On γ-deformations of type IIB Superstring Backgrounds

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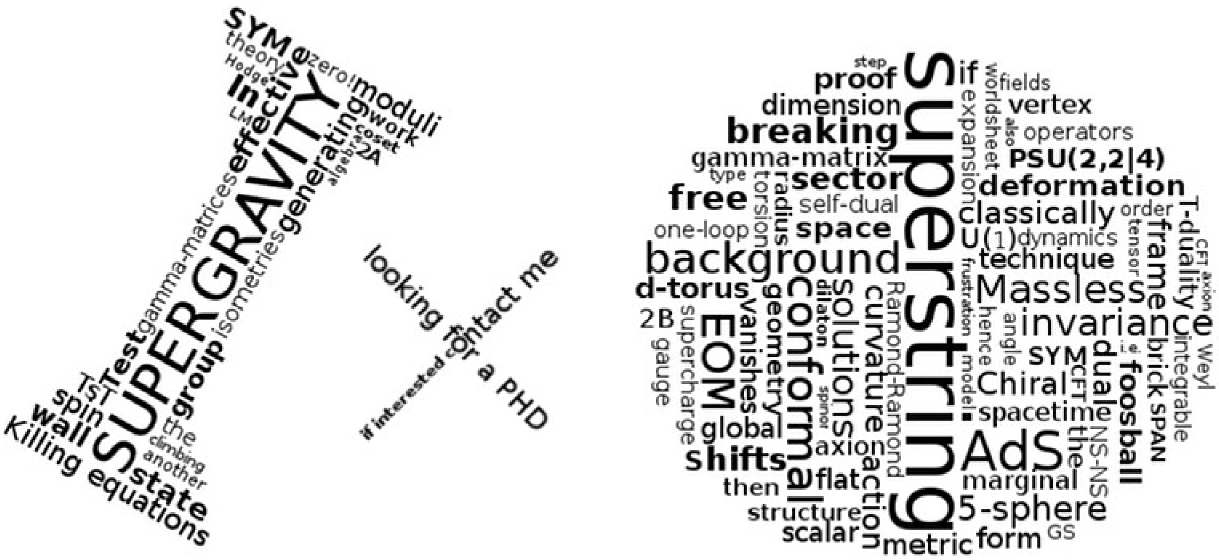

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Abstract

We consider type IIB superstring consistent backgrounds with $U(1)^d$ global isometry. The moduli-generating group for such backgrounds is $SO(d,d,\mathbb{R})$. We focus on the subgroup whose elements correspond to the γ-deformations introduced by O. Lunin and J. Maldacena. From the point of view of supergravity, these deformations can be used as a classical solution generating technique. The action of $SO(d,d,\mathbb{R})$ on the NS-NS and R-R bosonic fields of type IIB supergravity is presented. In this framework, we obtain the γ-deformed $AdS_5 \times S^5$ solution, known as the Lunin-Maldacena solution, and show subsequently that it satisfies the equations of motion derived from a type IIB supergravity covariant action. Finally, we consider several γ-deformations of a $\frac{1}{4}$-supersymmetric $AdS_2 \times S^2 \times T^6$ type IIB solution. We present these new solutions and determine their amount of unbroken supersymmetries.

Contents

Chapter 1

Introduction

Bosonic string theory on non-trivial backgrounds is described by non-linear sigma models. In order for the string to propagate consistently, the non-linear sigma model needs to remain conformally invariant at the quantum level. This condition strongly constrains the possible backgrounds. In the low-energy limit of string theory, where one can ignore the internal structure of the string and consider it as a point particle, these constraints boil down to a set of equations of motion for the background fields. It is then possible to construct a low-energy effective action corresponding to these equations of motion. In this picture, a consistent background is seen as a classical solution of a low-energy effective field theory.

In this thesis, we consider the low-energy limit of type IIB superstring theory, namely, type IIB supergravity. By analogy with the bosonic string scenario, a consistent type IIB superstring background satisfies the equations of motions of type IIB supergravity. Furthermore, we restrict ourselves to type IIB backgrounds with a $U(1)^d$ global isometry realized geometrically. The group generating the moduli space for such backgrounds has been identified in [1] as the special indefinite orthogonal group $SO(d,d,\mathbb{R})$. This means that any element of $SO(d,d,\mathbb{R})$ maps a consistent background into another consistent background. At the level of type IIB supergravity, this can be seen as a classical solution generating technique. We focus especially on the subgroup corresponding to the γ-deformations introduced by O. Lunin and J. Maldecena in [2]. The most famous examples of γ-deformations are probably the one-parameter deformation considered in [2] and the three-parameter deformation of Frolov [3]. Both of them were applied on the 5-sphere of the $AdS_5 \times S^5$ maximally supersymmetric background. The first one leads to a $\frac{1}{4}$-supersymmetric background known as the Lunin-Maldacena (LM) background, while the second deformation produces a non-supersymmetric background, sometimes referred to as Frolov's solution.

Three main reasons motivate the study of γ-deformations. The first one is naturally that they allow to generate, in a systematic way, new classes of non-trivial solutions of type IIB supergravity equations of motion. This is a remarkably powerful technique considering the complexity of these equations. Secondly, these deformations can be studied in the scope of the AdS/CFT correspondence [4], as they are known to be holographically dual to the so-called β-deformations on the gauge theory side. In general, this enables new tests of the correspondence between a string theory on a γ-deformed background and the associated β-deformed field theory. For instance, various tests have already been performed in [5] and [6] between the string theory on the LM background and its dual counterpart: the $\frac{1}{4}$-supersymmetric marginal β-deformation of $\mathcal{N}=4$ conformal Super Yang-Mills (SYM), namely a $\mathcal{N}=1$ conformal SYM. Finally, it is interesting to study the integrability properties of the string sigma model on different γ-deformed backgrounds. This has been discussed for the case of the LM background in [3] and [7].

In this work, we choose to apply these γ-deformations to a $\frac{1}{4}$-supersymmetric type IIB background whose geometry is that of the direct product $AdS_2 \times S^2 \times T^6$. It was introduced, along with several type IIA backgrounds sharing the same geometry, by Sorokin, Tseitlyn, Wulff and Zarembo in [8]. This background exhibits seven $U(1)$ isometries realized geometrically and, therefore, allows for a wide range of different γ-deformations. The geometry and the properties of the new solutions obtained in this manner, strongly depend on which part of the $AdS_2 \times S^2 \times T^6$ space we choose to deform. In particular, we derive one $\frac{1}{4}$-supersymmetric solution and two non-supersymmetric ones. Furthermore, we show that certain γ-deformations lead to solutions with an axion that cannot be gauged away.

The outline of this thesis is as follows. In chapter 2, starting from the expression of the non-linear sigma model for the bosonic closed string, we discuss how the conditions for a consistent propagation of the string, in the low-energy limit, constrain the background to satisfy the equations of motion of a low-energy effective action. In a second section, we present the field content of type IIB supergravity as well as its covariant action. In chapter 3, we start by introducing the notion of moduli space. We then focus on the case of backgrounds with $U(1)^d$ global isometries and give a representation of their associated moduli-generating group $SO(d, d, \mathbb{R})$. In particular, we describe the group action on the NS-NS and R-R fields of type IIB supergravity. We then identify the embedding of the γ-deformations and the TsT-transformations in $SO(d, d, \mathbb{R})$. Chapter 4 is dedicated to the study of the LM deformation. Using techniques described in chapter 3, we first entirely rederive the LM background which is then explicitly shown to satisfy the equations of motion of type IIB supergravity derived from the covariant action presented in chapter 2. Chapter 5 deals with the γ-deformations of the supersymmetric $AdS_2 \times S^2 \times T^6$ background. As mentioned earlier, we derive three new deformed solutions and comment on their geometry and regularity. Finally, in chapter 6, we consider in details the issue of spacetime supersymmetry breaking by γ-deformations. We first explain how, in principle, one needs to solve the Killing spinor equations in order to determine the number of supersymmetries preserved by an arbitrary type IIB background. Due to the complexity of this task for the case of a γ-deformed background, we develop another approach in which one only needs to consider the super-isometry algebra of the initial background. This allows us to study in detail the amount of unbroken supersymmetries of LM background, Frolov's solution and γ-deformed solutions obtained in chapter 5.

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Chapter 2

Supergravity as a low-energy limit of superstring theory

2.1 The closed bosonic string in curved spacetime

In this chapter, we introduce the notion of non-trivial backgrounds for the bosonic closed string. We discuss how, in the low-energy limit, a consistent propagation of the string requires the massless background fields to satisfy the equations of motion of a low-energy effective action.

2.1.1 The non-linear sigma model

For a complete introduction to the bosonic string in flat spacetime, see for example [9].

We start from the generalization of the flat Polyakov action, denoted S_P, describing a bosonic closed oriented string propagating in a D-dimensional curved spacetime:

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) \, . \tag{2.1}$$

X^μ is the coordinate of the string on the spacetime manifold (or target space) and defines an embedding of the worldsheet into the target space. $G_{\mu\nu}(X)$ is the spacetime metric while $h_{\alpha\beta}$ is the worldsheet one. The worldsheet coordinates are denoted by τ and σ. The Regge slope $\alpha' = l_s^2$ has units of spacetime-length-squared and l_s denotes the characteristic length of the string. Actions of the form (2.1) are known for historical reasons as non-linear sigma models.

At this point, the following legitimate question usually arises. By quantizing the bosonic closed string in flat spacetime, one already obtained the graviton as a massless state of the string. The curved spacetime metric $G_{\mu\nu}(X)$ should then logically be constructed out of these gravitons. How can one make this relation more explicit? To answer this question, at least schematically, let us follow most of the books and reviews on the subject (see [10] for example) and assume the following expansion of the metric

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X) \, , \tag{2.2}$$

where $\chi_{\mu\nu}(X)$ is a small fluctuation around flat spacetime. With such an expansion, the partition function Z for the action (2.1) becomes

$$Z = \int \mathcal{D}[X]\mathcal{D}[h] e^{-S_P - V} = \int \mathcal{D}[X]\mathcal{D}[h] e^{-S_{\mathrm{P}}} \left(1 - V + \frac{V^2}{2} + ...\right) \, , \tag{2.3}$$

where

$$V = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \chi_{\mu\nu}(X) \, . \tag{2.4}$$

Here, V is known as the vertex operator of the graviton state. For a graviton with polarization $\xi_{\mu\nu}$ and momentum p_μ, described by a plane wave, the small fluctuation is proportional to

$$\chi_{\mu\nu}(X) \propto \xi_{\mu\nu} e^{ip_\mu X^\mu} , \tag{2.5}$$

where $\xi_{\mu\nu}$ is a symmetric, traceless tensor. Inserting a single copy of V into the path integral (2.3) corresponds to the introduction of a single graviton state, while inserting e^V corresponds to the introduction of a coherent state of gravitons. In this sense, we see that we literally build the metric $G_{\mu\nu}(X)$ out of small fluctuations corresponding to graviton states of the string. A curved spacetime is then a coherent background of gravitons.

This procedure suggests a natural generalization of the action (2.1). Indeed, one can also include coherent backgrounds of other massless states of the bosonic closed string by exponentiating their corresponding vertex operators. The two other massless states contain the degrees of freedom of a 2-form $B_{\mu\nu}$ and a scalar ϕ called the dilaton. A detailed construction of their vertex operators is given in [11]. In the end, the action describing the propagation of a bosonic string in a background of massless fields $G_{\mu\nu}(X), B_{\mu\nu}(X)$ and $\phi(X)$, is given by

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{h} \left[\partial_\alpha X^\mu \partial_\beta X^\nu \left(h^{\alpha\beta} G_{\mu\nu}(X) + \epsilon^{\alpha\beta} B_{\mu\nu}(X)\right) + \alpha' R^{(2)} \phi(X)\right] . \tag{2.6}$$

Here, $R^{(2)}$ is the Ricci scalar of the worldsheet and $\epsilon^{\alpha\beta}$ is the antisymmetric 2-tensor normalized to $\sqrt{h}\epsilon^{01} = +1$. From now on, we define a bosonic closed string background as the collection of massless fields (also referred to as background fields) $\{G_{\mu\nu}(X), B_{\mu\nu}(X), \phi(X)\}$.

The natural question that arises now is wether a string background can be chosen arbitrarily. In other words, are there some constraints to be imposed on the background fields for the action (2.6) to properly describe the propagation of a string on such a background? As we will see in the coming subsections, it turns out that the background fields are bounded to satisfy a set of field equations. Therefore, in some low-energy limit, one would expect some of the aforementioned constraints to take the form of Einstein's equations for the spacetime metric $G_{\mu\nu}(X)$. This is actually what happens but let us first say a few words about this limit.

2.1.2 The α'-expansion

In the conformal gauge, the flat Polyakov action is a free field theory. The non-linear sigma model (2.6) describes an interacting two-dimensional field theory since its couplings are now the spacetime-dependent background fields. Let us look in more details at one of these couplings, namely the metric $G_{\mu\nu}(X)$, and expand the action around a classical solution which we take to be a string sitting at a point x_0^μ. Thus, $X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{\alpha'} Y^\mu(\tau,\sigma)$. We restrict ourselves to the term of the action containing the metric, as similar results would be obtained with the full action. The fluctuations Y^μ are dimensionless quantities since $\sqrt{\alpha'}$ has dimension of a lenght.

$$G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu = \alpha' \left(G_{\mu\nu}(x_0) + \sqrt{\alpha'} G_{\mu\nu,\rho}(x_0) Y^\rho + \frac{\alpha'}{2} G_{\mu\nu,\rho\lambda}(x_0) Y^\rho Y^\lambda + ...\right) \partial_\alpha Y^\mu \partial_\beta Y^\nu , \tag{2.7}$$

where we assumed $Y \ll 1$ and where $G_{\mu\nu,\rho_1\rho_2...\rho_n}(x_0)$ denotes the nth-derivative of the metric evaluated at x_0. The coefficients $\sqrt{\alpha'} G_{\mu\nu,\rho}(x_0), \alpha' G_{\mu\nu,\rho\lambda}(x_0), ...$ appearing in the Taylor expansion are the dimensionless coupling constants for the interactions of the fluctuations. Therefore, the theory has an infinite number of coupling constants, all of them nicely packaged into the function $G_{\mu\nu}(X)$.

In the weakly coupled limit, the interacting two-dimensional quantum field theory defined by the path integral

$$Z_\sigma = \int \mathcal{D}[X]\mathcal{D}[h] e^{-S_\sigma} , \tag{2.8}$$

can be generically studied in perturbation theory. It should be clear from our previous discussion, that such a quantum field theory can be considered as weakly coupled when all the dimensionless coupling constants appearing in (2.7) are small. Let us denote the characteristic radius of curvature of the target space by R_c. The derivative of the metric then scales as

$$\frac{\partial G}{\partial X} \sim \frac{1}{R_c} \,. \tag{2.9}$$

The dimensionless coupling constants appearing in equation (2.7) therefore scale as

$$\sqrt{\alpha'}\frac{\partial G}{\partial X} \sim \frac{\sqrt{\alpha'}}{R_c}\,, \quad \alpha'\frac{\partial^2 G}{\partial X \partial X} \sim \frac{\alpha'}{R_c^2}\,, \ldots \tag{2.10}$$

As can be seen from (2.10), the effective dimensionless coupling of the theory is $\frac{\sqrt{\alpha'}}{R_c}$ and all of the coupling constants will be small when it satisfies

$$\frac{\sqrt{\alpha'}}{R_c} \ll 1\,. \tag{2.11}$$

This means that one can use perturbation theory to study the quantum field theory (2.8) when the spacetime metric varies on scales much greater than the characteristic length of the string $\sqrt{\alpha'} = l_s$. This should naturally also be satisfied by the other background fields B and ϕ. The perturbation series in $\frac{\alpha'}{R_c}$ is usually referred to as the (small) α'-expansion. In what follows, we will assume condition (2.11) to be satisfied in order to use the α'-expansion.

Let us make a few final remarks. There exists a second perturbative expansion which goes under the name of g_s-expansion, such that, a quantity computed in string theory is given by a double perturbative expansion: one in α' and one in g_s. Indeed, one should formally also include, in the partition function (2.8), a sum over all possible topologies of the worldsheet. In this series, a term corresponding to a worldsheet of genus g is weighted by a factor $g_s^{2(1-g)}$. The expansion parameter $g_s = e^{\phi_0}$ is known as the string coupling and ϕ_0 denotes the average value of the dilaton. A detailed presentation of this topic can be found in any textbook on string theory (see for example [11]). We will, however, not deal with the g_s-expansion here as it is irrelevant for our purposes. It is also important to precise that we already assumed (2.11) to hold in order to restrict ourselves to coherent backgrounds of massless string states. Massive string states are not created when the characteristic wavelength is long compared to the string scale.

In the limit (2.11), another very useful tool can be used. Since the characteristic length scale of the background is larger than l_s, the string can be considered as a point particle. It is then possible to forget about its internal structure and derive a low-energy effective action governing the dynamics of the massless background fields.

2.1.3 The low-energy effective action

At the classical level, the two-dimensional quantum field theory defined by (2.8) is conformally invariant[1]. For general backgrounds however, this is not the case at the quantum level. The action (2.6) will define a consistent string theory only if the quantum field theory remains conformally invariant. This condition has to be fulfilled in order to avoid gauge anomalies (see [20]). The breaking of conformal symmetry at the quantum level can be generically studied in the α'-expansion. A thorough description of this phenomenon would constitute a chapter on its own and will therefore not be provided here. We merely state the results and direct the reader to [11] for a rather detailed presentation of the subject.

The condition for conformal invariance of the theory, up to order $(\alpha')^n$ in the α'-expansion, is the vanishing of the so-called beta functionals $\beta^G_{\mu\nu}, \beta^B_{\mu\nu}, \beta^\phi$, computed up to order $(\alpha')^n$. These functionals have been extensively studied in the literature (see for example [12]). Their expressions, to leading non-trivial

[1] This is only true for constant dilaton.

order in α' (known as one-loop beta functions), are

$$\beta^\phi = \frac{D-26}{48\pi^2} + \frac{\alpha'}{16\pi^2}\left(4(\nabla\phi)^2 - 4\nabla^2\phi - \mathrm{R} + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + \mathcal{O}(\alpha'^2)\,, \tag{2.12}$$

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} - \frac{\alpha'}{4}H_{\mu\rho\lambda}H^{\rho\lambda}_\nu + 2\alpha'\nabla_\mu\nabla_\nu\phi + \mathcal{O}(\alpha'^2)\,, \tag{2.13}$$

$$\beta^B_{\mu\nu} = \alpha'\nabla_\lambda H^\lambda{}_{\mu\nu} - 2\alpha'(\nabla_\lambda\phi)H^\lambda{}_{\mu\nu} + \mathcal{O}(\alpha'^2)\,. \tag{2.14}$$

These expressions are given in string frame (cf. Appendix A). The operator ∇ denotes the spacetime covariant derivative, while R and $R_{\mu\nu}$ denote, respectively, the spacetime Ricci scalar and the spacetime Ricci tensor. The 3-form H, sometimes also called torsion, is the field strength associated to the background field B

$$H_{\mu\nu\rho} = (dB)_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} = \partial_\mu B_{\nu\rho} - \partial_\nu B_{\mu\rho} - \partial_\rho B_{\mu\nu}\,. \tag{2.15}$$

For an arbitrary background, the condition for conformal invariance and thus consistent string propagation, at least up to leading non-trivial order in α', is then given by

$$\beta^\phi = \beta^G_{\mu\nu} = \beta^B_{\mu\nu} = 0\,. \tag{2.16}$$

A background satisfying (2.16) is called a consistent background.

Let us now fix $D = 26$ and rewrite the one-loop beta functions in a more familiar form. One can add equation (2.12) to (2.13) in such a way that condition (2.16) becomes

$$\beta^G_{\mu\nu} + 8\pi^2 G_{\mu\nu}\beta^\phi = R_{\mu\nu} - \frac{G_{\mu\nu}}{2}\mathrm{R} - T_{\mu\nu} = 0\,, \tag{2.17}$$

$$\beta^\phi = \frac{1}{16\pi^2}\left(4(\nabla\phi)^2 - 4\nabla^2\phi - \mathrm{R} + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) = 0\,, \tag{2.18}$$

$$\beta^B_{\mu\nu} = \nabla_\lambda H^\lambda{}_{\mu\nu} - 2(\nabla_\lambda\phi)H^\lambda{}_{\mu\nu} = 0\,. \tag{2.19}$$

Equation (2.17) now takes the usual form of Einstein's equation for the spacetime metric with the following energy momentum tensor

$$T_{\mu\nu} = \frac{1}{4}\left(H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda} - \frac{G_{\mu\nu}}{6}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right) - 2\nabla_\mu\nabla_\nu\phi + 2G_{\mu\nu}\nabla^2\phi - 2G_{\mu\nu}(\nabla\phi)^2\,. \tag{2.20}$$

Here, $T_{\mu\nu}$ is a symmetric tensor and it must be conserved since the left hand side of Einstein's equations is conserved as a result of a Bianchi identity for the Ricci tensor. For general field strength H and dilaton ϕ, (2.20) has no reason to be conserved but it is possible to verify that if (2.17), (2.18) and (2.19) are satisfied, it follows that $\nabla^\mu T_{\mu\nu} = 0$.

Equations (2.17), (2.18), (2.19) can be viewed as equations of motion for the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. One can now write a $D = 26$ dimensional spacetime action which precisely yields these equations of motion via variational principle. This action is known as the low-energy effective action of the bosonic string and is denoted here by S_1. Its expression in string frame is

$$S_1 = \frac{1}{2\kappa^2}\int d^{26}x\, e^{-2\phi}\sqrt{-G}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right)\,, \tag{2.21}$$

where x denotes a point on the spacetime manifold. The constant κ has no physical significance since it can be changed by a redenition of the dilaton, cf. Appendix A. Here, on dimensional grounds alone, it scales as $\kappa^2 \sim l_s^{24}$. The equations of motion for the dilaton and the field B derived from action (2.21) exactly match equations (2.18) and (2.19). However, a variation with respect to the metric does not, at first sight, yield equation (2.17). Let us show a way to bypass this problem. The specific form of the action in string frame allows one to rewrite the dilaton term in infinitely many different ways up to some boundary terms. This freedom is due to the factor $e^{-2\phi}$. In order to expose this property, let us first recall a relation between the covariant derivative and the normal one that will turn out to be very useful

$$\begin{aligned}\sqrt{-G}\left(\nabla^2\phi\right) &= \sqrt{-G}G^{\mu\nu}\nabla_\mu\left(\partial_\nu\phi\right) = \sqrt{-G}G^{\mu\nu}\partial_\mu\left(\partial_\nu\phi\right) - \sqrt{-G}G^{\mu\nu}\Gamma^\rho_{\mu\nu}\left(\partial_\rho\phi\right) \\ &= \partial_\mu\left(\sqrt{-G}G^{\mu\nu}\left(\partial_\nu\phi\right)\right)\,, \end{aligned} \tag{2.22}$$

where $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols defined as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}G^{\rho\lambda}\left(\partial_\mu G_{\lambda\nu} + \partial_\nu G_{\lambda\mu} - \partial_\lambda G_{\mu\nu}\right)\,. \tag{2.23}$$

Relation (2.22) leads to the following consequence for the dilaton term of the low-energy effective action

$$\begin{aligned}\int d^{26}x\, e^{-2\phi}\sqrt{-G}(\nabla^2\phi) = \int d^{26}x\, e^{-2\phi}\sqrt{-G}G^{\mu\nu}\nabla_\mu(\partial_\nu\phi) &= \int d^{26}x\, e^{-2\phi}\partial_\mu\left(\sqrt{-G}G^{\mu\nu}(\partial_\nu\phi)\right) \\ &= \text{Boundary term} - \int d^D x\,(\partial_\mu e^{-2\phi})\sqrt{-G}G^{\mu\nu}(\partial_\nu\phi) \\ &= 2\int d^{26}x\, e^{-2\phi}\sqrt{-G}G^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) \\ &= 2\int d^{26}x\, e^{-2\phi}\sqrt{-G}(\nabla\phi)^2\,. \end{aligned} \tag{2.24}$$

We consistently drop boundary terms as they do not affect[2] the equations of motion. Bearing this result in mind, action (2.21) can be written as

$$S_1 = \frac{1}{2\kappa^2}\int d^{26}x\, e^{-2\phi}\sqrt{-G}\left(R + A(\nabla\phi)^2 + \frac{B}{2}\nabla^2\phi - \frac{1}{12}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right)\,, \tag{2.25}$$

as long as $A+B=4$. We now demand its variation with respect to $G^{\mu\nu}$ to vanish. Explicitly,

$$\frac{\delta S_1}{\delta G^{\mu\nu}} = \frac{1}{2\kappa^2}\int d^{26}x\, e^{-2\phi}\frac{\delta\sqrt{-G}}{\delta G^{\mu\nu}}\left(\mathrm{R} + A(\nabla\phi)^2 + \frac{B}{2}\nabla^2\phi - \frac{1}{12}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right) \tag{2.26}$$

$$+\int d^{26}x\, e^{-2\phi}\sqrt{-G}\frac{\delta\left(\mathrm{R} + A(\nabla\phi)^2 + \frac{B}{2}\nabla^2\phi - \frac{1}{12}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right)}{\delta G^{\mu\nu}} \tag{2.27}$$

$$\begin{aligned}&= \frac{1}{2\kappa^2}\int d^{26}x\, e^{-2\phi}\left[\frac{\delta\sqrt{-G}}{\delta G^{\mu\nu}}\left(\mathrm{R} - 4(\nabla\phi)^2 + 4\nabla^2\phi - \frac{1}{12}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right)\right. \\ &\left.+\sqrt{-G}\frac{\delta\left(\mathrm{R} + 2\nabla^2\phi - \frac{1}{12}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right)}{\delta G^{\mu\nu}}\right] \end{aligned}$$

$$= \frac{1}{2\kappa^2}\int d^{26}x\, e^{-2\phi}\sqrt{-G}\left[\beta^G_{\mu\nu} + 8\pi^2 G_{\mu\nu}\beta^\phi\right] = 0\,, \tag{2.28}$$

where we choosed $A=-4$, $B=8$ in (2.26) and $A=0$, $B=4$ in (2.27). In this way, we see that the equations of motion for the metric derived from the low-energy effective action precisely coincide with Einstein's equations (2.17).

The effective action (2.21) governs the low-energy dynamics of the background fields. The appellation "low-energy" refers to the fact that we have worked with one-loop beta functions, i.e. we assumed $\frac{\alpha'}{R_c}$ to be very small in order to neglect higher order terms in the α'-expansion. If one decides to consider more quantum corrections by pushing the α'-expansion further, new terms need to be added to (2.21) to account for the higher order terms of the beta functions.

[2]This is not the case when the spacetime manifold has boundaries. In this work, we will focus closed submanifolds.

2.2 Type IIB supergravity

For a detailed introduction to superstring theory see for example [13]. We only review a few of the main results.

Consider now the type II closed superstring propagating in a $D = 10$ dimensional flat spacetime. Its massless spectrum is finite and is divided into spacetime bosons and spacetime fermions. The bosons split again into two distinct sectors: the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector and the Ramond-Ramond (R-R) sector. The former contains, just as for the bosonic closed string, the graviton, the dilaton and the two-form B. The fermions also split into two sectors: the NS-R sector and the R-NS sector. At this point, one should impose the Gliozzi-Scherk-Olive projection which essentially leads to two different theories with spacetime supersymmetric and tachyonic-free spectra. These theories are type IIA and type IIB superstring theory and their massless spectra are, respectively, that of type IIA and type IIB supergravity in ten dimensions. These two spectra share the NS-NS sector[3] previously described but differ by the content of their fermionic sector and R-R sector. In what follows, we will mainly focus on the type IIB superstring.

The type IIB superstring massless spectrum includes the degrees of freedom of:

- Spacetime bosons:
  - NS-NS sector: the metric G associated to the graviton, the dilaton ϕ and the 2-form B.
  - R-R sector: a real scalar χ (or sometimes denoted $C_{(0)}$) known as the axion, a 2-form $C_{(2)}$ and a self-dual 4-form $C_{(4)}$.
- Spacetime fermions: two Majorana-Weyl spin-$\frac{3}{2}$ spinors known as gravitinos $(\psi_{1,\mu}, \psi_{2,\mu})$ and two Majorana-Weyl spin-$\frac{1}{2}$ spinors known as dilatinos (λ_1, λ_2). The presence of pairs of spacetime fermions indicates that this theory has $\mathcal{N} = 2$ spacetime supersymmetry. The gravitinos and the dilatinos have different chirality. However, the two gravitinos as well as the two dilatinos share the same chirality. Therefore, the fermions and the theory, are said to be chiral.

The following table summarizes the massless fields of the type II superstring (supergravity) theories. The fields $C_{(1)}$ and $C_{(3)}$ are, respectively, a 1-form and a 3-form. The two gravitinos $(\tilde{\psi}_{1,\mu}, \tilde{\psi}_{2,\nu})$ as well as the two dilatinos $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ have opposite chirality.

Theory	NS-NS bosonic	R-R bosonic	Chiral fermionic	Non-chiral fermionic
Type IIA	$G_{\mu\nu}, \phi, B_{\mu\nu}$	$C_{(1)\mu}, C_{(3)\mu\nu\rho}$	-	$(\tilde{\psi}_{1,\mu}, \tilde{\psi}_{2,\nu}), (\tilde{\lambda}_1, \tilde{\lambda}_2)$
Type IIB	$G_{\mu\nu}, \phi, B_{\mu\nu}$	$C_{(0)}, C_{(2)\mu\nu}$ $C_{(4)\mu\nu\rho\lambda}$	$(\psi_{1,\mu}, \psi_{2,\mu}), (\lambda_1, \lambda_2)$	-

We would like to find the low-energy effective field theory describing the dynamics of the massless degrees of freedom of the type IIB superstring. A possible approach would be the one used in the previous section for the bosonic closed string, namely, constructing an action describing the propagation of the superstring in general massless background fields and requiring conformal invariance of the associated quantum field theory to leading order in α'. Such a procedure is, however, a lot more complicated now since one has to include coherent backgrounds of R-R and fermionic massless states by exponentiating their associated vertex operators. Indeed, the inclusion of R-R massless superstring fields in the non-linear sigma model is a very difficult task and how to do it is only known in certain cases. We will, therefore, drop this approach and follow the historical path.

Another possibility would be to compute the scattering amplitudes for the massless modes of the superstring, take the limit $\alpha' \to 0$, then construct a field theory that precisely reproduces these amplitudes. In principle, such a field theory has an expansion in powers of α' but one usually only considers the lowest-order

[3] Also known as common sector.

terms, which are also of lower order in fields derivatives. In turns out that, using spacetime supersymmetry arguments and at the level of two derivatives in the fields, the low-energy effective action is uniquely fixed. It was constructed in the late 1970's as type IIB supergravity [14], [15].

