Special Lagrangian submanifolds and non-perturbative Type IIA Superstring Theory

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Abstract

In this thesis membrane instantons in type IIA superstring theory, compactified on a Calabi-Yau manifold, are studied. These membrane instantons can be represented using special Lagrangian submanifolds, or SLags, of this Calabi-Yau manifold. A SLag represents the image, or target space, of a membrane instanton. SLags can also have some interesting geometric properties.

The main purpose of the research I did, presented in this thesis, was to find the image of membrane instantons, in a few different cases. We will start with a detailed introduction of instantons in general. Instantons are important: determining instanton solutions of a model is needed for performing complete quantum corrections to the model. Then we will continue with a very detailed list of the needed mathematical definitions and relations, but also with some specific propositions, related to the later chapters. Then we will introduce type IIA superstring theory, D-branes, Euclidean D-Branes and membrane instantons, and we will give a first motivation for studying SLags. Then we will briefly introduce the quintic and the mirror quintic.

After that we will first give a formal definition of SLags, in a mathematical sense. Then the first examples we start with are SLags embedded in tori of complex dimension 1, 2 and 3, thus in T^2, T^4 and T^6. We will formally introduce the Borcea-Voisin product of two Calabi-Yau manifolds and the SLags they contain. We will discuss SLags embedded in the Fermat quintic, in the Fermat cubic, in the Fermat quartic K3 surface, in the singular K3 surface and in the Borcea-Voisin product of the Fermat quartic K3 surface and a flat torus. The Fermat quintic and the Borcea-Voisin product of the quartic and the torus are Calabi-Yau manifolds of complex dimension three, so these are manifolds we can use to compactify type IIA superstring theory on, and the SLags these Calabi-Yau manifolds contain represent membrane instantons. The collection of SLags of real dimension 3, thus images of membrane instantons, I found can be regarded as the most important result of this thesis. They are presented in the chapter of conclusions.

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Contents

1 Introduction

Motivation. This thesis should be regarded as a report of some research I did about instantons in string theory, a model in theoretical physics. In general, studying instantons is quite important for performing full quantum corrections to a theory. Perturbation theory already gives us many quantum corrections, starting with classical solutions and performing numerical procedures, but when we restrict to using perturbation theory, the result is far from complete. The non-perturbative corrections, described by instantons, will help us with performing complete quantum corrections, often related to tunnelling effects.

The reader will also find much mathematics in this thesis. A lot of mathematical tools are needed for studying string theory and instantons in string theory. This thesis ends with physical results, but, before reaching the end, I first did much research about the mathematical aspects of instantons in string theory. This will be found in the later chapters, before the conclusions.

What are instantons anyway? In general, the study of instantons is a wide field of research. In many physical models in quantum mechanics and quantum field theory the notion of instantons is mentioned.

A brief explanation of what instantons are: they are related to the model in question after changing from Minkowski space-time to Euclidean space. In many models this is performed by using the so-called Wick rotation, and changing the action and the corresponding equations of motion to the 'Euclidean action' and the corresponding 'Euclidean equations of motion'. Then the time variable will be complexified, and we will use the imaginary part instead of the real part. We could say that an instanton, only living in imaginary time, lives at one instant of real time. This mainly explains its name.

Then an instanton is one of the solutions of the classical Euclidean equations of motion, satisfying the following convention: it has a *finite* Euclidean action. This is a rather artificial construction, but it will help us explaining the tunnelling effect, which is one of the main effects caused by the laws of quantum mechanics the microscopic world satisfies. In classical mechanics this would not have any meaning.

In the next section we will first point out *what* instantons technically are, and later in this section we will explain *why* we are interested in instantons.

More about string theory. In string theory the basic physical objects are strings, instead of pointlike particles. The travelling path of a pointlike particle we are familiar with is called a world-line, and this is a 1-dimensional subset of the background space-time. The travelling 'area' of a string is called a world-sheet, and this is a 2-dimensional subset of the background space-time.

In general there are different kinds of string theory. There is bosonic string theory, living in a background space-time with 26 dimensions, and there is supersymmetric string theory, or *superstring theory*, living in a background space-time with 10 dimensions. Superstring theory is a string theory describing both bosonic and fermionic degrees of freedom. Then, because this theory is supersymmetric, the results of the theory will be invariant if specific bosonic and fermionic fields are smoothly exchanged.

There are a few different types of superstring theory, namely *type I*, *type IIA*, *type IIB* and two different types of *heterotic* string theory. I restricted to type IIA superstring theory.

When studying one kind of theory, we can also choose what kind of 10-dimensional background space-time we will use. We could, for example, choose a *flat* space-time, say $\mathbb{R}^{10}$. We could also choose a space-time which is partially *curved* and *compact*, for example a direct product of $\mathbb{R}^4$ and C, where C is a curved and compact 6-dimensional space. The word *curved* means that the space in question has a more complex geometry, thus cannot be described as a flat space. The word *compact* means that the space in question has a finite volume, but no boundary, just like a circle. As we will mention later, we also assume that the properties of C have extra restrictions when applied to string theory.

In any case we need to construct a relation to physical reality of 4 dimensions; to connect the 10-dimensional theory to physical reality of 4 dimensions, we first need to perform a procedure to make the 6 extra dimensions invisible. In general, there are some already known theoretical ideas how to do this. We have chosen the so-called *compactification* of these 6 extra dimensions, so that the total 10-dimensional space-time is a direct product of the space-time $\mathbb{R}^4$ and a curved and compact space of 6 dimensions. The size of this, still, theoretical compact space is assumed to be very small, smaller than the sub-atomic scale, so that indeed a relation to physical reality of 4 dimensions is possible.

To conclude, in this thesis I restricted to studying instantons in type IIA superstring theory, defined on a 10-dimensional background space-time being a direct product of $\mathbb{R}^4$, which is a 4-dimensional flat Minkowski space, and a very small curved and compact space. I worked out a detailed definition of type IIA theory, defined on such a space-time, in Chapter 3.

More about the curved and compact space of six dimensions. To be more precise: when studying the compactification we will restrict to so-called Calabi-Yau spaces, which are very small compact 6-dimensional spaces with special but complicated geometric properties which are mathematically interesting. This type of spaces needs a very detailed and advanced mathematical definition, which can be found in Chapter 2, the chapter about the definitions of needed mathematical tools. Especially see Section 2.7. Here, in this chapter, we do not really need the technical definitions of Calabi-Yau spaces yet.

Instantons in type IIA superstring theory: Membrane instantons. In quantum theory an instanton solution is a field described by a map from a domain space to a target space, and the domain space will be a space with a Euclidean metric. In string theory the target space, or image, will be a subset of the 10-dimensional background space-time. Then we say that it is *embedded* in space-time. We assume that in any case the domain space and the target space have the same dimension.

An instanton in string theory can be described by a map with a target space of arbitrary dimension. In type IIA theory we have, for example, instantons described by a map with a target space of dimension 3. I restricted to the study of these instantons, or so-called *membrane instantons*. The target space of each of these membrane instantons is located at a single point in the first 4 dimensions of physical reality, thus also at a single moment in time, so that we may indeed call these objects instantons in the 4-dimensional physical reality.

To conclude, in this thesis I restricted to studying membrane instantons, or, to be more precise, to the target spaces of these membrane instantons. These target spaces are *compact* 3-dimensional spaces, thus with finite volume and without boundary, and also with interesting structures and special geometric properties. These target spaces are called *special Lagrangian submanifolds*, or *SLags*, and they are also embedded in the Calabi-Yau space. I have studied the mathematics specifically needed for this, and I will work this out in Section 6.1. Here, in this chapter, we do not really need the technical definitions of SLags yet. There are some other types of instantons in type IIA theory, but different mathematics is needed then. Then the target spaces of these instantons are not represented by SLags.

The main goal of this thesis. The main goal of the research I did, and worked out in this thesis, was to find some of the SLags, if a specific Calabi-Yau space is chosen. Then, in case of a 6-dimensional Calabi-Yau space, these SLags represent the target spaces of membrane instantons. It depends on the chosen Calabi-Yau space what the membrane instantons look like, and what the structure of the whole collection of possible membrane instantons looks like.

As a preparation I studied the following needed knowledge, to be found in already existing literature: the theory, techniques, formal definitions and mathematics regarding instantons in general, type IIA superstring theory, Calabi-Yau spaces, membrane instantons and SLags.

In this thesis I will discuss some Calabi-Yau spaces of dimension 2, 4 and 6. In general, a Calabi-Yau space of dimension $2n$ contains SLags of dimension n. I will discuss already known examples of SLags, in different cases of Calabi-Yau spaces, and I will discuss other SLags I found by myself.

Only the 3-dimensional SLags can be used as the target space of a membrane instanton. In Section 7.2 and 8.3 I will discuss 3-dimensional SLags I found by myself. In Section 7.3, 8.1 and 8.2 I will discuss other SLags I found by myself, but these cannot be used as target spaces of membrane instantons, as these are of a lower dimension. In the other sections of Chapter 6,7 and 8 I will discuss already known SLags, to be found in other literature, and I also did this as a preparation.

Outlook. In the rest of this chapter I will give a more detailed introduction of instantons in general, with some examples. The background knowledge can be found in other literature. In short, instantons in quantum mechanics and quantum field theory are used for studying the so-called non-perturbative properties of the theory, as they are in general, and I will mention these in Section 1.1 and 1.2. Section 1.1 will be about instantons in quantum mechanics and I will, for example, explain in more detail what tunnelling, perturbation theory and non-perturbative methods are. Here I will also work out the example of instantons in the double-well potential. Section 1.2 will be about instantons in quantum field theory and I will work out the example of instantons in Yang-Mills theory. In Section 1.3 I will mention the relation between solitons and instantons, and in Section 1.4 I will give an introduction to instantons in superstring theory.

At the end of Section 1.4 I will give a description and a motivation of the rest of the chapters. Then I will also explain in much more detail in which chapters I mention the already known SLags, in which chapters I mention the SLags I found by myself and how I did it.

1.1 Instantons and tunnelling in quantum mechanics

In this section we will start with introducing instantons in quantum mechanics, but first we will define the notions of Euclidean space and Wick rotation in some more detail. Then we will also mention tunnelling, perturbation theory and non-perturbative effects.

Euclidean space. Here we will explain what 'Euclidean space' means in this context, but we will start with Minkowski space-time. In a simple model, supporting special relativity, we often say we are dealing with a Minkowski space-time. Then in four dimensions, which are three space dimensions and one time dimension, say $\vec{x}$ and t, the inner product is $\vec{x}^2 - t^2$. (There is an equivalent convention, where all the signs are reversed.) The reason to use this inner product, or 'Minkowski metric', is because certain quantities are invariant under Lorentz transformations. (The inner product itself is one of the invariant examples.)

If we have another space of dimension four, also with three space dimensions, but with a different time dimension, say $\vec{x}$ and τ, and with inner product $\vec{x}^2 + \tau^2$, then we are dealing with a *Euclidean space*.

Now we can say it is possible to find a relation between the Minkowski space-time and a Euclidean version of this space, as follows. Normally t lies in $\mathbb{R}$, the space of real numbers. However, we can as well say that $t \in \mathbb{C}$, the canonical complexified version of $\mathbb{R}$. If we first say that also $\tau \in \mathbb{C}$, then we can construct the relation

$$\tau = it, \tag{1.1}$$

so that $\tau^2 = -t^2$. Now, if we restrict to purely imaginary t, then τ is a real number. Then the Euclidean inner product $\vec{x}^2 + \tau^2$ is a positive real number, and we say that τ is the *Euclidean time variable*. However, it possibly is not clear yet why we should restrict to purely imaginary t. We will discuss this now.

The Wick rotation. We can say that the real time axis can be regarded as an integration contour in $\mathbb{C}$. A *Wick rotation* is a rotation of an integration contour by 90 degrees. The convention we use is that the real axis will be rotated clockwise around the origin, thus it will be rotated to the imaginary axis. Then any $t \in \mathbb{R}$ will be mapped to $-it$, lying in $i\mathbb{R} \subset \mathbb{C}$, so that it will be mapped to t. This is in harmony with (1.1): when considered pointwise, any real number t will be canonically mapped to the real number τ after this Wick rotation.

