \langle n|\phi\rangle\langle\phi|m\rangle\langle m| A |n\rangle \tag{A.3}$$

$$= \sum_n \langle n| P_\phi A |n\rangle \tag{A.4}$$

$$= \mathrm{Tr}(P_\phi A) \tag{A.5}$$

$$\text{where } P_\phi = |\phi\rangle\langle\phi| \tag{A.6}$$

We call this latter quantity the projection operator.We can use this to express the full expectation value and the density matrix:

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| = \sum_i \rho_i P_{\phi_i} \tag{A.7}$$

$$< A >= \sum_i p_i < A >_i = \sum_i p_i \, \mathrm{Tr}(P_{\phi_i} A) = \sum_i \mathrm{Tr}(\rho A) \tag{A.8}$$

with orthonormal states $|\phi_i\rangle$ and $p_i$ the probabilities with same properties stated in section 2.1.

# B Derivation of the Schwarzschild metric: an explanation

The Schwarzschild metric can be derived by trial and error, and by a more, full systematic way. [23] To begin, we discover the first.

Looking at the outside of a spherical body, in vacuum, one can see that Einstein's equations reduce to $R_{\mu\nu} = 0$. We look at a hypothetic source that is static and spherically symmetric. We define 'static' here as two obey two conditions: all metric components are independent of the time coordinate. Furthermore, there are no space-time cross terms in the metric to be found: $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$. Since we would like to find a time independent solution, it has to be invariant under time changes. Thus, the cross terms have to be ruled out.

We start our investigation by writing the Minkowski metric of the studied space in polar coordinates: $x^\mu = (t, r, \theta, \phi)$.By looking at the requirements and doing some nice mathematical tricks, we finally arrive at

$$\mathrm{d}s^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)} + e^{2\gamma(r)}r^2\mathrm{d}\Omega^2 \tag{B.1}$$

Since we want to preserve spherical symmetry, this is stated by the last term. Also, the spheres have to be perfectly round. This implies certain coefficients, which are only functions of the radial coordinate.However, one can choose its radial coordinates such that the factor $\gamma(r)$ will not exist. From then on, one can use the Einstein equations to solve the coefficients $\alpha(r)$ and $\beta(r)$. Finally, in this way, we arrive at the Schwarzschild metric.

A full derivation can be given by Birkhoff.According to Birkhoff and his theorem, the Schwarzschild metric is indeed a unique vacuum solution with spherical symmetry. He also states that a time-dependent solution cannot be defined. One can prove these statements by showing multiple things. First, one needs to show that all points of the metric manifest themselves on an unique sphere which is invariant due to the creation of spherical symmetry: put simply, a spherically symmetric spacetime can be seen as two-spheres. Second, one has to prove that the metric can be stated in the form.

$$\mathrm{d}s^2 = \mathrm{d}\tau^2(a, b) + r^2(a, b)\mathrm{d}\Omega^2(\theta, \phi) \tag{B.2}$$

Third, we should put the metric into the Einstein equations in order to see that it is an unique outcome. Finally, one will arrive at the conclusion that the unique solution is the Schwarzschild metric. We will not do the full derivation here, but it is all very cleverly done in [23].

# C Penrose diagrams: bringing infinity into finite distances

We can construct a simple example of how infinity is brought to finite distances.[23] Consider the usual flat Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{C.1}$$

Here we have null trajectories at $t = \pm r$ so $-\infty < t < \infty, 0 \leqslant r < \infty$. First, we define null coordinates

$$u = t - r \tag{C.2}$$

$$v = t + r \tag{C.3}$$

Then the ranges for this coordinate system are

$$-\infty < u < \infty \tag{C.4}$$

$$-\infty < v < \infty \tag{C.5}$$

$$u \leqslant v \tag{C.6}$$

This is also seen in the figure, where each point can be seen as a 2-sphere with radius $r = \frac{1}{2}(v - u)$.