The field content of type IIB supergravity should be clear from the previous table. The bosonic fields are the NS-NS fields $\{G_{\mu\nu}, \phi, B_{\mu\nu}\}$ and the R-R fields $\{\chi, C_{(2)\mu\nu}, C_{(4)\mu\nu\rho\lambda}\}$. The fermionic fields are the chiral spinors $\{\psi_{1,\mu}, \psi_{2,\mu}, \lambda_1, \lambda_2\}$. The action governing the low-energy dynamics of these fields can be decomposed as follows

$$S_{\rm IIB} = S_{\rm IIB,\ Boson} + S_{\rm IIB, Fermion}\,, \tag{2.29}$$

where $S_{\rm IIB, Fermion}$ describes the interactions of the spacetime fermions. In what follows, we will consider purely bosonic solutions of type IIB supergravity where all the fermionic fields are consistently set to zero. It is, therefore, unnecessary to describe the fermionic part of the action (2.29). The type IIB supergravity action in string frame reduces to

$$S_{\rm IIB} = S_{\rm IIB,\ Boson} = S_{\rm II}^{\rm (NS\text{-}NS)} + S_{\rm IIB}^{\rm (R\text{-}R)} + S_{\rm IIB}^{\rm (CS)}\,, \tag{2.30}$$

with

$$S_{\rm II}^{\rm (NS\text{-}NS)} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}e^{-2\phi}\left(R + 4\,(\nabla\phi)^2 - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right)\,, \tag{2.31}$$

$$S_{\rm IIB}^{\rm (R\text{-}R)} = -\frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}\left(\frac{1}{2}F_{(1)\mu}F_{(1)}^{\mu} + \frac{1}{12}F_{(3)\mu\nu\lambda}F_{(3)}^{\mu\nu\lambda} + \frac{1}{4.5!}F_{(5)\mu\nu\lambda\delta\gamma}F_{(5)}^{\mu\nu\lambda\delta\gamma}\right)\,, \tag{2.32}$$

$$S_{\rm IIB}^{\rm (CS)} = +\frac{1}{4\kappa^2}\int\, C_{(4)}\wedge H\wedge F_{(3)}\,. \tag{2.33}$$

Here, $F_{(3)}$ is the 3-form field strength associated to the R-R field $C_{(2)}$, while the 5-form $F_{(5)}$ and the 1-form $F_{(1)}$ are respectively the field strengths associated to the R-R fields $C_{(4)}$ and χ.

$$F_{(3)\mu\nu\rho} = \left(dC_{(2)} - \chi\wedge H\right)_{\mu\nu\rho} = \partial_{[\mu}C_{(2)\nu\rho]} - \chi H_{\mu\nu\rho}\,, \tag{2.34}$$

$$F_{(5)\mu\nu\rho\lambda\sigma} = \left(dC_{(4)} - C_{(2)}\wedge H\right)_{\mu\nu\rho\lambda\sigma} = \partial_{[\mu}C_{(4)\nu\rho\lambda\sigma]} - C_{(2)[\mu\nu}H_{\rho\lambda\sigma]}\,, \tag{2.35}$$

$$F_{(1)\mu} = (d\chi)_\mu = \partial_\mu\chi\,, \tag{2.36}$$

where H is the 3-form field strength defined in (2.15). This action comes with an extra requirement that has to be implemented by hand: the 5-form $F_{(5)}$ has to be self-dual

$$F_{(5)} = F_{(5)}^{\star}\,. \tag{2.37}$$

The star denotes he Hodge dual, which will be defined later in (4.17). The equations of motion derived from the covariant action (2.30) are consistent with condition (2.37) but they do not imply it. Moreover, this condition has to be imposed on the solutions of the equations of motion rather than on the action. Indeed, if the latter is imposed on the action the wrong equations of motion will result.

The $S_{\rm II}^{\rm (NS\text{-}NS)}$ part precisely coincides with the low-energy effective action for the bosonic closed string (2.21). This is not really surprising, as the NS-NS sector of the massless spectrum of type IIB superstring theory coincides with the massless spectrum of the bosonic closed string. Actually, this part is also common to other supergravities in ten dimensions. The topological term $S_{\rm IIB}^{\rm (CS)}$ is called the Chern-Simons (CS) term. It can be rewritten in terms of fields components by introducing the fully antisymmetric tensor in 10 dimensions $\epsilon^{\mu\nu\rho\lambda\sigma\delta\eta\zeta\iota\xi}$ normalized to $\epsilon^{012\ldots 9} = 1$.

$$\begin{aligned}\int C_{(4)} \wedge H \wedge F_{(3)} &= \frac{1}{4!3!3!} \int d^{10}x\, \epsilon^{\mu\nu\rho\lambda\sigma\delta\eta\zeta\iota\xi} C_{(4)\mu\nu\rho\lambda} H_{\sigma\delta\eta} F_{(3)\zeta\iota\xi} \\ &= \frac{1}{4!3!3!} \int d^{10}x\, \epsilon^{\mu\nu\rho\lambda\sigma\delta\eta\zeta\iota\xi} C_{(4)\mu\nu\rho\lambda} (dB)_{\sigma\delta\eta} (dC_{(2)})_{\zeta\iota\xi} \\ &= \frac{1}{4!3!3!} \int d^{10}x\, \epsilon^{\mu\nu\rho\lambda\sigma\delta\eta\zeta\iota\xi} C_{(4)\mu\nu\rho\lambda} \left(\partial_{[\sigma} B_{\delta\eta]}\right) \left(\partial_{[\zeta} C_{(2)\iota\xi]}\right) \\ &= \frac{9}{4!3!3!} \int d^{10}x\, \epsilon^{\mu\nu\rho\lambda\sigma\delta\eta\zeta\iota\xi} C_{(4)\mu\nu\rho\lambda} \left(\partial_{\sigma} B_{\delta\eta}\right) \left(\partial_{\zeta} C_{(2)\iota\xi}\right) . \end{aligned} \tag{2.38}$$

Let us then summarize the action (2.33) as follows,

$$S_{\text{IIB}} = \frac{1}{2\kappa^2} \int d^{10}x\, \mathcal{L} = \frac{1}{2\kappa^2} \int d^{10}x \left[\sqrt{-G} \left(\mathcal{L}' + \mathcal{L}_{\text{Ramond}}\right) + \mathcal{L}_{\text{CS}}\right] , \tag{2.39}$$

where

$$\mathcal{L}' = e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\rho\lambda\sigma} H^{\rho\lambda\sigma}\right) , \tag{2.40}$$

$$\mathcal{L}_{\text{Ramond}} = -\frac{1}{2} F_{(1)\mu} F^{\mu}_{(1)} - \frac{1}{12} F_{(3)\mu\nu\lambda} F'^{\mu\nu\lambda}_{(3)} - \frac{1}{4.5!} F_{(5)\mu\nu\rho\lambda\sigma} F^{\mu\nu\rho\lambda\sigma}_{(5)} , \tag{2.41}$$

$$\mathcal{L}_{\text{CS}} = \frac{1}{8.4!} \epsilon^{\mu\nu\rho\lambda\sigma\delta\eta\iota\zeta\xi} C_{(4)\mu\nu\rho\lambda} \left(\partial_{\sigma} B_{\delta\eta}\right) \left(\partial_{\iota} C_{(2)\zeta\xi}\right) . \tag{2.42}$$

This is the form of the action we will use to derive the equations of motion for the bosonic background fields in section 4.3.

It should now be clear, by analogy with the bosonic string case, that a consistent type IIB superstring background to leading order in α', has to satisfy the equations of motion of type IIB supergravity and respect (2.37). Therefore, throughout the rest of this thesis, we will constantly juggle between the two appelations: consistent superstring background and supergravity solution.

As mentioned multiple times already, all of our results have been presented in string frame by opposition to Einstein frame. This is not always the case in the literature. The metrics in the two frames are related by a conformal rescaling depending on the dilaton. The expression of the bosonic part of the type IIB supergravity action in Einstein frame is presented in Appendix A.

Chapter 3

Deformation procedure: the moduli-generating group

3.1 Moduli space and solution generating technique

For an introduction to conformal field theory methods in string theory, see for example [11]. In this section we reconsider the case of the bosonic closed oriented string.

As explained in the previous chapter, string theories use as building blocks conformal field theories (CFT). Consider a given lagrangian L, which is a CFT defined on a two-dimensional manifold parametrized by the coordinates τ and σ. The couplings of the CFT are denoted by k_i. In the language of string theory, this lagrangian is a non-linear sigma model of the form (2.6), the manifold is obviously the worldsheet and the couplings are the background fields. One would like to investigate wether there exist other CFTs[1], denoted L', in the neighborhood of L. A neighborhood is defined as

$$L' = L + \sum_{i=1} \tilde{k}_i O_i(\tau, \sigma) \,, \tag{3.1}$$

where $O_i(\tau, \sigma)$ are the operators in the spectrum of the theory L, and $\tilde{k}_i$ are appropriate couplings. The goal is to look for those new couplings $\tilde{k}_i$ that can be added to L such that L' remains a CFT.

The operators $O_i(\tau, \sigma)$ of the theory can be classified into three categories:

- Operators whose mass dimension is larger than 2. These are called "irrelevant" operators. Their associated couplings have negative mass dimension.
- Operators whose mass dimension is smaller than 2. These are called "relevant" operators and their associated couplings have positive mass dimension.
- Operators whose mass dimension is exactly 2. They are called "marginal" operators. Their associated couplings are dimensionless.

We will forget about the irrelevant and relevant operators as they cannot be added to L. Indeed, their associated couplings are not dimensionless and their insertion in the theory would therefore break scale invariance. We are left with the marginal operators. Their couplings may however change under renormalization and this causes the marginal operators to subdivide into three subcategories. In the end, two of these subcategories turn out to contain relevant and irrelevant operators as their associated couplings flow towards smaller or larger values in the infrared limit. We do not give more explanations about this feature and we point the reader to [9] for more details. We only retain the remaining subcategory containing the

[1] With the same central charge.

so-called “truly marginal” operators. If L' differs from L only by the addition of truly marginal operators with appropriate couplings, then L' is a CFT. The truly marginal operators are said to form a basis for a neighborhood of conformal field theories.

The space of different conformal theories in the neighborhood of L is spanned by the corresponding couplings $\tilde{k}_i$. The space of all CFT's connected to L by truly marginal deformations is called the (connected) moduli space $\mathcal{M}$ of L. Locally around L, the moduli space reduces to the neighborhood described above. Therefore, two different points in $\mathcal{M}$ correspond to two CFT's with two different set of couplings, connected by a family of truly marginal operators.

The key point is that, in some cases, one can span the moduli space $\mathcal{M}$ by applying a continuous group $\mathcal{G}$ to L. Therefore, an element $g \in \mathcal{G}$ takes a theory L_1 at one point in $\mathcal{M}$ parametrized by the couplings k_i, to a theory L_2 corresponding to another point in $\mathcal{M}$ parametrized by new couplings k_i'

$$g : L_1\left(k_i\right) \longrightarrow L_2\left(k_i'\right) \ , \quad g \in \mathcal{G} \, . \tag{3.2}$$

As mentioned above, in the context of string theory, the couplings are the background fields. Since L_1 and L_2 are CFT's, it should be clear from the discussion of chapter 2 that the two sets of couplings k_i and k_i' correspond to two consistent string backgrounds, or equivalently to two solutions of the equations of motion (2.17), (2.18), (2.19). In what follows, we will forget about the worldsheet point of view and focus purely on the action of $\mathcal{G}$ on the couplings. Schematically,

$$\begin{gathered} g : k_i \longrightarrow k_i' \, , \quad g \in \mathcal{G} \\ \simeq \\ g : \text{Consistent string background} \longrightarrow \text{New consistent string background} \, . \end{gathered} \tag{3.3}$$

The group $\mathcal{G}$ depends on the particular choice of $\mathcal{M}$ and therefore on the type of couplings k_i, or string background, that one is considering. Although a general classification of these groups is not yet known, in some cases the group $\mathcal{G}$ has been identified and extensively studied. Given a consistent background and the group $\mathcal{G}$ spanning the associated moduli space one can generate new consistent backgrounds. The group $\mathcal{G}$ can therefore be used as a solution generating technique.

3.2 $O(d, d, \mathbb{R})$ and its action

In this section, we specify the type of backgrounds we will consider and present the group generating the associated moduli space. We then describe the action of specific group elements on the background fields.

3.2.1 Backgrounds with $\mathrm{U}(1)^d$ isometry

The CFT we consider is the non-linear sigma model (2.6) describing the propagation of the bosonic closed string on a curved background. The number of truly marginal operators for a generic D-dimensional background is D^2. These D^2 operators are composed by the $\frac{D(D+1)}{2}$ operators

$$\sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \, , \tag{3.4}$$

and the $\frac{D(D-1)}{2}$ operators

$$\epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \, , \tag{3.5}$$

corresponding to the symmetric couplings $G_{\mu\nu}(X)$ and the antisymmetric couplings $B_{\mu\nu}(X)$. For now, we disregard the dilaton. These truly marginal operators are precisely the vertex operators for the massless states of the bosonic string, which is not a surprise. Indeed, the non-linear sigma model (CFT in curved spacetime) was, roughly speaking, obtained from the Polyakov action (CFT in flat spacetime) by adding vertex operators. This process fits in the picture described by (3.1). The existence of truly marginal operators on the worldsheet corresponds to the existence of massless states in the target space. The target space tachyon corresponds to what we called a relevant operator on the worldsheet and the massive states in target space correspond to irrelevant operators.

We now restrict ourselves to a very specific type of backgrounds, namely, those which have d global $U(1)$ isometries realized geometrically. Let us parametrize the spacetime manifold by the coordinates $X = (\phi_i, x_a)$ where $i \in \{1, ..., d\}$ and $a \in \{1, ..., D-d\}$. The aforementioned condition means that all the background fields are independent of ϕ_i but may depend on x_a, and that the d global $U(1)$ isometries are realized as constant shifts of ϕ_i. The geometries of such backgrounds basically contain a d-torus, parametrized by the coordinates ϕ_i, fibered over a $(D-d)$-dimensional manifold. The d coordinates ϕ_i are usually called angle coordinates or isometry angles. These backgrounds are sometimes referred to as backgrounds with d toroidal isometries. Although they are type IIB superstring backgrounds, we can already mention that the $AdS_5 \times S^5$ and the $AdS_2 \times S^2 \times T^6$ backgrounds are of this type.

At this point it is useful to arrange the D^2 couplings of the non-linear sigma model (2.6) into one matrix $\mathcal{E}$ whose symmetric part is the spacetime metric $G_{\mu\nu}(x_a)$ and whose antisymmetric part is $B_{\mu\nu}(x_a)$

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} = \begin{pmatrix} E_{ij} & K^{(1)}_{ib} \\ K^{(2)}_{aj} & N_{ab} \end{pmatrix}, \tag{3.6}$$

where $i, j, ... = \{1, ..., d\}$ are d-torus indices. The $D \times D$ matrix $\mathcal{E}$ is usually referred to as the background matrix.

3.2.2 The $O(d,d,\mathbb{R})$ group

It has been shown in [1], that the indefinite orthogonal group $O(d,d,\mathbb{R})$ spans the moduli space of backgrounds with d toroidal isometries. This means that an element $g \in O(d,d,\mathbb{R})$ acting, in a way that remains to be defined, on a consistent string background with d toroidal isometries, described by its background matrix $\mathcal{E}$ and the dilaton ϕ, generates another consistent[2] background with d toroidal isometries described by $\mathcal{E}'$ and ϕ'. Let us now present this group and its action on the background matrix.

The group $O(d,d,\mathbb{R})$, known as T-duality group, has dimension $2d^2 - d$, and can be represented as $(2d \times 2d)$-dimensional matrices g preserving the bilinear form L

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \mathbb{1}_{d\times d} \\ \mathbb{1}_{d\times d} & 0 \end{pmatrix}, \tag{3.7}$$

where $a, b, c, d,$ are $(d \times d)$-dimensional matrices, and

$$g^t L g = L \Longrightarrow a^t c + c^t a = 0, \;\; b^t d + d^t b = 0, \;\; a^t d + c^t b = \mathbb{1}_{d\times d}\,, \tag{3.8}$$

where the superscript t stands for transposed. A useful consequence of the above condition is that if $g \in O(d,d,\mathbb{R})$, then $g^t \in O(d,d,\mathbb{R})$.

An embedding in $O(D,D,\mathbb{R})$ is given by

$$\hat{g} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \tag{3.9}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are the $D \times D$ matrices of the form

$$\hat{a} = \begin{pmatrix} a & 0 \\ 0 & \mathbb{1}_{D-d\times D-d} \end{pmatrix}; \; \hat{b} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}; \; \hat{c} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}; \; \hat{d} = \begin{pmatrix} d & 0 \\ 0 & \mathbb{1}_{D-d\times D-d} \end{pmatrix}. \tag{3.10}$$

[2]We should mention, to be more precise, that an element $g \in O(d,d,\mathbb{R})$ maps a consistent background with d toroidal isometries onto the leading order in α' of a consistent background. However, this is enough for us since such backgrounds already solve the equations of motion derived from the low-energy effective action.

It was proved, again in [1], that $\hat{g}$ acts on the background matrix $\mathcal{E}$ (3.6) as a fractional linear transformation:

$$\hat{g}:\mathcal{E}\longrightarrow\mathcal{E}'=\frac{\hat{a}\mathcal{E}+\hat{b}}{\hat{c}\mathcal{E}+\hat{d}}=\begin{pmatrix} E' & (a-E'c)K^{(1)} \\ K(2)(cE+d)^{-1} & N-K^{(2)}(cE+d)^{-1}cK^{(1)}\end{pmatrix}, \tag{3.11}$$

with

$$E'=(aE+b)(cE+d)^{-1}\,. \tag{3.12}$$

We further restrict the type of backgrounds we consider. We assume that $K^{(1)}_{ib}=0$ and $K^{(2)}_{aj}=0$ such that we can view the background as two sectors: the internal sector, whose background matrix is E, fibered over the "non-compact" or external sector whose background matrix is N. The internal sector is basically the d-torus and our sloppy appellation for the non-compact sector can be justified by the fact that it contains the time direction, which is usually non-compact. Once again, the $AdS_5\times S^5$ and the $AdS_2\times S^2\times T^6$ backgrounds fall in this category. Such backgrounds, therefore, transform under $\hat{g}$ as

$$\hat{g}:\mathcal{E}\longrightarrow\mathcal{E}'=\begin{pmatrix} E' & 0 \\ 0 & N\end{pmatrix}. \tag{3.13}$$

Since the non-compact sector is invariant under $O(d,d,\mathbb{R})$ transformations, we will forget about it. From now on, we focus entirely on the d-dimensional internal sector, i.e. the d-torus. The element $g\in\ O(d,d,\mathbb{R})$ acts as

$$g:E=G+B\longrightarrow E'=(aE+b)(cE+d)^{-1}:=G'+B'\,, \tag{3.14}$$

where G' and B' are respectively the new (or deformed) spacetime metric and B field.

Let us now describe three types of elements that generate the group $O(d,d,\mathbb{R})$:

- SL(d,$\mathbb{R}$) transformations

$$g_A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1}\end{pmatrix},\quad g_A\in SO(d,d,\mathbb{R})\subset O(d,d,\mathbb{R})\,, \tag{3.15}$$

  with $A\in GL(d,\mathbb{R})$. Under such elements, the background matrix transforms as

$$E\longrightarrow E'=AEA^t\,. \tag{3.16}$$

  From the point of view of the coordinates, this can also be seen as a simple change of isometry angles. Indeed, the line element squared of the initial background can be written as

$$ds^2=(d\phi)^tG(d\phi)=(d\phi)^t)A^{-1}G'(A^t)^{-1}(d\phi)=(d\varphi)^tG'(d\varphi)\,, \tag{3.17}$$

  where $\varphi_i=\sum_{j=1}^{d}\left((A^t)^{-1}\right)_{ij}\phi_j$ are the new isometry angles. We can interpret E' in two different ways, either as the background matrix (3.16) of a deformed background expressed in the initial coordinate basis ϕ_i, or as the background matrix (3.17) of the initial background but expressed in a new coordinate basis φ_i. This second point of view is a bit misleading. Although we will use it to make a quick remark later on, we stick with the first point of view in which the background gets deformed.

- Θ-shifts

$$g_\Theta=\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} \mathbb{1}_{d\times d} & \Theta \\ 0 & \mathbb{1}_{d\times d}\end{pmatrix},\quad g_\Theta\in SO(d,d,\mathbb{R})\subset O(d,d,\mathbb{R})\,, \tag{3.18}$$

  with Θ being an antisymmetric $d\times d$ matrix. These elements shift the B field to B+Θ.

- Factorized dualities

$$g_{D_i}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} \mathbb{1}_{d\times d}-e_i & -e_i \\ -e_i & \mathbb{1}_{d\times d}-e_i\end{pmatrix}, \tag{3.19}$$

where the $d\times d$ matrix e_i is zero except for the ii component which is equal to 1. The element g_{D_i} acts on the background as a T-duality on the circle parametrized by ϕ_i. From (3.19) and (3.14), one can easily rederive Tim Buscher's T-duality rules [16] for the components of the metric G and the B field. Indeed, if one applies a T-duality on the circle parametrized by ϕ_1,

$$\begin{aligned} E' &= ((\mathbb{1}_{d\times d}-e_1)E-e_1)\times(-e_1E+(\mathbb{1}_{d\times d}-e_1))^{-1} \\ &= \left(\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{d-1\times d-1}\end{pmatrix}\begin{pmatrix} E_{11} & E_{1k} \\ E_{l1} & E_{lk}\end{pmatrix}-\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\right) \\ &\qquad\times\left(\begin{pmatrix} -1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} E_{11} & E_{1k} \\ E_{l1} & E_{lk}\end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{d-1\times d-1}\end{pmatrix}\right)^{-1} \\ &= \begin{pmatrix} -1 & 0 \\ E_{l1} & E_{lk}\end{pmatrix}\begin{pmatrix} -E_{11} & -E_{1k} \\ 0 & \mathbb{1}_{d-1\times d-1}\end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ E_{l1} & E_{lk}\end{pmatrix}\begin{pmatrix} \frac{-1}{E_{11}} & \frac{-E_{1k}}{E_{11}} \\ 0 & \mathbb{1}_{d-1\times d-1}\end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{E_{11}} & \frac{E_{1k}}{E_{11}} \\ -\frac{E_{l1}}{E_{11}} & E_{lk}-\frac{E_{l1}E_{1k}}{E_{11}}\end{pmatrix}, \end{aligned} \tag{3.20}$$

where the indices $l,k\in\{2,...,d\}$. By definition,

$$G'_{ij}=\frac{E'_{ij}+E'_{ji}}{2}, \qquad B'_{ij}=\frac{E'_{ij}-E'_{ji}}{2}. \tag{3.21}$$

Then, from (3.20),

$$G'_{11}=\frac{1}{G_{11}}, \quad G'_{1k}=G'_{k1}=\frac{B_{1k}}{G_{11}}, \quad G'_{lk}=G_{lk}-\frac{G_{1l}G_{1k}-B_{1l}B_{1k}}{G_{11}}, \tag{3.22}$$

$$B'_{1k}=-B'_{k1}=\frac{G_{1k}}{G_{11}}, \quad B'_{lk}=B_{lk}-\frac{G_{1l}B_{1k}-B_{1k}G_{1l}}{G_{11}}. \tag{3.23}$$

It is important to mention that the determinant of g_{D_i} is equal to -1 and therefore $g_{D_i}\notin SO(d,d,\mathbb{R})$.

3.3 $SO(d,d,\mathbb{R})$ deformations

We now turn to our main interests, which are type II consistent superstring backgrounds with d toroidal isometries, or equivalently, type II supergravity solutions with d toroidal isometries. The moduli space of such backgrounds is still spanned by the group $O(d,d,\mathbb{R})$ and is now made of two disconnected components, corresponding to type IIA and type IIB backgrounds. These are mapped into each other by $O(d,d,\mathbb{R})$ elements with determinant equal to -1. This is the case of factorized dualities (3.19), which are known to take solutions of type IIB and type IIA supergravity to solutions of type IIA and type IIB supergravity[3], respectively. In what follows, we will exclusively consider type IIB solutions and therefore restrict ourselves to $SO(d,d,\mathbb{R})$, which acts within IIA or IIB. It is important to note that any even number of T-dualities is part of $SO(d,d,\mathbb{R})$.

The spacetime dimension is now $D=10$ and an element $g\in SO(d,d,\mathbb{R})$ acts on the background matrix of the d-dimensional internal sector as the fractional linear transformation (3.14). The action of $SO(d,d,\mathbb{R})$ on the R-R fields will be presented in subsection 3.3.3.

3.3.1 γ-deformations

The γ-deformations are the elements $g_\gamma\in SO(d,d,\mathbb{R})$ defined as

$$g_\gamma(\Gamma)=\begin{pmatrix} \mathbb{1}_{d\times d} & 0 \\ \Gamma & \mathbb{1}_{d\times d}\end{pmatrix}, \tag{3.24}$$

[3] T-duality flips the chirality of one gravitino and one dilatino. It naturally also transforms type IIA R-R fields into type IIB R-R fields. All of these transformation rules are explicitly presented in [17].

where Γ is an antisymmetric $d \times d$ matrix[4]. Therefore, the most general γ-deformation is a real, $\frac{d(d-1)}{2}$ parameter, deformation. From (3.11), one can see that, under γ-deformation, the background matrix transforms as

$$g_\gamma : E \longrightarrow E' = E(\Gamma E + \mathbb{1}_{d\times d})^{-1} \, . \tag{3.25}$$

These are the deformations that Lunin and Maldacena[5], as well as Frolov applied to the $AdS_5 \times S^5$ type IIB background. We will present this deformations in chapter 4 and those of the $AdS_2 \times S^2 \times T^6$ type IIB background in chapter 5.

The composition of two γ-deformations $g_\gamma(\Gamma_1)$ and $g_\gamma(\Gamma_2)$, yields another γ-deformation

$$g_\gamma(\Gamma_1).g_\gamma(\Gamma_2) = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 \\ \Gamma_1 & \mathbb{1}_{d\times d} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{d\times d} & 0 \\ \Gamma_2 & \mathbb{1}_{d\times d} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 \\ \Gamma_1 + \Gamma_2 & \mathbb{1}_{d\times d} \end{pmatrix} = g_\gamma(\Gamma_1 + \Gamma_2) \, , \tag{3.26}$$

Furthermore,

$$(g_\gamma(\Gamma))^{-1} = g_\gamma(-\Gamma) \, , \qquad g_\gamma(\Gamma = 0) : E \longrightarrow E' = E \, . \tag{3.27}$$

The γ-deformations, therefore, form a subgroup of $SO(d, d, \mathbb{R})$. One of the nice features of these deformations is that, starting from a non-singular solution, they usually generate a class of non-singular solutions.

Let us finally turn back to the dilaton. We do not discuss this issue explicitly, as this was done in [18] and [1], and merely state the result. Under γ-deformations, the dilaton transforms as

$$\phi \longrightarrow \phi' = \phi - \frac{1}{2} \left(\det\left(\Gamma E + \mathbb{1}_{d\times d}\right)\right) \, , \tag{3.28}$$

where the abbreviation “det” stands for determinant.

3.3.2 TsT-transformations

The TsT-transformations are the one-parameter γ-deformations of the form

$$g_{(T_{\phi_i} s_{\phi_j} T_{\phi_i})} = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 \\ \Gamma_{(T_{\phi_i} s_{\phi_j} T_{\phi_i})} & \mathbb{1}_{d\times d} \end{pmatrix} \, , \tag{3.29}$$

where the matrix $\Gamma_{(T_{\phi_i} s_{\phi_j} T_{\phi_i})}$ is zero except for the ij and ji component which are, respectively, equal to $-\gamma$ and γ. Here, $\gamma \in \mathbb{R}$ is the parameter of the TsT-transformation. Just to be clear, let us give an example. The TsT-transformation $g_{(T_{\phi_1} s_{\phi_2} T_{\phi_1})}$ is

$$g_{(T_{\phi_1} s_{\phi_2} T_{\phi_1})} = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 \\ \Gamma_{(T_{\phi_1} s_{\phi_2} T_{\phi_1})} & \mathbb{1}_{d\times d} \end{pmatrix} \, , \quad \text{with} \quad \Gamma_{(T_{\phi_1} s_{\phi_2} T_{\phi_1})} = \left(\begin{array}{cc|c} 0 & -\gamma & 0 \\ \gamma & 0 & 0 \\ \hline 0 & 0 & 0_{d-2\times d-2} \end{array}\right) . \tag{3.30}$$

It is obvious that a TsT-transformation $g_{(T_{\phi_i} s_{\phi_j} T_{\phi_i})}$, with parameter γ, is the same as a TsT-transformation $g_{(T_{\phi_j} s_{\phi_i} T_{\phi_j})}$ with parameter $-\gamma$.

A general TsT-transformation (3.29) can be decomposed as follows

$$g_{(T_{\phi_i} s_{\phi_j} T_{\phi_i})} = g_{D_i} . g_{A_{(ij)}} . g_{D_i} \, , \tag{3.31}$$

where $g_{A_{(ij)}}$ is the $SL(d, \mathbb{R})$ transformation defined in (3.15) and $A_{(ij)} \in GL(d, \mathbb{R})$ is the identity matrix up to a non-vanishing ij-component equal to γ. Again, just to be clear, if $i = 1, j = 2$, then

$$A_{(12)} = \left(\begin{array}{cc|c} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1_{d-2\times d-2} \end{array}\right) . \tag{3.32}$$

[4]It should not be confused with the Christoffel symbols introduced in (2.23).

[5]They actually used a different approach, based on the decomposition $SO(2, 2, \mathbb{R}) \simeq SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, which will be briefly discussed later.

As mentioned earlier, the action of $g_{A_{(ij)}}$ on the metric is $g_{A_{(ij)}}: G \to G' = A_{(ij)} G A^t_{(ij)}$. The same deformation can also be obtained by the following shift of isometry angles

$$\phi_k \longrightarrow \sum_{l=1}^{d} \left(A^t_{(ij)}\right)_{kl} \phi_l \,, \quad \forall\ k \,, \tag{3.33}$$

which is equal to

$$\phi_j \longrightarrow \gamma\phi_i + \phi_j \,, \tag{3.34}$$

$$\phi_k \longrightarrow \phi_k \,, \forall k \neq i \neq j \,. \tag{3.35}$$

Indeed, under such a shift

$$ds^2 = (d\phi)^t G (d\phi) \longrightarrow ds'^2 = (d\phi)^t A_{(ij)} G A^t_{(ij)} (d\phi) = (d\phi)^t G' (d\phi) \,. \tag{3.36}$$

This is regarded in the literature as a shift of the isometry angle ϕ_j and, therefore, the element $g_{A_{(ij)}}$ is (a bit misleadingly) referred to as a shift. It should not be confused with the Θ-shift introduced in (3.18). A TsT-transformation (3.31) is then a factorized duality on the circle parametrized by ϕ_i, followed by the $SL(d,\mathbb{R})$ transformation (or "shift") $g_{A_{(ij)}}$, and another factorized duality on the circle parametrized by ϕ_j. This explains the appellation "TsT-transformation".

Let us make a final remark. As it is clear from (3.29), a chain of two TsT-transformations with arbitrary parameters boils down to

$$g_{(T_{\phi_i} s_{\phi_j} T_{\phi_i})} \cdot g_{(T_{\phi_k} s_{\phi_l} T_{\phi_k})} = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 \\ \left(\Gamma_{(T_{\phi_i} s_{\phi_j} T_{\phi_i})} + \Gamma_{(T_{\phi_k} s_{\phi_l} T_{\phi_k})}\right) & \mathbb{1}_{d\times d} \end{pmatrix} , \tag{3.37}$$

such that any γ-deformation (3.24) can always be decomposed into a chain of TsT-transformations. In particular the most general γ-deformation can be written as a chain of $\frac{d(d-1)}{2}$ TsT-transformations (each of them with a different parameter).