Relation (1.1) implies that $t = -i\tau$, so that also $dt = -id\tau$. Now, by using Cauchy's theorem, we can rewrite an integral as follows:

$$\int_{\mathbb{R}} dt f(t) = \int_{+i\infty}^{-i\infty} dt f(t) = -i \int_{\mathbb{R}} d\tau f(\tau). \tag{1.2}$$

(Actually, in pure mathematics, this only makes sense if the integrals in question will not diverge, and if the function f also has no poles in the parts of the domain of f we use when we continuously change the contours. Also $f(\tau)$ and $f(t)$ are not precisely the same functions. We can say that $f(t) = f_t(t)$ and that $f(\tau) = f_\tau(\tau)$.)

The first rewriting step comes from continuously rotating the contour, and assuming that f goes to zero if $|t| \to \infty$. Then the contours lying on the 'circle' with infinite radius have no effect: then the value obtained from the integral will remain constant. Then the second rewriting step comes from substituting $t = t(\tau)$.

The Wick rotation applied to the action of a model. The Wick rotation of the time variable can be used for redefining the action S, defined on Minkowski space-time, to the *Euclidean action* S_E, defined on the the Euclidean space.

Let $S[q]$ be an action of a model in quantum mechanics, and let L be its Lagrangian, so that

$$S[q(t)] := \int dt L(q(t), \dot{q}(t)) \quad , \quad L(q(t), \dot{q}(t)) = T(\dot{q}(t)) - V(q(t)) = \frac{1}{2} m\dot{q}^2(t) - V(q(t)). \tag{1.3}$$

Here q is a certain time-dependent quantity related to the model, and $\dot{q}$ is its time derivative: $\dot{q}(t) = dq/dt$. (In this case we start with the assumption that S, t, $q(t)$ and L are all real quantities.)

The equations of motion, or the 'Euler-Lagrange equations', are obtained by finding a critical point of the action. In general we write:

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \quad \Rightarrow \quad \frac{\partial V}{\partial q} + m\ddot{q} = 0. \tag{1.4}$$

Now (1.1) implies that we can rewrite q and $\dot{q}$, which implies the following relation, after complexifying the time variable. Here we will use that

$$q(\tau) \simeq q_\tau(\tau) = q_t(-i\tau) = q_t(t(\tau)) \quad , \quad \dot{q}(\tau) = \dot{q}_\tau(\tau) = \frac{\partial q_\tau}{\partial \tau}(\tau) \simeq -i\frac{\partial q_t}{\partial t}(-i\tau) = -i\dot{q}_t(t(\tau)),$$

so that

$$\dot{q}^2(\tau) \simeq (\dot{q}_\tau(\tau))^2 \simeq (-i\dot{q}_t(t(\tau)))^2 \simeq -\dot{q}^2(t(\tau)).$$

Then we can apply (1.2) to $S[q]$ to obtain the following:

$$\begin{aligned} S[q(t)] &= \int dt(\frac{1}{2}m\dot{q}^2(t) - V(q(t))) = \int \frac{dt}{d\tau} d\tau(\frac{1}{2}m\dot{q}^2(t(\tau)) - V(q(t(\tau)))) \\ &= -i\int d\tau(-\frac{1}{2}m\dot{q}^2(\tau) - V(q(\tau))) = i\int d\tau(\frac{1}{2}m\dot{q}^2(\tau) + V(q(\tau))) = iS_E[q(\tau)]. \end{aligned} \tag{1.5}$$

Then we can make a slight rewrite of (1.3). Here the Euclidean action $S_E[q]$ and the Euclidean Lagrangian L_E are defined as follows:

$$S_E[q(\tau)] := \int d\tau L_E(q(\tau), \dot{q}(\tau)) \quad , \quad L_E(q(\tau), \dot{q}(\tau)) = T(\dot{q}(\tau)) + V(q(\tau)) = \frac{1}{2}m\dot{q}^2(\tau) + V(q(\tau)). \tag{1.6}$$

Then we have the following relation between the actions (1.3) and (1.6):

$$\frac{i}{\hbar}S[q(t)] = -\frac{1}{\hbar}S_E[q(\tau)].$$

(This is in harmony with the relation between S and S_E as described in [3].)

We say that the Euclidean action *itself* is real and positive, or, to be more precise, it is a real and positive *functional*. This means that the value of $S_E[q(\tau)]$ is real and positive if we restrict to $q(\tau)$ itself being real. (We cannot a priori make sure that a real-valued function $q(t)$ implies that we obtain a real-valued function $q(\tau)$, after a direct rewrite.) This is why we change the path integral $\int \mathcal{D}q(t)$ to another path integral $\int \mathcal{D}q(\tau)$, restricting to real-valued functions $q(\tau)$. Then the path integrals W (or the 'transition function') and W_E (or the 'Euclidean transition function'), corresponding to S and S_E respectively, are related as follows:

$$W = \int \mathcal{D}q(t)\exp(\frac{i}{\hbar}S[q(t)]) \quad \longleftrightarrow \quad W_E = \int \mathcal{D}q(\tau)\exp(-\frac{1}{\hbar}S_E[q(\tau)]).$$

(In this case we again assume that S_E, τ, $q(\tau)$ and L_E are all real quantities.)

The Euclidean equations of motion. We can rewrite the equations of motion, see (1.4), to the Euclidean equations of motion:

$$\frac{\partial L_E}{\partial q} = \frac{d}{d\tau}\frac{\partial L_E}{\partial \dot{q}} \quad \Rightarrow \quad \frac{\partial V}{\partial q} - m\ddot{q} = 0. \tag{1.7}$$

Instantons: Formal definition. Traditionally an *instanton* is an exact (real valued) solution of the classical Euclidean equations of motion of a model, derived from the Euclidean action, in many cases obtained after doing the Wick rotation, and it must satisfy an extra condition: its Euclidean action has a finite (but non-zero) value. An instanton is also assumed to be localised.

Instantons: The double-well potential as an example. The double-well potential is defined by

$$V(q) := \frac{1}{4}(q^2 - 1)^2. \tag{1.8}$$

It satisfies $V(q) \geq 0$, its global minima are $q = \pm 1$ and its minimal value is zero, at both minima. It has a local maximum at $q = 0$, with $V(0) = 1/4$, and V has no other critical points. It goes to $+\infty$, if $|q| \to \infty$.

This double-well potential implies the following Lagrangian:

$$L(q(t), \dot{q}(t)) := \frac{1}{2}\dot{q}^2(t) - \frac{1}{4}(q^2(t) - 1)^2.$$

Then, applying (1.5) and (1.6) to this model, we obtain the following Euclidean action:

$$S_E[q(\tau)] = \int_{\mathbb{R}} d\tau (\frac{1}{2}\dot{q}^2(\tau) + \frac{1}{4}(q^2(\tau) - 1)^2). \tag{1.9}$$

Then the Euclidean equations of motion, see (1.7), induced by this model, are:

$$m\ddot{q} = \ddot{q} = \frac{\partial V}{\partial q} = q^3 - q.$$

We can present some non-trivial finite-action solutions of this equation:

$$q(\tau) = q_{\pm,\tau_0}(\tau) := \pm \tanh((\tau - \tau_0)/\sqrt{2}) \quad \Rightarrow \quad \dot{q}_{\pm,\tau_0}(\tau) = \pm(1 - q^2_{\pm,\tau_0}(\tau))/\sqrt{2}. \tag{1.10}$$

(The constant τ_0 represents a translation.) Note that we are dealing with a non-linear differential equation, thus a solution $q_{\pm,\tau_0}(\tau)$ does not imply another solution $q^{\lambda}_{\pm,\tau_0}(\tau) := \lambda q_{\pm,\tau_0}(\tau)$ in general, with arbitrary λ.

About the function 'tanh' itself: it is an odd function, and it satisfies $|\tanh(x)| < 1$ for all $x \in \mathbb{R}$, and $\tanh(0) = 0$. It also satisfies

$$\lim_{x\to\infty} \tanh(x) = 1 \quad , \quad \lim_{x\to-\infty} \tanh(x) = -1,$$

which implies that

$$\lim_{|x|\to\infty} \tanh^2(x) - 1 = 0.$$

(We note that these values ± 1 correspond to the global minima of $V(q)$.)

Then (1.10) implies that for these $q_{\pm,\tau_0}(\tau)$ we can rewrite the Euclidean action:

$$S_E[q_{\pm,\tau_0}(\tau)] = \int_{\mathbb{R}} d\tau \frac{1}{2}(q^2_{\pm,\tau_0}(\tau) - 1)^2 = \int_{\mathbb{R}} d\tau \frac{1}{2}(\tanh^2((\tau - \tau_0)/\sqrt{2}) - 1)^2.$$

We note that the integrand is a positive (but non-zero) integrable function, so that its integral over $\mathbb{R}$ is a finite (but non-zero) value. Then these functions $q_{\pm,\tau_0}(\tau)$ can be regarded as valid instanton solutions of this model of the double-well potential.

By convention we say that q_{+,τ_0} is an *instanton solution*, and that q_{-,τ_0} is an *anti-instanton solution*. When τ goes from $-\infty$ to $+\infty$, the instanton solution goes from value $q = -1$ to $q = +1$ and the anti-instanton solution goes from value $q = +1$ to $q = -1$. We conclude that an instanton describes a particle, travelling in Euclidean time, from one to the other minimum. To be more precise, these functions $q_{\pm,\tau_0}$ are called *1-instanton solutions*. (There also exist 'n-instanton solutions', and these solutions describe n 'jumps' between minima of the potential. In many cases of models these solutions are hard to construct exactly: numerical mathematics is needed then.)

Instantons: What are they used for? Instantons are mainly used for explaining and finding the following things related to the model and arising from the quantum effects:

- Tunnelling processes.
- Non-perturbative solutions.

To be more precise, tunnelling processes are rather specific examples of non-perturbative effects. In general the non-perturbative effects are used for describing a full quantum correction to the model, starting with the classical version. Now we will mention some details about these things.

Tunnelling. In many models the potential has a local maximum, and, classically, if the total energy of a particle is lower than this maximum, then the particle *cannot* pass the barrier related to this local maximum. However, processes of *tunnelling* are happening in the real world, which means that sometimes a particle *can* pass this barrier. Therefore these processes cannot be explained by classical mechanics; quantum mechanics can be used to explain them, and the study of instantons is closely related to these processes.

As an example: assume that the potential $V(q)$ equals the double-well potential, as defined in (1.8), which has a local maximum at $q = 0$, and that $q(t)$ is a quantity related to a 'particle' or 'object', described by this model. Then in classical mechanics it is impossible for this object to travel from $q < 0$ to $q > 0$ if it has an energy lower than $V(0)$. We know that the energy of an object in quantum mechanics has an uncertainty, which means that there *is* a certain probability for this object to pass the maximum at $q = 0$.

Tunnelling and instantons. By intuition it is possible to relate this to one of the instanton solutions. If we observe the Euclidean Lagrangian, then we see that

$$L_E(q(\tau), \dot{q}(\tau)) = T(\dot{q}(\tau)) + V(q(\tau)) = T(\dot{q}(\tau)) - (-V(q(\tau))).$$

This is related to a potential $V_E = -V$, which has two global *maxima*, at $q = \pm 1$, and a local *minimum*, at $q = 0$. Then any of the instanton solutions defined by (1.10) describes a classical trajectory of an object, starting at $q = -1$ and with zero velocity if $\tau = -\infty$, and ending at $q = 1$ and (again) with zero velocity if $\tau = +\infty$. (In case of the anti-instanton solutions this happens in the opposite direction.)

Thus, if we look at a local maximum of V, with respect to real time, then, after doing the Wick rotation and changing to imaginary time, we will see that it will change into a local minimum of V_E. Then it is 'easy' to pass the barrier.

Thus, the tunnelling is closely related to the classical trajectory happening in imaginary time, related to one of the instanton solutions. In other words, an instanton solution will help us finding the transition probability for the object to pass the barrier, by a tunnelling process. (A better and far more detailed description can be found in [3].)

As we can read in [22], a tunnelling process itself, related to an instanton solution, lives in Minkowski space-time, and it describes a trajectory from one *vacuum* (or *local extremum of the potential*), at time $t = t_1$, to another vacuum, at time $t = t_2 > t_1$.