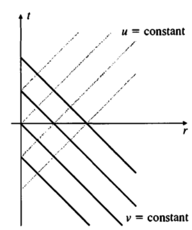



Figure 53: radial null coordinates in Minkowski space [23]

We take a step further and define coordinates to capture infinite coordinates into finite ones

$$U = \arctan(u) \tag{C.7}$$

$$V = \arctan(v) \tag{C.8}$$

These have ranges

$$-\frac{\pi}{2} < U < \frac{\pi}{2} \tag{C.9}$$

$$-\frac{\pi}{2} < V < \frac{\pi}{2} \tag{C.10}$$

$$U \leqslant V \tag{C.11}$$

After some calculations, the metric becomes

$$ds^2 = \frac{1}{4\cos^2(U)\cos^2(V)}[-2(dUdV + dVdU) + \sin^2(V - U)d\Omega^2] \tag{C.12}$$

Then, we define a time-like coordinate and a radial coordinate.

$$T = V + U \tag{C.13}$$

$$T = V - U \tag{C.14}$$

These have ranges

$$0 \leqslant R \leqslant \pi \tag{C.15}$$

$$|T| + R < \pi \tag{C.16}$$

Finally, by using the new coordinates, we arrive at the metric

$$ds^2 = w^{-2}(T, R)(-dT^2 + dR^2 + sin^2(R)d\Omega^2) \tag{C.17}$$

$$\text{where } w = \cos(T) + \cos(R) \tag{C.18}$$

Hence, we arrive at a metric where infinite coordinates have been described in terms of a finite coordinate system. Relating this to the original Minkowski metric, it seems similar except for the prefactor. In the final constructed metric, there appears to be curvature. Thus, this is something unphysical: a physical metric has to be flat spacetime, independent of the coordinates we choose. We can show the Minkowski space as a triangle. Here, the boundaries do not belong to the original spacetime. In the diagram for Minkowski spacetime, all time-like geodesics start at $i^-$ and end at $i^+$. All null geodesics start at $\mathbb{J}^-$ and will end at $\mathbb{J}^+$. The spacelike geodesics start and end at $i^0$. For Minkowski spacetime, this construction is nice, but we don't learn much new about this. Still, when considering curved spacetime, Penrose diagrams are a very useful tool to get an idea about the causal structure of the studied spacetime.

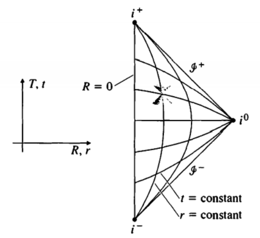



Figure 54: A Penrose diagram for Minkowski space.[23]

# D Black hole evaporation: an illustration

Here, we present an illustrative insight in the black hole evaporation. Stages in black hole evaporation are given by spatial slices in the corresponding Penrose diagram.

For the first figure: "After stellar collapse, the outside of the black hole is nearly stationary, but on the inside, the geometry continues to elongate in one direction while pinching toward zero size in the angular direction".[22]
In the second figure: "The Hawking process creates entangled pairs, one trapped behind the horizon and the other escaping to infinity where it is observed as (approximate) blackbody radiation. The black hole slowly shrinks as its mass is carried away by the radiation."[22]
In the third figure: "Eventually the angular directions shrink to zero size. This is the singularity. The event horizon also shrinks to zero."[22]
In the fourth figure: "At the end there is a smooth spacetime containing thermal Hawking radiation but no black hole. " [22]