3.3.3 γ-deformations of R-R fields

We describe here the action of $SO(d,d,\mathbb{R})$ on the R-R fields. It was realized in [19] that the R-R fields of type IIB supergravity combine with the NS-NS B-field in an appropriate way (see (2.36), (2.34) and (2.35)) to transform under the chiral spinor representation of $SO(d,d,\mathbb{R})$. We do not provide the details of the proof, but we rather give, following [18] very closely, a short summary of the results of [19]. However, our expressions of the field strengths $F_{(1)}, F_{(3)}, F_{(5)}$ slightly differ from those of [19] and [18] which is naturally also the case of our type IIB supergravity action. We stick with our conventions and translate their results.

First, we introduce F as the sum of the three field strengths of type IIB supergravity defined in (2.36), (2.34), (2.35):

$$F = F_{(1)} + F_{(3)} + F_{(5)} \,. \tag{3.38}$$

As a remark, one should mention the existence of the so-called democratic formulation that one sometimes encounters in the literature. It requires the introduction of the extra R-R field strengths $F_{(7)}$ and $F_{(9)}$. These are then treated as independent variables, but one should impose the following Hodge duality conditions

$$F_9^\star = F_{(1)} \,, \quad F_{(7)}^\star = F_{(3)}, \tag{3.39}$$

to match the right number of degrees of freedom. The advantage of this formulation is that one can rewrite the type IIB supergravity action[6] in a more compact and elegant way (see for example [20]). In this case, F should actually be defined as in [19]

$$F = F_{(1)} + F_{(3)} + F_{(5)} + F_{(7)} + F_{(9)} \,. \tag{3.40}$$

[6]By considering two extra R-R potentials in IIA, SUGRA one can actually cast both type IIA and type IIB action in a single expression.

However, we will consider type IIB solutions for which $F_{(1)} = F_{(3)} = 0$ and, therefore, $F_{(7)} = F_{(9)} = 0$. It will become apparent in what follows that the γ-defomations do not produce any non-vanishing $F_{(7)}$ or $F_{(9)}$ from these types of solution. Thus, equation (3.40) will always reduce to equation (3.38) and we will not deal with this alternative formulation of type IIB supergravity.

The second step is to introduce $2d$ fermionic operators, ψ_i and $\psi^{\dagger i}$, satisfying the usual anti-commutation relations

$$\{\psi_i, \psi^{\dagger j}\} = \delta_i{}^j \mathbf{1} \ , \ \ \{\psi_i, \psi_j\} = \{\psi^{\dagger i}, \psi^{\dagger j}\} = 0 \ , \qquad i, j, \in \{1, ..., d\} \, , \tag{3.41}$$

with $(\psi_i)^\dagger = \psi^{\dagger i}$. We construct a $2d$-dimensional Fock space $\mathcal{F}$ spanned by the states $|\alpha\rangle \in \mathcal{F}$ defined as

$$|\alpha\rangle = \psi^{\dagger i_1} .... \psi^{\dagger i_n} |0\rangle \ , \quad n \in \{0, ..., d\} \, , \tag{3.42}$$

where $|0\rangle$ is the vacuum, normalized to $\langle 0|0\rangle = 1$, such that $\psi_i|0\rangle = 0$. The operator $\mathbf{1}$ in (3.41) denotes the identity map on $\mathcal{F}$. In (3.42), α is a multi-index $\alpha = (i_1, ..., i_n)$ with $i_1 < ..... < i_n$.

The trick is now, still following [19], to use the one-to-one correspondence between the set of differential forms and the space of creation operators $\psi^{\dagger i}$, under which a general differential form Ω

$$\Omega = \sum_{n=1}^{d} \Omega_{i_1 ... i_n} d\phi_{i_1} \wedge ... \wedge d\phi_{i_n} = \sum_{q=1}^{10-n} \sum_{n=1}^{d} \Omega^{(q)}_{i_1 ... i_n} d\phi_{i_1} \wedge ... \wedge d\phi_{i_n} \, , \tag{3.43}$$

is mapped to the following operator

$$\mathbf{\Omega} \equiv \sum_{n=1}^{d} \Omega_{i_1 ... i_n} \psi^{\dagger i_1} \wedge ... \wedge \psi^{\dagger i_n} = \sum_{q=1}^{10-n} \sum_{n=1}^{d} \Omega^{(q)}_{i_1 ... i_n} \psi^{\dagger i_1} \wedge ... \wedge \psi^{\dagger i_n} \, , \tag{3.44}$$

where the superscript (q) indicates that $\Omega^{(q)}_{i_1 ... i_n}$ is a q-form for the non-compact indices $x_1, ..., x_{10-d}$. This gives an isomorphism as an algebra. It is then clear that we can associate a state $|\Omega\rangle \in \mathcal{F}$ to each differential form Ω as

$$|\Omega\rangle \equiv \mathbf{\Omega}|0\rangle \, . \tag{3.45}$$

The key result in [19] is that the state $|F\rangle$, corresponding to the differential form (3.38), transforms under $\Lambda \in SO(d, d, \mathbb{R})$ as

$$|F\rangle \longrightarrow |F'\rangle = \mathbf{\Lambda}|F\rangle \, , \tag{3.46}$$

where the operator $\mathbf{\Lambda}$ is defined as

$$\mathbf{\Lambda}|\beta\rangle = \sum_{\alpha} |\alpha\rangle S_{\alpha\beta}(\Lambda) \ , \qquad |\alpha\rangle, |\beta\rangle \in \mathcal{F} \, . \tag{3.47}$$

Here, $S_{\alpha\beta}(\Lambda)$ denotes the spinor representation. The operator $\mathbf{g}_\gamma(\Gamma)$ acting on $\mathcal{F}$, which corresponds to the γ-deformation $g_\gamma(\Gamma)$ defined in (3.24), was constructed in [19] as

$$\mathbf{g}_\gamma(\Gamma) = \exp\left(\frac{1}{2}\Gamma_{mn}\psi_m\psi_n\right) \ , \quad m, n \in \{1, ..., d\} \, . \tag{3.48}$$

It should be clear from the isomorphism mapping (3.43) to (3.44), that under the γ-deformation, $g_\gamma(\Gamma)$

$$|F\rangle \longrightarrow |F'\rangle = \mathbf{g}_\gamma(\Gamma)\, |F\rangle \ , \quad F \longrightarrow F' = \mathbb{g}_\gamma(\Gamma)\, F \, , \tag{3.49}$$

with

$$\mathbb{g}_\gamma(\Gamma) = \exp\left(\frac{1}{2}\Gamma_{mn}\iota_m\iota_n\right) \, , \tag{3.50}$$

where ι_m is the contraction with the isometry direction $\frac{\partial}{\partial\phi_m}$. Such a contraction takes a n-form to an $(n-1)$-form. For instance,

$$\iota_m \left(d\phi_1 \wedge .. \wedge d\phi_{m-1} \wedge d\phi_m \wedge d\phi_{m+1} \wedge .. \wedge d\phi_n\right) = (-1)^{m-1} \left(d\phi_1 \wedge .. \wedge d\phi_{m-1} \wedge d\phi_{m+1} \wedge .. \wedge d\phi_n\right) \ , \quad (3.51)$$

where here m is fixed. In particular $\iota_m \iota_m = 0$.

As mentioned earlier, we will consider the $AdS_5 \times S^5$ and $AdS_2 \times S^2 \times \mathrm{T}^6$ type IIB solutions which both have $\chi = C_{(2)} = 0$ but non-vanishing $C_{(4)}$. Hence, $F = F_{(5)}$. After a γ-deformation, the new R-R field strengths $F'_{(1)}, F'_{(3)}, F'_{(5)}$, are by definition,

$$\mathfrak{g}_\gamma U = \exp\left(\frac{1}{2}\Gamma_{mn}\iota_m\iota_n\right) F_{(5)} = F' := F'_{(1)} + F'_{(3)} + F'_{(5)} \, . \quad (3.52)$$

Expanding the exponential yields

$$\exp\left(\frac{1}{2}\Gamma_{mn}\iota_m\iota_n\right) F_{(5)} = \sum_{p=0}^{\infty} \frac{\left(\Gamma_{mn}\iota_m\iota_n\right)^p}{2^p.p!} F_{(5)} = \left(1 + \frac{1}{2}\Gamma_{mn}\iota_m\iota_n + \frac{1}{8}\Gamma_{kl}\Gamma_{mn}\iota_k\iota_l\iota_m\iota_n\right) F_{(5)} \, . \quad (3.53)$$

Higher order terms in the expansion vanish when contracted with the 5-form $F_{(5)}$. Therefore, the expressions of the γ-deformed R-R field strengths are

$$F'_{(3)} = \frac{1}{2}\Gamma_{mn}\iota_m\iota_n F_{(5)} \ ; \quad F'_{(1)} = \frac{1}{8}\Gamma_{kl}\Gamma_{mn}\iota_k\iota_l\iota_m\iota_n F_{(5)} \, . \quad (3.54)$$

One would think, from (3.53), that the 5-form remains invariant under γ-deformations. This is not exactly the case as the spacetime metric gets deformed and $F_{(5)}$ depends implicitly on the metric via the self-duality condition (2.37). Let us write $F_{(5)} = w + w^\star$, with w being a 5-form, such that (2.37) is automatically satisfied. The expression of the γ-deformed 5-form field strength is then

$$F'_{(5)} = w + w^{\star'} \, , \quad (3.55)$$

where the primed star denotes the Hodge dual taken with respect to the deformed spacetime metric.

Chapter 4

γ-deformations of the $AdS_5 \times S^5$ background

In this chapter, we start by presenting the consistent type IIB $AdS_5 \times S^5$ background. We then consider two different γ-deformations in order to derive two type IIB supergravity solutions known as the LM solution and Frolov's solution. Finally, we verify that the LM solution satisfies the equations of motion derived from the covariant action presented in chapter 2.

4.1 The $AdS_5 \times S^5$ background

In order to discuss deformations, it is first necessary to give a detailed description of the $AdS_5 \times S^5$ background. Its geometry is defined, in string frame, by the line element squared

$$ds_S^2 = ds_{AdS_5}^2 + ds_{S^5}^2 \,.$$

The Anti-de Sitter space AdS_5, is the five-dimensional maximally symmetric space with negative constant curvature. We do not give a detailed construction[1] of AdS_5 as we will exclusively consider deformations of the 5-sphere S^5. Nevertheless, we choose to parametrize AdS_5 by the coordinates $(x_0,\, x_1,\, x_2,\, x_3, x_4)$ such that

$$ds_{AdS_5}^2 = \frac{R^2}{x_0^2}\left(dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2\right) , \tag{4.1}$$

where R denotes the radius of AdS_5 in string frame.

We view the five-dimensional sphere of radius R as an embedding in a six-dimensional Euclidean space. In terms of Euclidean coordinates $(X_1, X_2, X_3, X_4, X_5, X_6)$, the 5-sphere is defined by the constraint

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = R^2 \,. \tag{4.2}$$

We now switch to three pairs of polar coordinates (r_i, ϕ_i), $i \in \{1, 2, 3\}$, defined as

$$X_i = Rr_i\cos(\phi_i)\,, \quad x_{i+3} = Rr_i\sin(\phi_i)\,. \tag{4.3}$$

Condition (4.2) then becomes

$$\sum_{i=1}^{3} r_i^2 = 1\,. \tag{4.4}$$

The line element squared of the Euclidean space is given by

$$ds^2 = \sum_{p=1}^{6} dX_p^2 = R^2\left(\sum_{i=1}^{3} dr_i^2 + r_i^2 d\phi_i^2\right) , \tag{4.5}$$

[1] This can be easily be found in multiple books and reviews. See for example [20].

and when condition (4.4) is implemented, this also describes the line element squared of the 5-sphere $ds^2_{S^5}$. This embedding will prove to be useful for some computations, but in other cases it will turn out to be more convenient to deal with a set of five local coordinates $(\alpha, \theta, \phi_1, \phi_2, \phi_3)$, defined as

$$r_1 = \cos(\alpha) = c_\alpha\,, \quad r_2 = \sin(\alpha)\cos(\theta) = s_\alpha c_\theta\,, \quad r_3 = \sin(\alpha)\sin(\theta) = s_\alpha s_\theta\,, \tag{4.6}$$

which automatically satisfy (4.4). Substituting (4.6) into (4.5) yields

$$ds^2_{S^5} = R^2\left(d\alpha^2 + s^2_\alpha d\theta^2 + c^2_\alpha d\phi_1^2 + s^2_\alpha c^2_\theta d\phi_2^2 + s^2_\alpha s^2_\theta d\phi_3^2\right)\,. \tag{4.7}$$

These are the notations suggested in [2]. In the end, the $AdS_5 \times S^5$ manifold is parametrized by the local coordinates $(x_0, x_1, x_2, x_3, x_4, \alpha, \theta, \phi_1, \phi_2, \phi_3)$ and its metric in string frame can be read from

$$ds^2_S = ds^2_{AdS_5} + ds^2_{S^5} \tag{4.8}$$

$$= \frac{R^2}{x_0^2}\left(dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2\right) + R^2\left(d\alpha^2 + s^2_\alpha d\theta^2 + c^2_\alpha d\phi_1^2 + s^2_\alpha c^2_\theta d\phi_2^2 + s^2_\alpha s^2_\theta d\phi_3^2\right)\,. \tag{4.9}$$

In the rest of this chapter we adopt the, possibly awkward, convention in which ten-dimensional spacetime indices $\mu, \nu, \rho, ... \in \{x_0, x_1, x_2, x_3, x_4, \alpha, \theta, \phi_1, \phi_2, \phi_3\}$ such that, for example, $G_{\theta\theta} = R^2 s^2_\alpha$. We also introduce, for latter convenience, the hatted indices $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, ...$ and the indices $a, b, c, d, e, ...$ which are, respectively, AdS_5 and S^5 indices. For instance $G_{\hat{a}a} = 0$. Furthermore, the determinant of the metric factorizes as $G = G_{AdS_5}.G_{S^5}$

The Ramond-Ramond field $C_{(4)}$ on $AdS_5 \times S^5$ reads

$$C_{(4)} = 4R_E^4\left(w_4 + w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right)\,, \tag{4.10}$$

where the 4-form w_4 and the 1-form w_1 are defined via the following relations

$$w_{AdS_5} = dw_4 = \sqrt{-G_{AdS_{5_1}}}\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\,, \tag{4.11}$$

$$dw_1 = c_\alpha s^3_\alpha s_\theta c_\theta\, d\alpha \wedge d\theta = \sqrt{G_{S^5_1}}\, d\alpha \wedge d\theta\,. \tag{4.12}$$

Here, w_{AdS_5} is the volume form on AdS_5 and the tiny subscript 1 in $G_{AdS_{5_1}}$ and $G_{S^5_1}$ means that $R = 1$ for these determinants. We also have that

$$R_E^4 e^{\phi_0} = R^4\,, \tag{4.13}$$

where R_E denotes the radius in Einstein frame.

The dilaton is constant and denoted by ϕ_0 while all of the other fields vanish

$$\chi = B = C_{(2)} = 0\,.$$

The associated field strengths, therefore, also vanish

$$H = dB = 0\,, \quad F_{(3)} = dC_{(2)} = 0\,, \quad F_{(1)} = d\chi = 0\,. \tag{4.14}$$

However, the 5-form field strength $F_{(5)}$ associated to $C_{(4)}$ doesn't vanish and is equal to

$$F_{(5)} = dC_{(4)} - C_{(2)} \wedge dB = dC_{(4)} = 4R_E^4\left(w_{AdS_5} + w_{S^5}\right)\,, \tag{4.15}$$

where w_{S^5} is the volume form on S^5

$$w_{S^5} = dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 = \sqrt{G_{S^5_1}}\, d\alpha \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\,. \tag{4.16}$$

The Hodge dual of a p-form w on $AdS_5 \times S^5$ is the $(10-p)$-form $w^\star$, defined as

$$w^\star_{\mu_1....\mu_p} = \frac{-1}{p!\sqrt{-G}} \epsilon^{\nu_1....\nu_p\nu_{p+1}....\nu_{10}} G_{\mu_1\nu_1}....G_{\mu_p\nu_p} w_{\nu_{p+1}....\nu_{10}} \,, \tag{4.17}$$

where the indices $\mu_1, ..., \mu_p, \nu_1, ..., \nu_{10} \in \{x_0, x_1, x_2, x_3, x_4, \alpha, \theta, \phi_1, \phi_2, \phi_3\}$ and where ϵ is the fully antisymmetric tensor in ten dimensions normalized to $\epsilon^{x_0x_1x_2x_3x_4\alpha\theta\phi_1\phi_2\phi_3} = 1$. Using (4.11) and (4.17), one can easily show that

$$w^\star_{\mathrm{AdS}_5} = w_{\mathrm{S}^5} \,. \tag{4.18}$$

Therefore,

$$F_{(5)} = 4R_E^4 \left(w_{AdS_5} + w^\star_{AdS_5} \right) \,, \tag{4.19}$$

is clearly self-dual as required from (2.37). In terms of components, (4.15) reads

$$F_{(5)\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} = 4R_E^4 \sqrt{-G_{AdS_{5_1}}}\, \epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} \,, \quad F_{(5)abcde} = 4R_E^4 \sqrt{G_{S_1^5}}\, \epsilon_{abcde} \,, \tag{4.20}$$

where the fully antisymmetric tensors $\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ and ϵ_{abcde} are normalized to $\epsilon_{x_0x_1x_2x_3x_4} = 1$ and $\epsilon_{\alpha\theta\phi_1\phi_2\phi_3} = 1$. One could have been wondering why AdS_5 and S^5 share the same radius R. It should now be clear it has to be the case in order for the 5-form $F_{(5)}$ to be self-dual.

Finally, let us mention that this background is a type IIB supergravity solution and is therefore a consistent background. This can be proven using the type IIB equations of motion derived in section 4.3. Due to the relative simplicity of the $AdS_5 \times S^5$ background, we leave this task to the reader. We can now apply the solution generating technique described in section 3.2 to generate new type IIB solutions.

4.2 γ-deformations of $\mathrm{AdS}_5 \times \mathrm{S}^5$

The 5-sphere possesses three $U(1)$ isometries realized as constant shifts of the angle coordinates ϕ_i , $i \in \{1, 2, 3\}$, which parametrize a 3-torus

$$\phi_i \longrightarrow \phi_i + \alpha_i \,, \quad \alpha_i \in \mathbb{R} \,. \tag{4.21}$$

Actually, these three $U(1)$ isometries are symmetries of the whole $AdS_5 \times S^5$ solution since the transformations (4.21) leave all the fields defined in section 4.1 invariant. Only the invariance of the $C_{(4)}$ field defined in (4.10) might appear a bit puzzling. However, definitions (4.11) and (4.12) imply that all the components of the 4-form w_4 and the 1-form w_1 are independent of the isometry angles ϕ_i. This immediately proves the invariance of $C_{(4)}$ under (4.21). The geometry of the background contains, as explained earlier, a 3-torus (or internal sector) parametrized by the isometry angles, fibered over a non-compact sector parametrized by the coordinates $x_0, x_1, x_2, x_3, x_4, \alpha, \theta$.

Following our discussion in chapter 3, it is now possible to generate new solutions of type IIB supergravity by deforming the $AdS_5 \times S^5$ background with $SO(3,3,\mathbb{R})$ elements. The AdS_5 space also exhibits $U(1)$ isometries and it would, therefore, also be possible to deform along its isometry directions. Although this is feasible[2], we will not perform such deformations here, as their consequence is obviously the breaking of conformal symmetry of AdS_5. In what follows, we will focus entirely on γ-deformations of the 5-sphere. In particular, we will study two main deformed solutions. The first one, chronologically, has been obtained in [2] by a very specific one-parameter γ-deformation and is known in the literature as the LM solution (or LM background), while the second one, known as Frolov's solution, has been obtained later by the three-parameter γ-deformation in [3].

4.2.1 The Lunin-Maldacena background

We will follow a different approach to the original derivation [2] of the LM background. A few remarks on the the original approach will be made later. Here, we will make use of the solution generating technique described in chapter 3.

[2]As long as T-duality is not performed on the non-compact time direction.

NS-NS sector

Let us first focus on the deformation of the NS-NS fields. We denote the background matrix of the internal sector (i.e. the 3-torus), defined in (3.6), by $E_{(\phi)}$

$$E_{(\phi)} = \begin{pmatrix} G_{\phi_1\phi_1} & G_{\phi_1\phi_2} + B_{\phi_1\phi_2} & G_{\phi_1\phi_3} + B_{\phi_1\phi_3} \\ G_{\phi_1\phi_2} - B_{\phi_1\phi_2} & G_{\phi_2\phi_2} & G_{\phi_2\phi_3} B_{\phi_1\phi_3} \\ G_{\phi_1\phi_3} - B_{\phi_1\phi_3} & G_{\phi_2\phi_3} - B_{\phi_2\phi_3} & G_{\phi_3\phi_3} \end{pmatrix} = \begin{pmatrix} R^2 r_1^2 & 0 & 0 \\ 0 & R^2 r_2^2 & 0 \\ 0 & 0 & R^2 r_3^2 \end{pmatrix} . \tag{4.22}$$

The deformation of [2] is a TsT-transformation applied on a very specific 2-torus. This particular choice of the torus is motived by spacetime supersymmetry arguments. Indeed, the LM solution is known to preserve $\frac{1}{4}$ of the supersymmetries of the $AdS_5 \times S^5$ solution. This probably sounds a bit confusing at this point and we point the reader to chapter 6 for detailed explanations.

Let us now follow the train of thought of [3] to obtain the deformed background matrix of the LM background. The first step is to make the following change of angle coordinates,

$$\varphi_i = \sum_{j=1}^{3} \left((A^t_{(LM)})^{-1} \right)_{ij} \phi_j \,, \quad \text{with} \quad \left(A^t_{(LM)} \right)^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} , \tag{4.23}$$

where the coordinates $(\varphi_1, \varphi_2, \varphi_3)$ provide another parameterization of the 3-torus. The three $U(1)$ isometries are now realized as constant shifts of the new isometry angles $\varphi_1, \varphi_2, \varphi_3$. From our discussion of equation (3.17), the background matrix of the internal sector of $AdS_5 \times S^5$ becomes

$$E_{(\varphi)} = g_{A_{(LM)}} E_{(\phi)} = \begin{pmatrix} G_{\varphi_1\varphi_1} & G_{\varphi_1\varphi_2} + B_{\varphi_1\varphi_2} & G_{\varphi_1\varphi_3} + B_{\varphi_1\varphi_3} \\ G_{\varphi_1\varphi_2} - B_{\varphi_1\varphi_2} & G_{\varphi_2\varphi_2} & G_{\varphi_2\varphi_3} B_{\varphi_1\varphi_3} \\ G_{\varphi_1\varphi_3} - B_{\varphi_1\varphi_3} & G_{\varphi_2\varphi_3} - B_{\varphi_2\varphi_3} & G_{\varphi_3\varphi_3} \end{pmatrix} , \tag{4.24}$$

where the element $g_{A_{(LM)}} \in SO(3, 3, \mathbb{R})$ has been defined in (3.15). Recall that $SO(d, d, \mathbb{R})$ elements act on the background matrix as fractional linear transformations (3.14). The second step is to perform a TsT-transformation, with parameter γ, on the 2-torus parametrized by the angle coordinates (φ_1, φ_2). The background matrix then gets deformed to

$$E_{(\varphi)} \longrightarrow E'_{(\varphi)} = g_{(T_{\varphi_1} s_{\varphi_2} T_{\varphi_1})} E_{(\varphi)} \,, \tag{4.25}$$

where the expression of $g_{(T_{\varphi_1} s_{\varphi_2} T_{\varphi_1})}$ is given by (3.30). The last step is to switch back to our initial isometry angles ϕ_1, ϕ_2, ϕ_3. The deformed background matrix in the initial angle coordinates system is then

$$E'_{(\phi)} = \left(g_{A_{(LM)}} \right)^{-1} E'_{(\varphi)} \,. \tag{4.26}$$

Patching all these steps together gives the expression of the one-parameter γ-deformation used in [2]

$$E_{(\phi)} \longrightarrow E'_{(\phi)} = g_{\gamma_{(LM)}} E_{(\phi)} \,, \tag{4.27}$$

where

$$\begin{aligned} g_{\gamma_{(LM)}} &= \left(g_{A_{(LM)}} \right)^{-1} \cdot g_{(T_{\varphi_1} s_{\varphi_2} T_{\varphi_1})} \cdot g_{A_{(LM)}} \\ &= g_{A^{-1}_{(LM)}} \cdot g_{(T_{\varphi_1} s_{\varphi_2} T_{\varphi_1})} \cdot g_{A_{(LM)}} \\ &= \begin{pmatrix} A^{-1}_{(LM)} & 0 \\ 0 & A^t_{(LM)} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1}_{3\times 3} & 0 \\ \Gamma_{(T_{\phi_1} s_{\phi_2} T_{\phi_1})} & \mathbb{1}_{3\times 3} \end{pmatrix} \cdot \begin{pmatrix} A_{(LM)} & 0 \\ 0 & (A^t_{(LM)})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1}_{3\times 3} & 0 \\ \left(A^t_{(LM)} \cdot \Gamma_{(T_{\phi_1} s_{\phi_2} T_{\phi_1})} \cdot A_{(LM)} \right) & \mathbb{1}_{3\times 3} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{3\times 3} & 0 \\ \Gamma_{(LM)} & \mathbb{1}_{3\times 3} \end{pmatrix} , \end{aligned} \tag{4.28}$$

with

$$\Gamma_{(LM)} = \begin{pmatrix} 0 & -\gamma & \gamma \\ \gamma & 0 & -\gamma \\ -\gamma & \gamma & 0 \end{pmatrix} . \tag{4.29}$$

The LM deformation $g_{\gamma_{(LM)}}$ is then equivalent to three TsT-transformations, each of them with the same parameter γ, on the three 2-torus parametrized by the pairs of isometry angles (ϕ_1, ϕ_2), (ϕ_3, ϕ_1) and (ϕ_2, ϕ_3). After some simple algebra, equation (4.27) gives

$$E'_{(\phi)} = R^2 J \begin{pmatrix} r_1^2(1 + R^4\gamma^2 r_2^2 r_3^2) & r_1^2(R^2\gamma r_2^2 + R^2\gamma^2 r_2^2 r_3^2) & r_1^2(-R^2\gamma r_3^2 + R^4\gamma^2 r_2^2 r_3^2) \\ r_2^2(-R^2\gamma r_1^2 + R^4\gamma^2 r_1^2 r_3^2) & r_2^2(1 + R^4\gamma^2 r_1^2 r_3^2) & r_2^2(R^2\gamma r_3^2 + R^4\gamma^2 r_1^2 r_3^2) \\ r_3^2(R^2\gamma r_1^2 + R^4\gamma^2 r_1^2 r_2^2) & r_3^2(-R^2\gamma r_2^2 + R^4\gamma^2 r_1^2 r_2^2) & r_3^2(1 + R^4\gamma^2 r_1^2 r_2^2) \end{pmatrix} , \tag{4.30}$$

with

$$J = \left(1 + \gamma^2 R^4 (r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2)\right)^{-1} . \tag{4.31}$$

Identifying the symmetric and antisymmetric part of the deformed background matrix $E'_{(\phi)}$ yields, respectively, the internal sector components of the deformed metric G' and the deformed field B'

$$G'_{\phi_1\phi_1} = R^2 J r_1^2 (1 + R^4\gamma^2 r_2^2 r_3^2) = \frac{R^2 c_\alpha^2 + R^6\gamma^2 c_\alpha^2 s_\alpha^4 s_\theta^2 c_\theta^2}{1 + \gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)} , \tag{4.32}$$

$$G'_{\phi_2\phi_2} = R^2 J r_2^2 (1 + R^4\gamma^2 r_1^2 r_3^2) = \frac{R^2 s_\alpha^2 c_\theta^2 + R^6\gamma^2 c_\alpha^2 s_\alpha^4 s_\theta^2 c_\theta^2}{1 + \gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)} , \tag{4.33}$$

$$G'_{\phi_3\phi_3} = R^2 J r_3^2 (1 + R^4\gamma^2 r_2^2 r_1^2) = \frac{R^2 s_\alpha^2 s_\theta^2 + R^6\gamma^2 c_\alpha^2 s_\alpha^4 s_\theta^2 c_\theta^2}{1 + \gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)} , \tag{4.34}$$

$$G'_{\phi_1\phi_2} = G_{\phi_1\phi_3} = G_{\phi_3\phi_2} = R^6 J \gamma^2 r_1^2 r_2^2 r_3^2 = \frac{R^6\gamma^2 c_\alpha^2 s_\alpha^4 s_\theta^2 c_\theta^2}{1 + \gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)} , \tag{4.35}$$

$$B'_{\phi_1\phi_2} = R^4\gamma J r_1^2 r_2^2 = \frac{R^4\gamma c_\alpha^2 s_\alpha^2 c_\theta^2}{1 + \gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)} , \tag{4.36}$$

$$B'_{\phi_3\phi_1} = R^4\gamma J r_3^2 r_1^2 = \frac{R^4\gamma c_\alpha^2 s_\alpha^2 s_\theta^2}{1 + \gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)} , \tag{4.37}$$

$$B'_{\phi_2\phi_3} = R^4\gamma J r_2^2 r_3^2 = \frac{R^2\gamma s_\alpha^4 s_\theta^2 c_\theta^2}{1 + \gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)} . \tag{4.38}$$

As explained in 3.2, the other components of the metric and the B field are not affected by the γ-deformation. One can then write

$$B' = R^4\gamma J \left(r_1^2 r_2^2 \, d\phi_1 \wedge d\phi_2 + r_3^2 r_1^2 \, d\phi_3 \wedge d\phi_1 + r_1^2 r3^2 \, d\phi_2 \wedge d\phi_3\right) . \tag{4.39}$$

From (3.28), we deduce that, under $g_{\gamma_{(LM)}}$, the dilaton ϕ_0 transforms as

$$\phi_0 \longrightarrow \phi' = \phi_0 - \frac{1}{2}\ln\left(\det(\Gamma_{(LM)} E_{(\phi)} + \mathbb{1}_{3\times 3})\right) = \phi_0 - \frac{1}{2}\ln(J^{-1}) = \phi_0 + \frac{1}{2}\ln(J) . \tag{4.40}$$

In what follows, we will denote the deformed 5-sphere by S^5_γ. Note that the determinant of its metric is $G_{S^5_\gamma} = J^2 G_{S^5}$.