Perturbation theory. Before we will discuss the notion of non-perturbative solutions of models, we will first discuss perturbation theory. In general, perturbation theory is a numerical procedure, in classical and quantum models, to construct new solutions if some basic solutions (of a simpler model) are already known exactly. If we perform an n-step numerical procedure, with a finite n, then we often find approximate solutions only. Perturbation theory is often used if we cannot exactly find (a vast majority of) the solutions of the model.

In quantum mechanics and quantum field theory we often try to find (exact) classical solutions of a simpler model first, to be used as basic solutions, and perform perturbation theory 'around' these classical solutions to find quantum solutions. However, perturbation theory can be regarded as a *continuous* procedure of changing the basic solution of the simplified model to one of the approximate solutions of the full model.

Non-perturbative methods. In general, the full quantum model often has a whole family of other solutions, which cannot be approximated by using this continuous procedure of perturbation. In many cases there are still other methods for finding these other solutions, and we will call these methods 'non-perturbative'. (The whole family of the non-perturbative solutions, related to instantons, brings corrections to the path integral of the model: this family can be regarded as the collection of the leading quantum corrections, applied to the classical behaviour related to the model.)

In some cases we can exactly find such a non-perturbative solution. Especially if we are studying 1-instanton solutions of the Euclidean version of the model, then we exactly find solutions we cannot find by using the techniques of perturbation theory. (Then we can still apply numerical procedures to find n-instanton solutions from this basic one.)

The tunnelling processes we already discussed are nice examples of processes which *cannot* be described or explained by using perturbation theory. We can regard the procedures we used to find them anyway, as an example of finding a non-perturbative solution. This is why we say that the quest for instantons can be regarded as a *non-perturbative* procedure.

A comment. We will end this section with the notion that $q(t)$ was mentioned as a one-dimensional quantity. We can replace this q by a quantity of an arbitrary higher dimension. Then we can do similar things, describing the instantons in such models in quantum mechanics.

1.2 Instantons and tunnelling in quantum field theory

Until now we mainly discussed instantons in quantum mechanics. In this case we studied quantities q, of arbitrary dimension, and only depending on time. Now we will also discuss instantons in quantum field theory.

Instantons in quantum field theory. Also in quantum field theory instantons are related to the model in question after doing a change from Minkowski space-time to Euclidean space. Also in this case we can use the Wick rotation: $t \in \mathbb{R}$ can be mapped to $-it \in i\mathbb{R}$, and then we obtain a real value $\tau = it$.

Let now $\phi(t, x^j)$ be a field, defined on a space-time of D dimensions, and let $S[\phi]$ be an action of a model in quantum field theory. The action S and the Lagrangian density $\mathcal{L}$, with respect to Minkowski space-time, are defined as follows:

$$\begin{aligned}
S[\phi(t,x^j)] &:= \int dt \int d^{D-1}x\mathcal{L}(\phi(t,x^j),\dot{\phi}(t,x^j),\partial_k\phi(t,x^j)),\\
\mathcal{L}(\phi(t,x^j),\dot{\phi}(t,x^j),\partial_k\phi(t,x^j)) &= T(\dot{\phi}(t,x^j),\partial_k\phi(t,x^j)) - V(\phi(t,x^j))\\
&= \frac{1}{2}(\frac{\partial\phi}{\partial t}(t,x^j))^2 - \frac{1}{2}\sum_{k=1}^{D-1}(\frac{\partial\phi}{\partial x^k}(t,x^j))^2 - V(\phi(t,x^j)),
\end{aligned}$$

or, using compact notation:

$$S[\phi] = \int dt \int d^{D-1}x \mathcal{L}(\phi, \dot{\phi}, \nabla\phi) \quad , \quad \mathcal{L}(\phi, \dot{\phi}, \nabla\phi) = T(\dot{\phi}, \nabla\phi) - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi).$$

(We say that S and $\mathcal{L}$ describe a field theory in D dimensions.)

Then we can change the action to the Euclidean action $S_E[\phi(\tau, x^j)]$:

$$\begin{aligned} S_E[\phi(\tau, x^j)] &:= \int d\tau \int d^{D-1}x \mathcal{L}_E(\phi(\tau, x^j), \dot{\phi}(\tau, x^j), \partial_k\phi(\tau, x^j)), \\ \mathcal{L}_E(\phi, \dot{\phi}, \partial_k\phi) &= T(\dot{\phi}, \partial_k\phi) + V(\phi) = \frac{1}{2}(\frac{\partial\phi}{\partial\tau}(\tau, x^j))^2 + \frac{1}{2}\sum_{k=1}^{D-1}(\frac{\partial\phi}{\partial x^k}(\tau, x^j))^2 + V(\phi(\tau, x^j)). \end{aligned}$$

Then the variables (τ, x^j) are elements of a Euclidean space of dimension D. We again have the following relation between the actions:

$$\frac{i}{\hbar}S[\phi(t, x^j)] = -\frac{1}{\hbar}S_E[\phi(\tau, x^j)].$$

Then the path integrals W and W_E, corresponding to S and S_E respectively, are related as follows:

$$W = \int \mathcal{D}\phi(t, x^j)\exp(\frac{i}{\hbar}S[\phi(t, x^j)]) \quad \longleftrightarrow \quad W_E = \int \mathcal{D}\phi(\tau, x^j)\exp(-\frac{1}{\hbar}S_E[\phi(\tau, x^j)]).$$

The perturbative and non-perturbative methods in quantum field theory are similar to the perturbative and non-perturbative methods in quantum mechanics, as introduced in the previous section.

Instantons in Yang-Mills theory as an example. Yang-Mills theory is an example of a gauge theory, based on the non-abelian symmetry group SU(N), for some arbitrary $N \geq 2$. Here we will restrict to $N = 2$, and the corresponding Lie group SU(2) has dimension 3. We assume that we are dealing with a fundamental representation of the Lie group, and that the corresponding matrices are complex-valued and 2-by-2. Then the corresponding Lie algebra is generated by 3 complex-valued traceless anti-hermitian 2-by-2 matrices. It has three generators T_a, satisfying the following relations:

$$T_a = -\frac{i}{2}\tau_a \quad , \quad \mathrm{tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}, \tag{1.11}$$

where τ_a are the Pauli matrices. (We say that T_a are generators of a fundamental representation of the Lie algebra of SU(2).)

The Yang-Mills field itself is a matrix-valued space-time vector $A_\mu(t, x^j)$, with respect to the 4-dimensional Minkowski space-time, and the matrices are lying in the Lie algebra representation. With respect to T_a it is expressed as

$$A_\mu(t, x^j) = A_\mu^a(t, x^j)T_a.$$

Its field strength $F = F(A)$ is expressed as

$$F_{\mu\nu}(t, x^j) = F_{\mu\nu}^a(t, x^j)T_a \quad , \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Thus, A_μ and $F_{\mu\nu}$ are matrix-valued quantities, and their full components A_μ^a and $F_{\mu\nu}^a$ are scalar values.

The related gauge fixed action and Lagrangian density are given by:

$$S[A] = S_{gf}[A] := \int dt \int d^3x \mathcal{L}_{gf}(F(A)) \quad , \quad \mathcal{L}(F(A)) = \mathcal{L}_{gf}(F) := c\,\mathrm{tr}(F^2).$$

Here c is some constant, and the 'value' of F^2 itself is a contraction of the matrix-valued space-time tensor $F_{\mu\nu}$ with itself:

$$F_{\mu\nu} = F_{\mu\nu}^a T_a \quad \Rightarrow \quad F^2 = F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}^a T_a \cdot F^{\mu\nu b}T_b = F_{\mu\nu}^a F^{\mu\nu b} \cdot T_a T_b.$$

Then we can compute $\mathrm{tr}(F^2)$, using (1.11):

$$\mathrm{tr}(F^2) \;=\; F^a_{\mu\nu}F^{\mu\nu b}\,\mathrm{tr}(T_aT_b) = -\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu b}\delta_{ab} = -\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu a}.$$

(Note that here μ and ν are indices of space-time vectors, so that the Minkowski metric is used for raising or lowering these indices. The indices a and b satisfy the Euclidian metric, as usual in the Lie algebra. According to a rule in theoretical physics, we then can say that an upper index and a lower index are equivalent.)

Now we can transform the Minkowski space to a Euclidean space, and change the action. In ([22]) we can read how the action will change, and how to find 1-instanton solutions of the corresponding Euclidean action. We will shortly represent it here. The Euclidean action is written as:

$$S_E[A] = S_{gf,E}[A] := \int d^4x\mathcal{L}_{gf,E}(F(A)) \quad , \quad \mathcal{L}_E(F(A)) = \mathcal{L}_{gf,E}(F) := -\frac{1}{2g^2}\,\mathrm{tr}(F^2)_E.$$

(Here $x^4 \simeq \tau$.) Here F is a function of Euclidean variables *and* with Euclidean indices, and g is the coupling constant. Then we write $F^{\mu\nu} \simeq F_{\mu\nu}$, so that we can write

$$\mathrm{tr}(F^2)_E = \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) = -\frac{1}{2}F^a_{\mu\nu}F^a_{\mu\nu} \quad\Rightarrow\quad S_E[A] = \frac{1}{4g^2}\int d^4xF^a_{\mu\nu}F^a_{\mu\nu}.$$

We should note that $F^a_{\mu\nu}F^a_{\mu\nu}$ is positive, so that the Euclidean action itself is also positive.

Let now D_μ be a covariant derivative, with respect to the Lie algebra. Then the Euclidean action yields the following classical Euclidean equations of motion:

$$D_\mu F_{\mu\nu} := \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0.$$

We can solve these equations by rewriting the action. Let first $\tilde{F}$ be the tensor field dual to F:

$$\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}.$$

If we take the dual of the dual of F, then we obtain F itself again. To show this we can use some identities:

$$\epsilon_{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\rho\sigma} = 2(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}), \quad F_{\mu\nu} = -F_{\nu\mu} \Rightarrow \quad \tilde{\tilde{F}} = F.$$

Similarly we can show that $\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} = F_{\mu\nu}F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}F_{\mu\nu} = F_{\mu\nu}\tilde{F}_{\mu\nu}$. Now we can find minima (thus some of the extrema) of the action, if we rewrite the action as follows:

$$\begin{aligned}
(F \mp \tilde{F})^2 \;&=\; F^2 + \tilde{F}^2 \mp F\tilde{F} \mp \tilde{F}F = 2(F^2 \mp F\tilde{F}) \Rightarrow F^2 = \frac{1}{2}(F \mp \tilde{F})^2 \pm F\tilde{F} \Rightarrow \\
S_E[A] \;&=\; -\frac{1}{2g^2}\int d^4x\,\mathrm{tr}(F^2)_E = -\frac{1}{2g^2}\int d^4x\,\mathrm{tr}(\frac{1}{2}(F \mp \tilde{F})^2 \pm F\tilde{F}) \\
&=\; -\frac{1}{4g^2}\int d^4x\,\mathrm{tr}(F \mp \tilde{F})^2 \mp \frac{1}{2g^2}\int d^4x\,\mathrm{tr}\,F\tilde{F} \geq \mp\frac{1}{2g^2}\int d^4x\,\mathrm{tr}\,F\tilde{F}.
\end{aligned}$$

Thus, we have a minimum of the action, which is only reached if $F \mp \tilde{F} = 0$. We say in general that a field F is *selfdual* if it satisfies $\tilde{F} = F$, or *anti-selfdual* if it satisfies $\tilde{F} = -F$. To conclude, if F is (anti-) selfdual, then it satisfies $F \mp \tilde{F} = 0$, so that we can rewrite and 'simplify' the action:

$$S_E[A] = \mp\frac{1}{2g^2}\int d^4x\,\mathrm{tr}\,F\tilde{F} = \frac{8\pi^2}{g^2}|Q| \quad , \quad Q := -\frac{1}{16\pi^2}\int d^4x\,\mathrm{tr}\,F\tilde{F}. \tag{1.12}$$

This A has a finite action, and is a solution of the Euclidean equations of motion, thus it represents instantons. The field strength F is selfdual in case of instanton solutions, and it is anti-selfdual in case of anti-instanton solutions. The quantity Q, called the *Pontryagin index*, the *winding number*, or the *topological charge*, actually is an integer number, and it can be positive or negative. We can use this index to indicate the instanton or anti-instanton we are dealing with. The action has the same value in case of an instanton with number Q and an anti-instanton with number $-Q$.