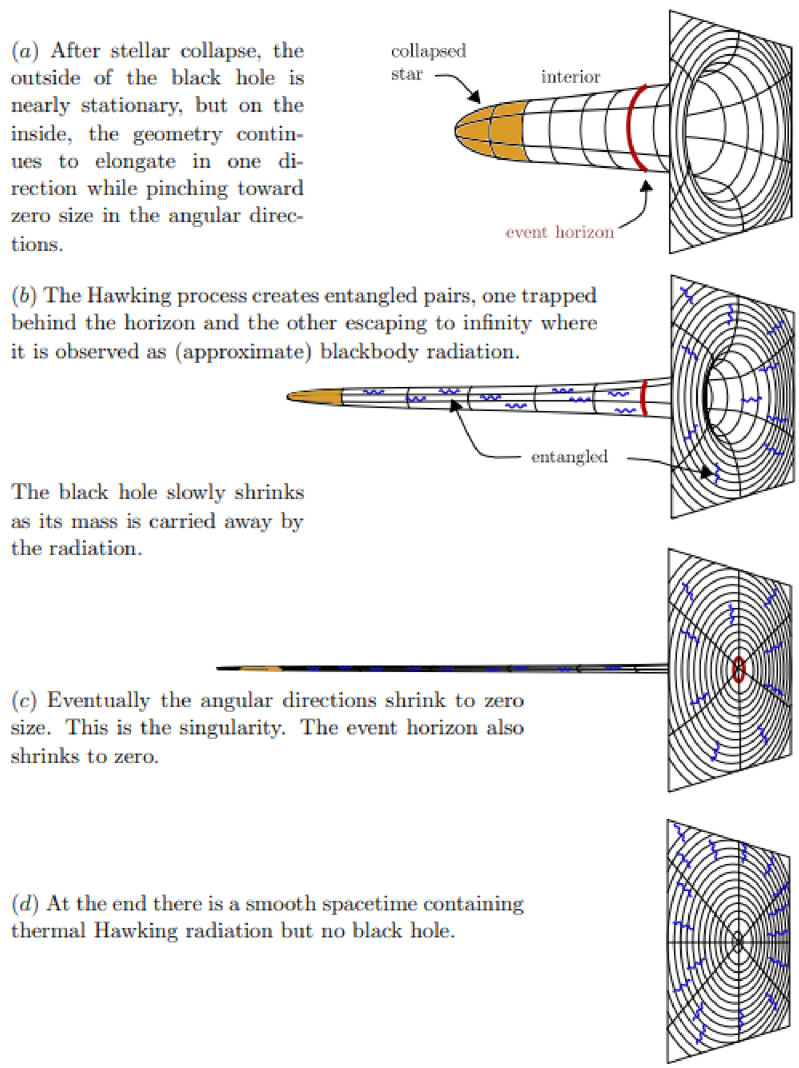



Figure 55: Stages in black hole evaporation. By the particle-antiparticle creation Hawking discovered, the black hole is able to evaporate. After complete evaporation, what is left is just flat space.[22]

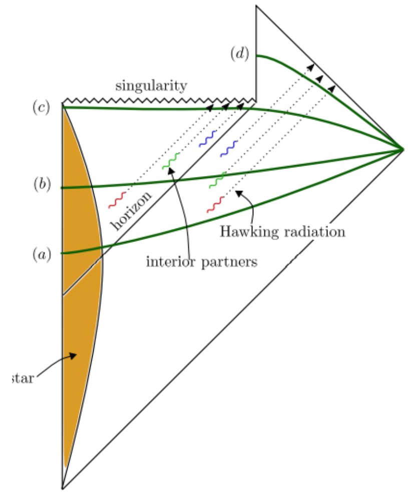



Figure 56: A Penrose diagram for an evaporating black hole that has been fromed by stellar collapse. (a) - (d) correspond to spatial slices, defining stages in black hole evaporation. Eventually, by the Hawking particle-antiparticle creation, the black hole will vanish. The Hawking particles will head to the future $\mathbb{J}^+$, where we have causal Minkowski space. [22]

# E Raju's argument

In order to give an explanation of the argument, we will follow the lines of thought given by Raju [38] in his article. Consider a system that has a density matrix

$$\rho_E = \frac{1}{W} \sum_{E_0 - \Delta < E < E_0 + \Delta} |E\rangle\langle E| \tag{E.1}$$

Here we have a sum around a certain energy $E_0$, with a with of $2\Delta$ and an energy $E$. W is a normalization that finds its origin by the number of states, and it requires the trace to be equal to one $\mathrm{Tr}(\rho_E) = 1$. Now we define a pure state. We look how close a pure state is to a mixed state, so we have to define 'closeness' more precisely.