R-R sector: Field strengths

We now turn to the action of $g_{\gamma_{(LM)}}$ on the R-R fields. For the $AdS_5 \times S^5$ background, the sum of the field strengths F, introduced in (3.38), is

$$F = F_{(5)} = 4R_E^4 \left(\sqrt{-G_{AdS_1^5}} \, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + \sqrt{G_{S_1^5}} \, d\alpha \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right) . \tag{4.41}$$

Under $g_{\gamma_{(LM)}}$,

$$F \longrightarrow F' = \mathbb{g}_{\gamma_{(LM)}} F = \exp\left(\frac{1}{2}(\Gamma_{(LM)})_{mn}\iota_m\iota_n\right) U = \left(1 - \gamma(\iota_{\phi_1}\iota_{\phi_2} - \iota_{\phi_3}\iota_{\phi_1} + \iota_{\phi_2}\iota_{\phi_3})\right) F$$
$$= F_{(5)} - \gamma(\iota_{\phi_1}\iota_{\phi_2} + \iota_{\phi_3}\iota_{\phi_1} + \iota_{\phi_2}\iota_{\phi_3})F_{(5)}$$
$$\equiv F'_{(5)} + F'_{(3)} + F'_{(1)} \,. \tag{4.42}$$

Clearly, $F'_{(1)} = d\chi' = 0$ and therefore χ' is pure gauge. Using (4.41), one obtains

$$F'_{(3)} = -\gamma(\iota_{\phi_1}\iota_{\phi_2} - \iota_{\phi_3}\iota_{\phi_1} + \iota_{\phi_2}\iota_{\phi_3})F_{(5)} = 4R_E^4\gamma\sqrt{G_{S^5_1}}\, d\alpha \wedge d\theta \wedge (d\phi_1 + d\phi_2 + d\phi_3)$$
$$= 12R_E^4\gamma dw_1 \wedge d\psi \,, \tag{4.43}$$

where the 2-form dw_1 was defined in (4.12) and we also introduced the angle $\psi := \frac{1}{3}\left(d\phi_1 + d\phi_2 + d\phi_3\right)$. It was noticed in [3], that our expression for $F'_{(3)}$ differs by a minus sign from the one of [2]. This difference is due to the choice of T-duality rules. We add here a minus sign to $F_{(3)}$ to exactly match the results of [2]. The deformed 5-form field strength is

$$F'_{(5)} = 4R_E^4\left(w_{AdS_5} + w^{\star'}_{AdS_5}\right) \,, \tag{4.44}$$

where the Hodge dual is taken with respect to deformed metric. From the definition of the Hodge duality and the expression of $F_{(5)}$, one can easily notice that the only non-vanishing component of $w^{\star'}_{AdS_5}$ is

$$(w^{\star'}_{AdS_5})_{\alpha\theta\phi_1\phi_2\phi_3} = \frac{-1}{5!\sqrt{-G'}}\epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}} G'_{\alpha\mu_1} G'_{\theta\mu_2} G'_{\phi_1\mu_3} G'_{\phi_2\mu_4} G'_{\phi_3\mu_5} F_{(5)\mu_6\mu_7\mu_8\mu_9\mu_{10}}$$
$$= \frac{-1}{5!\sqrt{-G'}}\epsilon^{\alpha\theta\phi_1\phi_2\phi_3\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} G'_{\alpha\alpha} G'_{\theta\theta} G'_{\phi_1\phi_1} G'_{\phi_2\phi_2} G'_{\phi_3\phi_3}\sqrt{-G_{AdS_{5_1}}}\,\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$$
$$= \frac{1}{5!}\cdot\frac{+1}{\sqrt{-G'}} G_{S^5_\gamma} 5!\sqrt{-G_{AdS_{5_1}}} = \frac{1}{\sqrt{-G_{AdS_5}.G_{S^5_\gamma}}} J^2 G_{S^5}\sqrt{-G_{AdS_{5_1}}}$$
$$= \frac{1}{J\sqrt{-G_{AdS_5}.G_{S^5}}} J^2 G_{S^5}\sqrt{-G_{AdS_{5_1}}} = \frac{J}{\sqrt{R^{20}}\sqrt{-G_{AdS_{5_1}} G_{S^5_1}}} R^{10} G_{S^5_1}\sqrt{-G_{AdS_{5_1}}}$$
$$= J\sqrt{G_{S^5_1}} \,. \tag{4.45}$$

Therefore, we have

$$F'_{(5)} = 4R'_E\left(w_{AdS_5} + w^{\star'}_{AdS_5}\right) = 4R'_E\left(w_{AdS_5} + J\sqrt{G_{S^5_1}}\, d\alpha \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right)$$
$$= 4R'_E\left(w_{AdS_5} + Jw_{S^5}\right) \,, \tag{4.46}$$

and the self-duality of $F'_{(5)}$ is ensured by construction.

Bianchi identities

A good method to check the consistency of our expressions for the deformed field strengths is to verify wether the following Bianchi identities are satisfied

$$dH' = d\left(dB\right) = 0 \,, \tag{4.47}$$
$$dF'_{(3)} = d\left(dC'_{(2)} - H' \wedge \chi'\right) = H' \wedge F'_{(1)} \,, \tag{4.48}$$
$$dF'_{(5)} = d\left(dC'_{(4)} - H' \wedge C'_{(2)}\right) = H' \wedge F'_{(3)} + H' \wedge H' \wedge \chi' = H' \wedge F'_{(3)} \,, \tag{4.49}$$
$$dF'_{(1)} = d\left(d\chi'\right) 0 \,. \tag{4.50}$$

The last identity is trivially satisfied as $F'_{(1)} = 0$. One can compute the field strength H from (4.39),

$$\begin{aligned} H = dB = \gamma R^4 \big(&\partial_\alpha(Jr_1^2r_2^2)\, d\alpha \wedge d\phi_1 \wedge d\phi_2 + \partial_\alpha(Jr_3^2r_1^2)\, d\alpha \wedge d\phi_3 \wedge d\phi_1 + \partial_\alpha(Jr_2^2r_3^2))\, d\alpha \wedge d\phi_2 \wedge d\phi_3 \\ &+\partial_\theta(Jr_1^2r_2^2)\, d\theta \wedge d\phi_1 \wedge d\phi_2 + \partial_\theta(Jr_3^2r_1^2)\, d\theta \wedge d\phi_3 \wedge d\phi_1 + \partial_\theta(Jr_2^2r_3^2)\, d\theta \wedge d\phi_2 \wedge d\phi_3\big)\,. \end{aligned} \tag{4.51}$$

The first identity is satisfied by construction since the exterior derivative is nilpotent. Explicitly,

$$\begin{aligned} dH = \gamma R^4 \big(&(\partial_\theta\partial_\alpha(Jr_1^2r_2^2) + \partial_\theta\partial_\alpha(Jr_3^2r_1^2) + \partial_\theta\partial_\alpha(Jr_2^2r_3^2))\, d\theta \wedge d\alpha \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \\ &+(\partial_\alpha\partial_\theta(Jr_1^2r_2^2) + \partial_\alpha\partial_\theta(Jr_3^2r_1^2) + \partial_\alpha\partial_\theta(Jr_2^2r_3^2))\, d\alpha \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\big) = 0\,. \end{aligned} \tag{4.52}$$

The third identity is also satisfied since both sides vanish. This can be immediately seen by closely considering the expression of $F'_{(5)}$ and by realizing that the wedge product on the right hand side necessarily vanishes as the components of $F'_{(3)}$ and H' always overlap along one spacetime direction. In the second identity, the right hand side vanishes since $F'_{(1)} = 0$, while once more, it is straightforward to notice that the left hand side is also null by looking at the expression of $F'_{(3)}$.

R-R sector: Gauge fields

Let us now derive the expressions of the deformed R-R fields, $C'_{(2)}$ and $C'_{(4)}$. We first choose to gauge χ away,

$$\chi = 0\,. \tag{4.53}$$

Bearing this result in mind and using (4.43) with the necessary sign flip, one obtains the relation

$$F'_{(3)} = dC'_{(2)} - \chi' \wedge H' = dC'_{(2)} = -4R_E^4\gamma R_E^4 dw_1 \wedge (d\phi_1 + d\phi_2 + d\phi_3)\,, \tag{4.54}$$

from which we deduce that

$$C'_{(2)} = -4R_E^4 w_1 \wedge (d\phi_1 + d\phi_2 + d\phi_3) = -12R_E^4 w_1 \wedge d\psi\,. \tag{4.55}$$

In order to derive $C'_{(4)}$, we first compute some useful expressions

$$\begin{aligned} dC'_{(2)} \wedge B' &= -4R_E^4 J(\gamma^2 R^4(r_1^2r_2^2 + r_3^2r_1^2 + r_2^2r_3^2))\, dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \\ &= -4R_E^4 J(J^{-1} - 1)\, dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 = -4R_E^4(1 - J)\, dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\,, \end{aligned} \tag{4.56}$$

and,

$$\begin{aligned} d(C'_{(2)} \wedge B') &= d(-4R_E^4 J(\gamma^2 R^4(r_1^2r_2^2 + r_3^2r_1^2 + r_2^2r_3^2))\, w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3) \\ &= d(-4R_E^4(1 - J)\, w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3) \\ &= 4R_E^4\, ((\partial_\alpha J)d\alpha + (\partial_\theta J)d\theta) \wedge w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \\ &\quad - 4R_E^4(1 - J)\, dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\,. \end{aligned} \tag{4.57}$$

The exterior derivative of $C'_{(4)}$ can be written, using (4.56) and (4.57), as

$$\begin{aligned} dC'_{(4)} &= F'_{(5)} + C'_{(2)} \wedge H' = F'_{(5)} + C'_{(2)} \wedge dB' = F'_{(5)} + d\left(C'_{(2)} \wedge B'\right) - dC'_{(2)} \wedge B' \\ &= 4R_E^4\, (w_{\mathrm{AdS}_5} + J dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3) + 4R_E^4\, ((\partial_\alpha J)d\alpha + (\partial_\theta J)d\theta) \wedge w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \\ &\quad - 4R_E^4(1 - J)\, dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 - \left(-4R_E^4(1 - J)\, dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right) \\ &= 4R_E^4\, (dw_4 + (J dw_1 + (\partial_\alpha J)d\alpha \wedge w_1 + (\partial_\theta J)d\theta \wedge w_1) \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3) \\ &= 4R_E^4\, (dw_4 + d\,(J w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3)) = d\left(4R_E^4(w_4 + J w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3)\right)\,. \end{aligned} \tag{4.58}$$

Hence,

$$C'_{(4)} = 4R_E^4\, (w_4 + J w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3)\,. \tag{4.59}$$

The LM background

Here, we sum up all of the results and drop the prime on the deformed fields. The LM background, in string frame, is

$$ds_S^2 = ds^2_{AdS_5} + ds^2_{S^5_\gamma} = ds^2_{AdS_5} + R^2 \sum_{i=1}^{3} \left(dr_i^2 + J r_i^2 d\phi_i^2\right) + R^6 \gamma^2 J r_1^2 r_2^2 r_3^2 (\sum_{i=1}^{3} d\phi_i)(\sum_{j=1}^{3} d\phi_j)\,, \tag{4.60}$$

$$J = \frac{1}{1+\gamma^2 R^4 (r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2)} = \frac{1}{1+\gamma^2 R^4 (c_\alpha^2 s_\alpha^2 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_\theta^2 + s_\alpha^4 c_\theta^2 s_\theta^2)}\,, \tag{4.61}$$

$$e^{2\phi} = e^{2\phi_0} J\,, \tag{4.62}$$

$$B = \gamma R^4 J (r_1^2 r_2^2\, d\phi_1 \wedge d\phi_2 + r_3^2 r_1^2\, d\phi_3 \wedge d\phi_1 + r_2^2 r_3^2\, d\phi_2 \wedge d\phi_3)\,, \tag{4.63}$$

$$C_{(2)} = -12\gamma R_E^4 w_1 \wedge d\psi = -4\gamma R_E^4 (w_1 \wedge d\phi_1 + w_1 \wedge d\phi_2 + w_1 \wedge d\phi_3)\,, \tag{4.64}$$

$$C_{(4)} = 4R_E^4 \left(w_4 + J\, w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right)\,, \tag{4.65}$$

$$F_{(5)} = 4R_E^4 \left(w_{AdS_5} + J\, w_{S^5}\right)\,, \tag{4.66}$$

$$\chi = 0\,. \tag{4.67}$$

It coincides exactly with the expressions of [2]. For $\gamma = 0$, one immediately recovers the $AdS_5 \times S^5$ background.

4.2.2 Frolov's solution

In [3], Frolov generated a new solution of type IIB supergravity by applying the following three-parameter γ-deformation, also referred to as γ_i-deformation, on $AdS_5 \times S^5$

$$g_{\gamma_{(F)}} = \begin{pmatrix} \mathbb{1}_{3\times3} & 0 \\ \Gamma_{(F)} & \mathbb{1}_{3\times3} \end{pmatrix}\,, \text{ with } \Gamma_{(F)} = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}\,. \tag{4.68}$$

This γ_i-deformation corresponds to a chain of three TsT-transformations: one with parameter γ_3 on the 2-torus parametrized by (ϕ_1, ϕ_2), another one with parameter γ_2 on the 2-torus parametrized by (ϕ_3, ϕ_1) and finally, one with parameter γ_1 on the 2-torus parametrized by (ϕ_2, ϕ_3). When the three parameters are equal, Frolov's solution obviously boils down to the LM solution.

We do not provide the complete derivation of Frolov's solution as it is basically the same as for the LM solution. Instead, we just state the result in string frame

$$ds_S^2 = ds^2_{AdS_5} + ds^2_{S^5_\gamma} = ds^2_{AdS_5} + R^2 \sum_{i=1}^{3} \left(dr_i^2 + J r_i^2 d\phi_i^2\right) + R^6\; J r_1^2 r_2^2 r_3^2 (\sum_{i=1}^{3} \gamma_i d\phi_i)(\sum_{j=1}^{3} \gamma_j d\phi_j)\,, \tag{4.69}$$

$$J = \frac{1}{1+R^4 (\gamma_3^2 r_1^2 r_2^2 + \gamma_2^2 r_3^2 r_1^2 + \gamma_1^2 r_2^2 r_3^2)} = \frac{1}{1+R^4 (\gamma_3^2 c_\alpha^2 s_\alpha^2 c_\theta^2 + \gamma_2^2 c_\alpha^2 s_\alpha^2 s_\theta^2 + \gamma_1^2 s_\alpha^4 c_\theta^2 s_\theta^2)}\,, \tag{4.70}$$

$$e^{2\phi} = e^{2\phi_0} J\,, \tag{4.71}$$

$$B = R^4 J (\gamma_3^2 r_1^2 r_2^2\, d\phi_1 \wedge d\phi_2 + \gamma_2^2 r_3^2 r_1^2\, d\phi_3 \wedge d\phi_1 + \gamma_1^2 r_2^2 r_3^2\, d\phi_2 \wedge d\phi_3)\,, \tag{4.72}$$

$$C_{(2)} = -4\gamma R_E^4 (w_1 \wedge \left(\sum_{i=1}^{3} \gamma_i d\phi_i\right)\,, \tag{4.73}$$

$$C_{(4)} = 4R_E^4 \left(w_4 + J\, w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right)\,, \tag{4.74}$$

$$F_{(5)} = 4R_E^4 \left(w_{AdS_5} + J\, w_{S^5}\right)\,, \tag{4.75}$$

$$\chi = 0\,. \tag{4.76}$$

Frolov's deformation is known to break all the supersymmetries of the maximally supersymmetric $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Hence, Frolov's solution is a non-supersymmetric one. For more details, see chapter 6.

4.2.3 Original approach and AdS/CFT correspondance

In this subsection, we present a very short summary of the original approach of Lunin and Maldacena. In [2], they introduced the TsT-transformations, in the context of the AdS/CFT correspondence [4], as the holographic duals of the, sometimes called, β-deformations on the gauge theory side.

Consider a $U(N)$ conformal gauge theory with $U(1) \times U(1)$ global symmetry. The β-deformations arise as a redefinition of the product of fields in the lagrangian

$$f_{(1)} \star f_{(2)} = e^{i\pi\gamma\left(Q^1_{f_{(1)}} Q^2_{f_{(2)}} - Q^2_{f_{(1)}} Q^1_{f_{(2)}}\right)} f_{(1)} f_{(2)} \,, \tag{4.77}$$

where $f_{(1)} f_{(2)}$ is the ordinary product and (Q^1, Q^2) are the $U(1) \times U(1)$ charges of the fields $f_{(1)}$ and $f_{(2)}$. The gravity duals of such gauge theories exhibit two $U(1)$ isometries realized geometrically, such that their geometries contain a 2-torus fibered over an eight-dimensional manifold. The real components of the metric and the B field on the 2-torus are denoted by G_{11}, G_{12}, G_{22} and B_{12}. The holographic description of the deformation (4.77) is given by the following transformation of the Kähler modulus ρ of the 2-torus

$$\rho = B_{12} + i\sqrt{V_T} \longrightarrow \rho' = \frac{\rho}{1+\gamma\rho} \equiv B'_{12} + i\sqrt{V'_T} \,, \tag{4.78}$$

where $V_T = G_{11}G_{22} - G_{12}^2$ is the volume of the 2-torus and where $\gamma \in \mathbb{R}$. As explained in [2], this deformation can then be used to generate new supergravity solutions. This is not surprising as one can show that (4.78) precisely corresponds to a TsT-transformation.

As explained in chapter 3, for the case of a type IIB background containing a 2-torus, the solution generating group is $SO(2,2,\mathbb{R})$. A possible decomposition is $SO(2,2,\mathbb{R}) \simeq SL(2\mathbb{R})_\tau \times SL(2,\mathbb{R})_\rho$, where the subindexes τ and ρ refer to the fact that the first $SL(2,\mathbb{R})$ acts on the complex structure modulus τ of the torus, while the second $SL(2,\mathbb{R})$ acts on its Kähler modulus ρ. More details can be found in any textbook on string theory (see for instance [11]). We will not deal here with the $SL(2,\mathbb{R})_\tau$ as it plays no role in the type of deformations we consider. On the other hand, $SL(2,\mathbb{R})_\rho$ acts on the Kähler modulus as

$$\rho \longrightarrow \rho' = \frac{\tilde{a}\rho + \tilde{b}}{\tilde{c}\rho + \tilde{d}} \,, \quad \text{with} \quad \tilde{a}\tilde{b} - \tilde{c}\tilde{d} = 1 \,, \tag{4.79}$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}$. An embedding of $SL(2,\mathbb{R})_\rho$ into $SO(2,2,\mathbb{R})$ in terms of 4×4 matrices acting as fractional linear transformations on the background matrix of the 2-torus, is given in [18] by

$$g_\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tilde{a} & 0 & 0 & -\tilde{b} \\ 0 & \tilde{a} & \tilde{b} & 0 \\ 0 & -\tilde{c} & \tilde{d} & 0 \\ \tilde{c} & 0 & 0 & \tilde{d} \end{pmatrix} \in SO(2,2,\mathbb{R}) \,, \tag{4.80}$$

$$g_\rho : E \longrightarrow E' = \frac{aE+b}{cE+d} \quad \text{with} \quad E = \begin{pmatrix} G_{11} & G_{12}+B_{12} \\ G_{12}-B_{12} & G_{22} \end{pmatrix} . \tag{4.81}$$

It is now easy to see that deformation (4.78), introduced in [2], is the specific $SL(2,\mathbb{R})_\rho$ transformation (4.79) for which $\tilde{a} = \tilde{d} = 1, \tilde{b} = 0$ and $\tilde{c} = \gamma$. The corresponding embedding (4.80) precisely reduces to the expression of a TsT-transformation defined in (3.29).

Let us make two closing remarks. The first one concerns the AdS/CFT correspondence. It is widely believed that type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ conformal Super Yang-Mills (SYM). The LM deformation (4.27) of the 5-sphere is, as explained above, dual to a β-deformation. Since the deformation on the gravity side does not affect AdS_5, it should not be surprising that the corresponding β-deformation does not break conformal symmetry[3] on the field theory side. Furthermore, the LM deformation breaks $\frac{3}{4}$ of the

[3] These types of deformation are called marginal.

spacetime supersymmetries of the $AdS_5 \times \mathrm{S}^5$ background. The type IIB string theory on the LM background is then dual to a $\mathcal{N}=1$ conformal SYM. Various tests of this duality have already been performed (see for example [5] and [6]).

$$
\begin{array}{llcl}
\textbf{Gravity:} & \text{type IIB strings on } AdS_5 \times S^5 & \xrightarrow{\textit{LM deformation}} & \text{type IIB strings on } AdS_5 \times S^5_\gamma \\
 & \Updownarrow & \Uparrow & \Updownarrow \\
 & & \textbf{AdS}_5/\textbf{CFT}_4 \textbf{ correspondence} & \\
 & & \Downarrow & \\
\textbf{Field theory:} & \mathcal{N}=4 \text{ conformal SYM} & \xrightarrow[\textit{Marginal } \beta\textit{-deformation}]{} & \mathcal{N}=1 \text{ conformal SYM}
\end{array}
$$

As a second remark, let us mention that type IIB supergravity is originally invariant under $SL(2,R)_s$ transformations associated to S-duality. We point the reader to [21] for the derivation of a $SL(2,\mathbb{R})_s$ invariant type IIB supergravity action. When compactified on a 2-torus, type IIB supergravity becomes invariant under $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})_\tau$. The $SL(3,\mathbb{R})$ symmetry is the result of the combination of the $SL(2,\mathbb{R})_s$ and $SL(2,\mathbb{R})_\rho$ symmetries. In [2], Lunin and Maldacena also used $SL(3,\mathbb{R})$ transformations to generate more general type IIB solutions.

4.3 The equations of motion

Our goal is now to verify that the LM background is indeed a consistent background. In what follows, we will first derive the complete equations of motion for the six spacetime fields arising from the action (2.39), then verify that they are satisfied if one plugs in the LM background. They will be presented in a way that allows immediate numerical treatment, c.f Appendix B. In particular, we will prefer normal derivatives to covariant ones throughout the whole computation. For simplicity, we also set $\phi_0 = 0$.

4.3.1 The axion

This is the simplest equation of motion,

$$\partial_\mu \left(\sqrt{-G} \frac{\partial \mathcal{L}_{\text{Ramond}}}{\partial(\partial_\mu \chi)} \right) = \sqrt{-G} \frac{\partial \mathcal{L}_{\text{Ramond}}}{\partial \chi} \, . \tag{4.82}$$

Developping both side yields

$$\partial_\mu \left(-\sqrt{-G} F^\mu_{(1)} \right) = \partial_\mu \left(-\sqrt{-G} (\partial^\mu \chi) \right) = \frac{\sqrt{-G}}{6} F_{(3)\nu\rho\lambda} H^{\nu\rho\lambda} \, . \tag{4.83}$$

For the LM background, the right hand side vanishes since $\chi = 0$. The left hand side also vanishes as the contraction of the two field strengths is null.

4.3.2 The dilaton

The equation of motion for the dilaton is naturally the same one as in the bosonic string case

$$\partial_\mu \left(\sqrt{-G} \frac{\partial \mathcal{L}'}{\partial \left(\partial_\mu \phi \right)} \right) = \sqrt{-G} \frac{\partial \mathcal{L}'}{\partial \phi} \, . \tag{4.84}$$

The left hand side of (4.84) is equal to

$$\begin{aligned}
\partial_\mu \left(8\sqrt{-G} e^{-2\phi} G^{\mu\nu} \left(\partial_\nu \phi \right) \right) &= -16\sqrt{-G} e^{-2\phi} G^{\mu\nu} \left(\partial_\mu \phi \right) \left(\partial_\nu \phi \right) + 8e^{-2\phi} \partial_\mu \left(\sqrt{-G} G^{\mu\nu} \left(\partial_\nu \phi \right) \right) \\
&= -16\sqrt{-G} e^{-2\phi} G^{\mu\nu} \left(\partial_\mu \phi \right) \left(\partial_\nu \phi \right) + 4e^{-2\phi} \sqrt{-G} \left(\partial_\mu G_{\rho\delta} \right) G^{\rho\delta} G^{\mu\nu} \left(\partial_\nu \phi \right) \\
&+ 8e^{-2\phi} \sqrt{-G} \left(\partial_\mu G^{\mu\nu} \right) \left(\partial_\nu \phi \right) + 8e^{-2\phi} \sqrt{-G} G^{\mu\nu} \partial_\mu \left(\partial_\nu \phi \right) \, ,
\end{aligned} \tag{4.85}$$

and the right hand side is equal to

$$-8\sqrt{-G}e^{-2\phi}G^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right)-2\sqrt{-G}e^{-2\phi}\left(\mathrm{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right). \tag{4.86}$$

The equation of motion for the dilaton (4.84) then becomes

$$2\left(\partial_\mu G_{\rho\delta}\right)G^{\rho\delta}G^{\mu\nu}\left(\partial_\nu\phi\right)+4G^{\mu\nu}\partial_\mu\left(\partial_\nu\phi\right)-4G^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right)+4\left(\partial_\mu G^{\mu\nu}\right)\left(\partial_\nu\phi\right)=\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}-\mathrm{R}. \tag{4.87}$$

We now substitute the LM background in (4.87). An immediate consequence is that the last term of the left hand side drops out. The final form of the equation that we obtain and need to check is the following one

$$\frac{\left(\partial_\mu G_{\rho\delta}\right)G^{\rho\delta}G^{\mu\nu}\left(\partial_\nu J\right)}{J}+2\frac{G^{\mu\nu}\partial_\mu\left(\partial_\nu J\right)}{J}-3\frac{G^{\mu\nu}\left(\partial_\mu J\right)\left(\partial_\nu J\right)}{J^2}=\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}-\mathrm{R}, \tag{4.88}$$

where,

$$\begin{aligned}H_{\mu\nu\rho}H^{\mu\nu\rho}=&6\left(H_{\alpha\phi_1\phi_2}H^{\alpha\phi_1\phi_2}+H_{\theta\phi_1\phi_2}H^{\theta\phi_1\phi_2}+H_{\alpha\phi_2\phi_3}H^{\alpha\phi_2\phi_3}\right.\\&\left.+H_{\theta\phi_2\phi_3}H^{\theta\phi_2\phi_3}+H_{\alpha\phi_1\phi_3}H^{\alpha\phi_1\phi_3}+H_{\theta\phi_1\phi_3}H^{\theta\phi_1\phi_3}\right).\end{aligned} \tag{4.89}$$

The Ricci scalar is computed numerically. Equation (4.88) is satisfied.

4.3.3 The $C_{(4)}$ field

The only parts of the lagrangian contributing to the equations of motion for the $C_{(4)}$ field are $\mathcal{L}_{\text{Ramond}}$ and $\mathcal{L}_{\text{CS}}$. Hence,

$$\partial_\mu\left(\sqrt{-G}\frac{\partial\mathcal{L}_{\text{Ramond}}}{\partial\left(\partial_\mu C_{(4)\nu\rho\lambda\sigma}\right)}\right)=\sqrt{-G}\frac{\partial\mathcal{L}_{\text{CS}}}{\partial C_{(4)\nu\rho\lambda\sigma}}. \tag{4.90}$$

With (2.41), the left hand side yields

$$\begin{aligned}\frac{-2}{4.5!}\partial_\mu\left(\sqrt{-G}F_{(5)}^{\zeta\iota\delta\eta\xi}\frac{\partial F_{(5)\zeta\iota\delta\eta\xi}}{\partial\left(\partial_\mu C_{(4)\nu\rho\lambda\sigma}\right)}\right)&=\frac{-2}{4.5!}\partial_\mu\left(\sqrt{-G}F_{(5)}^{\zeta\iota\delta\eta\xi}\frac{\partial(\mathrm{d}C_{(4)}-C_{(2)}\wedge H)_{\zeta\iota\delta\eta\xi}}{\partial\left(\partial_\mu C_{(4)\nu\rho\lambda\sigma}\right)}\right)\\&=\frac{-1}{2.5!}\partial_\mu\left(\sqrt{-G}F_{(5)}^{\zeta\iota\delta\eta\xi}\frac{\partial(\partial_{[\zeta}C_{(4)\iota\delta\eta\xi]})}{\partial\left(\partial_\mu C_{(4)\nu\rho\lambda\sigma}\right)}\right)\\&=\frac{-1}{2.4!}\partial_\mu\left(\sqrt{-G}F_{(5)}^{\zeta\iota\delta\eta\xi}\frac{\partial(\partial_\zeta C_{(4)\iota\delta\eta\xi})}{\partial\left(\partial_\mu C_{(4)\nu\rho\lambda\sigma}\right)}\right)\\&=\frac{-1}{2.4!}\partial_\mu\left(\sqrt{-G}F_{(5)}^{\mu\nu\rho\lambda\sigma}\right).\end{aligned} \tag{4.91}$$

Using (2.42) in (4.90), the equations of motion for $C_{(4)}$ simplify to

$$\begin{aligned}\sqrt{-G}\left[\frac{1}{2}\left(\partial_\mu G_{\eta\zeta}\right)G^{\eta\zeta}F_{(5)}^{\mu\nu\rho\lambda\sigma}+\partial_\mu F_{(5)}^{\mu\nu\rho\lambda\sigma}\right]&=-\frac{1}{4}\epsilon^{\nu\rho\lambda\sigma\delta\zeta\eta\iota\xi\tau}\left(\partial_\delta B_{\zeta\eta}\right)\left(\partial_\iota C_{(2)\xi\tau}\right)\\&=0.\end{aligned} \tag{4.92}$$

The term on the right hand side vanishes as H and $F_{(3)}$ only have non-vanishing components on the deformed sphere. Let us now focus on the left hand side. Since this expression is fully antisymmetric in ν,ρ,σ,δ we are left with 210 equations to check.

- Mixed indices (e.g. $\nu,\rho\in AdS_5$, $\lambda,\sigma\in S^5_\gamma$) : the right hand side vanishes as $F_{(5)}^{\mu\nu\rho\lambda\sigma}=0$.
- $\nu,\rho,\lambda,\sigma\in AdS_5$: this case splits again into two possibilities. If one of the indices is equal to x_0, the right

hand side vanishes as the components of $F_{(5)}$ and G are independent of x_1, x_2, x_3, x_4. We are then left with the second possibility $\nu = x_1, \rho = x_2, \lambda = x_3$ and $\sigma = x_4$. The right hand side becomes

$$\begin{aligned}
\frac{1}{2}\left(\partial_{x_0} G_{\eta\zeta}\right) G^{\eta\zeta} F_{(5)}^{x_0x_1x_2x_3x_4} + \partial_{x_0} F_{(5)}^{x_0x_1x_2x_3x_4} &= \frac{1}{2}\left(\frac{-10}{x_0}\right) F_{(5)}^{x_0x_1x_2x_3x_4} + \partial_{x_0} F_{(5)}^{x_0x_1x_2x_3x_4} \\
&= \left(\frac{-5}{x_0} + \partial_{x_0}\right)\left(\frac{-x_0^{10}}{R^{10}} 4R^4 \sqrt{-G_{\mathrm{AdS}5_1}}\right) \\
&= \left(\frac{-5}{x_0} + \partial_{x_0}\right)\left(\frac{-4x_0^5}{R^6}\right) = 0\,. \qquad (4.93)
\end{aligned}$$

- $\underline{\nu, \rho, \lambda, \sigma \in S^5_\gamma}$: this case splits into three possibilities. If two indices are equal to α and θ, then the right hand side vanishes as the components of G and $F_{(5)}$ are independent of ϕ_1, ϕ_2, ϕ_3. The two remaining possibilities are

$$\frac{1}{2}\left(\partial_\alpha G_{\eta\zeta}\right) G^{\eta\zeta} F_{(5)}^{\alpha\theta\phi_1\phi_2\phi_3} + \partial_\alpha F_{(5)}^{\alpha\theta\phi_1\phi_2\phi_3} = 0\,, \qquad (4.94)$$

$$\frac{1}{2}\left(\partial_\theta G_{\eta\zeta}\right) G^{\eta\zeta} F_{(5)}^{\theta\alpha\phi_1\phi_2\phi_3} + \partial_\theta F_{(5)}^{\theta\alpha\phi_1\phi_2\phi_3} = 0\,. \qquad (4.95)$$

These relations are checked with mathematica and are indeed satisfied.