Now we can represent some examples of 1-instanton solutions in Yang-Mills theory, with respect to SU(2), to be found in more detail in [22]. How to represent these solutions depends on which gauge is in use. We will shortly represent solutions in the regular gauge ($A^a_{\mu,R}$), and in the singular gauge ($A^a_{\mu,S}$):

$$A^a_\mu(X;X_0,\rho)_R = 2\frac{\eta^a_{\mu\nu}(X-X_0)_\nu}{(X-X_0)^2+\rho^2} \quad , \quad A^a_\mu(X;X_0,\rho)_S = -\bar{\eta}^a_{\mu\nu}\partial_\nu \log(1+\frac{\rho^2}{(X-X_0)^2}). \tag{1.13}$$

(Here $X \simeq (\tau, x^j) = (\tau, x^1, x^2, x^3) \simeq (x^1, x^2, x^3, x^4)$.) Each of these solutions is only correctly defined on its own domain, and these domains are not the same. (The symbols $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are some standard matrices with constant components, and can be found in other literature. They are called *'t Hooft η tensors*.)

We can simply construct 1-anti-instanton solutions by exchanging the role of the symbols $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ in (1.13). In Yang-Mills theory n-instanton solutions can be exactly constructed, and they are mentioned in [3]. In short: an n-instanton solution corresponds to a $Q = n$ solution of (1.12), and an n-anti-instanton solution corresponds to a $Q = -n$ solution of (1.12).

Each of these instantons corresponds to a tunnelling transition from one vacuum to another vacuum. These vacua also have a Pontryagin index each, say N and $N' = N + Q$ respectively.

1.3 Solitons and instantons: a relation

A *soliton* is a special solution of a (non-linear) classical field equation. It has a finite energy and has a localised energy density. Its shape is preserved in time and it travels at a constant velocity. The dimension D of the space of variables satisfies $D \geq 2$. In this section we will discuss the relation between solitons and instantons. (As we can read in [3], there is no universal convention for the exact definition of solitons. The solitons we will discuss here should actually be called *solitary waves*, and real solitons can have extra constraints, and we will not use these here.)

Solitons. Let M be a Minkowski space-time of dimension D, with vectors $(t, x^j) \in M$. Then x^j is an element of a space of dimension $D-1$. Let $\mathcal{H}(\phi)$ be the Hamiltonian density related to a classical field equation, where ϕ is a function with domain space M. Then the total Hamiltonian is

$$H[\phi] := \int d^{D-1}x\mathcal{H}(\phi(t,x^j)).$$

If now ϕ satisfies the equations of motion, then ϕ represents a soliton if its energy density satisfies the following relation:

$$\mathcal{H}(\phi(t,x^j)) =: \epsilon(t,x^j) \simeq \epsilon(x^j - v^j t), \tag{1.14}$$

where v^j is some velocity vector.

Static solitons. A *static soliton* ϕ is a soliton, thus satisfying (1.14), with $v^j = 0$. Thus, its energy density only depends on x^j:

$$\mathcal{H}(\phi(t,x^j)) = \epsilon(x^j).$$

(Any soliton can be transformed to a static soliton, after a change of coordinate frame of the observer.)

A relation between solitons and instantons. In many cases an instanton in D dimensions is equivalent to a static soliton in $D+1$ dimensions. If the instanton ϕ in dimension D has the action

$$S_E[\phi] = \int d\tau \int d^{D-1}x\mathcal{L}_E(\phi,\dot{\phi},\nabla\phi) = \int d\tau \int d^{D-1}x(T(\dot{\phi},\nabla\phi) + V(\phi)) \simeq \int d^D x(T+V)$$

(here we again replaced τ by x^D), then ϕ can be regarded as a static soliton solution. Let $\psi(t, x^j)$ be the corresponding static soliton defined on a space-time of dimension $D+1$. Then we know that $\psi(t, x^j) = \psi(t', x^j)$ for all t and t', so that we can write $\psi(t, x^j) = \phi(x^j)$. Then the extra kinetic energy term related to the t-variable will be zero:

$$\begin{aligned} T_{(D+1)} &= \frac{1}{2c^2}(\partial_t \psi(t, x^j))^2 = 0 \Rightarrow \\ S_E[\phi] &= \int d^D x(T+V) \simeq \int d^D x(T_{(D+1)} + T + V) \simeq \int d^D x(T(\partial_t \psi, \partial_j \psi) + V(\psi)) = H[\psi]. \end{aligned}$$

Now knowing that the instanton has a finite action, we conclude that the static soliton has a finite energy. (This construction of static solitons is valid for instantons both in quantum mechanics *and* in quantum field theory.)

Solitons related to instantons in the double-well potential. The Euclidean action of the double-well potential, see (1.9), can be used to construct a model in dimension 2. If we replace the symbol τ in this action by x, and replace $q(\tau)$ by $\phi(t, x)$, then the instanton corresponds to a static soliton of a model with the following Hamiltonian:

$$H[\phi] = \int_{\mathbb{R}} dx (\frac{1}{2c^2}(\frac{\partial \phi}{\partial t})^2 + \frac{1}{2}(\frac{\partial \phi}{\partial x})^2 + \frac{1}{4}(\phi^2 - 1)^2).$$

The instantons $q_\pm$ mentioned in (1.10) are directly related to static soliton solutions $\phi_\pm$:

$$\phi_\pm(t, x) := q_{\pm, \tau_0}(\tau)|_{\tau = x}.$$

Then a ϕ_+, related to an instanton solution, is called a *kink*, and a ϕ_-, related to an anti-instanton solution, is called an *antikink*. (Note that the instanton in quantum mechanics corresponds to a static soliton in 2-dimensional classical *field* theory.)

1.4 Instantons in string theory: outlook

As already mentioned, the rest of this thesis will mainly be a report of studying geometric properties of several special Lagrangian submanifolds, or SLags. A SLag is a submanifold (thus a smooth subset) of the Calabi-Yau manifold. Any SLag can be regarded as the image of a membrane instanton, which is an instanton in type IIA superstring theory, one of the string theoretic models we know. Here, in this section, we will briefly introduce the context of type IIA theory and membrane instantons. At the end of this section, I will give a description and a motivation of the rest of the chapters.

A compact description of the string theory we will study. The type of string theory we will study is *type IIA superstring theory*, which is a supersymmetric model living in a Minkowski space-time of 10 dimensions. A string itself is a 1-dimensional object, sweeping through time, and the equations of motion it satisfies are (locally) preserved under Lorentz transformations. Thus, the string can also be described as a 2-dimensional object in 10-dimensional space-time.

To make a connection to physical reality, we need to get rid of the 6 extra dimensions, so that we nicely end up with 3 space dimensions, and one time dimension. One method of dealing with the 6 extra dimensions is to 'compactify' them: we make some kind of 6-dimensional 'loops' of these dimensions. This will be done on the sub-atomic scale, so that, effectively, these dimensions are invisible at the scale of everyday life. We say a certain Calabi-Yau space K, of real dimension 6, or complex dimension 3, can help us. It has a list of rather complex mathematical and geometrical properties, and these will be listed in Chapter 2. In mathematics this space K is called a complex manifold of complex dimension 3 with Calabi-Yau 's properties, or, in short, K is a Calabi-Yau 3-fold. This Calabi-Yau 3-fold has a non-trivial curvature. We must introduce these mathematical properties before we can make any use of them, and this mathematics needs some complex introductory lecture itself.

In Chapter 3 and 4 we will work out the main definitions of string theory, 10-dimensional supergravity and membrane instantons in more details. These definitions can also be found in many already existing literature. We will see that string theory and supergravity are very closely related. Massless string theory is equivalent to supergravity, and type IIA supergravity also more generally contains objects like odd-dimensional p-branes, which are objects of p spatial dimensions and one time dimension. (Then p itself is an even number.)

In fact type IIA supergravity also supports so-called Euclidean p-branes, which are objects of $p+1$ spatial dimensions and *no* time dimension: they only exist at a moment in time. Thus, if we study instantons in massless string theory, then we will study Euclidean p-branes. Note that restricting to only studying massless strings is enough for studying instantons, thus it is enough to effectively study type IIA supergravity.

Now, if we restrict to Euclidean p-branes embedded in the Calabi-Yau 3-fold, then this object is automatically of finite volume, thus with a finite action. The ones that also have a minimized volume and have some supersymmetry are serious candidate instantons in string theory. In type IIA theory we will restrict to so-called *membrane instantons*, and these are objects of (real) dimension 3.

However, until now we have mentioned some standard examples of instantons. The instantons mentioned here should need some more general approach before we can continue, as these are not quite standard.

Domain space and target space of a model. Until now we mainly gave some examples where the instanton is described by a function $q(\tau)$ or a field $\phi(\tau, x^j)$, and where τ, each component of x^j, q and ϕ are just real numbers. Thus, until now we were dealing with a trivial *domain space*, a set of points $\tau \in \mathbb{R}$ and $x^j \in \mathbb{R}^{D-1}$, and a trivial *target space*, the set of points, lying in $\mathbb{R}$ or $\mathbb{C}$ (or with higher dimension), the points lying in the domain space get mapped to, by q or ϕ.

In general we are dealing with arbitrary domain spaces and target spaces. We note that $\mathbb{R}^n$ has a trivial topology (it is a contractible space), but in general the domain space and target space of the model do not necessarily have trivial topological properties. Especially in string theory we are dealing with non-trivial topological spaces.

What about tunnelling in string theory? In this case we will not really search for tunnelling processes, just because these processes are not fully understood yet in string theory. Also note that, when restricting to the first 4 coordinates, a membrane instanton remains at the same location. But, we can at least look at some of the non-perturbative aspects of string theory, expressed in more general language. (At the end of Chapter 4 we will discuss the effect of the membrane instantons: there will be a correction of the geometry of the hypermultiplet moduli space.)

A Calabi-Yau 3-fold already has a Euclidean metric and a finite volume, thus, as a consequence, its subspaces also have a Euclidean metric. The relevant subspaces mentioned are special Lagrangian submanifolds, which are spaces without boundary, with a minimized finite volume, and of real dimension 3. Here we claim that each of these spaces can directly be regarded as the target space of an instanton, so that we are dealing with a solution with minimized action. Thus, we are dealing with *membrane instantons* and the target spaces are SLags. (In Section 4.2 we will explain this.)

More comments about membrane instantons. Now we can assume that we can pull back the Euclidean metric, defined on the SLag L, to the domain space Σ of the membrane instanton. In this sense we could say that we are dealing with Euclidean variables defined on Σ. In this case it would be enough if the map $\iota : \Sigma \to L$ is surjective (not an isomorphism), so that multiple wrappings are supported. (Note that we assume that also Σ has a finite volume.)

In the study of instantons in string theory we will mainly focus on the action: its volume. We will ignore the quest for (Euclidean) time variables and Lagrangians. The action of a membrane instanton wrapping a SLag is directly related to the volume of the SLag, the target space in question. The SLag itself already has a minimal volume, so that we can indeed use it as the correct target space of a membrane instanton.

A SLag, embedded in a Calabi-Yau 3-fold, is a real manifold of real dimension 3, and it is *not* a Calabi-Yau manifold. It also has a list of properties, and the mathematical definitions and techniques, as listed in

Chapter 2, will also help us defining these SLags. The SLags themselves will be introduced in much detail in Section 6.1, and the SLag properties will be explained there.

More comments about the rest of the chapters. Here I will discuss the rest of the chapters. I will shortly explain what they are about, what parts can already be found in other, already existing, literature, and what parts discuss my own result, of independent studying.

Chapter 2 should mainly be regarded as a self-contained chapter. It lists the needed mathematics, including brief explanations, but the reader can easily skip this. However, we will refer to formulas of this chapter in later chapters. Some specific examples are also mentioned, and we will reuse them in the later chapters. The final, most important, part of this chapter is about the main definition of Calabi-Yau spaces, see Section 2.7. In fact there are some equivalent definitions of these Calabi-Yau spaces, but we will not use them all. We have Calabi-Yau spaces with an extra property, and these Calabi-Yau spaces will be called *strict* Calabi-Yau spaces. These strict ones are important when we will use them in physical models. Most of the knowledge and basic definitions, mentioned in this chapter, can be found in already existing literature, however, I used my own style of explanation. According to my opinion this style contains essential and necessary parts, which often misses in other literature. In this chapter I also mentioned some partial results of my own research. Especially one large part of Section 2.7, about the three properties of a holomorphic top-form, should be regarded as one of these results. The proof of the third property can also be found in [4] and [10], but there it contains some obscurities and a small mistake, and to my opinion I cleared it up. One large part of Section 2.2, about the special property of the Laplace operator, see (2.25), should also be regarded as one of these results. The proof of this relation is already worked out in other literature, but in many cases a compact but more abstract mathematics is needed. Here I used a method of brute-force, using very compact notation and a modular structure of proofs.