Since the meaning of physical observations involves the probabilities of numerous measurements, which are given by the expectation value of projection operators stated earlier. Hence, it is useful to understand how much probability distributions of measurements vary between the density matrix and the pure state defined earlier. For a projection operator $P$, we take a look at the quantity

$$|\phi\rangle = \sum_E a_E |E\rangle \tag{E.2}$$

We can estimate the deviation between a pure state and a (microcanonical) mixed state by

$$\int d\mu_\phi \langle\phi| P |\phi\rangle = \frac{1}{W} \sum \langle E| P |E\rangle = \mathrm{Tr}(\rho_E P) \tag{E.3}$$

with $d\mu_\phi = \frac{1}{V}\delta(\sum_E |a_E|^2 - 1) \prod_E d^2 a_E$ a natural probability distribution with V a normalization factor, and $a_E$ a complex coefficient that can vary in an uncorrelated way if the norm of the state is unit. Finally, after doing some mathematics, the average size of the deviation is also

$$\int d\mu_\phi [\mathrm{Tr}(\rho_E P) - \langle\Phi| P |\Phi\rangle]^2 \leqslant \frac{1}{W+1} \tag{E.4}$$

Since these results have been exact all the time, one can look at a system that has a large number of degrees of freedom $W = e^S$ with S an entropy. According to Raju [38], this tells us that "random pure states are exponentially close to mixed states in a system with a large number of degrees of freedom". [38]

# F Spheres and hyperboloids

In order to get some extra insight in forms of space, which are related to the de Sitter and anti-de Sitter space, and so in the AdS/CFT-correspondence, we will take a small look at some basic manifolds here: spheres and hyperboloids. An

(a) (b)

Figure 57: On the left: a spherical model of space. Geodesics are found by the intersection of the sphere and two-planes that move through the origin. On the right: a model of space that is hyperbolic. The red surface, $\mathbb{H}^3$ is space-like. Once again, geodesics can be found by the intersection of $\mathbb{H}^3$ and two-planes through the center.[42]

$n$-dimensional sphere, $\mathbb{S}^n$ with a radius $L$, centered at the origin. It has a set of points $(x_1, ..., x_{n+1})$ in $n+1$-dimensional Euclidean space $\mathbb{E}^{n+1}$. Then its metric is defined as

$$ds^2 = \sum_{i=1}^{n+1} dx_i^2 \tag{F.1}$$

which satisfies the equation

$$\sum_{i}^{n+1} x_i^2 = L^2 \tag{F.2}$$

For a more general central point than the origin, take a point $(c_1, c_2, ..., c_{n+1})$, and this equation takes the form

$$\sum_{i}^{n+1} (x_i - c_i)^2 = L^2 \tag{F.3}$$

The $n$-dimensional sphere, geodesics can be described by the intersection of the sphere and two-planes that move through the center of the sphere. In this geometry, straight lines from flat space become maximal circles. [42]. The sphere has a positive curvature.

However, for negative curvature, we can look at an upper sheet for two layers of hyperboloid sheets $\mathbb{H}$. We define this as

$$\mathbb{H}^3 = x \in \mathbb{M}^4, x_0 - \sum_{i=1}^{3} x_i = a^2 \tag{F.4}$$

for $a \in \mathbb{R}$ In this geometry, the light cone does not meet the manifold. Hence, this surface is spacelike. Here, geodesics are hyperbolae, which are the intersection of $\mathbb{H}$ and two-planes.

# G Islands in black holes: extra illustrations

The island solution is an interesting proposal to the black hole information paradox, and it is believed that it solves the information problem. [12][10] Here, we take a look at some extra illustration that help to understand the concept.