4.3.4 The $C_{(2)}$ field

As for the $C_{(4)}$ field, only $\mathcal{L}_{\mathrm{Ramond}}$ and $\mathcal{L}_{\mathrm{CS}}$ contribute to the equations of motion

$$\partial_\mu\left(\frac{\partial(\sqrt{-G}\mathcal{L}_{\mathrm{Ramond}} + \mathcal{L}_{\mathrm{CS}})}{\partial\left(\partial_\mu C_{(2)\nu\rho}\right)}\right) = \sqrt{-G}\frac{\partial\mathcal{L}_{\mathrm{Ramond}}}{\partial C_{(2)\nu\rho}}\,. \qquad (4.96)$$

Let us first focus on the left hand side

$$\begin{aligned}
\partial_\mu\left(\frac{\partial(\sqrt{-G}\mathcal{L}_{\mathrm{Ramond}} + \mathcal{L}_{\mathrm{CS}})}{\partial\left(\partial_\mu C_{(2)\nu\rho}\right)}\right) &= \partial_\mu\left(\frac{-\sqrt{-G}}{6} F_{(3)}^{\lambda\sigma\delta}\frac{\partial F_{(3)\lambda\sigma\delta}}{\partial\left(\partial_\mu C_{(2)\nu\rho}\right)}\right) + \partial_\mu\left(\frac{1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\eta\zeta\tau\mu\nu\rho} C_{(4)\lambda\sigma\delta\iota}(\partial_\eta B_{\zeta\tau})\right) \\
&= \partial_\mu\left(\frac{-\sqrt{-G}}{6} F_{(3)}^{\lambda\sigma\delta}\frac{\partial(\partial_{[\lambda} C_{(2)\sigma\delta]})}{\partial\left(\partial_\mu C_{(2)\nu\rho}\right)}\right) + \frac{1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\eta\zeta\tau\mu\nu\rho}(\partial_\mu C_{(4)\lambda\sigma\delta\iota})(\partial_\eta B_{\zeta\tau}) \\
&= \frac{-\sqrt{-G}}{2}\left(\frac{1}{2}\left(\partial_\mu G_{\eta\zeta}\right) G^{\eta\zeta} F_{(3)}^{\mu\nu\rho} + \partial_\mu F_{(3)}^{\mu\nu\rho}\right) \\
&\quad + \frac{1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\eta\zeta\tau\mu\nu\rho}(\partial_\mu C_{(4)\lambda\sigma\delta\iota})(\partial_\eta B_{\zeta\tau})\,. \qquad (4.97)
\end{aligned}$$

Expanding the right hand side gives

$$\begin{aligned}
\sqrt{-G}\frac{\partial\mathcal{L}_{\mathrm{Ramond}}}{\partial C_{(2)\nu\rho}} &= -\frac{\sqrt{-G}}{2.5!} F_{(5)}^{\lambda\sigma\delta\eta\iota}\frac{\partial F_{(5)\lambda\sigma\delta\eta\iota}}{\partial C_{(2)\nu\rho}} = \frac{\sqrt{-G}}{2.5!} F_{(5)}^{\lambda\sigma\delta\eta\iota}\frac{\partial(C_{(2)[\lambda\sigma}\partial_\delta B_{\eta\iota]})}{\partial C_{(2)\nu\rho}} \\
&= \frac{\sqrt{-G}}{8} F_{(5)}^{\nu\rho\delta\eta\iota}(\partial_\delta B_{\eta\iota})\,. \qquad (4.98)
\end{aligned}$$

With (4.97) and (4.98), the equations of motion (4.96) take the final form

$$\sqrt{-G}\left(\frac{-1}{4}\left(\partial_\mu G_{\eta\zeta}\right) G^{\eta\zeta} F_{(3)}^{\mu\nu\rho} - \frac{1}{2}\partial_\mu F_{(3)}^{\mu\nu\rho} - \frac{1}{8} F_{(5)}^{\nu\rho\delta\eta\iota}(\partial_\delta B_{\eta\iota})\right) = \frac{-1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\eta\zeta\tau\mu\nu\rho}(\partial_\mu C_{(4)\lambda\sigma\delta\iota})(\partial_\eta B_{\zeta\tau})\,. \qquad (4.99)$$

This equation is antisymmetric in ν and ρ, i.e., we are left with 45 equations to check. Let us then start and plug in the LM background. Once again, the study splits into three main cases. Two of them can be treated by hand.

• $\nu, \rho \in AdS_5$: the components of the field strength $F_{(3)}$ are zero and we are left with

$$\sqrt{-G}F^{abcde}(\partial_c B_{de})) = \frac{1}{4!}\epsilon^{\hat{b}\hat{c}\hat{d}\hat{e}cde\hat{a}ab}(\partial_{\hat{a}} C_{(4)\hat{b}\hat{c}\hat{d}\hat{e}})(\partial_c B_{de})\,, \tag{4.100}$$

since $\partial_\mu C_{(4)\lambda\sigma\delta\iota} = 0$ for mixed indices. Both sides of (4.100) vanish since B doesn't have components on AdS_5.

• $\nu \in AdS_5,\ \rho \in S^5_\gamma$: for the same reasons (supplemented by $F^{\nu\rho\delta\eta\iota}_{(5)} = 0$ for mixed indices), all the terms vanish again independently.

• $\nu, \rho \in S^5_\gamma$: for $\nu = \alpha$, $\rho = \theta$, all the terms vanish independently since the components of both the metric and the B field are independent of ϕ_1, ϕ_2, ϕ_3. For $\nu, \rho = \{\phi_1, \phi_2, \phi_3\}$, it is possible to prove, with the same types of arguments, that all terms are once again null. We are finally left with the possibilities $\nu \in \{\alpha, \theta\}$ and $\rho \in \{\phi_1, \phi_2, \phi_3\}$ which correspond to 6 equations of motion. These are treated with mathematica. Let us simplify, as an example, the case $\nu = \alpha$, $\rho = \phi_1$. The left hand side of (4.99) becomes

$$\sqrt{-G}\left(\frac{-1}{4}(\partial_\theta G_{\eta\zeta})G^{\eta\zeta}F^{\theta\alpha\phi_1}_{(3)} - \frac{1}{2}\partial_\theta F^{\theta\alpha\phi_1}_{(3)} - \frac{1}{8}F^{\theta\alpha\phi_1\phi_2\phi_3}_{(5)}(2.\partial_\theta B_{\phi_2\phi_3})\right)\,, \tag{4.101}$$

while the right hand side becomes

$$\begin{aligned}\frac{1}{8.4!}\epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\theta\phi_2\phi_3\alpha\phi_1}(\partial_{\hat{a}} C_{(4)\hat{b}\hat{c}\hat{d}\hat{e}})(2.\partial_\theta B_{\phi_2\phi_3}) &= -\frac{4.2.R^4\sqrt{-G_{\mathrm{AdS}_{5_1}}}}{5.8.4!}\epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\alpha\theta\phi_1\phi_2\phi_3}\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\partial_\theta B_{\phi_2\phi_3})\\ &= -R^4\sqrt{-G_{\mathrm{AdS}_{5_1}}}(\partial_\theta B_{\phi_2\phi_3})\,,\end{aligned} \tag{4.102}$$

where we used that $5\partial_{\hat{a}} C_{(4)\hat{b}\hat{c}\hat{d}\hat{e}} = F_{(5)\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$. It is possible to prove numerically that both sides are equal.

4.3.5 The B field

The equations of motion for the B field take the following form

$$\partial_\mu\left(\frac{\partial(\sqrt{-G}(\mathcal{L}' + \mathcal{L}_{\mathrm{Ramond}}) + \mathcal{L}_{\mathrm{CS}})}{\partial(\partial_\mu B_{\nu\rho})}\right) = 0\,. \tag{4.103}$$

With (2.40), (2.41) and (2.42), the left hand side yields

$$\begin{aligned}\partial_\mu\left(\frac{\partial(\sqrt{-G}(\mathcal{L}' + \mathcal{L}_{\mathrm{Ramond}}) + \mathcal{L}_{\mathrm{CS}})}{\partial(\partial_\mu B_{\nu\rho})}\right) &= \partial_\mu\left(\sqrt{-G}\left(\frac{-e^{-2\phi}H^{\lambda\sigma\delta}}{6}\frac{\partial H_{\lambda\sigma\delta}}{\partial(\partial_\mu B_{\nu\rho})} - \frac{F^{\lambda\sigma\delta\eta\iota}_{(5)}}{2.5!}\frac{\partial F_{(5)\lambda\sigma\delta\eta\iota}}{\partial(\partial_\mu B_{\nu\rho})}\right)\right.\\ &\quad\left.+\frac{1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\mu\nu\rho\eta\zeta\tau}C_{(4)\lambda\sigma\delta\iota}(\partial_\eta C_{(2)\zeta\tau})\right)\\ &= \partial_\mu\left(\sqrt{-G}\left(\frac{-e^{-2\phi}}{2}H^{\mu\nu\rho} + \frac{1}{8}F^{\lambda\sigma\mu\nu\rho}_{(5)}C_{(2)\lambda\sigma}\right)\right)\\ &\quad + \frac{1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\mu\nu\rho\eta\zeta\tau}(\partial_\mu C_{(4)\lambda\sigma\delta\iota})(\partial_\eta C_{(2)\zeta\tau})\\ &= \frac{\sqrt{-G}}{8}\left(\frac{1}{2}(\partial_\mu G_{\eta\zeta})G^{\eta\zeta}F^{\lambda\sigma\mu\nu\rho}_{(5)}C_{(2)\lambda\sigma} + \partial_\mu F^{\lambda\sigma\mu\nu\rho}_{(5)}C_{(2)\lambda\sigma}\right) \quad (4.104)\\ &\quad + \sqrt{-G}\left(\frac{-1}{4}(\partial_\mu G_{\eta\zeta})G^{\eta\zeta}e^{-2\phi}H^{\mu\nu\rho} + \frac{1}{8}F^{\lambda\sigma\mu\nu\rho}_{(5)}(\partial_\mu C_{(2)\lambda\sigma})\right)\\ &\quad + \sqrt{-G}\left((\partial_\mu\phi)e^{-2\phi}H^{\mu\nu\rho} - \frac{e^{-2\phi}}{2}\partial_\mu H^{\mu\nu\rho}\right)\\ &\quad + \frac{1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\mu\nu\rho\eta\zeta\tau}(\partial_\mu C_{(4)\lambda\sigma\delta\iota})(\partial_\eta C_{(2)\zeta\tau})\,.\end{aligned} \tag{4.105}$$

The two terms in (4.104) cancel against each other as they precisely correspond to the equations of motion for the $C_{(4)}$ field (4.92). Bearing this in mind, the equations of motion (4.103) become

$$\begin{aligned}\sqrt{-G}\left(\frac{-1}{4}\left(\partial_\mu G_{\eta\zeta}\right)G^{\eta\zeta}e^{-2\phi}H^{\mu\nu\rho}+\frac{1}{8}F_{(5)}^{\lambda\sigma\mu\nu\rho}(\partial_\mu C_{(2)\lambda\sigma})+(\partial_\mu\phi)e^{-2\phi}H^{\mu\nu\rho}-\frac{e^{-2\phi}}{2}\partial_\mu H^{\mu\nu\rho}\right)\\ =-\frac{1}{8.4!}\epsilon^{\lambda\sigma\delta\iota\mu\nu\rho\eta\zeta\tau}(\partial_\mu C_{(4)\lambda\sigma\delta\iota})(\partial_\eta C_{(2)\zeta\tau})\,.\end{aligned} \tag{4.106}$$

These equations are antisymmetric in ν and ρ such that, as for the $C_{(2)}$ field, one has to verify here 45 equations. We substitute the LM background. We will not go again over the different cases in details as the arguments are the same as for the $C_{(4)}$ and $C_{(2)}$ fields (with the additional observation that the field strength H only has components on S^5_γ). It turns out, after a quick analysis of (4.106), that if ν and/or ρ belong to AdS_5, all the terms vanish independently. The same goes for $\nu,\rho\in\{\alpha,\theta\}$ and $\nu\in\{\alpha,\theta\},\ \rho\in\{\phi_1,\phi_2,\phi_3\}$. Therefore, we are this time left with 3 equations of motion to compute numerically. They correspond to $\nu,\rho\in\{\phi_1,\phi_2,\phi_3\}$. As an example, we give the explicit form of (4.106) for the case $\nu=\phi_1,\ \rho=\phi_2$. The left hand side is

$$\begin{aligned}\sqrt{-G}\Bigg(\frac{-1}{4J}\left(\partial_\alpha G_{\eta\zeta}\right)G^{\eta\zeta}H^{\alpha\phi_1\phi_2}-\frac{1}{4J}\left(\partial_\theta G_{\eta\zeta}\right)G^{\eta\zeta}H^{\theta\phi_1\phi_2}+\frac{2}{8}F_{(5)}^{\alpha\theta\phi_1\phi_2\phi_3}F_{(3)\alpha\theta\phi_3}\\ +\frac{1}{2J^2}\left((\partial_\alpha J)H^{\alpha\phi_1\phi_2}+(\partial_\theta J)H^{\theta\phi_1\phi_2}\right)-\frac{1}{2J}\left(\partial_\alpha H^{\alpha\phi_1\phi_2}+\partial_\theta H^{\theta\phi_1\phi_2}\right)\Bigg)\,,\end{aligned} \tag{4.107}$$

while the right hand side reads

$$\begin{aligned}-\frac{1}{8.4!}\epsilon^{\hat a\hat b\hat c\hat d\hat e\phi_1\phi_2\alpha\theta\phi_3}(\partial_{\hat a}C_{(4)\hat b\hat c\hat d\hat e})(2.F_{(3)\alpha\theta\phi_3}) &= -\frac{4.2.R^4\sqrt{-G_{\mathrm{AdS}_{5_1}}}}{5.8.4!}\epsilon^{\hat a\hat b\hat c\hat d\hat e\alpha\theta\phi_1\phi_2\phi_3}\epsilon_{\hat a\hat b\hat c\hat d\hat e}F_{(3)\alpha\theta\phi_3}\\ &= -R^4\sqrt{-G_{\mathrm{AdS}_{5_1}}}F_{(3)\alpha\theta\phi_3}\,.\end{aligned} \tag{4.108}$$

Numerics show that (4.107) cancels precisely against (4.108).

4.3.6 Einstein's equations

We have already derived Einstein's equations for a bosonic string background from the appropriate action (2.21). Comparing the latter with the action (2.39), one immediately notices that Einstein's equations of type IIB supergravity will be those of (2.17) with an additional term for the energy-momentum tensor coming from $\mathcal{L}_{\mathrm{Ramond}}$.

$$R_{\mu\nu}-\frac{1}{2}G_{\mu\nu}\mathrm{R}=T_{\mu\nu}\,, \tag{4.109}$$

$$T_{\mu\nu}=\frac{1}{4}\left(H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda}-\frac{G_{\mu\nu}}{6}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right)-2\nabla_\mu\nabla_\nu\phi+2G_{\mu\nu}\nabla^2\phi-2G_{\mu\nu}(\nabla\phi)^2-\frac{e^{2\phi}}{\sqrt{-G}}\frac{\partial\left(\sqrt{-G}\mathcal{L}_{\mathrm{Ramond}}\right)}{\partial G^{\mu\nu}}\,. \tag{4.110}$$

The factor $e^{2\phi}$ in front of the new term is probably a bit puzzling for the reader at this point. Its presence is due to the fact that the rest of the energy-momentum tensor's terms were derived from $\mathcal{L}'$, which contains a factor $e^{-2\phi}$. With (2.41), one gets

$$-\frac{e^{2\phi}}{\sqrt{-G}}\frac{\partial\left(\sqrt{-G}\mathcal{L}_{\mathrm{Ramond}}\right)}{\partial G^{\mu\nu}}=e^{2\phi}\left(\frac{1}{2}(\partial_\mu\chi)(\partial_\nu\chi)+\frac{1}{4}F_{(3)\mu\rho\lambda}F_{(3)\nu}{}^{\rho\lambda}+\frac{1}{4.4!}F_{(5)\mu\rho\lambda\sigma\delta}F_{(5)\nu}{}^{\rho\lambda\sigma\delta}\right) \tag{4.111}$$

$$+\frac{1}{2}G_{\mu\nu}e^{2\phi}\left(-\frac{1}{2}\left(\partial_\rho\chi\right)\left(\partial^\rho\chi\right)-\frac{1}{12}F_{(3)\rho\lambda\sigma}F_{(3)}^{\rho\lambda\sigma}-\frac{1}{4.5!}F_{(5)\rho\lambda\sigma\delta\eta}F_{(5)}^{\rho\lambda\sigma\delta\eta}\right)\,. \tag{4.112}$$

The last term of (4.112) drops out as the 5-form field strength $F_{(5)}$ is self-dual. It was, however, necessary to keep it up to this point in order to obtain the last term of (4.111). The explicit form of the $\nabla_\mu \nabla_\nu \phi$ term is

$$\nabla_\mu \nabla_\nu \phi = \nabla_\mu (\partial_\nu \phi) = \partial_\mu (\partial_\nu \phi) - \Gamma^\rho_{\mu\nu} (\partial_\rho \phi) \,, \qquad \text{with} \qquad \Gamma^\rho_{\mu\nu} = \frac{1}{2} G^{\rho\lambda} \left(\partial_\mu G_{\nu\lambda} + \partial_\nu G_{\mu\lambda} - \partial_\lambda G_{\mu\nu} \right) \,. \quad (4.113)$$

The final form of the energy momentum tensor is therefore

$$\begin{aligned} T_{\mu\nu} = G_{\mu\nu} & \left(2\partial_\rho (\partial^\rho \phi) - 2G^{\rho\lambda} \Gamma^\sigma_{\rho\lambda} (\partial_\sigma \phi) - 2(\partial_\rho \phi)(\partial^\rho \phi) - \frac{1}{24} H_{\rho\lambda\sigma} H^{\rho\lambda\sigma} - \frac{e^{2\phi}}{4} F_{(1)\rho} F^\rho_{(1)} - \frac{e^{2\phi}}{24} F_{(3)\rho\lambda\sigma} F^{\rho\lambda\sigma}_{(3)} \right) \\ & - 2\partial_\mu (\partial_\nu \phi) + 2\Gamma^\rho_{\mu\nu} (\partial_\rho \phi) + \frac{1}{4} H_{\mu\rho\lambda} H_\nu{}^{\rho\lambda} + \frac{e^{2\phi}}{2} F_{(1)\mu} F_{(1)\nu} + \frac{e^{2\phi}}{4} F_{(3)\mu\rho\lambda} F_{(3)\nu}{}^{\rho\lambda} + \frac{e^{2\phi}}{4.4!} F_{(5)\mu\rho\lambda\sigma\delta} F_{(5)\nu}{}^{\rho\lambda\sigma\delta} \end{aligned} \quad (4.114)$$

<u>Remark</u> : As a proof of consistency, one should check that these equations are satisfied for the $AdS_5 \times S^5$ background with constant dilaton $\phi_0 \neq 0$. In this case, the Ricci scalar is zero as well as the field strengths H and $F_{(3)}$. Einstein's equations reduce to

$$R_{\mu\nu} = -\frac{4}{R^2} G_{\mu\nu} = \frac{e^{2\phi_0}}{4.4!} F_{(5)\mu\rho\lambda\sigma\delta} F_{(5)\nu}{}^{\rho\lambda\sigma\delta} \,. \quad (4.115)$$

If $\mu \neq \nu$, both sides trivially vanish. For $\mu = \nu$ and $\mu \in AdS_5$,

$$\begin{aligned} \frac{e^{2\phi_0}}{4.4!} F_{(5)\mu\rho\lambda\sigma\delta} F_{(5)\nu}{}^{\rho\lambda\sigma\delta} &= \frac{e^{2\phi_0}}{4} \left(F_{(5)x_0x_1x_2x_3x_4} \right)^2 \frac{G^{-1}_{\text{AdS}_5}}{G^{\mu\nu}} = 4e^{2\phi_0} R_E^8 \left(\sqrt{-G_{\text{AdS}_{5_1}}} \right)^2 \frac{G^{-1}_{\text{AdS}_5}}{G^{\mu\nu}} \\ &= -4R^8 \frac{1}{R^{10} G^{\mu\nu}} = -\frac{4}{R^2} G_{\mu\nu} \,. \end{aligned} \quad (4.116)$$

For $\mu \in S^5$, the proof is basically the same.

Finally, let us turn back to the LM background (and set back $\phi_0 = 0$). Two immediate consequences follow. Firstly, the Ricci scalar does not vanish anymore. Secondly, all terms containing the axion drop out. Although most cases will require numerical treatment, it is still possible to simplify a few of them by hand. Einstein's equations are symmetric in μ and ν and, therefore, correspond to 55 independent equations.

The Ricci tensor vanishes for mixed indices. Thus, the 25 equations corresponding to $\mu \in AdS_5,\ \nu \in S^5$ are trivially satisfied as both sides vanish. Two main cases remain:

• <u>$\mu, \nu \in AdS_5$</u> : the Ricci tensor simplifies to $R_{\mu\nu} = -\frac{4}{R^2} G_{\mu\nu}$, which leads to the following Einstein's equations

$$\begin{aligned} -\frac{4}{R^2} G_{\mu\nu} - \frac{1}{2} G_{\mu\nu} \mathrm{R} = G_{\mu\nu} & \left(2\partial_\rho (\partial^\rho \phi) - 2G^{\rho\lambda} \Gamma^\sigma_{\rho\lambda} (\partial_\sigma \phi) - 2(\partial_\rho \phi)(\partial^\rho \phi) - \frac{1}{24} H_{\rho\lambda\sigma} H^{\rho\lambda\sigma} - \frac{e^{2\phi}}{24} F_{(3)\rho\lambda\sigma} F^{\rho\lambda\sigma}_{(3)} \right) \\ & + \frac{e^{2\phi}}{4.4!} F_{(5)\mu\rho\lambda\sigma\delta} F_{(5)\nu}{}^{\rho\lambda\sigma\delta} \,. \end{aligned} \quad (4.117)$$

If $\mu \neq \nu$ all the terms on both sides vanish. For the 5 cases corresponding to $\mu = \nu$, one should definitely use mathematica. To this purpose, we give here all the non-vanishing components (on S^5_γ only) of the symmetric 2-tensors $\Gamma^\alpha_{\rho\lambda}$ and $\Gamma^\theta_{\rho\lambda}$

$$\Gamma^\alpha_{\theta\theta} = -\frac{1}{2} G^{\alpha\alpha} (\partial_\alpha G_{\theta\theta}) \,, \quad (4.118)$$

$$\Gamma^\theta_{\theta\alpha} = \frac{1}{2} G^{\theta\theta} (\partial_\alpha G_{\theta\theta}) \,, \quad (4.119)$$

$$\Gamma^\theta_{\phi_i\phi_j} = -\frac{1}{2} G^{\theta\theta} (\partial_\theta G_{\phi_i\phi_j}) \,, \quad (4.120)$$

$$\Gamma^\alpha_{\phi_i\phi_j} = -\frac{1}{2} G^{\alpha\alpha} (\partial_\alpha G_{\phi_i\phi_j}) \,, \quad (4.121)$$

where $\phi_i, \phi_j \in \{\phi_1, \phi_2, \phi_3\}$. Numerics, once more, show that the equations (4.117) are satisfied.

• $\underline{\mu, \nu \in \mathrm{S}^5_\gamma}$: the Ricci tensor is null when $\mu \in \{\alpha, \theta\}$, $\nu \in \{\phi_1, \phi_2, \phi_3\}$. With this information it is possible to see that both sides of (2.39) vanish. We have now reached the last step: 9 equations have to be worked out. This is performed numerically. We choose here to expand entirely the case $\mu = \nu = \theta$, as none of the terms vanish in the expression of the energy momentum tensor (4.114),

$$\begin{aligned} R_{\theta\theta} - \frac{G_{\theta\theta}\mathrm{R}}{2} =& G_{\theta\theta}\left[\partial_\alpha\left(\frac{\partial^\alpha J}{J}\right) + \partial_\theta\left(\frac{\partial^\theta J}{J}\right) - G^{\theta\theta}\Gamma^\alpha_{\theta\theta}\frac{\partial_\alpha J}{J} - G^{\phi_i\phi_j}\Gamma^\alpha_{\phi_i\phi_j}\frac{\partial_\alpha J}{J} - G^{\phi_i\phi_j}\Gamma^\theta_{\phi_i\phi_j}\frac{\partial_\theta J}{J}\right. \\ &\left. - \frac{1}{2J^2}\left((\partial_\alpha J)(\partial^\alpha J) + (\partial_\theta J)(\partial^\theta J)\right) - \frac{1}{24}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma} - \frac{J}{3}F_{(3)\alpha\theta\phi_i}F^{\alpha\theta\phi_i}_{(3)}\right] - \partial_\theta\left(\frac{\partial_\theta J}{J}\right) \\ &+ \Gamma^\alpha_{\theta\theta}\frac{\partial_\alpha J}{J} + \frac{1}{2}\left(H_{\theta\phi_1\phi_2}H_\theta{}^{\phi_1\phi_2} + H_{\theta\phi_1\phi_3}H_\theta{}^{\phi_1\phi_3} + H_{\theta\phi_2\phi_3}H_\theta{}^{\phi_2\phi_3}\right) + \frac{J}{2}F_{(3)\theta\alpha\phi_i}F_{(3)\theta}{}^{\alpha\phi_i} \\ &+ \frac{J}{4}F_{(5)\theta\alpha\phi_1\phi_2\phi_3}F^{\theta\alpha\phi_1\phi_2\phi_3}_{(5)}\,, \end{aligned} \tag{4.122}$$

where the explicit expression of $H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}$ has been given in (4.89). The reader should be able to work out the remaining 8 equations. With two months of work and a probably a little bit of luck, numerics show that (4.122) is satisfied.

This ends our "quest" in verifying the consistency of the Lunin-Maldacena background. All the equations of motion of type IIB supergravity have been shown to hold. We can now move on and apply γ-deformations to the more recently discovered $AdS_2 \times S^2 \times T^6$ background.

Chapter 5

γ-deformations of the $AdS_2 \times S^2 \times T^6$ background

We now consider type II superstring theory on $AdS_2 \times S^2 \times T^6$ as introduced in [8]. There exist several solutions of type IIA and type IIB supergravity with the geometry of $AdS_2 \times S^2 \times T^6$, which are related by T-duality. We focus, not surprisingly, on a type IIB solution which has all its fields turned to zero except the R-R field strength $F_{(5)}$. The deformations discussed earlier will be applied in order to obtain new supergravity solutions. Although we are not ultimately interested in the superstring theory itself, let us still highlight some of its features that motivated the study of $AdS_2 \times S^2 \times T^6$ backgrounds in the first place.

The initial purpose for studying such a superstring theory is to better understand the $AdS_2 \times S^2$ background. The latter is of great importance as it describes the near-horizon geometry of extremal four-dimensional Reisner-Nordström black holes. This could therefore shed some light on the actual problems encountered when trying to understand the $\text{AdS}_2/\text{CFT}_1$ duality. The meaning of a CFT_1, or (super)conformal quantum mechanics is, indeed, still not very well understood. The second reason to study such a superstring theory is the issue of integrability. It is, in principle, always feasible, using the Green-Schwarz (GS) formalism, to construct an action (known as GS action) for a superstring theory on a R-R background. This is, however, a very complicated task. Nevertheless, in certain cases, it is possible to bypass this difficulty by observing that the GS action is equivalent to a supercoset sigma model. The most famous example is probably the superstring theory on $AdS_5 \times S^5$ described by $PSU(2,2|4)/SO(1,4) \times SO(5)$. The same applies to our case and it turns out that a four-dimensional GS superstring action on $AdS_2 \times S^2$ is given by the supercoset sigma model $PSU(1,1|2)/SO(1,1) \times U(1)$. The latter is known to be fully integrable at the classical level due to the $\mathbb{Z}_4$-structure of the $\mathfrak{psu}(1,1|2)$ superalgebra. However, the non-vanishing components of the field strength $F_{(5)}$ introduce a mixing between the "flat directions" of T^6 and the coset directions of $AdS_2 \times S^2$. The consequence, as explained in [8], is that one cannot represent the full string sigma model as the direct sum of the $PSU(1,1|2)/SO(1,1) \times U(1)$ supercoset and additional bosonic and fermionic degrees of freedom associated to T^6. One has, therefore, to rely on the GS approach. In [8], the GS action was constructed up to quadratic order in fermions. Furthermore, a Lax connection was derived using the (super)symmetric Noether currents associated with the unbroken (super)symmetries of the $AdS_2 \times S^2 \times T^6$ background. This led to the proof of the classical integrability of the full superstring theory. This specific approach to integrability, based on the amount of supersymmetries of the background, is particularly interesting to us since the γ-deformations tend to break some of these supersymmetries. We will come back, shortly, to this idea in the conclusion.