Chapter 3 is the main chapter introducing the physics, starting with the basic theory of the classical string, and finishing with the bosonic sector of the massless superstring theory in the general setting of a curved space-time. In Section 3.1 I will start with a short introduction of superstring theory, defined in a flat Minkowski space-time of 10 dimensions. There I will also give a short introduction of *supersymmetry* and of massless type II superstring theory. In Section 3.2 I will change from a flat Minkowski space-time to curved space-time. There I will mention that massless type IIA superstring theory is equivalent to type IIA supergravity. At the end of Section 3.2 I give a short introduction of *supermultiplets*, *vector multiplets* and *hypermultiplets*. These are the degrees of freedom of the low-energy effective action in 4 dimensions. I mainly followed the line of the already existing literature.

In Chapter 4 we specifically introduce the idea of D-branes, Euclidean D-branes and instantons. These will give a physical motivation for studying the special Lagrangian submanifolds, or *supersymmetric 3-cycles*, embedded in the Calabi-Yau manifold. The Euclidean D2-branes, or membrane instantons, play a role in the non-perturbative sector of the massless type IIA superstring theory. Here we will mainly work on a physical motivation for the SLag conditions, related to the preservation of supersymmetry. Before we mention a complete and formal definition of SLags, as I do in Section 6.1, we can say that a SLag is also called a *supersymmetric 3-cycle*, see [10]. Compactness and the SLag conditions guarantee that these spaces support a supersymmetric theory of membrane instantons. Especially in this chapter I will work this out in the physical context, and for this I will mainly follow the way how it is mentioned in [10]. In Section 4.1 we mainly follow [20]. Here we mainly list the basics of the D-branes and Euclidean D-branes. In Section 4.2 we mainly follow [10]. Here we will specifically mention membrane instantons, and how the supersymmetric conditions are related to the SLag conditions. At the end of Section 4.2 we will shortly discuss the non-perturbative corrections of the low-energy effective action in 4 dimensions.

In the next chapters we will mainly look at specific SLags, thus we will look at specific membrane instantons, *not* at the total correction to the moduli space.

In Chapter 5 we will introduce some general knowledge about one of the Calabi-Yau 3-folds we will mainly study in the later chapters: the Fermat quintic. The quintic is one of the best known Calabi-Yau 3-folds, a manifold which can directly be used to compactify type IIA superstring theory on. I will also shortly mention the mirror quintic here, but I will not continue about it in a later chapter. I mainly followed the line of the already existing knowledge about these objects.

In Chapter 6 we will discuss the formal definition of special Lagrangian submanifolds, or SLags, in a mathematical sense. In general, if M is a Calabi-Yau space, a complex manifold with complex dimension m, then the real dimension of a SLag L embedded in M equals m. A SLag itself will always be described as a real manifold. We will start with very simple basic examples of SLags, also in Calabi-Yau manifolds with lower dimensions. Finally we will introduce the formal construction of a so-called Borcea-Voisin manifold, a product of lower dimensional strict Calabi-Yau manifolds, so that the result is again a strict Calabi-Yau manifold. If we have a strict Calabi-Yau m-fold M and a strict Calabi-Yau n-fold N, then we can construct a so-called Borcea-Voisin $(m + n)$-fold, which is again a strict Calabi-Yau manifold of complex dimension $m + n$. The SLags embedded in M and N can be used to construct SLags embedded in the Borcea-Voisin product of M and N. This chapter should be regarded as a chapter mainly following the line of the article by Halmagyi, Melnikov and Sethi [21], one of the main articles I studied for this thesis. In Section 6.1 I worked out the formal definition and its features, as introduced in [21], but I especially added far more detailed explanations and mathematical proofs of all the features, which I worked out myself, independently. We can already find examples of SLags in T^2 and T^6, and a short introduction of the Borcea-Voisin construction in [21]. I slightly rewrote the needed explanations in all these sections, and I added an independent (though rather trivial) example of SLags in T^4. (We note that T^2 is a Calabi-Yau 1-fold, T^4 is a Calabi-Yau 2-fold and T^6 is a Calabi-Yau 3-fold.)

In Chapter 7 I will work out some more research in much detail, about a collection of SLags embedded in the Fermat quintic. The Fermat quintic is a Calabi-Yau 3-fold, thus this manifold itself can be used to compactify type IIA superstring theory on. The SLags it contains then represent membrane instantons. I have found 625 SLags in total, and these are all related. All the SLags are diffeomorphic to $\mathbb{R}\mathrm{P}^3$, the real projective space of dimension 3. (Being 'diffeomorphic' means that there exists a pointwise 1-to-1 relation between all these sets, including $\mathbb{R}\mathrm{P}^3$, preserving the so-called smooth structure.) They even all share the same geometry: distances are measured the same way on each of these sets. The main line, the result and the conclusion of Section 7.1 can be found in the article by Becker, Becker and Strominger [10] (also one of the main articles I studied for this thesis), but I used a far more detailed approach to fully explain these results. Here just one of the 625 SLags is mentioned. Section 7.2 should be regarded as one of the results of my own research: here I mention the 624 other SLags, using some kind of copying technique, to make duplications of the first SLag, introduced in Section 7.1. Section 7.3 is about an analog situation, where I study SLags embedded in a much simpler Calabi-Yau space, of one complex dimension. This Calabi-Yau space is also called the 'cubic torus', and the SLags embedded in it are also circles, similar to the SLags in the flat torus, mentioned in Section 6.2. In this case I found 9 SLags, and it should also be regarded as one of the results of my own research. (We note that the cubic torus is a Calabi-Yau 1-fold, just like the flat torus T^2.)

In Chapter 8 we will discuss SLags in K3 surfaces. We will finally introduce the Borcea-Voisin 3-fold, made from a Fermat quartic K3 surface and a flat torus, and shortly mention some SLags embedded in this 3-fold. Also this manifold itself can be used to compactify type IIA superstring theory on, and the SLags it contains represent membrane instantons. In Section 8.1 I will start with the Fermat quartic, which is a smooth K3 surface and a Calabi-Yau 2-fold, and I will introduce 24 SLags, embedded in this surface. I used a similar method as used in Section 7.2 to find them. In Section 8.2 I will shortly mention a basic example of a singular K3 surface, a K3 surface with singularities. This singular K3 surface can be regarded as a Borcea-Voisin product of the flat torus with itself. Here I use the Borcea-Voisin construction to find SLags, and to do this I explicitly used SLags embedded in the flat torus. Here I explain a relation between SLags embedded in the singular K3 surface and SLags embedded in the smooth Fermat quartic K3 surface. In Section 8.3 I will again explicitly work out the result of applying the Borcea-Voisin construction, applied to the product of the smooth K3 surface and the flat torus, and the SLags embedded in these spaces. The Borcea-Voisin construction used here is quite similar to the construction used in Section 8.2. This complete chapter should also be regarded as one of the results of my own research.

Chapter 9 is a summary of the mathematical results of Chapter 6,7 and 8. We can say that the sections, in these chapters, all have their own result, and we will list these results here. These results can be used for the conclusions of this thesis, in the next chapter.

Chapter 10 is about the conclusions of this thesis. The SLags represented in Section 7.2 and 8.3 are 3-dimensional spaces, so I can turn these in the main physical context of membrane instantons.

Disclaimer. I cannot claim that all of my own results should be regarded as *really* new results, but at least I can say that these are the results of my own thinking, instead of reusing the results from other literature.

2 Mathematical definitions

Preface. This chapter should be regarded as a self-contained part of the thesis, and people who are already familiar with the basics should skip reading that part and go on with the more advanced concepts. Any graduate student in theoretical physics should have an intuitive idea about the concepts described in the basics, however I think it is important to pay a little attention to how it is built up in a mathematical framework.

In Section 2.1 we will really introduce the basics of mathematics. In Section 2.2 some more advanced concepts will be introduced, like differential forms, the exterior derivative, de Rham cohomology groups, the Künneth formula, metrics, Riemannian geometry, vielbeins, the Hodge star operator, the exterior co-derivative, the Laplace operator and harmonic forms. In Section 2.3 we will continue with complex manifolds, complex differential forms, holomorphic and anti-holomorphic exterior derivatives, Dolbeault cohomology groups and Hodge numbers. In Section 2.4 we will show some important examples. In later chapters we will refer to examples introduced here in this section. In Section 2.5 we will continue with the subject of Section 2.3. Here we will introduce complex geometry, Kähler manifolds, holomorphic and anti-holomorphic exterior co-derivatives, more about Laplace operators on compact Kähler manifolds and the Künneth formula applied to Hodge numbers. In Section 2.6 we have some more examples. In Section 2.7 we will finally introduce the Calabi-Yau manifolds, which are very specific compact complex Kähler manifolds, with special properties. We can use these Calabi-Yau manifolds to compactify the 6 extra dimensions in superstring theory. (The notions introduced in Section 2.3 and 2.5 are needed for Section 2.7.) In Section 2.8 we have another self-contained part, this time of Chapter 2 itself. Then we will work out some definitions of, for example, involutions and isometries, also needed in later Chapters.

2.1 The basics

Topological spaces. Let M be an unspecified set, and let $\mathcal{T}$ be a collection of subsets of M. These subsets of M are now called *elements* of $\mathcal{T}$. If any union and any finite intersection of elements of $\mathcal{T}$ regarded as subsets of M, is again an element of $\mathcal{T}$, and if the empty set and M itself is an element of $\mathcal{T}$, then $\mathcal{T}$ is said to be a *topology* of M. Then, the pair $(M, \mathcal{T})$ forms a *topological space*. By defining a topology $\mathcal{T}$ of M, we gain knowledge of which subsets of M we can declare to be *open*, and we can say whether a subset of M, or M itself is compact.

Let now $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ be a pair of topological spaces. A surjective map $f : M \to N$, if existing, is said to be *continuous* if for any $U \subset N$, which is declared open with respect to $\mathcal{T}_N$, the set $f^{-1}(U) \subset M$ is open with respect to $\mathcal{T}_M$. If f is not surjective we can restrict the notion of continuity to the map $f : M \to f(M) \subset N$. The map f is called a *homeomorphism* if it is continuous and at the same time a bijection.

With these definitions all kinds of exotic topological structures are possible, but from now on we are only interested in the standard topology needed to describe real spaces. There is a straightforward definition of a very useful topology on $\mathbb{R}^m$. Without mentioning too much detail, we can intuitively define a *basis* for this topology. Let $\|\cdot\|$ be the Euclidean distance function, which equips $\mathbb{R}^m$ with a *metric*, turning it into a *metric space*. For any $x \in \mathbb{R}^m$ and any $\delta > 0$, define $B(x; \delta) := \{y \in \mathbb{R}^m | \|y - x\| < \delta\}$. Our intuition tells us that this is nothing more than an open ball, or *disk*, in $\mathbb{R}^m$. The collection of all possible open disks $B(x; \delta)$ is the basis for a topology of $\mathbb{R}^m$, in the sense that any open set, according to the classical definition, is the union of such open disks. We call this topology the *standard Euclidean metric topology.* This topology satisfies the *Hausdorff condition*, which tells us that for any pair of distinct points $a, b \in \mathbb{R}^m$, there exist open sets $A, B \subset \mathbb{R}^m$ such that $a \in A$, $b \in B$ and $A \cap B = \varnothing$.

Topological manifolds and homology. A topological space $(M, \mathcal{T})$ is called a *topological manifold* of dimension m, if $\mathcal{T}$ satisfies the Hausdorff condition, and if M is locally homeomorphic to $\mathbb{R}^m$. Here m is a positive integer or zero. The last property means that for any point $p \in M$ there is an open neighbourhood $U \subset M$ and a homeomorphism $\kappa : U \to V$, where V is an open subset of $\mathbb{R}^m$.