## G.1 Vanishing and non-vanishing surface

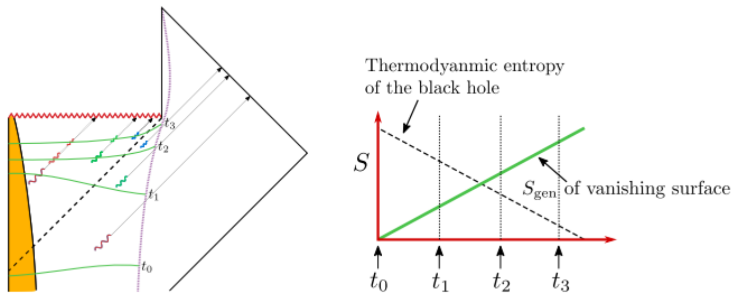



Figure 58: The contribution to the black hole entropy from the vanishing surface. The entropy grows, since more Hawking particles escape the black hole. Just as the other figures, particles that are entangled behind and outside the horizon have the same colours. The green lines represent the area spanned by the extremal surface and the cutoff surface at different times. If nothing else would happen, unitarity would be violated in this way, looking at the curve on the right. [22]

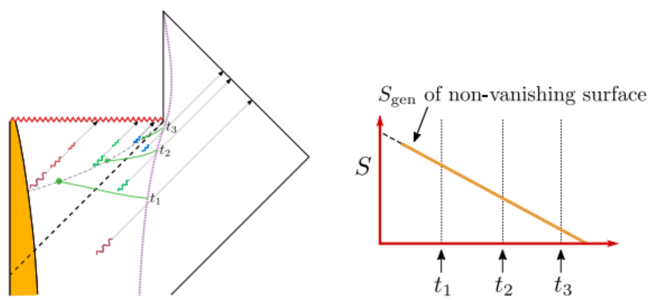



Figure 59: Around the Page time, the non-vanishing surface describes the right curve for the entropy of the black hole. At the start of black hole evaporation, it is located around the black hole horizon. At later times, the surface moves up to the event horizon in a spacelike direction. The generalized entropy will shirnk, since the quantum contribution $S_{semi-cl}$ can be neglected, and the black hole area decreases. [22]

## G.2 Non-island and island contribution

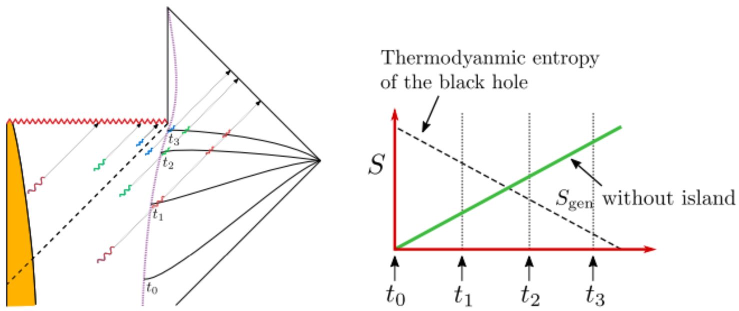



Figure 60: The contribution to the entropy when no island appears. The entropy will grow until the black hole has completely evaporated. [22]

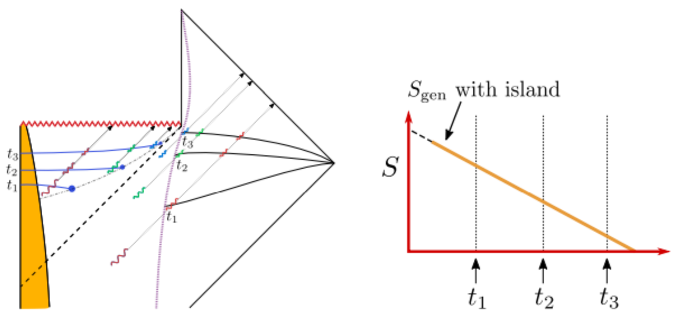



Figure 61: The contribution to the entropy when island appear. By the island formalism, the entropy decreases. It follows a same curve as the thermodynamic entropy of the black hole. [22]

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