5.1 The supersymmetric $AdS_2 \times S^2 \times T^6$ type IIB solution

We consider the type IIB supergravity solution discussed in [8]. We choose the metric, in string frame, of the $AdS_2 \times S^2 \times T^6$ manifold as

$$\begin{aligned} ds_s^2 &= ds^2_{\mathrm{AdS}_2} + ds^2_{\mathrm{S}^2} + ds^2_{\mathrm{T}^6} \\ &= R^2\left(-\cosh^2(x_1)dx_0^2 + dx_1^2\right) + R^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) + \sum_{i=1}^{6} dy_i^2\,, \end{aligned} \tag{5.1}$$

where the coordinates (x_0, x_1), (θ, φ) and $(y_1, y_2, y_3, y_4, y_6)$ parametrize, respectively, AdS_2, S^2 and T^6. Once again, $\mu, \nu, \rho, ... \in \{x_0, x_1, \theta, \varphi, y_1, y_2, y_3, y_4, y_6\}$ are ten-dimensional spacetime indices such that, for instance, $G_{\varphi\varphi} = R^2\sin^2(\theta)$. Note that AdS_2 and S^2 share the same radius R, while the flat torus T^6 is the direct product of six circles with unit radius. The constant dilaton is by denoted ϕ_0 and the B field vanishes. Therefore, the torsion also vanishes

$$H = dB = 0\,. \tag{5.2}$$

The only non-vanishing R-R field strength is the 5-form $F_{(5)}$. Its expression is given in [8] in terms of the zehnbein. We choose a diagonal zehnbein as well as the “mostly plus” convention for the signature of the Minkowski metric of the ten-dimensional tangent space. This yields

$$F_{(1)} = d\chi = 0\,, \tag{5.3}$$

$$F_{(2)} = dC_{(2)} - H \wedge \chi = dC_{(2)} = 0\,, \tag{5.4}$$

$$F_{(5)} = dC_{(4)} - H \wedge C_{(2)} = dC_{(4)} = -e^{-\phi_0} R\cosh(x_1)\, dx_0 \wedge dx_1 \wedge \mathrm{Re}\left(\Omega_{(3)}\right) + \text{Hodge dual}\,, \tag{5.5}$$

where $\Omega_{(3)} = dz_1 \wedge dz_2 \wedge dz_3$ is the holomorphic 3-form on T^6 and where z_1, z_2, z_3 are the following holomorphic coordinates

$$z_1 = y_1 + iy_2\;; \quad z_2 = y_3 + iy_4\;; \quad z_3 = y_5 + iy_6\,. \tag{5.6}$$

One therefore obtains

$$F_{(5)} = w_{(5)} + \text{Hodge dual}\,, \tag{5.7}$$

with

$$\begin{aligned} w_{(5)} = -e^{-\phi_0} R\cosh(x_1)\, dx_0 \wedge dx_1 \wedge (& dy_1 \wedge dy_3 \wedge dy_5 - dy_1 \wedge dy_4 \wedge dy_6 \\ & - dy_2 \wedge dy_4 \wedge dy_5 - dy_2 \wedge dy_3 \wedge dy_6)\,. \end{aligned} \tag{5.8}$$

As in the $AdS_5 \times S^5$ case, the Hodge dual of a p-form w on $AdS_2 \times S^2 \times T^6$ is the $(10-p)$-form $w^\star$, defined as

$$w^\star_{\mu_1 ....\mu_p} = \frac{-1}{p!\sqrt{-G}} \epsilon^{\nu_1 ....\nu_p \nu_{p+1} ....\nu_{10}} G_{\mu_1\nu_1} ....G_{\mu_p\nu_p} w_{\nu_{p+1}....\nu_{10}}\,, \tag{5.9}$$

where the indices $\mu_1, ..., \mu_p, \nu_1, ..., \nu_{10} \in \{x_0, x_1, \theta, \varphi, y_1, y_2, y_3, y_4, y_6\}$ and ϵ is the fully antisymmetric tensor in ten dimensions, normalized to $\epsilon^{x_0 x_1 \theta, \varphi y_1 y_2 y_3 y_4 y_6} = 1$. As an example, we compute one of the non-vanishing components of $w^\star_{(5)}$

$$\begin{aligned} w^\star_{(5)\theta\varphi y_2 y_4 y_6} &= \frac{-1}{5!\sqrt{-G}} \epsilon^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9\nu_{10}} G_{\theta\nu_1} G_{\varphi\nu_2} G_{y_2\nu_3} G_{y_4\nu_4} G_{y_6\nu_5} w_{(5)\nu_6\nu_7\nu_8\nu_9\nu_{10}} \\ &= \frac{-1}{5!\sqrt{-G}} \epsilon^{\theta\varphi y_2 y_4 y_6 \nu_6\nu_7\nu_8\nu_9\nu_{10}} \left(R^4\sin^2(\theta)\right) w_{(5)\nu_6\nu_7\nu_8\nu_9\nu_{10}} \\ &= \frac{-1}{\sqrt{-G}} \left(R^4\sin^2(\theta)\right)\left(-e^{\phi_0} R\cosh(x_1)\right) = \frac{e^{\phi_0} R^5\cosh(x_1)\sin^2(\theta)}{\sqrt{R^8\cosh^2(x_1)\sin^2(\theta)}} \\ &= e^{-\phi_0} R\sin(\theta)\,. \end{aligned} \tag{5.10}$$

In the same manner, one obtains $w^{\star}_{(5)\theta\varphi y_2 y_3 y_5} = w^{\star}_{(5)\theta\varphi y_1 y_3 y_6} = w^{\star}_{(5)\theta\varphi y_1 y_4 y_5} = -e^{-\phi_0} R\sin(\theta)$ such that

$$\begin{aligned} w^{\star}_{(5)} = e^{-\phi_0} R\sin(\theta) d\theta \wedge d\varphi \wedge & \left(dy_2 \wedge dy_4 \wedge dy_6 - dy_2 \wedge dy_3 \wedge dy_5 \right. \\ & \left. -dy_1 \wedge dy_3 \wedge dy_6 - dy_1 \wedge dy_4 \wedge dy_5 \right) . \end{aligned} \tag{5.11}$$

The complete expression of the 5-form field strength is then

$$F_{(5)} = w_{(5)} + w^{\star}_{(5)} \,, \tag{5.12}$$

where the expressions of $w_{(5)}$ and $w^{\star}_{(5)}$ are, respectively, given by (5.8) and (5.11). Note that such a 5-form field strength makes this solution non-trivial, as it couples AdS_2 and T^6 as well as S^2 and T^6. Finally, it is important to mention that this solution preserves 8 spacetime supersymmetries.

In order to test the compatibility of our conventions and the ones from [8], we now show that the above expressions satisfy the type IIB supergravity equations of motion derived in section 4.3. It is, arguably, easy to see that the equations of motion for the axion, the dilaton, the B field and the $C_{(2)}$ field are satisfied as all the terms vanish independently. This is due to (5.2), (5.3), (5.4) and the fact that the dilaton is constant. We are then left with the equations of motion for the $C_{(4)}$ field and Eintein's equations.

Einstein's equations:

The Ricci tensor for $AdS_2 \times S^2 \times T^6$ can be computed easily and is equal to

$$R_{\mu\nu} = \begin{cases} -\frac{G_{\mu\nu}}{R^2} & \text{if } \mu,\nu \in \text{AdS}_2 \\ \frac{G_{\mu\nu}}{R^2} & \text{if } \mu,\nu \in \text{S}^2 \\ 0 & \text{otherwise .} \end{cases} \tag{5.13}$$

Therefore, the Ricci scalar vanishes, R$= G^{\mu\nu} R_{\mu\nu} = -2 + 2 = 0$. The energy momentum tensor (4.114), for the $AdS_2 \times S^2 \times T^6$ solution, reduces to

$$T_{\mu\nu} = \frac{e^{2\phi_0}}{4.4!} F_{(5)\mu\rho\lambda\sigma\delta} F_{(5)\nu}{}^{\rho\lambda\sigma\delta} \,. \tag{5.14}$$

Hence, Einstein's equations take the following simple form

$$R_{\mu\nu} = \frac{e^{2\phi_0}}{4.4!} F_{(5)\mu\rho\lambda\sigma\delta} F_{(5)\nu}{}^{\rho\lambda\sigma\delta} \,. \tag{5.15}$$

We split these 55 equations into three categories:

• $\underline{\mu \neq \nu}$: due to the structure of the 5-form $F_{(5)}$ the contraction $F_{(5)\mu\rho\lambda\sigma\delta} F_{(5)\nu}{}^{\rho\lambda\sigma\delta}$ vanishes. As mentioned in (5.13), the left hand side also vanishes.

• $\underline{\mu = \nu \text{ and } \mu,\nu \in T^6}$: let us consider the case $\mu = \nu = y_1$ in details. The remaining five cases are basically the same. The right hand side reads

$$\begin{aligned} T_{y_1 y_1} = \frac{e^{2\phi_0}}{4.4!} F_{(5) y_1 \rho\lambda\sigma\delta} F_{(5) y_1}{}^{\rho\lambda\sigma\delta} &= \frac{4! e^{2\phi_0}}{4.4!} \Big(\left(F_{y_1 x_0 x_1 y_3 y_5}\right)^2 G^{x_0 x_0} G^{x_1 x_1} G^{y_3 y_3} G^{y_5 y_5} \\ & \quad + \left(F_{y_1 x_0 x_1 y_4 y_6}\right)^2 G^{x_0 x_0} G^{x_1 x_1} G^{y_4 y_4} G^{y_6 y_6} \\ & \quad + \left(F_{y_1 \theta\varphi y_3 y_6}\right)^2 G^{\theta\theta} G^{\varphi\varphi} G^{y_3 y_3} G^{y_6 y_6} \\ & \quad + \left(F_{y_1 \theta\varphi y_4 y_5}\right)^2 G^{\theta\theta} G^{\varphi\varphi} G^{y_4 y_4} G^{y_5 y_5} \Big) \\ &= \frac{e^{2\phi_0}}{4} \left(2 \left(-e^{-\phi_0} R\cosh(x_1)\right)^2 \left(\frac{-1}{R^4 \cosh^2(x_1)} \right) \right. \\ & \quad \left. + 2 \left(e^{-\phi_0} R\sin(\theta)\right)^2 \left(\frac{1}{R^4 \sin^2(\theta)} \right) \right) \\ &= 0 \,. \end{aligned} \tag{5.16}$$

The corresponding Einstein's equation is satisfied since $R_{y_1 y_1} = 0$.

• $\mu = \nu$ and $\mu, \nu \in AdS_2 \times S^2$: we choose $\mu = \nu = \varphi$. Once more, the remaining three cases are basically the same. The right hand side is

$$\begin{aligned} T_{\varphi\varphi} = \frac{e^{2\phi_0}}{4.4!} F_{(5)\varphi\rho\lambda\sigma\delta} F_{(5)\varphi}{}^{\rho\lambda\sigma\delta} = \frac{4! e^{2\phi_0}}{4.4!} \Big(& \left(F_{\varphi\theta y_2 y_4 y_6}\right)^2 G^{\theta\theta} G^{y_2 y_2} G^{y_4 y_4} G^{y_6 y_6} \\ & + \left(F_{\varphi\theta y_2 y_3 y_5}\right)^2 G^{\theta\theta} G^{y_2 y_2} G^{y_3 y_3} G^{y_5 y_5} \\ & + \left(F_{\varphi\theta y_1 y_3 y_6}\right)^2 G^{\theta\theta} G^{y_1 y_1} G^{y_3 y_3} G^{y_6 y_6} \\ & + \left(F_{\varphi\theta y_1 y_4 y_5}\right)^2 G^{\theta\theta} G^{y_1 y_1} G^{y_4 y_4} G^{y_5 y_5} \Big) \\ = \frac{e^{2\phi_0}}{4} & \left(4 \left(-e^{-\phi_0} R \sin(\theta)\right)^2 \left(\frac{1}{R^2}\right)\right) \\ = \sin^2(\theta) \,. & \end{aligned} \tag{5.17}$$

The left hand side is $R_{\varphi\varphi} = \frac{G_{\varphi\varphi}}{R^2} = \sin^2(\theta)$ and the Einstein's equation is therefore satisfied.

The $C_{(4)}$ field:

For the $AdS_2 \times S^2 \times T^6$ solution, one can see from (4.92) that the equations of motion for the $C_{(4)}$ field reduce to

$$\frac{1}{2} \left(\partial_\mu G_{\eta\zeta}\right) G^{\eta\zeta} F_{(5)}^{\mu\nu\rho\lambda\sigma} + \partial_\mu F_{(5)}^{\mu\nu\rho\lambda\sigma} = 0 \,. \tag{5.18}$$

Both terms on the left hand side vanish trivially except for the two following non-trivial cases.

• $\nu = x_0, \rho, \lambda, \sigma \in T^6$: we pick $\rho = y_1, \lambda = y_3, \sigma = y_5$. The rest of the cases can be worked out similarly.

$$\begin{aligned} \frac{1}{2} \left(\partial_\mu G_{\eta\zeta}\right) G^{\eta\zeta} F_{(5)}^{\mu x_0 y_1 y_3 y_5} + \partial_\mu F_{(5)}^{\mu x_0 y_1 y_3 y_5} &= \frac{1}{2} \left(\partial_{x_1} G_{x_0 x_0}\right) G^{x_0 x_0} F_{(5)}^{x_1 x_0 y_1 y_3 y_5} + \partial_{x_1} F_{(5)}^{x_1 x_0 y_1 y_3 y_5} \\ &= \left(\tanh(x_1) + \partial_{x_1}\right) F_{(5)}^{x_1 x_0 y_1 y_3 y_5} \\ &= \left(\tanh(x_1) + \partial_{x_1}\right) \left(\frac{e^{-\phi_0} R \cosh(x_1)}{-R^4 \cosh^2(x_1)}\right) \\ &= 0 \,. \end{aligned} \tag{5.19}$$

• $\nu = \varphi, \rho, \lambda, \sigma \in T^6$: this time, we pick $\rho = y_2, \lambda = y_4, \sigma = y_6$. Once more, the rest of the cases can be worked out similarly.

$$\begin{aligned} \frac{1}{2} \left(\partial_\mu G_{\eta\zeta}\right) G^{\eta\zeta} F_{(5)}^{\mu\varphi y_2 y_4 y_6} + \partial_\mu F_{(5)}^{\mu\varphi y_2 y_4 y_6} &= \frac{1}{2} \left(\partial_\theta G_{\varphi\varphi}\right) G^{\varphi\varphi} F_{(5)}^{\theta\varphi y_2 y_4 y_6} + \partial_\theta F_{(5)}^{\theta\varphi y_2 y_4 y_6} \\ &= \left(\tan^{-1}(\theta) + \partial_\theta\right) F_{(5)}^{\theta\varphi y_2 y_4 y_6} \\ &= \left(\tan^{-1}(\theta) + \partial_\theta\right) \left(\frac{e^{-\phi_0} R \sin(\theta)}{R^4 \sin^2(\theta)}\right) \\ &= 0 \,. \end{aligned} \tag{5.20}$$

This proves that the type IIB $AdS_2 \times S^2 \times T^6$ background is indeed a consistent background. Let us now start applying γ-deformations in order to obtain new type IIB supergravity solutions.

5.2 Deformations of $AdS_2 \times S^2 \times T^6$

The $AdS_2 \times S^2 \times T^6$ manifold possesses seven[1] $U(1)$ global isometries realized as constant shifts of the angle coordinates

$$\varphi \longrightarrow \varphi + \alpha_0 \,, \qquad y_i \longrightarrow y_i + \alpha_i \,, \;\; i = \{1, ..., 6\} \,, \qquad \alpha_0, \alpha_1, ..., \alpha_6 \in \mathbb{R} \,. \tag{5.21}$$

These isometries are also symmetries of the whole $AdS_2 \times S^2 \times T^6$ solution as the expression of the 5-form field strength (5.12) is invariant under the transformations (5.21). The geometry of the solution contains an internal sector parametrized by the isometry angles $\varphi, y_1, y_2, y_3, y_4, y_5, y_6$ which is fibered over a non-compact sector parametrized by the coordinates x_0, x_1, θ. The internal sector can be seen as the 7-torus resulting from the direct product $S^1_{(\varphi)} \times T^6$, where $S^1_{(\varphi)} \subset S^2$ is parametrized by the angle φ. The 7×7 background matrix of the internal sector is therefore

$$E = G + B = \mathrm{diag}\left(G_{\varphi\varphi}, G_{y_1y_1}, G_{y_2y_2}, G_{y_3y_3}, G_{y_4y_4}, G_{y_5y_5}, G_{y_6y_6}\right) = \left(\begin{array}{c|c} R^2 \sin^2(\theta) & 0 \\ \hline 0 & \mathbb{1}_{6\times 6} \end{array}\right) . \tag{5.22}$$

As explained in chapter 3, we can now use elements of the moduli-generating group $SO(7,7,\mathbb{R})$ to deform the internal sector and obtain new solutions of type IIB supergravity. We will restrict ourselves to three different one-parameter γ-deformations, as it turns to be enough to capture all the important features of most γ-deformed $AdS_2 \times S^2 \times T^6$ backgrounds.

5.2.1 TsT-transformations of $AdS_2 \times S^2 \times T^6$

We start by applying two types of TsT-transformations.

TsT-transformations involving S^2 and T^6

The first type of TsT-transformation that we consider acts on the 2-torus $S^1_{(\varphi)} \times S^1_{(y_i)}$, where $S^1_{(y_i)} \subset T^6$ is parametrized by the isometry angle $y_{(i)}$. The choice of $S^1_{(y_i)}$ is arbitrary, as different circles in T^6 lead to new solutions with very similar structure. We choose to perform a TsT-transformation with parameter γ on the 2-torus parametrized by (φ, y_1). The corresponding $SO(7,7,\mathbb{R})$ element,

$$g_{(T_\varphi s_{y_1} T_\varphi)} = \begin{pmatrix} \mathbb{1}_{7\times 7} & 0 \\ \Gamma_{(T_\varphi s_{y_1} T_\varphi)} & \mathbb{1}_{7\times 7} \end{pmatrix} \quad \text{with} \;\; \Gamma_{(T_\varphi s_{y_1} T_\varphi)} = \left(\begin{array}{cc|c} 0 & -\gamma & 0 \\ \gamma & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_{5\times 5} \end{array}\right) , \tag{5.23}$$

acts on the background matrix (5.22) as a fractional linear transformation. Hence, the deformed background matrix E' reads

$$\begin{aligned} E \longrightarrow E' = E\left(\Gamma_{(T_\varphi s_{y_1} T_\varphi)} E + \mathbb{1}_{7\times 7}\right)^{-1} &= \left(\begin{array}{cc|c} R^2 J \sin^2(\theta) & R^2 J\gamma \sin^2(\theta) & 0 \\ -R^2 J\gamma \sin^2(\theta) & J & 0 \\ \hline 0 & 0 & \mathbb{1}_{5\times 5} \end{array}\right) \\ &= G' + B' \,, \end{aligned} \tag{5.24}$$

with

$$J = \frac{1}{1 + R^2\gamma^2 \sin^2(\theta)} \,. \tag{5.25}$$

Symmetrizing and antisymmetrizing the deformed background matrix (5.24) yields, respectively, the expressions of the deformed metric and B field

$$G' = \left(\begin{array}{cc|c} R^2 J \sin^2(\theta) & 0 & 0 \\ 0 & J & 0 \\ \hline 0 & 0 & \mathbb{1}_{5\times 5} \end{array}\right) , \quad B' = \left(\begin{array}{cc|c} 0 & R^2 J\gamma \sin^2(\theta) & 0 \\ -R^2 J\gamma \sin^2(\theta) & 0 & 0 \\ \hline 0 & 0 & 0_{5\times 5} \end{array}\right) . \tag{5.26}$$

[1] As discussed in the previous section, we do not consider here the $U(1)$ isometry of AdS_2 as γ-deformations along this direction would probably break conformal symmetry.

The dilaton transforms, according to (3.28), as

$$\phi \longrightarrow \phi' = \phi_0 - \frac{1}{2}\ln\left(\det\left(\Gamma_{(T_\varphi s_{y_1} T_\varphi)}E + \mathbb{1}_{7\times 7}\right)\right) = \phi_0 + \frac{1}{2}\ln(J)\,. \tag{5.27}$$

According to (3.52), the R-R field strengths transform as follows

$$\begin{aligned} F \longrightarrow F' = \mathfrak{g}_{(T_\varphi s_{y_1} T_\varphi)}F &= \exp\left(\frac{1}{2}\left(\Gamma_{(T_\varphi s_{y_1} T_\varphi)}\right)_{mn}\iota_m\iota_n\right)F \\ &= \exp\left(-\gamma\iota_\varphi\iota_{y_1}\right)F_{(5)} = \left(1-\gamma\iota_\varphi\iota_{y_1}\right)F_{(5)} \\ &= F_{(5)} - \gamma\iota_\varphi\iota_{y_1}F_{(5)} = F'_{(5)} + F'_{(3)} + F'_{(1)}\,. \end{aligned} \tag{5.28}$$

Obviously, $F'_{(1)} = 0$. With (5.11), one obtains

$$\begin{aligned} F'_{(3)} = -\gamma\iota_\varphi\iota_{y_1}F_{(5)} &= -\gamma\iota_\varphi\iota_{y_1}w^\star_{(5)} \\ &= -e^{-\phi_0}R\gamma\sin(\theta)\iota_\varphi\iota_{y_1}\left(-d\theta\wedge d\varphi\wedge dy_1\wedge dy_3\wedge dy_6 - d\theta\wedge d\varphi\wedge dy_1\wedge dy_4\wedge dy_5\right) \\ &= -e^{-\phi_0}R\gamma\sin(\theta)\left(dy_3\wedge dy_6 + \wedge dy_4\wedge dy_5\right)\,, \end{aligned} \tag{5.29}$$

and

$$F'_{(5)} = w_{(5)} + w^{\star'}_{(5)}\,. \tag{5.30}$$

The computation of $w^{\star'}_{(5)}$ is very similar to (5.10) and we do not perform it here explicitly. One obtains

$$\begin{aligned} w^\star_{(5)} = e^{-\phi_0}R\sin(\theta)d\theta\wedge d\varphi\wedge&\left(dy_2\wedge dy_4\wedge dy_6 - dy_2\wedge dy_3\wedge dy_5\right. \\ &\left.-Jdy_1\wedge dy_3\wedge dy_6 - Jdy_1\wedge dy_4\wedge dy_5\right)\,. \end{aligned} \tag{5.31}$$

Let us now sum up our results. The TsT-transformed solution, dropping the prime on the deformed fields, is

$$ds^2_S = ds^2_{\mathrm{AdS}_2} + R^2d\theta^2 + R^2J\sin^2(\theta)d\varphi^2 + Jdy_1^2 + \sum_{i=2}^{6}dy_i^2\,, \tag{5.32}$$

$$J = \left(1 + R^2\gamma^2\sin(\theta)^2\right)^{-1}\,, \tag{5.33}$$

$$e^{2\phi} = e^{2\phi_0}J\,, \tag{5.34}$$

$$B = \gamma R^2J\sin(\theta)\,d\varphi\wedge dy_1\,, \tag{5.35}$$

$$F_{(1)} = 0\,, \tag{5.36}$$

$$F_{(3)} = e^{-\phi_0}R\gamma\sin(\theta)\left(dy_3\wedge dy_6 + \wedge dy_4\wedge dy_5\right)\,, \tag{5.37}$$

$$F_{(5)} = w_{(5)} + w^{\star'}_{(5)}\,, \tag{5.38}$$

with the necessary sign flip of the 3-form field strength discussed in section 4.2. Although we do not present explicit computations here, we have verified, in the gauge $\chi = 0$, that this deformed background satisfies all the equations of motion of type IIB supergravity of section 4.3. Furthermore, it will be argued in chapter 6 that the TsT-transformation applied here breaks all the spacetime supersymmetries of the $AdS_2 \times S^2 \times T^6$ solution. This deformed solution is therefore non-supersymmetric. Finally, the geometry of the initial background has been deformed, as the metric (5.32) does not describe[2] the direct product $AdS_2 \times S^2 \times T^6$ anymore. Indeed, the 2-sphere is now deformed by J along the φ direction and the 6-torus is now fibered over the deformed 2-sphere as the radius of $S^1_{y_1} \subset T^6$ is now equal to $\sqrt{J}$.

[2]Although locally this is still the case.

TsT-transformations on $T^2 \subset T^6$

We now perform a TsT-transformation on the 2-torus $S^1_{(y_i)} \times S^1_{(y_j)}$. This time, different choices of the circles will lead to solutions with vanishing or non-vanishing 3-form field strength $F_{(3)}$. Looking at the deformation procedure of the R-R field strengths (3.52) and at the components of $F_{(5)}$, one sees that a TsT-transformation applied on any of the three 2-torus parametrized by the isometry angles (y_1, y_2), (y_3, y_4) or (y_5, y_6) leads to a solution with vanishing $F_{(3)}$. All other choices for the 2-torus lead to solutions with non-vanishing $F_{(3)}$. We choose to apply a TsT-transformation with parameter γ on the 2-torus parametrized by (y_2, y_3). The corresponding $SO(7, 7, \mathbb{R})$ element is

$$g_{(T_{y_2} s_{y_3} T_{y_2})} = \begin{pmatrix} \mathbb{1}_{7\times7} & 0 \\ \Gamma_{(T_{y_2} s_{y_3} T_{y_2})} & \mathbb{1}_{7\times7} \end{pmatrix}, \quad \text{with } \Gamma_{(T_{y_2} s_{y_3} T_{y_2})} = \left(\begin{array}{c|cc|c} 0_{2\times2} & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma & 0 \\ 0 & \gamma & 0 & 0 \\ \hline 0 & 0 & 0 & 0_{3\times3} \end{array}\right). \tag{5.39}$$

After some very simple algebra, one finds the deformed background matrix

$$E' = E\left(E\Gamma_{(T_{y_2} s_{y_3} T_{y_2})} + \mathbb{1}_{7\times7}\right)^{-1} = \left(\begin{array}{cc|cc|c} R^2\sin^2(\theta) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & J & \gamma J & 0 \\ 0 & 0 & -\gamma J & J & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbb{1}_{3\times3} \end{array}\right), \text{ with } J' = \frac{1}{1+\gamma^2}. \tag{5.40}$$

As usual, the dilaton becomes

$$\phi' = \phi_0 + \frac{1}{2}\ln(J'). \tag{5.41}$$

and the R-R field strengths transforms as

$$\begin{aligned} F \longrightarrow F' = \mathfrak{g}_{(T_{y_2} s_{y_3} T_{y_2})} F &= \exp\left(\frac{1}{2}\left(\Gamma_{(T_{y_2} s_{y_3} T_{y_2})}\right)_{mn} \iota_m \iota_n\right) F \\ &= F_{(5)} - \gamma \iota_{y_2} \iota_{y_3} F_{(5)}. \end{aligned} \tag{5.42}$$

The TsT-transformed solution, dropping the prime on deformed fields, is therefore

$$ds^2_S = ds^2_{\text{AdS}_2} + ds^2_{\text{S}^2} + dy_1^2 + J\left(dy_2^2 + dy_3^2\right) + \sum_{i=4}^{6} dy_i^2, \tag{5.43}$$

$$J' = \left(1+\gamma^2\right)^{-1}, \tag{5.44}$$

$$e^{2\phi} = e^{2\phi_0} J, \tag{5.45}$$

$$B = \gamma J\, dy_2 \wedge dy_3, \tag{5.46}$$

$$F_{(1)} = 0, \tag{5.47}$$

$$F_{(3)} = e^{-\phi_0} R\gamma \left(\sin(\theta)\, d\theta \wedge d\varphi \wedge dy_5 - \cosh(x_1)\, dx_0 \wedge dx_1 \wedge dy_6\right), \tag{5.48}$$

$$F_{(5)} = w_{(5)} + w^{\star'}_{(5)}, \tag{5.49}$$

where we flipped, as usual, the sign of $F_{(3)}$ and where

$$\begin{aligned} w^{\star'}_{(5)} = e^{-\phi_0} R\sin(\theta) d\theta \wedge d\varphi \wedge \big(&dy_2 \wedge dy_4 \wedge dy_6 - J dy_2 \wedge dy_3 \wedge dy_5 \\ &-dy_1 \wedge dy_3 \wedge dy_6 - \frac{1}{J} dy_1 \wedge dy_4 \wedge dy_5\big). \end{aligned} \tag{5.50}$$

Once more, we have verified that this new background satisfies all the equations of motion of type IIB supergravity. This TsT-transformation does not affect the geometry of our initial background. Indeed, the deformed metric still describes the direct product $AdS_2 \times S^2 \times T^6$. However, the two circles $S^1_{(y_2)} \subset T^6$ and $S^1_{(y_3)} \subset T^6$, have now a γ-dependent radius equal to $\sqrt{J}$. This time, the TsT-transformation we applied does not break any spacetime supersymmetries of the initial $AdS_2 \times S^2 \times T^6$ solution. Therefore, the deformed solution preserves 8 supersymmetries.

5.2.2 γ-deformation of $AdS_2 \times S^2 \times T^6$

We consider, in this last subsection, a slightly more complicated one-parameter γ-deformation. Our goal is to generate a new solution with non-vanishing 1-form field strength $F_{(1)}$. Looking back at the definition (3.54), one realizes that this was not possible in the $AdS_5 \times S^5$ case, as a minimum of four isometries is required. Thus, we choose the following γ-deformation with parameter γ

$$g_\gamma = \begin{pmatrix} \mathbb{1}_{7\times 7} & 0 \\ \Gamma_\gamma & \mathbb{1}_{7\times 7} \end{pmatrix} \quad \text{with } \Gamma_\gamma = \begin{pmatrix} 0 & -\gamma & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 & 0 \end{pmatrix}, \tag{5.51}$$

which can be decomposed as a chain of two TsT-transformations with the same parameter γ

$$g_\gamma = g_{(T_\varphi s_{y_1} T_\varphi)} \cdot g_{(T_{y_3} s_{y_6} T_{y_3})} \,. \tag{5.52}$$

The deformed metric and B field are respectively obtained by symmetrizing and antisymmetrizing the deformed background matrix

$$E' = \begin{pmatrix} R^2 J \sin^2(\theta) & R^2\gamma J \sin^2(\theta) & 0 & 0 & 0 & 0 & 0 \\ -\gamma J \sin^2(\theta) & J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J' & 0 & 0 & \gamma J' \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\gamma J' & 0 & 0 & J' \end{pmatrix}, \tag{5.53}$$

where J and J' are respectively as in (5.33) and (5.44). Bearing (5.52) in mind and considering our previous results (**??**) and (5.40), one should not be surprised by the structure of the deformed background matrix (5.53). The deformed dilaton is

$$\phi' = \phi_0 + \frac{1}{2}\ln(JJ') \,. \tag{5.54}$$

With (3.54), the deformed R-R field strengths read

$$F'_{(3)} = \frac{1}{2}(\Gamma_\gamma)_{mn}\iota_m\iota_n F_{(5)} = (-\gamma\iota_\varphi\iota_{y_1} - \gamma\iota_{y_3}\iota_{y_6})F_{(5)} \,, \tag{5.55}$$

$$F'_{(1)} = \frac{1}{8}(\Gamma_\gamma)_{mn}(\Gamma_\gamma)_{pq}\iota_p\iota_q\iota_m\iota_n F_{(5)} = \gamma^2\iota_\varphi\iota_{y_1}\iota_{y_3}\iota_{y_6}F_{(5)} \,, \tag{5.56}$$

$$F'_{(5)} = w_{(5)} + w^{\star'}_{(5)} \,, \tag{5.57}$$

The γ-deformed solution therefore reads

$$ds^2_S = ds^2_{\mathrm{AdS}_2} + R^2 d\theta^2 + R^2 J \sin^2(\theta) d\varphi^2 + J dy_1^2 + dy_2^2 + J' dy_3^2 + dy_4^2 + dy_5^2 + J' dy_6^2 \,, \tag{5.58}$$

$$J = \left(1 + R^2\sin^2(\theta)\gamma^2\right)^{-1} \quad ; \quad J' = \left(1+\gamma^2\right)^{-1} \,, \tag{5.59}$$

$$e^{2\phi} = e^{2\phi_0} JJ' \,, \tag{5.60}$$

$$B = \gamma\left(R^2 J \sin^2(\theta)\, d\varphi \wedge dy_1 + J'\, dy_3 \wedge dy_6\right) \,, \tag{5.61}$$

$$F_{(1)} = -e^{-\phi_0} R\gamma^2 \sin(\theta)\, d\theta \,, \tag{5.62}$$

$$F_{(3)} = e^{-\phi_0} R\gamma \left(\sin(\theta)\, d\theta \wedge (d\varphi \wedge dy_1 + dy_3 \wedge dy_6 + dy_4 \wedge dy_5)\right) \tag{5.63}$$

$$\quad - e^{-\phi_0} R\gamma \cosh(x_1)\, dx_0 \wedge dx_1 \wedge dy_2 \,, \tag{5.64}$$

$$F_{(5)} = w_{(5)} + w^{\star'}_{(5)} \,, \tag{5.65}$$

where the sign of $F_{(3)}$ has been flipped and where the Hodge dual of the 5-form $w_{(5)}$ is now

$$w^{\star'}_{(5)} = e^{-\phi_0} R \sin(\theta) d\theta \wedge d\varphi \wedge \Big(dy_2 \wedge dy_4 \wedge dy_6 - J dy_2 \wedge dy_3 \wedge dy_5 \\ - J J' dy_1 \wedge dy_3 \wedge dy_6 - \frac{J}{J'} dy_1 \wedge dy_4 \wedge dy_5 \Big) \, . \tag{5.66}$$

We expect this new background to satisfy all the type IIB supergravity equations of motion, although we only checked the ones for the NS-NS fields. The main feature of this new solution is the presence of a non-vanishing 1-form field strength $F_{(1)}$. This is due to the specific γ-deformation we applied and to the structure of the initial 5-form $F_{(5)}$. For instance, applying similar γ-deformations, such as $g_\gamma = g_{(T_\varphi s_{y_1} T_\varphi)} g_{(T_{y_3} s_{y_2} T_{y_3})}$ or $g_\gamma = g_{(T_{y_3} s_{y_1} T_{y_3})} g_{(T_{y_4} s_{y_6} T_{y_4})}$, would lead to solutions with vanishing $F_{(1)}$. In the current case, the geometry of the deformed background is different from the direct product $AdS_2 \times S^2 \times T^6$ as the 6-torus is, once again, fibered over a deformed 2-sphere. Finally, we will argue in the next chapter that this solution breaks all the supersymmetries of $AdS_2 \times S^2 \times T^6$.