For any topological manifold M of dimension m, and for any $0 \leq k \leq m$ we can define the k-th *homology group*, which we denote by $H_k(M;G)$. Here G can be any abelian group. Any homology group is abelian by definition, and it has a finite basis if M is compact. I prefer the following definition of homology groups of a topological manifold. It is called the singular homology group, and the corresponding group G is $\mathbb{Z}$. It is defined by

$$H_k(M;\mathbb{Z}) := \mathrm{Ker}(\partial : C_k(M) \longrightarrow C_{k-1}(M))/\,\mathrm{Im}(\partial : C_{k+1}(M) \longrightarrow C_k(M)),$$

where, for all k, the group $C_k(M;\mathbb{Z}) = \mathbb{Z}S_k(M)$ is the space of *singular chains* of dimension k in M. The groups $C_k(M;\mathbb{Z})$ are called the *singular chain groups* of M, and ∂ is the boundary operator between them. The sequence of these groups and the maps ∂ inbetween is called the *singular chain complex*. Every boundary operator is a group homomorphism, and it is nothing more than an operator which detects the boundary of a given chain. For example, a filled triangle in M is a trivial example of an element of $C_2(M)$, and the boundary operator finds the triangle itself, which is an element of $C_1(M)$. One subtlety is that we should assign an orientation to the building blocks of a chain, and a rule for how ∂ acts on these oriented building blocks, such that a boundary, which in case of this example is just the triangle, itself has no boundary. In general we should find that ∂ is a nilpotent operator, thus $\partial^2 = 0$. This is the reason why we can construct a homology theory of the singular chain complex of M at all, based on arguments from homological algebra.

Any homology group of a compact topological manifold is a finitely generated abelian group, and as such it satisfies the following isomorphism:

$$H_k(M;\mathbb{Z}) \simeq \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r_k} \oplus \mathbb{Z}_{p(k,1)} \oplus \cdots \oplus \mathbb{Z}_{p(k,l)}.$$

The *rank* of $H_k(M;\mathbb{Z})$ is defined as the integer dimension of the free part of $H_k(M;\mathbb{Z})$, or just using symbols: $\mathrm{rank}(H_k(M;\mathbb{Z})) = r_k$. Later we will refer to r_k as the k-th *Betti number* of M, denoted by $b_k(M)$.

A special case of interest is the first homology group $H_1(M;\mathbb{Z})$. It can be identified with the abelianized *fundamental group* $\pi_1(M,p)$. A topological space is called *simply connected* when its fundamental group is trivial. This implies that $H_1(M;\mathbb{Z})$ itself is trivial, which in turn implies that the first Betti number $b_1(M)$ is zero. This implication cannot always be reversed: if $b_1(M) = 0$, then $H_1(M;\mathbb{Z})$ may still contain a non-free part, for example when M is the real projective space $\mathbb{R}P^2$. Another interesting case is $H_0(M;\mathbb{Z})$, as it counts the number of connected components of M. If M is (*arcwise*) *connected*, then $H_0(M;\mathbb{Z}) \simeq \mathbb{Z}$, thus $b_0(M) = 1$.

Another simple but nevertheless important case is the k-th homology group of $\mathbb{R}^m$. Any of these $\mathbb{R}^m$ are contractible spaces, in the sense of homotopy equivalence to a single point. This can be explained intuitively by realizing that any boundary of a chain in $\mathbb{R}^m$ can be contracted to a point, meaning that all boundaries are trivial, in the sense of homology classes. This means that for all m, $b_0(\mathbb{R}^m) = 1$ and $b_k(\mathbb{R}^m) = 0$ for all $k > 0$.

A cell complex of a compact topological manifold. As we will discuss cell complexes of some manifolds later, I would like to explain here what they are. For any compact topological manifold M we can choose a finite set of k-cells, which equals M after gluing them together. By a k-cell in M we mean a subset of M which is homeomorphic to the standard open k-dimensional unit disk in $\mathbb{R}^k$. Note that this disk is declared to be open according to the standard Euclidean metric topology which satisfies the Hausdorff condition. With respect to this topology all k-cells are contractible spaces. The only exception of this description is the 0-cell: it is just a point. However, we note that a point is always contractible by definition.

We denote the k-cells by C_k, and k is called the dimension of the cell. However, this notation does not tell us anything about the exact shape and location of C_k. The cells must be chosen such that they are mutually disjoint. The cell complex is this set, together with relations which describe how the k-cells are glued together. The gluing always happens on the boundary of the cells, and is denoted by the $\cup$-symbol. The space M itself equals the union of these k-cells. The union of cells of the same dimension can be denoted by

$$C_k^n := C_k \cup C_k \cup \cdots \cup C_k.$$

As an example, let's have a look at a cell complex of the sphere S^2. The sphere can be obtained by shrinking the boundary of an open 2-dimensional disk to a point. This gives us the cell complex $C_0 \cup C_2$. For an arbitrary compact topological manifold M of dimension m, a cell complex $C(M)$ has the following form in general:

$$C(M) = C_0^{n_0} \cup C_1^{n_1} \cup \cdots \cup C_m^{n_m} = \bigcup_{k=0}^{m} C_k^{n_k}.$$

Note that how the cells are glued together is not described by this notation, but it turns out that for our purposes this is not important.

A cell complex of a space is far from unique. It is obvious that we are free to choose how these cells exactly are defined. We can slightly deform any cell, as long as the total cell complex stays disjoint. We can even split up one cell C_k into the disjoint union of 2 C_k's and one C_{k-1}. Especially this last feature makes plausible that we can describe a topological invariant using the data derived from the cell complex. This invariant turns out to be the well-known Euler number of M, denoted by $\chi(M)$. When a cell complex of M is known, its Euler number $\chi(M)$ can be shown to satisfy the following relation:

$$\chi(M) \equiv \chi(C(M)) = \chi(\bigcup_{k=0}^{m} C_k^{n_k}) = \sum_{k=0}^{m} \chi(C_k^{n_k}) = \sum_{k=0}^{m} (-1)^k n_k.$$

(At least we can do some fast check of the Euler number, with this plain and very simple technique.) Note that this has the same form as an alternative definition in terms of the Betti numbers $b_k(M)$:

$$\chi(M) = \sum_{k=0}^{m} (-1)^k b_k(M).$$

This is of course no coincidence, as the Betti numbers and the n_k are related by the Morse inequalities (see [2]).

For any two disjoint M_1 and M_2 we can easily show that $\chi(M_1 \cup M_2) = \chi(M_1) + \chi(M_2)$. Another useful identity is the product rule for the Euler number. When M_1 and M_2 are compact topological manifolds, then $M := M_1 \times M_2$ is again a compact topological manifold. Let $C(M_i)$ be cell complexes of M_i:

$$C(M_1) = \bigcup_{k=0}^{m_1} C_k^{n_{1,k}} \quad , \quad C(M_2) = \bigcup_{k=0}^{m_2} C_k^{n_{2,k}}. \tag{2.1}$$

Now we should note that the direct product of two cells is homeomorphic to another cell: $C_k \times C_l \simeq C_{k+l}$. Then (2.1) defines a cell complex of M:

$$\begin{aligned}
C(M) &= C(M_1 \times M_2) = C(M_1) \times C(M_2) = \bigcup_{k=0}^{m_1} \bigcup_{l=0}^{m_2} C_k^{n_{1,k}} \times C_l^{n_{2,l}} \simeq \bigcup_{k=0}^{m_1} \bigcup_{l=0}^{m_2} C_{k+l}^{n_{1,k} n_{2,l}} \Rightarrow \\
\chi(M) &= \chi(C(M)) = \chi(\bigcup_{k=0}^{m_1} \bigcup_{l=0}^{m_2} C_{k+l}^{n_{1,k} n_{2,l}}) = \sum_{k=0}^{m_1} \sum_{l=0}^{m_2} (-1)^{k+l} n_{1,k} n_{2,l} \\
&= \sum_{k=0}^{m_1} (-1)^k n_{1,k} \sum_{l=0}^{m_2} (-1)^l n_{2,l} = \chi(C(M_1))\chi(C(M_2)) = \chi(M_1)\chi(M_2).
\end{aligned} \tag{2.2}$$

(This implies that $\chi(M^k) = \chi(M)^k$.) Later we will discuss a relation called the Künneth formula, which relates the Betti numbers of M, M_1 and M_2, and this formula yields an alternative argument to conclude the validity of relation (2.2).

An atlas for a topological manifold. A *chart* on M consists of a pair (U, κ), where U is an open subset of M, and $\kappa : U \to \mathbb{R}^m$ is a homeomorphism from U to an open subset of $\mathbb{R}^m$. We say that the map κ provides a *coordinate* on U. If (V, λ) is another chart, and $U \cap V$ is not empty, then we have two charts for $U \cap V$. The map

$$\lambda \circ \kappa^{-1} : \kappa(U \cap V) \longrightarrow \lambda(U \cap V)$$

is a homeomorphism between open subsets of $\mathbb{R}^m$, which we will call a *chart transition*, or *coordinate transformation*. The integer m is called the (real) dimension of M, written as $\dim_{\mathbb{R}}(M)$, or simply $\dim(M)$. A collection of charts $\mathcal{A} = \{(U_k, \kappa_k)\}$ for M is called an *atlas* if $\{U_k\}_k$ covers M totally.

Smooth manifolds. An atlas $\mathcal{A} = \{(U_k, \kappa_k)\}$ for M is called a *smooth* atlas if all coordinate transformations $\psi_{kl} := \kappa_l \circ \kappa_k^{-1}$ are smooth. This means that for every ψ_{kl}, its inverse ψ_{lk} is also a coordinate transformation. Thus, all ψ_{kl} are smooth (i.e. C^∞) diffeomorphisms. When $U \subset M$ is open, then a function $f : U \to \mathbb{R}$ is called smooth with respect to $\mathcal{A}$ if, for all k, $f \circ \kappa_k^{-1} : \kappa_k(U_k \cap U) \to \mathbb{R}$ is smooth. We say that f is smooth on M with respect to $\mathcal{A}$ if f is smooth with respect to $\mathcal{A}$ on any open $U \subset M$. Another atlas $\mathcal{B}$ of M is called equivalent to $\mathcal{A}$ if they both determine the same notion of smooth functions. The equivalence class of $\mathcal{A}$, also known as a *maximal atlas*, is called a smooth structure of M. When M is equipped with such a smooth structure, we call M a *smooth manifold*.

If both $\mathcal{A}$ and $\mathcal{A}'$ are atlases of M, then $\mathcal{A}'$ is called a *subatlas* of $\mathcal{A}$ if all (U_k, κ_k) lying in $\mathcal{A}'$ also lie in $\mathcal{A}$ and if all diffeomorphisms ψ_{kl} corresponding to $\mathcal{A}'$ also correspond to $\mathcal{A}$.

We will always assume that we are dealing with smooth manifolds without boundary. This means that a *compact* manifold and a *closed* manifold are the same thing.

Smooth submanifolds. Let M and N be smooth manifolds. Then N is called a smooth submanifold of M if $N \subset M$ and if a smooth map $\iota : N \to M$ exists. This map ι is called a canonical smooth embedding: it smoothly embeds N into M.