As a closing remark for this chapter, let us quickly consider the regularity of our deformed solutions. The only problems that could arise would be divergences associated with possible poles of J and J'. However, one immediately notices from (5.59) that the denominator of these quantities never vanishes as $\gamma \in \mathbb{R}$. All of the three deformed solutions are therefore regular.

Chapter 6

Supersymmetry breaking

6.1 Unbroken (super)symmetry

In this section, we discuss the issue of supersymmetry preserved by generic backgrounds. In a first part, following [22], we introduce the notions of (super)symmetric solution, (super)-isometry group and (super)-isometry algebra. We then explain the effects of γ-deformations on the supersymmetries of the $AdS_5 \times S^5$ and $AdS_2 \times S^2 \times T^6$ backgrounds.

6.1.1 Purely bosonic theories

The equations of motion of a given theory (in our case type IIB supergravity) are invariant under certain symmetries forming a symmetry group G. Usually, the solutions (in our case consistent type IIB superstring backgrounds) break most of these symmetries: those which remain are called residual or unbroken symmetries and they also form a symmetry group $H \subset G$. These solutions are said to be symmetric. The symmetries of the theories that are broken by the solution can then be used to generate new solutions. This is precisely the solution generating technique we have been using from the beginning. The equations of motion of type IIB supergravity, assuming solutions with $U(1)^d$ isometry, are invariant under $SO(d, d, \mathbb{R})$ transformations. However, the same does not apply to the solutions.

A usual example is given by general relativity. Einstein's equations are invariant under the infinite-dimensional group of general coordinate transformations (GCT). The solutions are given metrics g which are only invariant under a finite-dimensional group of isometries. This isometry group H is therefore a subgroup of the group of GCTs. An infinitesimal isometry which, by definition, leaves the metric (i.e. the solution) invariant, acts as

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu} = -2(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = 0\,, \tag{6.1}$$

where the solutions are $\xi^\mu(x) = \xi k^\mu(x)$. ξ is an infinitesimal parameter and $k^\mu(x)$ is a Killing vector[1]. Equation (6.1) is known as the Killing equation. Each Killing vector $k_{(I)}$ is associated to a generator $P_{(I)}$ of the isometry algebra $\mathfrak{H}$. As can be seen from the Killing vector equation, the action of these generators on the metric is represented by the Lie derivative $-\mathcal{L}_{k_{(I)}}$.

For more complicated theories, which contain the metric but also other fields (as $B, \phi, \chi, C_{(2)}, C_{(4)}$ for type IIB supergravity), only those isometries that leave invariant the metric and all of the other fields, form the isometry group[2] of a given solution.

As a closing remark for this subsection, let us mention that those solutions which preserve a maximal number of symmetries are usually called vacua. This is due to the fact that they correspond to possible

[1] Here x denotes a point on the spacetime manifold.
[2] We will systematically refer to the symmetry group of a solution as the isometry group since its elements always leave the metric invariant.

vacuum states in the quantum field theory. These states will be annihilated by the operators associated to the unbroken symmetries in the quantum theory. In the case of pure general relativity in $d = 4$, without cosmological constant, the only vacuum is Minkowski spacetime. Its isometry group is the Poincaré group (ten isometries). With a (negative) positive cosmological constant, the only vacuum is the (anti-) de Sitter spacetime which has ($SO(2,3)$) $SO(1,4)$ as an isometry group (still ten-dimensional) .

6.1.2 Supersymmetric supergravity solutions

The equations of motion of a supergravity theory are invariant under the (infinite) group of local spacetime supersymmetry transformations. As expected from our previous discussions, the solutions are usually not invariant under these local supersymmetry transformations (3.14). Those which remain invariant under a finite number of residual (or unbroken) supersymmetries are said to be supersymmetric. The action of an infinitesimal supersymmetry transformation on the bosonic (B) and fermionic (F) spacetime fields can be represented schematically by

$$\delta_\epsilon B \sim \epsilon F\,, \tag{6.2}$$

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon\,, \tag{6.3}$$

where $\epsilon(x)$ is the infinitesimal local supersymmetry parameter and ∂ is a differential operator. Although these schematic expressions hold for all supergravity theories, their exact expressions naturally differ from one theory to another. As mentioned earlier, we have only been interested in purely bosonic solutions of the bosonic equations of motion and we have therefore consistently set all fermionic fields to zero. It is clear, from (6.2) and (6.3), that a bosonic supergravity solution is supersymmetric if it satisfies

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon = 0 \tag{6.4}$$

for some parameter $\epsilon(x)$. In the absence of fermionic fields, (6.2) vanishes automatically. Equation (6.4) is called a the Killing spinor equation. Its solutions can be written as $\epsilon(x) = \epsilon\kappa(x)$ which is the product of an infinitesimal anticommuting number ϵ and a Killing spinor $\kappa(x)$. The Killing spinors are the generators of the supersymmetries. Therefore, for a given solution, one has to solve the Killing spinor equation in order to determine the number of unbroken supersymmetries.

In the same spirit as in the previous subsection, the unbroken supersymmetries of a solution form a finite-dimensional supergroup S called super-isometry group. Each Killing spinor is associated with a fermionic generator of the super-isometry algebra $\mathfrak{S}$. The supergroup S is part of the infinite-dimensional supergroup of all local supersymmetry transformations, GCTs, and other possible symmetries of the supergravity theory. Therefore, it is not surprising, that the bosonic generators of $\mathfrak{S}$ are the ones generating the unbroken isometries of the solutions. Hence the name super-isometry group for S. To sum up, given a supergravity solution, the bosonic and fermionic elements of its super-isometry algebra are the generators of its isometries and supersymmetries, respectively.

6.1.3 Type IIB supergravity Killing spinors equations

Let us now focus on the case of type IIB supergravity. The fermionic fields of the theory are one Weyl complex gravitino ψ_μ and one Weyl complex dilatino λ. When discussing supersymmetry and Killing spinor equations, it is convenient to think of a complex spinor as a doublet of real spinors. Following [23] we represent ψ_μ as a $SO(2)$ doublet of Majorana-Weyl gravitinos $\psi_{1,\mu}, \psi_{2,\mu}$ and λ as a $SO(2)$ doublet of Majorana-Weyl dilatinos λ_1, λ_2. Explicitly, we write

$$\psi_\mu = \begin{pmatrix} \psi_{1,\mu} \\ \psi_{2,\mu} \end{pmatrix}\,, \qquad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\,. \tag{6.5}$$

To be exact, $\psi_{1,\mu}$ and $\psi_{2,\mu}$ are 32-component spinors projected onto one chirality by $\frac{1}{2}(1+\Gamma_{11})$. The gamma matrix Γ_{11} is defined as $\Gamma_{11} = \Gamma^0\Gamma^1.....\Gamma^9$, where the constant $D = 10$ gamma-matrices Γ satisfy the Clifford algebra

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{rs}\,, \quad \text{with} \quad \eta^{AB} = \text{diag}(-,+,+,+,+,+,+,+,+,+)\,. \tag{6.6}$$

This projection reduces the number of non-zero components of $\psi_{1,\mu}, \psi_{2,\mu}$ to 16. The gravitino ψ_μ is therefore considered as a 32-component chiral spinor. The same happens to the dilatino, but projected by $\frac{1}{2}(1-\Gamma_{11})$. Thus, the gravitino and the dilatino have opposite chirality. Finally, the supersymmetry parameter is described in the same way as the gravitino

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad \text{with} \quad \frac{1}{2}(1+\Gamma_{11})\epsilon_k = \epsilon_k \,, \tag{6.7}$$

where $k = 1, 2$ is an $SO(2)$ index. The supersymmetry parameter ϵ is, therefore, also considered as a 32-component chiral spinor which shares the gravitino's chirality.

There are obviously two Killing spinor equations for type IIB supergravity corresponding to the schematic expression (6.4). One is algebraic and arises from requiring that the variation of the dilatino λ vanishes. The second one is a differential equation and arises by requiring the same for the variation of the gravitino ψ_μ. Although we won't need them explicitly, we present the Killing spinor equations in string frame

$$\begin{aligned} \delta_\epsilon \psi_\mu &:= \tilde{\nabla}_\mu \epsilon + \frac{1}{8} H_{\mu\nu\rho}\Gamma^{\nu\rho}\sigma_3\epsilon - \frac{e^{-\phi}}{8}\left(iF_{(1)\nu}\Gamma^\nu\sigma_2 - \frac{1}{3}F_{(3)\nu\rho\lambda}\Gamma^{\nu\rho\lambda}\sigma_1 + \frac{i}{2.5!}F_{(5)\nu\rho\lambda\xi\delta}\Gamma^{\nu\rho\lambda\xi\delta}\sigma_2\right)\Gamma_\mu\epsilon = 0\,, \\ \delta_\epsilon\lambda &:= (\partial_\mu\phi)\Gamma^\mu\epsilon + \frac{1}{12}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\sigma_3\epsilon - e^{-\phi}\left(\frac{1}{12}F_{(3)\mu\nu\rho}\Gamma^{\mu\nu\rho}\sigma_1 - iF_{(1)\mu}\Gamma^\mu\sigma_2\right)\epsilon = 0\,, \end{aligned} \tag{6.8}$$

where all the indices are spacetime indices. All the spacetime fields have been introduced in chapter 3. The object $\Gamma^{\mu_1\mu_2.....\mu_n}$ is defined as the entirely antisymmetrized product of n gamma-matrices

$$\Gamma^{\mu_1\mu_2.....\mu_n} := \Gamma^{[\mu_1}\Gamma^{\mu_2}.....\Gamma^{\mu_n]}\,. \tag{6.9}$$

The spacetime vector Γ^μ is related to the tangent space vector Γ^A introduced in (6.6) by $\Gamma^A = e^A_\mu\Gamma^\mu$, where e^A_μ is the zehnbein and A a tangent space vector index. The matrices σ_1, σ_2 and σ_3 are the Pauli matrices and they act on the $SO(2)$ index k of the spinors. Let us finally mention that the differential operator $\tilde{\nabla}$ is the sum of the usual covariant derivative ∇ and a term depending on the spin connection. Detailed treatments of the Killing spinor equations for different type II supergravities can be found in [22] and [17].

To sum up, the general procedure to determine the amount of unbroken supersymmetries and the super-isometry algebra of any given type IIB supergravity solution boils down to two steps. One should first solve the Killing vector and Killing spinor equations (6.8) for the given solution. As mentioned earlier, only the Killing vectors generating the isometries under which all fields are invariant should be kept. The second step is to associate a bosonic generator $P_{(I)}$ of the super-isometry algebra to each Killing vector $k_{(I)}$ and to associate a fermionic generator (also called supercharge) $Q_{(M)}$ with each Killing spinor $\kappa_{(M)}$. A representation of the super-isometry algebra is determined by the structure constants ${f_{IJ}}^K, {f_{MI}}^N$ and ${f_{MN}}^I$ appearing in the commutators

$$\left[P_{(I)}, P_{(J)}\right] = {f_{IJ}}^K P_{(K)}\,, \qquad \left[Q_{(M)}, P_{(I)}\right] = {f_{MI}}^N Q_{(N)}\,, \qquad \left[Q_{(M)}, Q_{(N)}\right] = {f_{MN}}^I P_{(I)}\,. \tag{6.10}$$

This is unfortunately easier said than done, as solving Killing spinor equations quickly becomes a difficult task when the solutions become "non trivial". However, this has been done for the solutions we consider. In particular, the Killing spinor equations have been solved for the $AdS_5 \times S^5$ solution (see for exemple [24]), which is known to be a maximally supersymmetric solution preserving 32 supersymmetries. Its super-isometry algebra is known to be the $\mathfrak{psu}(2,2|4)$ superalgebra. The only other maximally supersymmetric solution of type IIB supergravity is the Minkowski solution. Furthermore, the Killing spinor equations for the $AdS_2 \times S^2 \times T^6$ solution have also been solved. As mentioned earlier, it has been shown in [8] that this solution preserves 8 supersymmetries and that its super-isometry algebra is an extended version of the $\mathfrak{psu}(1,1|2)$ algebra.

6.1.4 TsT-transformations and unbroken supersymmetries

We have now introduced the necessary material in order to ask the question: how much supersymmetries are preserved by the different γ-deformed backgrounds[3]? There is at least one obvious method to answer

[3] By that, we mean the LM solution, Frolov's solution and our γ-deformed solutions of $AdS_2 \times S^2 \times T^6$.

this question which is to solve the Killing spinor equations for the deformed backgrounds. This is precisely what we want to avoid since the expressions of these backgrounds are too complicated to be plugged in (6.8). We would like to find another method that does not require to solve the Killing spinor equations.

We have seen that the deformations we consider can always be decomposed as a chain of TsT-transformations. It turns out that the effects of T-duality (which can then be generalized for TsT-transformations) on the supersymmetries of a supergravity solution have been studied before. Since we know the super-isometry algebra of our initial solutions ($AdS_5 \times S^5$ and $AdS_2 \times S^2 \times T^6$) we can deduce, from these effects, how much supersymmetries will remain after a specific TsT-transformation.

As usual, we restrict ourselves to type IIB supergravity solutions with d $U(1)$ global isometries realized as shifts of the angle coordinates $(\phi_1, ..., \phi_d)$. It is was shown in [25] that if one applies a T-duality on a circle parametrized by the angle coordinate ϕ_i, the Killing spinors of the solution with explicit dependence on ϕ_i will lead to broken supersymmetries, while T-duality will commute with the supersymmetries generated by the Killing spinors independent of ϕ_i. A TsT-transformation, by definition, involves two different circles parametrized by (ϕ_i, ϕ_j). The shift, in between the two T-duality, "mixes" the two circles. The second T-duality therefore affects both circles[4]. The above statement then naturally generalizes to: supersymmetries generated by Killing spinors without explicit dependence on ϕ_i and ϕ_j will commute with such a TsT-transformation. From the point of view of the super-isometry algebra $\mathfrak{S}$ of the solution, a Killing spinor independent of ϕ_i and ϕ_j corresponds to a supercharge Q satisfying

$$[P_{\phi_i}, Q] = \left[P_{\phi_j}, Q\right] = 0\,, \qquad Q, P_{\phi_i}, P_{\phi_j} \in \mathfrak{S}\,, \tag{6.11}$$

where P_{ϕ_i} and P_{ϕ_j} are, respectively, the generators of the U(1) isometries realized as a shifts of the angle coordinate ϕ_i and ϕ_j.

It is clear that a supersymmetry that commutes with a TsT-transformation will remain a supersymmetry of the deformed solution (obtained by such a TsT). This can be seen in the following very naive and schematic way without specifying any representation. If one denotes the TsT-transformation with parameter γ by $Ts_\gamma T$ and a supersymmetry of the solution A by S_A, then the action of the TsT-transformation on the solution, $Ts_\gamma T.A = B$, generates the deformed solution B. The action of the supersymmetry, by definition, leaves the solution invariant $S_A.A = A$. If $[Ts_\gamma T, S_A]\,.A = 0$, then

$$[Ts_\gamma T, S_A]\,.A = Ts_\gamma T.A - S_A.B = B - S_A.B = 0\,, \tag{6.12}$$

which means that $S_A.B = B$. Therefore S_A is also a supersymmetry of the deformed solution. On the other hand, if $[Ts_\gamma T, S_A] \neq 0$, then obviously, $S_A.B \neq B$. At first sight one would say that, in this case, S_A is not a supersymmetry of the deformed solution anymore. However, this case turns out to be a lot more tricky. S_A is still a supersymmetry but it is, in some sense, "hidden". One says that it is realized non-locally. A proper explanation of this phenomenon calls for worldsheet arguments that we will not provide here. This is explained in details in [26], [27], [28] and [29]. In what follows, we will not consider it as a supersymmetry of the deformed solution as it does not appear in the way we initially defined it. This is also the point of view adopted by almost all authors when discussing γ-deformations.

To summarize, if a supersymmetry of the initial solution is invariant under the $U(1) \times U(1)$ isometry realized as shift of the angle coordinates ϕ_i, ϕ_j, then it will remain a supersymmetry of the deformed solution obtained by a TsT-transformation on the 2-torus parametrized by (ϕ_i, ϕ_j). This is precisely the condition stated in [2]. This condition is satisfied, at the level of the super-isometry algebra of the initial solution, if equation (6.11) holds. This is the point of view we will adopt. Finally, if the supersymmetry is not invariant under the $U(1) \times U(1)$, it will then be broken by the TsT-transformation.

[4] This has to be the case otherwise any TsT-transformation would always reduce to the shift. Indeed, applying two T-duality on the same circle reduces to the identity map.

6.2 The $AdS_5 \times S^5$ case

In this section, we investigate the effects of the different γ-deformations studied in chapter 4 on the supersymmetries of the $AdS_5 \times S^5$ background. In other words we determine the amount of unbroken supersymmetries (in the sense of locally realized as discussed earlier) of the deformed backgrounds.

6.2.1 The $(P)SU(2,2|4)$ super-isometry group

The $AdS_5 \times S^5$ solution preserves 32 supersymmetries and its super-isometry group is $PSU(2,2|4)$[5]. The super-isometry algebra $\mathfrak{su}(2,2|4)$ can be represented in terms of 8×8 matrices. An element $M \in \mathfrak{su}(2,2|4)$ can be decomposed in terms of 4×4 matrices as follows. We refer to [30] for a more detailed presentation.

$$M = \begin{pmatrix} m & \eta \\ \tilde{\eta} & n \end{pmatrix}, \quad \mathrm{str} M = \mathrm{tr}(m) - \mathrm{tr}(n) = 0\,, \tag{6.13}$$

where the even elements m and n respectively span the unitary subalgebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$ and therefore satisfy the following conditions

$$m^\dagger = -\Sigma m \Sigma\,, \qquad n^\dagger = -n\,, \qquad \Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{6.14}$$

As explained in [30], $\mathfrak{su}(2,2|4)$ also contains the $\mathfrak{u}(1)$-generator $i\mathbb{1}$. Therefore, the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)\,. \tag{6.15}$$

The off-diagonal odd elements η and $\tilde{\eta}$ of M are the fermionic generators of the group and are constrained to

$$\tilde{\eta} = -\Sigma \eta^\dagger\,. \tag{6.16}$$

The superalgebra $\mathfrak{psu}(2,2|4)$ is obtained by taking the quotient algebra $\mathfrak{su}(2,2|4)$ over its $\mathfrak{u}(1)$-factor. It is important to note that there does not exist a representation of $\mathfrak{psu}(2,2|4)$ in terms of 8×8 matrices anymore. We will however continue to work with this representation as we won't have to consider $\mathfrak{u}(1)$ elements. In our case, to sum up, $m \in \mathfrak{su}(2,2)$ and $n \in \mathfrak{su}(4)$ are respectively the generators of the AdS_5 and S^5 isometries and the 4×4 matrices η are the 16 complex (32 real) supercharges.

The idea is now to work with an embedding of $AdS_5 \times S^5$ into $SU(2,2|4)$. Let $g(z) \in SU(2,2) \times SU(4) \subset SU(2,2|4)$ be such an embedding. The coordinates $z = (x_a, y_a)$, with $a = 1, ..., 5$ parametrize S^5 and AdS_5, respectively. Following [7] and [31], we can define G as

$$\mathrm{G} = g(z) K_8 g(z)^t = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix}, \text{ with } K_8 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \tag{6.17}$$

where K are 4×4 matrices satisfying $K^2 = -\mathbb{1}_4$. The 4×4 matrices $g_a \in SU(2,2)$ and $g_s \in SU(4)$ then provide another parameterization of AdS_5 and S^5. For a certain choice of K, the 5-sphere can be parametrized as follows (see [32])

$$g_s(r_i, \phi_i) = \begin{pmatrix} 0 & u_3 & u_1 & u_2 \\ -u_3 & 0 & u_2^\star & -u_1^\star \\ -u_1 & -u_2^\star & 0 & u_3^\star \\ -u_2 & u_1^\star & -u_3^\star & 0 \end{pmatrix}, \quad \text{with } |u_1|^2 + |u_2|^2 + |u_3|^2 = 1\,, \tag{6.18}$$

where $u_i = r_i e^{i\phi_i}$ with $i = 1, 2, 3$ and (r_i, ϕ_i) are the coordinates we introduced in (4.3).

[5]We actually consider the group $SU(2,2|4)$ as will be explained below.

Our goal, for now, is to explicitly construct the three elements $p_{\phi_1}, p_{\phi_2}, p_{\phi_3} \in \mathfrak{su}(2,2|4)$ generating the three commuting isometries that act as shifts of the angle coordinates ϕ_1, ϕ_2, ϕ_3 of S^5. It is clear that

$$p_{\phi_i} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}_{\phi_i} \end{pmatrix}, \tag{6.19}$$

where $\tilde{p}_{\phi_i} \in \mathfrak{su}(4)$. Let us denote the associated group elements by $P_{\phi_i}(\alpha)$

$$P_{\phi_i}(\alpha) = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & \tilde{P}_{\phi_i}(\alpha) \end{pmatrix} \in SU(2,2|4)\,, \quad \tilde{P}_{\phi_i}(\alpha) = e^{\alpha p_{\phi_i}} \in SU(4)\,, \tag{6.20}$$

where α is the shift parameter such that under $P_{\phi_i}(\alpha)$ the angles transform as $\phi_i \to \phi_i + \alpha$.

A generic transformation $h \in SU(2,2|4)$ acts on $g(z) \in SU(2,2|4)$ by left multiplication $g \to hg$. From (6.17), we then deduce that for $h = P_{\phi_i}(\alpha)$,

$$G \to (P_{\phi_i}(\alpha))\, G\, (P_{\phi_i}(\alpha))^t\,, \tag{6.21}$$

$$\begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix} \to \begin{pmatrix} g_a & 0 \\ 0 & \left(\tilde{P}_{\phi_i}(\alpha)\right) g_s \left(\tilde{P}_{\phi_i}(\alpha)\right)^t \end{pmatrix}. \tag{6.22}$$

Since, under $P_{\phi_i}(\alpha)$, we see that $g_s(r_i,\phi_i) \to \left(\tilde{P}_{\phi_i}(\alpha)\right)(g_s(r_i,\phi_i))\left(\tilde{P}_{\phi_i}(\alpha)\right)^t$ it is then clear that we can pick

$$\tilde{p}_{\phi_1} = \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \tilde{p}_{\phi_2} = \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \tilde{p}_{\phi_3} = \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.23}$$

This is a correct choice as these three generators are linearly independent and satisfy

$$\mathrm{str}(p_{\phi_i}) = 0 - \mathrm{tr}(\tilde{p}_{\phi_i}) = 0\,. \tag{6.24}$$

As desired, they generate the shifts of the angle variables ϕ_1, ϕ_2, ϕ_3 of the 5-sphere. As an exemple, let us look at the action of $P_{\phi_1}(\alpha)$

$$\begin{aligned} \left(\tilde{P}_{\phi_1}(\alpha)\right)(g_s(r_i,\phi_i))\left(\tilde{P}_{\phi_1}(\alpha)\right)^t &= \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{-i\alpha}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{i\alpha}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{-i\alpha}{2}} \end{pmatrix} g_s(r_i,\phi_i) \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{-i\alpha}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{i\alpha}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{-i\alpha}{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & u_3 & u_1 e^{i\alpha} & u_2 \\ -u_3 e^{i\alpha} & 0 & u_2^\star & -u_1^\star \\ -u_1 & -u_2^\star & 0 & u_3^\star e^{-i\alpha} \\ -u_2 & u_1^\star & -u_3^\star e^{-i\alpha} & 0 \end{pmatrix} = g_s(r_i, \phi_1+\alpha, \phi_2, \phi_3)\,. \end{aligned} \tag{6.25}$$

It is also important to notice, from the expression of the generators p_{ϕ_i}, that all the supersymmetries transform under all the isometries P_{ϕ_i}. One can say that all the "fermions" are charged under these isometries. This can easily be seen by computing the following commutator at the level of the super-isometry algebra

$$\begin{aligned} [p_{\phi_i}, Q] &= \begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}_{\phi_i} \end{pmatrix}\begin{pmatrix} 0 & \eta \\ -\Sigma\eta^\dagger & 0 \end{pmatrix} - \begin{pmatrix} 0 & \eta \\ -\Sigma\eta^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}_{\phi_i} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \eta\tilde{p}_{\phi_i} \\ -\Sigma\tilde{p}_{\phi_i}\eta^\dagger & 0 \end{pmatrix}, \end{aligned} \tag{6.26}$$

where $Q \in \mathfrak{su}(2,2|4)$ denotes an arbitrary supercharge. Since the $\tilde{p}_{\phi_i}$'s do not have vanishing eigenvalues, it is clear that for any $Q \neq 0$, the commutator (6.26) never vanishes.

6.2.2 The LM deformation

We have set our framework and will now explain why the one-parameter deformation of Lunin and Maldacena breaks $\frac{3}{4}$ of the supersymmetries of the $AdS_5 \times S^5$ background. Let us follow exactly the approach of [3], as was already done in 4.2.1, and look at the implications of each step on the super-isometry algebra $\mathfrak{su}(2,2|4)$.

The first step is to make the following change of angle coordinates

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = (A^t_{(LM)})^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}. \tag{6.27}$$

The coordinates $(r_1, r_2, r_3, \varphi_1, \varphi_2, \varphi_3)$ along with condition (6.18) provide a new parameterization of S^5. The generators of the three commuting isometries $P_{\varphi_1}, P_{\varphi_2}, P_{\varphi_3}$, acting as shift of the new angle coordinates, are therefore given by

$$p_{\varphi_i} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}_{\varphi_i} \end{pmatrix} \in \mathfrak{su}(2,2|4) \quad \text{with} \quad \begin{pmatrix} \tilde{p}_{\varphi_1} \\ \tilde{p}_{\varphi_2} \\ \tilde{p}_{\varphi_3} \end{pmatrix} = (A^t_{(LM)})^{-1} \begin{pmatrix} \tilde{p}_{\phi_1} \\ \tilde{p}_{\phi_2} \\ \tilde{p}_{\phi_3} \end{pmatrix} \in \mathfrak{su}(4)\,, \tag{6.28}$$

such that

$$\tilde{p}_{\varphi_1} = \frac{i}{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \tilde{p}_{\varphi_2} = \frac{i}{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \tilde{p}_{\varphi_3} = \frac{i}{6} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The key point is that some fermions are now uncharged under the two isometries $P_{\varphi_1}, P_{\varphi_2}$ because of the vanishing eigenvalues of $\tilde{p}_{\varphi_1}$ and $\tilde{p}_{\varphi_2}$. It is indeed obvious that there are four complex (8 real) independent non-trivial supercharges Q satisfying

$$[p_{\varphi_1}, Q] = [p_{\varphi_2}, Q] = 0\,. \tag{6.29}$$

The second step is to make a TsT-transformation with parameter γ on the two circles parametrized by φ_1 and φ_2. As explained in the previous section, after this TsT-transformation, the remaining unbroken supersymmetries are the ones left invariant by the $U(1) \times U(1)$ isometry realized as shifts of the angle coordinates φ_1, φ_2. It is then clear from (6.29) that the LM background only preserves eight real supersymmetries. The other $32 - 8 = 24$ supersymmetries, initially preserved by the $AdS_5 \times S^5$ background have been broken by the TsT-transformation. From the point of view of the super-isometry algebra, this TsT-transformation reduces the super-isometry algebra $\mathfrak{su}(2,2|4)$ to a subalgebra of $\mathfrak{su}(2,2|4)$ that we denote by $\mathfrak{J}$. The bosonic part of $\mathfrak{J}$ naturally contains the whole $\mathfrak{su}(2,2)$, as our TsT-transformation does not deform the AdS_5 space. Also contained in the bosonic part of $\mathfrak{J}$ is a subalgebra of $\mathfrak{su}(4)$[6]. The fermionic part of $\mathfrak{J}$ only contains the eight supercharges that commute with p_{φ_1} and p_{φ_2}[7]. $\mathfrak{J}$ is the super-isometry algebra of the LM background and is known as the $\mathcal{N} = 1$ superconformal algebra.

The third step, which is now obsolete for us as we already determined the amount of supersymmetry preserved by the LM background, is to rotate back the angle coordinates to the initial angles (ϕ_1, ϕ_2, ϕ_3). This is needed in order to obtain the expression of the LM background in the (r_i, ϕ_i) coordinates as presented in [2]. The super-isometry algebra $\mathfrak{J}$ remains obviously the same one, but the Cartan generators of the deformed 5-sphere are rotated back to $p_{\phi_1}, p_{\phi_2}, p_{\phi_3}$. It is probably worth noting that none of the eight remaining real supercharges commute with the generators $p_{\phi_1}, p_{\phi_2}, p_{\phi_3}$ as this was not the case in the $\mathfrak{su}(2,2|4)$ super-isometry algebra.

[6] It is clear that $\mathfrak{su}(4)$ is partially broken as we deformed S^5. The generators p_{φ_i} are, however, part of $\mathfrak{J}$ as the LM background is still invariant under shifts of the angle coordinates.

[7] However, they do not commute with P_{φ_3} which does not have vanishing eigenvalues.

6.2.3 Frolov's deformation

We refer to section 4.2 for its detailed expression. Frolov's solution is obtained by a three-parameter deformation of the $AdS_5 \times S^5$ background. We recall the corresponding element $g_{\gamma_{(F)}} \in SO(3,3,\mathbb{R})$ acting as a fractional linear transformation on the background matrix $E_{(\phi)}$ defined in (4.22)

$$g_{\gamma_{(F)}} = \begin{pmatrix} \mathbb{1}_3 & 0 \\ \Gamma_{(F)} & \mathbb{1}_3 \end{pmatrix} \in SO(3,3,\mathbb{R})\,, \quad \Gamma_{(F)} = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{6.30}$$

Here, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ are the 3 parameters of the deformation. This deformation can be decomposed into three TsT-transformations, each of them with a different parameter, and each of them involving two different circles. As explained in section 4.2, if one performs an arbitrary change of angle coordinates in the same spirit as in (6.27), $\varphi_i = \sum_{j=1}^{3} (A^t)^{-1}_{ij}\, \phi_j$, the expression of the deformation $g_{\gamma_{(F)}}$ acting now on the background matrix $E_{(\varphi)} = g_A E_{(\phi)}$ is

$$g_{\gamma_{(F)}} = \begin{pmatrix} \mathbb{1}_3 & 0 \\ (A^t)^{-1}\,\Gamma_{(F)}\,(A)^{-1} & \mathbb{1}_3 \end{pmatrix}, \qquad \det(A) \neq 0\,. \tag{6.31}$$

It is maybe obvious to the reader, but nevertheless very important to realize, that after such change of angle coordinates, this deformation will, by definition, always remain a 3 parameters deformation. Therefore, regardless of the basis (or set of angle coordinates) in which we choose to express the deformation, it will always boil down to a chain of three TsT-transformations with different parameters. Since such a deformation always involves a 3-torus parametrized by three isometry angles of S^5, it should be clear that the only supersymmetries preserved by the deformed background are the ones left invariant by the $U(1)^3$ isometry realized as shifts of the three angle coordinates. Picking the basis (ϕ_1, ϕ_2, ϕ_3), then from (6.23), we see that due to the absence of vanishing eigenvalues for $\tilde{p}_{\phi_1}, \tilde{p}_{\phi_2}, \tilde{p}_{\phi_3}$, the deformation (6.30) will break all the supersymmetries of the $AdS_5 \times S^5$ background.