The tangent space and the tangent bundle. For any point $p \in M$ there is a linear space T_pM which is called the *tangent space* of M at p. This is a vectorspace over the field $\mathbb{R}$. When M is described as a space smoothly embedded in $\mathbb{R}^n$, the space T_pM can be interpreted as the m-dimensional hyperplane that is tangent to M at p. The formal definition however is somewhat more abstract. For a specific point $p \in M$ of interest, we can choose a chart κ around p. For any abstract curve $\gamma : \mathbb{R} \to M$, with $\gamma(0) = p$, the map $\gamma_{(\kappa)} = \kappa \circ \gamma : \mathbb{R} \to \mathbb{R}^m$ is an ordinary curve, which can be smooth (at least at $\kappa(p)$). However, if $\gamma_{(\kappa)}$ is smooth, then for any second chart λ, $\gamma_{(\lambda)}$ is also smooth. This is because $\gamma_{(\kappa)}$ and $\gamma_{(\lambda)}$ differ by composition with a coordinate transformation, which is a smooth local diffeomorphism. To check whether the abstract curve γ itself is smooth (at p), it is thus sufficient to pick an arbitrary coordinate κ and look what the ordinary curve $\gamma_{(\kappa)}$ is doing at $\kappa(p)$. We can define the following canonical equivalence relation between curves. Let γ and γ' be two smooth curves, with $\gamma(0) = \gamma'(0) = p$, then they are equivalent if

$$\frac{d}{dt}\gamma_{(\kappa)}|_{t=0} = \frac{d}{dt}\gamma'_{(\kappa)}|_{t=0}.$$

Every equivalence class obtained this way can be denoted by a *vector* $V_{(\kappa)}$, which is unique. The subscript (κ) reminds us that the components of this vector depend on the chart. The curves γ and γ' are both equivalent to the curve $\kappa^{-1}(\kappa(p) + tV_{(\kappa)})$. This can be written as $[\gamma] = [\gamma'] = V$. The set of all V's can be given the structure of a (real) vectorspace of the same dimension as M. The only thing needed is a correct definition of vector operations. For any two smooth curves γ_1 and γ_2 through p, and for any pair of real numbers λ_1 and λ_2, there exists a third smooth curve γ_3 which can be interpreted as the vector sum $[\gamma_3] = \lambda_1[\gamma_1] + \lambda_2[\gamma_2]$, or $V_3 = \lambda_1 V_1 + \lambda_2 V_2$. The last identity of course also holds for the components of all vectors.

For any chart κ we can introduce a basis for T_pM, denoted by $(\partial/\partial\kappa^\mu)|_p$. This choice of basis may seem arbitrary, but it has a meaning I will make somewhat more clear in the next item about the cotangent space.

Any vector $V \in T_pM$ can be written with respect to this basis as $V = V_{(\kappa)}{}^{\mu}(\partial/\partial\kappa^{\mu})|_p$. From now on the (κ)-subscript and the $|_p$ will be dropped when not needed.

The *tangent bundle* is the disjoint union of all tangent spaces of M. It is denoted by TM, and it is defined by $TM = \coprod_{p \in M} T_pM$. When we are talking about a *vector field*, we are always looking at a smooth section of the tangent bundle. A section maps any $p \in M$ to a pair $(V, p) \in TM$. Note that the word *section* refers to the notion of a *vector bundle*. The vector bundle corresponding to the tangent bundle is the canonical projection which maps any (V, p) in TM to its *basepoint* p in M. This projection itself is smooth. For convenience we ignore the basepoint from now on, and we can write a vector field as a smooth map $V : p \mapsto V(p) \in T_pM$ for any $p \in M$. This can be expressed as $V(p) = V^{\mu}(p)\partial/\partial\kappa^{\mu}$, and $V^{\mu}(p)$ can be interpreted as a vector-valued function which is smooth on the whole chart domain, for any chart κ.

For a pair of smooth manifolds M and N, and a smooth map $f : M \to N$, we see that any smooth curve on M (and its equivalence class) is mapped to a smooth curve on N. This implies that any such f induces a map between tangent spaces in a canonical way. We will call this the *tangent map* of f, written as $D_pf : T_pM \to T_{f(p)}N$. A special case is when $N = \mathbb{R}$. In this case we can write $df_p : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$. We call this the *differential* of f at p. What is striking is that the direction of the mapping arrow is preserved when going from f to D_pf.

The cotangent space and the cotangent bundle. The *cotangent space* at p, denoted by T_p^*M, is the space of linear maps, also called *covectors*, from T_pM to $\mathbb{R}$, and if M is finite dimensional, the dimension of T_p^*M is the same as the dimension of the tangent space itself. When a chart around p is given, the cotangent space can also be equipped with a basis, which is denoted by $d\kappa^{\mu}$. This basis is dual to the basis of T_pM in the sense that $d\kappa^{\mu}(\partial/\partial\kappa^{\nu}) = \delta^{\mu}_{\nu}$. The *cotangent bundle* is defined in a similar way as the tangent bundle: $T^*M = \coprod_{p \in M} T_p^*M$.

Let now f be a real-valued function on M, then we should note that in fact df_p is an element of T_p^*M. Then we can define a natural action of a vector V on f, denoted by

$$V[f] = V^{\mu}_{(\kappa)} \frac{\partial(f \circ \kappa^{-1})}{\partial\kappa^{\mu}} = \frac{\partial(f \circ \kappa^{-1})}{\partial\kappa^{\mu}} d\kappa^{\mu}(V) = df_p(V). \tag{2.3}$$

For a smooth map $f : M \to N$, we can again induce a map between cotangent spaces, which we should call the *cotangent map*. This time however it is only possible to define a map $D_p^*f : T^*_{f(p)}N \to T_p^*M$. What is striking now is that the direction of the mapping arrow is reversed when going from f to D_p^*f.

Coordinate transformations. When we have two different coordinates (U, κ) and (V, λ) around p, there are also two different bases for T_pM and T_p^*M each. The relation between these bases can be derived from the coordinate transformations. Instead of looking at the bases, we can look at how the components of vectors and covectors with respect to these bases are related. The components are related by something called the differential map of the coordinate transformations. Let $W := U \cap V$ and $\psi = \lambda \circ \kappa^{-1} : \kappa(W) \to \lambda(W)$, then for every $p \in W$ it induces canonical maps $\psi_* : T_{\kappa(p)}\kappa(W) \to T_{\lambda(p)}\lambda(W)$, and $\psi^* : T^*_{\lambda(p)}\lambda(W) \to T^*_{\kappa(p)}\kappa(W)$. These maps ψ_* and ψ^* can be derived from ψ in terms of classical calculus. Note however that ψ has an inverse, thus the pullback can really be regarded as the inverse of some other map $\phi_* : T^*_{\kappa(p)}\kappa(W) \to T^*_{\lambda(p)}\lambda(W)$. Using the composition rule, and a testfunction f, we obtain the following relation:

$$V[f] = V^{\mu}_{(\kappa)} \frac{\partial(f \circ \kappa^{-1})}{\partial\kappa^{\mu}} = V^{\mu}_{(\kappa)} \frac{\partial(f \circ \lambda^{-1} \circ \lambda \circ \kappa^{-1})}{\partial\kappa^{\mu}} = \frac{\partial(\lambda \circ \kappa^{-1})^{\nu}}{\partial\kappa^{\mu}} V^{\mu}_{(\kappa)} \frac{\partial(f \circ \lambda^{-1})}{\partial\lambda^{\nu}} = V^{\nu}_{(\lambda)} \frac{\partial(f \circ \lambda^{-1})}{\partial\lambda^{\nu}}.$$

In the context of differentiation, it is common use, at least in most of the physics literature, to drop all the inverse chart functions in the notation. For example $(\lambda \circ \kappa^{-1})^{\mu}$ becomes λ^{μ}. We then obtain the following relation for the components after throwing away the testfunction:

$$V^{\mu}_{(\lambda)} = \frac{\partial\lambda^{\mu}}{\partial\kappa^{\nu}} V^{\nu}_{(\kappa)}. \tag{2.4}$$

In a similar way we obtain a relation for the components of a covector by using (2.3). For any $\omega \in T_p^*M$:

$$\omega_\mu^{(\kappa)} = \frac{\partial \lambda^\nu}{\partial \kappa^\mu} \omega_\nu^{(\lambda)} \Longrightarrow \omega_\mu^{(\lambda)} = \frac{\partial \kappa^\nu}{\partial \lambda^\mu} \omega_\nu^{(\kappa)}. \tag{2.5}$$

Thus, the components of vectors and covectors transform as (2.4) and (2.5) under coordinate transformations. We will follow mathematical conventions and say that any object transforming like a vector has a *covariant* (or upper) index, and any object transforming like a covector has a *contravariant* (or lower) index. This convention is motivated by category theory which is the origin of a sensible meaning of the adjectives covariant and contravariant. For any $p \in M$ and for $\mathcal{A}$ the atlas in use for M, we can choose a finite collection

$$\mathcal{A}_p := \{(U_i, \kappa_i) \in \mathcal{A} \mid \forall_i : p \in U_i\}$$

of charts. This $\mathcal{A}_p$ can be interpreted as an atlas for $\bigcup_i U_i$ and it contains at least one chart. However, we are interested in an $\mathcal{A}_p$ with multiple charts, thus we are looking for a suitable p which lies in multiple chart domains. Define $U := \bigcap_i U_i$. This is a nonempty open subset of M as it is a finite intersection of open sets. For this U we can define a category $\mathcal{CT}(U)$ whose objects are the charts κ_i restricted to U. The coordinate transformations ψ_{ij} should then be regarded as morphisms between these objects. Then for any $q \in U$ there is a canonical covariant functor from $\mathcal{CT}(U)$ to $\mathcal{CT}(T_qU)$, and a canonical contravariant functor from $\mathcal{CT}(U)$ to $\mathcal{CT}(T_q^*U)$. The objects of, for example, $\mathcal{CT}(T_qU)$ are the vectorspaces $\mathcal{V}_{q,i} := T_{\kappa_i(q)}\kappa_i(U)$, and the morphisms are the tangent maps $(\psi_{ij})_* : \mathcal{V}_{q,i} \to \mathcal{V}_{q,j}$ evaluated in the point $\kappa_i(q)$.

Orientation. When a chart is given around a point $p \in M$, we automatically obtain an ordered basis for T_pM, which is $\partial/\partial\kappa^\mu$, for $\mu = 1, \cdots, m$. If $\partial/\partial\lambda^\mu$ is another ordered basis for T_pM, belonging to another coordinate, then the matrix $\partial\lambda^\mu/\partial\kappa^\nu$ has a determinant which we denote by $D_{(\kappa\to\lambda)}$. The coordinate transformation is named *orientation preserving* if $D_{(\kappa\to\lambda)} > 0$, and *orientation reversing* if $D_{(\kappa\to\lambda)} < 0$. We always have the freedom to redefine the chart λ such that the coordinate transformation is orientation preserving. What is more interesting is whether it is possible to define a chain of chart domains which covers the whole manifold, such that all coordinate transformations are orientation preserving. A manifold is called *orientable* if this is possible, and *non-orientable* otherwise. However, this doesn't say anything about the actual atlas in use. The atlas of an orientable manifold can still be unoriented. Such an atlas can be redefined however to be oriented.

Tensors. A *tensor* is a generalization of objects like vectors and covectors. Locally, the bases $\partial/\partial\kappa^\mu$ of T_pM and $d\kappa^\mu$ of T_p^*M can be extended to a basis

$$\frac{\partial}{\partial \kappa^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial \kappa^{\mu_r}} \otimes d\kappa^{\nu_1} \otimes \cdots \otimes d\kappa^{\nu_s} \tag{2.6}$$

for the space

$$\mathcal{T}_s^r(M)_p := \underbrace{T_pM \otimes \cdots \otimes T_pM}_{r} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{s},$$

for arbitrary r and s. It is clear that this basis spans a vectorspace of dimension m^{r+s}. Any tensor $A \in \mathcal{T}_s^r(M)_p$ can be written in the following way:

$$A = A_{(\kappa)}{}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \frac{\partial}{\partial \kappa^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial \kappa^{\mu_r}} \otimes d\kappa^{\nu_1} \otimes \cdots \otimes d\kappa^{\nu_s}.$$

The basis defined by (2.6) changes under coordinate transformations. This causes the components of A to change also. Now A has r covariant and s contravariant indices, thus the components should change under a coordinate transformation $(\kappa \to \lambda)$ as follows:

$$A_{(\kappa)}{}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \mapsto A_{(\lambda)}{}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = \frac{\partial \lambda^{\mu_1}}{\partial \kappa^{\rho_1}} \cdots \frac{\partial \lambda^{\mu_r}}{\partial \kappa^{\rho_r}} \frac{\partial \kappa^{\sigma_1}}{\partial \lambda^{\nu_1}} \cdots \frac{\partial \kappa^{\sigma_s}}{\partial \lambda^{\nu_s}} A_{(\kappa)}{}^{\rho_1\cdots\rho_r}{}_{\sigma_1\cdots\sigma_s}.$$

Tensors and maps between manifolds. If $f : M \to N$ is a smooth map between manifolds, then also vectors and tensors can be mapped from M to N. In general we can only map purely covariant vectors and tensors from M to N, and purely contravariant vectors and tensors from N to M. The map f induces a map $f_* : T_pM \to T_{f(p)}N$ and a map $f^* : T^*_{f(p)}N \to T^*_pM$. (Note that f_* equals D_pf and that f^* equals D^*_pf, which are operators already shortly introduced earlier as the *tangent map* and the *cotangent map.*)

These maps f_* and f^* are again linear, and they commute with taking tensor products of the input variables. See the identities

$$f_*(V \otimes W) = f_*V \otimes f_*W \quad , \quad f^*(\alpha \otimes \beta) = f^*\alpha \otimes f^*\beta,$$

which hold for any pair of vectors $V, W \in T_pM$ and any pair of covectors $\alpha, \beta \in T^*_{f(p)}N$.