Finally, one could wonder wether this conclusion is indeed consistent up to the change of angle coordinates (6.31). This should obviously be the case as the deformation remains the same. To this purpose, let us try to find a set of new angle coordinates $(\varphi_1, \varphi_2, \varphi_3)$ such that a certain number of supercharges $Q \in \mathfrak{su}(2,2|4)$ satisfy

$$[p_{\varphi_1}, Q] = [p_{\varphi_2}, Q] = [p_{\varphi_3}, Q] = 0 \;\text{ with }\; \begin{pmatrix} \tilde{p}_{\varphi_1} \\ \tilde{p}_{\varphi_2} \\ \tilde{p}_{\varphi_3} \end{pmatrix} = (A^t)^{-1} \begin{pmatrix} \tilde{p}_{\phi_1} \\ \tilde{p}_{\phi_2} \\ \tilde{p}_{\phi_3} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} \tilde{p}_{\phi_1} \\ \tilde{p}_{\phi_2} \\ \tilde{p}_{\phi_3} \end{pmatrix}, \tag{6.32}$$

where $A \in GL(3,\mathbb{R})$ and with p_{ϕ_i} and $\tilde{p}_{\phi_i}$ given by (6.19) and (6.23), respectively. This is only possible if $\tilde{p}_{\varphi_1}, \tilde{p}_{\varphi_2}$ and $\tilde{p}_{\varphi_3}$ share at least one vanishing eigenvalue. A possible choice of A satisfying this condition is

$$(A^t)^{-1} = \begin{pmatrix} (-a_2 - a_3) & a_2 & a_3 \\ (-b_2 - b_3) & b_2 & b_3 \\ (-c_2 - c_3) & c_2 & c_3 \end{pmatrix}. \tag{6.33}$$

This is, however, not an acceptable change of basis as $\det((A^t)^{-1})$=0. All other choices of $(A^t)^{-1}$ satisfying (6.32) have vanishing determinant. This confirms that no fermions can be uncharged under the three commuting isometries at the same time and that Frolov's solution is therefore non-supersymmetric. Another way to see that the condition (6.32) can never be satisfied is to realize that it is impossible to find three linearly independent diagonal 4×4 matrices $\tilde{p}_{\varphi_1}, \tilde{p}_{\varphi_2}$ and $\tilde{p}_{\varphi_3}$, sharing a vanishing eigenvalue and satisfying $\mathrm{tr}(\tilde{p}_{\varphi_i}) = 0$. As a closing remark, let us point that a two-parameter deformation of the form of (6.30) would also break all supersymmetries as it would already involve the 3-torus parametrized by (ϕ_1, ϕ_2, ϕ_3).

6.3 The $AdS_2 \times S^2 \times T^6$ case

Our goal, in this section, is to determine the amount of supersymmetry preserved by the γ-deformed solutions obtained in chapter 5. To do so, it is necessary to study the super-isometry algebra of the $\frac{1}{4}$-supersymmetric type IIB $AdS_2 \times S^2 \times T^6$ solution. In particular, we are interested in the following commutators

$$[p_\varphi, Q] \ , \qquad [p_{y_i}, Q] \ , \ \text{ for } \ i = \{1,2,3,4,5,6\}, \tag{6.34}$$

where Q denotes the $\frac{32}{4} = 8$ supercharges generating the unbroken supersymmetries of the solution and where p_φ and p_{y_i} are respectively the generators of the $U(1)$ isometries acting as shifts of the angles coordinates φ and y_i. Fortunately, the super-isometry algebra, as well as a convenient representation, have already been provided by Sorokin *et al.* in [8]. We adopt their conventions and give, in a first part, a very brief review of their results. This allows, in a second part, to study in details the commutators (6.34) and determine the amount of unbroken supersymmetries of our γ-deformed solutions.

6.3.1 Enlarged $\mathfrak{psu}(1,1|2)$ algebra

The first step is to construct a representation of the $D = 10$ dimensional gamma-matrices[8] Γ^A satisfying the Clifford algebra (6.6). The indices $A, B, ... = (a, \hat{a}, a')$ are ten-dimensional tangent space vector indices, while the indices $a, b, ... \in \{0,1\}$, $\hat{a}, \hat{b}, .. \in \{2,3\}$ and $a', b', ... \in \{4,5,6,7,8,9\}$ are, respectively, AdS_2, S^2 and T^6 tangent space vector indices. As mentioned earlier, we choose the zehnbein as

$$e^A_\mu = \text{diag}\left(R\cosh(x_1), R, R, R\sin(\theta), 1,1,1,1,1,1\right) \ , \ \text{ with } \ \eta_{AB} = \text{diag}\left(-,+,+,...,+\right) \ , \tag{6.35}$$

such that the expression of the spacetime metric, $G_{\mu\nu} = e^A_\mu \eta_{AB} e^B_\nu$, coincides with (5.1). For instance, $\Gamma_\varphi = e^A_\varphi \Gamma_A = e^3_\varphi \Gamma_3 = R\sin(\theta)\Gamma_3$.

The four-dimensional, 4×4 gamma-matrices $\gamma^{\underline{a}}$ associated with $AdS_2 \times S^2$ are

$$\{\gamma^{\underline{a}}, \gamma^{\underline{b}}\} = 2\eta^{\underline{ab}} \ , \ \text{ with } \ \eta^{\underline{ab}} = \text{diag}\left(-,+,+,+\right) \ , \tag{6.36}$$

$$\hat{\gamma}^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ , \quad \hat{\gamma}^5\hat{\gamma}^5 = \mathbb{1}_{4\times 4} \ , \quad \{\hat{\gamma}^5, \gamma^{\underline{a}}\} = 0 \ , \tag{6.37}$$

where $(\underline{a}, \underline{b}, ..) = (a, b, ..; \hat{a}, \hat{b}, ..) \in \{0,1,2,3\}$ are $AdS_2 \times S^2$ tangent space vector indices. The components of the these gamma-matrices are denoted by $(\gamma^{\underline{a}})_{\alpha\beta}$, with $\alpha, \beta \in \{1,2,3,4\}$ being the indices of a four-dimensional spinor representation of $SO(3,1)$. These 4×4 matrices can be decomposed in terms of 2×2 AdS_2 gamma-matrices ρ^a and 2×2 S^2 gamma-matrices $\rho^{\hat{a}}$

$$\gamma^a = \rho^a \otimes \mathbb{1}_{2\times 2} \ , \quad \gamma^{\hat{a}} = \gamma \otimes \rho^{\hat{a}} \ , \quad \gamma = \rho^0\rho^1 \ , \tag{6.38}$$

where we choose $\rho^0 = i\sigma_1, \rho^1 = \sigma_2, \rho^2 = \sigma_1, \rho^3 = \sigma_3$. The six-dimensional 8×8, gamma-matrices $\gamma^{a'}$ associated with T^6 are

$$\{\gamma^{a'}, \gamma^{b'}\} = 2\delta^{a'b'} \ , \quad \text{with} \quad \delta^{a'b'} = \text{diag}\left(+,+,+,+,+,+\right) \ , \tag{6.39}$$

$$\gamma^7 = i\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8\gamma^9 \ , \quad \gamma^7\gamma^7 = \mathbb{1}_{8\times 8} \ , \quad \{\gamma^7, \gamma^{a'}\} = 0 \, . \tag{6.40}$$

The components of the gamma-matrices are denoted $(\gamma^{a'})_{\alpha'\beta'}$, with $\alpha'\beta' \in \{1,2,3,4,5,6,7,8\}$ being the indices of an eight-dimensional spinorial representation of $SO(6)$. We choose the following realization

$$\gamma^4 = \sigma_1 \otimes \mathbb{1}_{2\times 2} \otimes \mathbb{1}_{2\times 2} \ , \quad \gamma^5 = \sigma_2 \otimes \mathbb{1}_{2\times 2} \otimes \mathbb{1}_{2\times 2} \ , \quad \gamma^6 = \sigma_3 \otimes \sigma_1 \otimes \mathbb{1}_{2\times 2} \ , \tag{6.41}$$

$$\gamma^7 = \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}_{2\times 2} \ , \quad \gamma^8 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \ , \quad \gamma^9 = \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \, . \tag{6.42}$$

Finally, using (6.36) and (6.39), one can construct the $D = 10$, 32×32 gamma-matrices Γ^A as

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \ , \qquad \Gamma^A = \left(\Gamma^{\underline{a}}, \Gamma^{a'}\right) \ , \tag{6.43}$$

$$\Gamma^{\underline{a}} = \gamma^{\underline{a}} \otimes \mathbb{1}_{8\times 8} \ , \quad \Gamma^{a'} = \hat{\gamma}^5 \otimes \gamma^{a'} \ , \quad \Gamma^{11} = \hat{\gamma}^5 \otimes \gamma^7 \ , \quad \{\Gamma^{11}, \Gamma^A\} = 0 \, . \tag{6.44}$$

[8]These should not be confused with the antisymmetric Γ matrix introduced in (3.24).

In this realization, a 32-component spinor $\Theta_{\alpha\alpha'}$ is labeled by the 4-component $AdS_2 \times \mathrm{S}^2$ spinor index α and the 8-component T^6 spinor index α'. The components of the gamma-matrices are, for instance, $(\Gamma^{\underline{a}})_{\alpha\beta,\alpha'\beta'} = (\gamma^{\underline{a}})_{\alpha\beta}(\mathbb{1}_{8\times 8})_{\alpha'\beta'}$. Before moving one to the super-isometry algebra itself, it will turn out to be useful to introduce a spinor projection operator $\mathcal{P}_8$ which projects onto an eight-dimensional subspace of the thirty-two-dimensional space of spinors. It was defined in [8] as

$$\mathcal{P}_8 = \frac{1}{4}\left(\mathbb{1}_{32\times 32} - i\left(\Gamma^4\Gamma^5 + \Gamma^6\Gamma^7 + \Gamma^8\Gamma^9\right)\tilde{\gamma}^7\right) , \tag{6.45}$$

where $\tilde{\gamma}^7 = (\hat{\gamma}^5)^6 \otimes \gamma^7 = \mathbb{1}_{4\times 4} \otimes \gamma^7$. Using the representation introduced above, one finds

$$\begin{aligned} \mathcal{P}_8 &= \frac{1}{4}\left(\mathbb{1}_{32\times 32} - i\mathbb{1}_{4\times 4} \otimes \left(\gamma^4\gamma^5 + \gamma^6\gamma^7 + \gamma^8\gamma^9\right)\gamma^7\right) \\ &= \frac{1}{4}\mathbb{1}_{4\times 4} \otimes \left(\mathbb{1}_{8\times 8} - \gamma^6\gamma^7\gamma^8\gamma^9 - \gamma^4\gamma^5\gamma^8\gamma^9 - \gamma^4\gamma^5\gamma^6\gamma^7\right) , \end{aligned} \tag{6.46}$$

and $\gamma^6\gamma^7\gamma^8\gamma^9 + \gamma^4\gamma^5\gamma^8\gamma^9 + \gamma^4\gamma^5\gamma^6\gamma^7 = \mathrm{diag}(-3, 1, 1, 1, 1, 1, 1, 1, -3)$. Therefore

$$\mathcal{P}_8 = \mathbb{1}_{4\times 4} \otimes \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0_{6\times 6} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \equiv \mathbb{1}_{4\times 4} \otimes p_8 \, . \tag{6.47}$$

The super-isometry algebra of the full $AdS_2 \times S^2 \times T^6$ solution presented in chapter 5 is an extended version of the $\mathfrak{psu}(1, 1|2)$ superalgebra. The bosonic elements of the $\mathfrak{psu}(1, 1|2)$ superalgebra are the generators of the isometries of $AdS_2 \times S^2$, while its fermionic elements are the eight supercharges Q. The extended version of $\mathfrak{psu}(1, 1|2)$ is enlarged by the additional bosonic generators of the $U(1)$ isometries and $SO(6)$ rotations in T^6. We point the reader to [8] where this enlarged super-isometry algebra is presented in details. The eight supercharges are represented by the 32-component spinors $Q_{\alpha\alpha'}$ subject to the eight-dimensional projection

$$Q = \mathcal{P}_8 Q \, . \tag{6.48}$$

Therefore $Q_{\alpha i} = 0, i \in \{2, 3, 4, 5, 6, 7\}$ and we write an arbitrary supercharge as

$$Q = (a_1, a_2, a_3, a_4) \otimes (b_1, 0, 0, 0, 0, 0, 0, b_2) , \tag{6.49}$$

where $a_1, a_2, a_3, a_4, b_1, b_2 \in \mathbb{R}$ are independent parameters. We can directly read from [8] the result of the commutator we are interested in

$$[p_A, Q] = \frac{i}{2R} Q\tilde{\gamma}\tilde{\gamma}^7\Gamma_A\mathcal{P}_8 \, , \quad \text{with} \quad \tilde{\gamma} = \Gamma^0\Gamma^1 \, . \tag{6.50}$$

Here, $p_{a'}$ are the generators of the $U(1)$ isometries realized as shifts of the angle coordinates of T^6, while $p_3 = (R\sin(\theta))^{-1}p_\varphi$ is the generator of the $U(1)$ isometry realized as a shift of φ.

6.3.2 Supersymmetries of the deformed $AdS_2 \times S_2 \times T^6$ solutions

Let us first pick the case $A = a'$. One can see easily that the following commutators vanish

$$\left[\tilde{\gamma}^7, \mathcal{P}_8\right] = \mathbb{1}_{4\times 4} \otimes \left[\gamma^7, p_8\right] = 0 \, , \qquad \left[\tilde{\gamma}, \mathcal{P}_8\right] = \left[\gamma^0\gamma^1 \otimes \mathbb{1}_{8\times 8}, \mathbb{1}_{4\times 4} \otimes p_8\right] = 0 \, , \tag{6.51}$$

$$\{\tilde{\gamma}^7, \Gamma_{a'}\} = \hat{\gamma}^5 \otimes \{\gamma^7, \gamma_{a'}\} = 0 \, , \qquad [\tilde{\gamma}, \Gamma_{a'}] = \left[\gamma^0\gamma^1, \hat{\gamma}^5\right] \otimes \gamma_{a'} = 0 \, , \tag{6.52}$$

although the first one requires to compute the explicit form of γ^7. The commutator (6.50), keeping in mind the constraint (6.48), then becomes

$$[p_{a'}, Q] = \frac{i}{2R} Q\tilde{\gamma}\tilde{\gamma}^7\Gamma_{a'}\mathcal{P}_8 = -\frac{i}{2R} Q\tilde{\gamma}\Gamma_{a'}\mathcal{P}_8\tilde{\gamma}^7 = -\frac{i}{2R} Q\Gamma_{a'}\mathcal{P}_8\tilde{\gamma}^7\tilde{\gamma} \, . \tag{6.53}$$

With some simple algebra, one can verify that $Q\Gamma_{a'}\mathcal{P}_8 = 0$ for any value of the index a'. As an example, we pick $a' = 4$

$$Q\Gamma_{a'}\mathcal{P}_8 = Q\left(\hat{\gamma}^5 \otimes \gamma^4 p_8\right) = (a_1, a_2, a_3, a_4)\,\hat{\gamma}^5 \otimes (b_1, 0, 0, 0, 0, 0, 0, b_2)\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0_{4\times 4} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$= 0\,. \tag{6.54}$$

We now turn to the case $A = 3$. The commutator (6.50) reads

$$[p_3, Q] = Q\frac{i}{2R}Q\tilde{\gamma}\tilde{\gamma}^7\Gamma_3\mathcal{P}_8 = Q\left(\gamma^0\gamma^1\gamma^3 \otimes \gamma^7 p_8\right) \tag{6.55}$$

$$= (a_1, a_2, a_3, a_4)\begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} \otimes (b_1, 0, 0, 0, 0, 0, 0, b_2)\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0_{6\times 6} & 0 \\ \hline 0 & 0 & -1 \end{array}\right). \tag{6.56}$$

This commutator never vanishes.

The eight supercharges of the extended $\mathfrak{psu}(1,1|2)$ superalgebra therefore satisfy

$$[p_\varphi, Q] \neq 0\,, \qquad [p_{y_i}, Q] = 0\,, \quad \text{for} \quad i \in \{1, 2, 3, 4, 5, 6\}\,. \tag{6.57}$$

This allows us to conclude that a TsT-transformation involving the circle $S^1_{(\varphi)} \subset S^2$ parametrized by the isometry angle φ breaks all the supersymmetries of the initial $AdS_2 \times S^2 \times T^6$ solution. On the other hand, a TsT-transformation applied on any of the 2-torus $T^2 \subset T^6$ does not break any supersymmetries. This explains why the first and the second TsT-transformed solution derived in chapter 5 are respectively a non-supersymmetric solution and $\frac{1}{4}$-supersymmetric solution. The third solution is the result of a chain of two TsT-transformations $g_\gamma = g_{(T_\varphi s_{y_1} T_\varphi)} \cdot g_{(T_{y_3} s_{y_6} T_{y_3})}$. One might fear that, as for the LM deformation, this precise combination of TsT-transformations preserves in a subtle way some of the supersymmetries of the initial solution. Such a phenomenon was possible in the LM case since all 32 supercharges were charged under the three $U(1)$ isometries of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution. Our case is much simpler. The TsT-transformation $g_{(T_{y_3} s_{y_6} T_{y_3})}$ does not affect any of the 8 supersymmetries while $g_{(T_\varphi s_{y_1} T_\varphi)}$ breaks all of them. The third solution is, therefore, non-supersymmetric.

Finally, let us mention that several $\frac{1}{4}$-supersymmetric type IIA supergravity solutions with $AdS_2 \times S^2 \times T^6$ geometry were presented in [8]. They can be obtained by T-dualizing our type IIB solution along one of the T^6 directions. The fact that all of these type IIA solutions remain supersymmetric is consistent with our previous conclusions.

Chapter 7

Conclusion

In the two first chapters, we explained why and how the γ-deformations can be used as a type IIB supergravity solution generating technique. We applied them to the $AdS_5 \times S^5$ background in order to rederive the LM background. We then proved explicitly that the latter satisfies the equations of motion derived from a type IIB supergravity covariant action. In chapter 5, we considered the recently discovered $\frac{1}{4}$-supersymmetric $AdS \times S^2 \times T^6$ type IIB background. Applying three different γ-deformations led us to three new regular solutions. An interesting feature of one of the solutions was the presence of a non-vanishing R-R 1-form field strength. Finally, we studied the effects of the γ-deformations on the supersymmetries of the backgrounds. We proved that all the supersymmetries of the initial $AdS \times S^2 \times T^6$ background break as soon as the 2-sphere gets deformed. This allowed us to conclude that two of our new deformed solutions still preserve 8 supersymmetries while one of them is non-supersymmetric. Let us also mention that we performed the most general γ-deformation, namely, a $7(7-1)/2 = 21$ parameter deformation. However, we did not present those results here as the expressions of the deformed background fields turn out to be too complicated. For arbitrary parameters, this deformed solution is non-supersymmetric.

These new results suggest two main directions for further research. A first idea would be to look for the β-deformed gauge theories dual to the string theories on the γ-deformed $AdS_2 \times S^2 \times T^6$ backgrounds. Since the γ-deformations we applied do not affect the AdS_2 space, one would expect the deformed gauge theories to remain conformally invariant. Depending on the type of γ-deformations one is considering, it would be interesting to perform tests of the $\text{AdS}_2/\text{CFT}_1$ correspondence between (non)-supersymmetric theories.

γ-deformation

Gravity: IIB strings on $AdS_2 \times S^2 \times T^6$ $\longrightarrow$ IIB strings on (non)-supersymmetric γ-deformed $AdS_2 \times S^2 \times T^6$

$\Updownarrow$ $\Uparrow$ **$\text{AdS}_2/\text{CFT}_1$ correspondence** $\Downarrow$ $\Updownarrow$

Field theory: CFT_1 $\longrightarrow$ (non)-supersymmetric β-deformed CFT_1

Marginal β-deformation

A second idea would be to look at the integrability properties of the superstring theory on the $AdS_2 \times S^2 \times T^6$ γ-deformed backgrounds. As previously mentioned, it was shown in [8] that the type IIB GS superstring propagating on the $AdS_2 \times S^2 \times T^6$ background is classically integrable. This was done by constructing a zero-curvature Lax connection (up to second order in the fermions) from the components of the conserved currents of the GS action. In particular, the existence of a supersymmetric current is necessary to ensure that the curvature of such a Lax-connection vanishes. Since the γ-deformations tend to break some of the supersymmetries of the string backgrounds, it would be interesting to analyze their direct implications on the possibility of constructing zero-curvature Lax connections for the superstrings on γ-deformed backgrounds.

Appendix A

String frame vs. Einstein frame

We start from the low-energy effective action for the bosonic string (2.21) in D spacetime dimensions, expressed in the so-called string frame

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G} e^{-2\phi} \left(-\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 (\nabla_\mu \phi)(\nabla^\mu \phi) \right) . \tag{A.1}$$

The first term was actually absent in (2.21), as we fixed $D = 26$. Here we will keep the value of D arbitrary.

One can notice that the action (A.1) does not look exactly like the Einstein-Hilbert one, as it contains a strange factor $e^{-2\phi}$. The kinetic terms are not canonically normalized and the dilaton term seems to have the wrong sign. In order to rewrite the action in a more familiar form we make a field redefinition. To this purpose, it is first necessary to distinguish between the average value ϕ_0 of the dilaton and the part that varies $\tilde{\phi}$. We set the average value of the dilaton to zero such that

$$\tilde{\phi} = \phi - \phi_0 = \phi . \tag{A.2}$$

Secondly, in D dimensions, we define the Einstein frame metric g as a conformal rescaling of the string frame metric G by the dilaton

$$g_{\mu\nu}(x) \equiv e^{\frac{-4\tilde{\phi}}{D-2}} G_{\mu\nu}(x) = e^{\frac{-4\phi}{D-2}} G_{\mu\nu}(x) . \tag{A.3}$$

It is useful to precise the associated transformation for the inverse metric as well as for the determinant of the metric G

$$g^{\mu\nu}(x) = e^{\frac{4\phi}{D-2}} G^{\mu\nu}(x) , \quad \sqrt{-g} = e^{\frac{-2D\phi}{D-2}} \sqrt{-G} . \tag{A.4}$$

This redefinition of the metric obviously leads to a, not so obvious, new Ricci tensor $\tilde{R}$. From [11], we read

$$R = e^{\frac{-4\phi}{D-2}} \tilde{R} + 4\frac{D-1}{D-2} (\nabla_\mu \phi)(\nabla^\mu \phi) - 4\frac{D-1}{D-2} \left(\nabla^2 \phi\right) . \tag{A.5}$$

Plugging the transformations (A.3), (A.4) and (A.5) in the action (A.1) yields

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g}\, e^{\frac{2D\phi}{D-2}} e^{-2\phi} \left(-\frac{2(D-26)}{3\alpha'} + R - e^{\frac{-12\phi}{D-2}} \frac{1}{12} H_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} + 4 e^{\frac{-4\phi}{D-2}} (\nabla_\mu \phi)(\tilde{\nabla}^\mu \phi) \right) , \tag{A.6}$$

where a tilde means that the indices have been lifted with the Einstein frame metric g. Let us focus on the Ricci scalar and more precisely on the term containing $\nabla^2\phi$.

$$\begin{aligned}
-4\frac{D-1}{D-2}\int d^Dx\,\sqrt{-G}e^{-2\phi}\nabla^2\phi &= -4\frac{D-1}{D-2}\int d^Dx\,\sqrt{-G}G^{\mu\nu}e^{-2\phi}\nabla_\mu\left(\partial_\nu\phi\right) \\
&= -4\frac{D-1}{D-2}\int d^Dx\,e^{-2\phi}\partial_\mu\left(\sqrt{-G}G^{\mu\nu}\left(\partial_\nu\phi\right)\right) \qquad (A.7)\\
&= -4\frac{D-1}{D-2}\int d^Dx\,e^{-2\phi}\partial_\mu\left(e^{\frac{(2D-4)\phi}{D-2}}\sqrt{-g}g^{\mu\nu}\left(\partial_\nu\phi\right)\right) \\
&= -4\frac{D-1}{D-2}\int d^Dx\,\frac{2D-4}{D-2}e^{-2\phi}e^{\frac{(2D-4)\phi}{D-2}}\sqrt{-g}g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) \\
&\quad -4\frac{D-1}{D-2}\int d^Dx\,\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\left(\partial_\nu\phi\right)\right) \qquad (A.8)\\
&= \frac{-8(D-1)}{(D-2)}\int d^Dx\,\sqrt{-G}e^{-2\phi}\left(\nabla_\mu\phi\right)\left(\nabla^\mu\phi\right)\,, \qquad (A.9)
\end{aligned}$$

where the term of (A.8) drops out as a total derivative and we used the identity (2.22) in line (A.7). With (A.5) and (A.9), the Ricci scalar term in the action now becomes

$$\begin{aligned}
\int d^Dx\,\sqrt{-G}\,e^{-2\phi}\,R &= \int d^Dx\,\sqrt{-G}\,e^{-2\phi}\left(e^{\frac{-4\phi}{D-2}}\tilde{R}-4\frac{D-1}{D-2}\left(\nabla_\mu\phi\right)\left(\nabla^\mu\phi\right)\right) \\
&= \int d^Dx\,\sqrt{-g}\,e^{\frac{2D\phi}{D-2}}e^{-2\phi}\left(e^{\frac{-4\phi}{D-2}}\tilde{R}-4\frac{D-1}{D-2}g^{\mu\nu}e^{\frac{-4\phi}{D-2}}\left(\nabla_\mu\phi\right)\left(\nabla_\nu\phi\right)\right) \\
&= \int d^Dx\,\sqrt{-g}\left(\tilde{R}-4\frac{D-1}{D-2}\left(\tilde{\nabla}^\mu\phi\right)\left(\nabla_\nu\phi\right)\right)\,. \qquad (A.10)
\end{aligned}$$

Substituting (A.10) in (A.6) gives the action in Einstein frame

$$S_E = \frac{1}{2\kappa_E^2}\int d^Dx\,\sqrt{-g}\left(-\frac{2\left(D-26\right)}{3\alpha'}e^{\frac{4\phi}{D-2}}+\tilde{R}-e^{\frac{-8\phi}{D-2}}\frac{1}{12}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda}-\frac{4}{D-2}\left(\nabla_\mu\phi\right)\left(\tilde{\nabla}^\mu\phi\right)\right)\,, \qquad (A.11)$$

where $\kappa_E=\kappa$. However, if one does not set $\phi_0=0$, one obtains $\kappa_E=\kappa e^{\phi_0}$. The coefficient in front of the Einstein-Hilbert term $\sqrt{-g}\tilde{R}$ is usually identified with Newton's constant G_N as

$$\kappa_E^2 = 8\pi G_N\,. \qquad (A.12)$$

Note however, that this is Newton's constant in D arbitrary dimensions and its value will, therefore, differ from the one measured in a four-dimensional world. In $D=26$ dimensions, one obtains the usual action for the bosonic string

$$S_{1,E} = \frac{1}{2\kappa_E^2}\int d^{26}x\,\sqrt{-g}\left(\tilde{R}-e^{\frac{-\phi}{3}}\frac{1}{12}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda}-\frac{1}{6}\left(\nabla_\mu\phi\right)\left(\tilde{\nabla}^\mu\phi\right)\right)\,.$$

Let us express the bosonic part of the type IIB supergravity action in Einstein frame. We recall its expression in string frame

$$S_{\mathrm{IIB}} = S_{\mathrm{II}}^{(\text{NS-NS})}+S_{\mathrm{IIB}}^{(\text{R-R})}+S_{\mathrm{IIB}}^{(\mathrm{CS})}\quad, \qquad (A.13)$$

with

$$\begin{aligned}
S_{\mathrm{II}}^{(\text{NS-NS})} &= \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}e^{-2\phi}\left(R+4\left(\nabla\phi\right)^2-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right)\,, \\
S_{\mathrm{IIB}}^{(\text{R-R})} &= -\frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}\left(\frac{1}{2}F_{(1)\mu}F_{(1)}^{\mu}+\frac{1}{12}F_{(3)\mu\nu\lambda}F_{(3)}^{\mu\nu\lambda}+\frac{1}{4.5!}F_{(5)\mu\nu\lambda\delta\gamma}F_{(5)}^{\mu\nu\lambda\delta\gamma}\right)\,, \qquad (A.14)\\
S_{\mathrm{IIB}}^{(\mathrm{CS})} &= \frac{1}{4\kappa^2}\int\,C_{(4)}\wedge H\wedge F_{(3)}\,.
\end{aligned}$$

The $S_{\mathrm{II}}^{(\text{NS-NS})}$ part is the same as (A.1) without the first term and with $D = 10$. It is then clear that its expression in Einstein frame can be immediately deduced from (A.11). The $S_{\mathrm{IIB}}^{(\mathrm{CS})}$ part remains the same in Einstein frame as it does not depend on the metric. The expression of $S_{\mathrm{IIB}}^{(\text{R-R})}$ in Einstein frame is obtained by using (A.3) in (A.14)

$$S_{\mathrm{IIB},E}^{(\text{R-R})} = -\frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-g}e^{\frac{5\phi}{2}}\left(\frac{1}{2}e^{\frac{-\phi}{2}}F_{(1)\mu}\tilde{F}^{\mu}_{(1)} + \frac{1}{12}e^{\frac{-3\phi}{2}}F_{(3)\mu\nu\lambda}\tilde{F}^{\mu\nu\lambda}_{(3)} + \frac{1}{4.5!}e^{\frac{-5\phi}{2}}F_{(5)\mu\nu\lambda\delta\gamma}\tilde{F}^{\mu\nu\lambda\delta\gamma}_{(5)}\right) .$$

The complete expression of the bosonic part of the type IIB supergravity action in Einstein frame is then

$$S_E = \frac{1}{2\kappa_E^2}\int d^{10}x\,\sqrt{-g}\,\Big(\tilde{R} - \frac{1}{2}\left(\nabla_\mu\phi\right)(\tilde{\nabla}^\mu\phi) - \frac{1}{2}e^{2\phi}F_{(1)\mu}\tilde{F}^{\mu}_{(1)} - e^{-\phi}\frac{1}{12}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} - \frac{1}{12}e^{\phi}F_{(3)\mu\nu\lambda}\tilde{F}'^{\mu\nu\lambda}_{(3)}$$
$$+\frac{1}{4.5!}F_{(5)\mu\nu\lambda\delta\gamma}\tilde{F}^{\mu\nu\lambda\delta\gamma}_{(5)}\Big) + \frac{1}{4\kappa^2}\int\ C_{(4)}\wedge H\wedge H' \,.$$

Appendix B

Mathematica programs

These programs, written in Mathematica, check that the LM background satisfies all type IIB supergravity equations of motion. Naturally, the same programs were used to verify the consistency of the three γ-deformed $AdS_2 \times S^2 \times T^6$ backgrounds. Those interested in these codes should contact me at the following e-mail address: *ciceriffranz@gmail.com*.

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