These maps f_* and f^* are defined as follows. Let κ be a coordinate around p and let λ be a coordinate around $f(p)$. Now let $V \in T_pM$ be expanded in components as ${V_{(\kappa)}}^\mu \partial/\partial\kappa^\mu$, with respect to coordinate κ, and let $\alpha \in T^*_{f(p)}N$ be expanded in components as ${\alpha^{(\lambda)}}_\mu d\lambda^\mu$, with respect to coordinate λ. Then we have

$$\begin{aligned} f_*V &= f_*(V^\mu_{(\kappa)} \frac{\partial}{\partial\kappa^\mu}) = V^\mu_{(\kappa)} f_*(\frac{\partial}{\partial\kappa^\mu}) = V^\mu_{(\kappa)} \frac{\partial\lambda^\nu}{\partial\kappa^\mu}\frac{\partial}{\partial\lambda^\nu} = V^\mu_{(\kappa)} \frac{\partial(\lambda\circ f\circ\kappa^{-1})^\nu}{\partial\kappa^\mu}\frac{\partial}{\partial\lambda^\nu}, \\ f^*\alpha &= f^*(\alpha^{(\lambda)}_\mu d\lambda^\mu) = \alpha^{(\lambda)}_\mu f^*(d\lambda^\mu) = \alpha^{(\lambda)}_\mu \frac{\partial\lambda^\mu}{\partial\kappa^\nu} d\lambda^\nu = \alpha^{(\lambda)}_\mu \frac{\partial(\lambda\circ f\circ\kappa^{-1})^\mu}{\partial\kappa^\nu} d\lambda^\nu. \end{aligned}$$

Thus, in short, if $W := f_*V \in T_{f(p)}N$ and $\beta := f^*\alpha \in T^*_pM$, then we have $W = {W_{(\lambda)}}^\nu \partial/\partial\lambda^\nu$ and $\beta = {\beta^{(\kappa)}}_\nu d\kappa^\nu$, with

$$W^\nu_{(\lambda)} = \frac{\partial\lambda^\nu}{\partial\kappa^\mu} V^\mu_{(\kappa)} = \frac{\partial(\lambda\circ f\circ\kappa^{-1})^\nu}{\partial\kappa^\mu} V^\mu_{(\kappa)} \quad , \quad \beta^{(\kappa)}_\nu = \frac{\partial\lambda^\mu}{\partial\kappa^\nu}\alpha^{(\lambda)}_\mu = \frac{\partial(\lambda\circ f\circ\kappa^{-1})^\mu}{\partial\kappa^\nu}\alpha^{(\lambda)}_\mu. \tag{2.7}$$

We see that a relation holds: for any $V \in T_pM$ and $\alpha \in T^*_{f(p)}N$ we can write

$$(f^*\alpha)(V) = \alpha(f_*V). \tag{2.8}$$

In the specific case of f being a diffeomorphism, tensors of any type can be mapped from M to N and vice versa. In this case we also have the map $f^{-1} : N \to M$. Let now A again be a tensor in $\mathcal{T}^r_s(M)_p$, thus with r covariant and s contravariant indices, and define $B := f_*A$. Then A will transform as follows:

$${A_{(\kappa)}}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \mapsto {B_{(\lambda)}}^{\rho_1\cdots\rho_r}{}_{\sigma_1\cdots\sigma_s} = \frac{\partial\lambda^{\rho_1}}{\partial\kappa^{\mu_1}}\cdots\frac{\partial\lambda^{\rho_r}}{\partial\kappa^{\mu_r}}\frac{\partial\kappa^{\nu_1}}{\partial\lambda^{\sigma_1}}\cdots\frac{\partial\kappa^{\nu_s}}{\partial\lambda^{\sigma_s}}{A_{(\kappa)}}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}.$$

Let for example $r = s = 1$. Then we write

$${A_{(\kappa)}}^\mu{}_\nu \mapsto {B_{(\lambda)}}^\rho{}_\sigma = \frac{\partial\lambda^\rho}{\partial\kappa^\mu}\frac{\partial\kappa^\nu}{\partial\lambda^\sigma}{A_{(\kappa)}}^\mu{}_\nu = \frac{\partial(\lambda\circ f\circ\kappa^{-1})^\rho}{\partial\kappa^\mu}\frac{\partial(\kappa\circ f^{-1}\circ\lambda^{-1})^\nu}{\partial\lambda^\sigma}{A_{(\kappa)}}^\mu{}_\nu.$$

Tensor fields. A *tensor field* is a generalization of objects like vector fields and covector fields. It is a smooth section of the following tensor bundle:

$$\mathcal{T}^r_s(M) := \coprod_{p\in M}\mathcal{T}^r_s(M)_p \longrightarrow M.$$

If A is a section of this tensor bundle, then its components ${A_{(\kappa)}}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(p)$ depend smoothly on p, and it is called an (r, s)-tensor field.

2.2 Some more advanced concepts

Differential forms. A (smooth) *differential form* of *degree* k, or simply a k-form, on a manifold M of dimension m, is a smooth $(0,k)$-tensor field ω which is also antisymmetric. The last property is best understood when feeding ω with a set of vector fields $V_1, \cdots, V_k$. The antisymmetry property then reads

$$\omega(V_1, V_2, V_3, \cdots, V_k) = -\omega(V_2, V_1, V_3, \cdots, V_k) = \omega(V_2, V_3, V_1, \cdots, V_k) = \text{etc.}$$

The space of k-forms on M is denoted as $\Omega^k(M)$, and it can be interpreted as an abelian group with standard addition as the group operation. The components of ω at p, with respect to some coordinate κ, are obtained by feeding it with basis vectors $(\partial/\partial\kappa^\mu)|_p$:

$$\omega^{(\kappa)}_{\mu_1\cdots\mu_k}(p) = \omega(\frac{\partial}{\partial\kappa^{\mu_1}}|_p, \cdots, \frac{\partial}{\partial\kappa^{\mu_k}}|_p).$$

Thus, the components themselves are antisymmetric in all indices. It is then sufficient to use $d\kappa^{\mu_1}\otimes\cdots\otimes d\kappa^{\mu_k}$ as a basis for $\Omega^k(M)$. However, we will do operations with k-forms and their components, and it turns out that it is more convenient then to use an alternative basis instead, which is written as

$$\frac{1}{k!} d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k}. \tag{2.9}$$

These basis tensors themselves are antisymmetric, and pointwise they span a linear space of dimension $\binom{m}{k}$. As an example, when $k = 2$, the basis reads $\frac{1}{2}d\kappa^\mu \wedge d\kappa^\nu = \frac{1}{2}(d\kappa^\mu \otimes d\kappa^\nu - d\kappa^\nu \otimes d\kappa^\mu)$. This basis automatically antisymmetrizes the components. When the components are already antisymmetric, the basis (2.9) is equivalent to the basis $d\kappa^{\mu_1}\otimes\cdots\otimes d\kappa^{\mu_k}$. Another option is to use compact notation $K := \mu_1, \cdots, \mu_k$ and $|K| = k$, and write $d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k} = d\kappa^K$. Then we write

$$\omega = \frac{1}{k!}\omega_K d\kappa^K = \frac{1}{k!}\omega^{(\kappa)}_{\mu_1\cdots\mu_k} d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k}.$$

We will call such an index K a *multi-index*.

Another reason, maybe the main reason, to use the basis (2.9) is that it becomes straightforward to define a multiplication operation. This operation is called the *exterior product* or *wedge product*, which to any (ordered) pair of differential forms $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, should assign a $(k+l)$-form ω, and we write $\omega = \alpha \wedge \beta$. When the components of α and β, with respect to some coordinate, are known, we can compute ω:

$$\begin{aligned} \omega &= \alpha \wedge \beta = \frac{1}{k!l!}\alpha_{\mu_1\cdots\mu_k}\beta_{\nu_1\cdots\nu_l} d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k} \wedge d\kappa^{\nu_1} \wedge \cdots \wedge d\kappa^{\nu_l} \\ &= \frac{1}{(k+l)!}\omega_{\rho_1\cdots\rho_{k+l}} d\kappa^{\rho_1} \wedge \cdots \wedge d\kappa^{\rho_{k+l}}. \end{aligned}$$

This induces a relation between the components of α, β and ω. The order of the factors of the wedge product does matter, however the only difference is an extra minus sign in some cases. In general we have $\alpha\wedge\beta = (-1)^{kl}\beta\wedge\alpha$. In particular, when k is odd, also k^2 is odd, thus in that case $\alpha\wedge\alpha = -\alpha\wedge\alpha \Rightarrow \alpha\wedge\alpha = 0$. Using compact notation we write $d\kappa^{KL} = d\kappa^K \wedge d\kappa^L$. For $\alpha = \frac{1}{k!}\alpha_K d\kappa^K$ and $\beta = \frac{1}{l!}\beta_L d\kappa^L$, with $k = |K|$ and $l = |L|$, we then write

$$\alpha \wedge \beta = \frac{1}{k!l!}\alpha_K \beta_L d\kappa^{KL}. \tag{2.10}$$

Differential forms: A pullback. Let $f : M \to N$ be a smooth map. Then we know that a pushforward $f_* : T_pM \to T_{f(p)}N$ and a pullback $f^* : T^*_{f(p)}N \to T^*_pM$ exist. Let now ω be a k-form defined on N. Then an extension of (2.8) tells us that

$$(f^*\omega)(V_1, \cdots, V_k) = \omega(f_*V_1, \cdots, f_*V_k),$$

where $V_j \in T_pM$.

Differential forms on direct products of manifolds. When $M = M_1 \times M_2$, $\alpha \in \Omega^k(M_1)$ and $\beta \in \Omega^l(M_2)$, we can naturally lift these forms to an $\alpha \in \Omega^k(M)$ and a $\beta \in \Omega^l(M)$. Using coordinates on M induced from the coordinates on M_i, we can write the components of $\omega := \alpha \wedge \beta \in \Omega^{k+l}(M)$ as in (2.10), where now the multi-index K only takes values on the M_1-part of the product, and L only on the M_2-part. In other words, K is said to be an index tangent to M_1 only, and L is said to be an index tangent to M_2 only. For example, when $\dim(M_1) = \dim(M_2) = 2$, and $\alpha \in \Omega^1(M_1)$ and $\beta \in \Omega^1(M_2)$, then $\alpha \wedge \beta = \alpha_\mu \beta_\nu d\kappa^\mu \wedge d\kappa^\nu$, where $\mu \in \{1, 2\}$ and $\nu \in \{3, 4\}$. From now on, when we examine direct products of manifolds, $M = M_1 \times M_2$, we automatically assume the coordinates on M being induced from the coordinates on M_i.

The exterior derivative. The *exterior derivative* is a linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$, and it can be interpreted as a group homomorphism between abelian groups. For any $\omega \in \Omega^k(M)$, locally expressed as $\frac{1}{k!}\omega_{\mu_1\cdots\mu_k} d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k}$, the exterior derivative reads

$$d\omega = \frac{1}{k!}\left(\frac{\partial}{\partial\kappa^\nu}\omega_{\mu_1\cdots\mu_k}\right) d\kappa^\nu \wedge d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k}. \tag{2.11}$$

Using notation $\partial_\mu = \partial/\partial\kappa^\mu$, we see that $\partial_\nu \omega_{\mu_1\cdots\mu_k}$ is not totally antisymmetric by itself, but when contracting with the antisymmetric basis (2.9), only its antisymmetric part survives. Thus, we are again dealing with a differential form $d\omega$, which has degree $k+1$. Using compact notation, we write $d\kappa^\lambda \wedge d\kappa^K = d\kappa^{\lambda K}$, thus

$$d\omega = \frac{1}{k!}\partial_\lambda \omega_K d\kappa^{\lambda K}.$$

We can also compute the exterior derivative of $\alpha \wedge \beta$:

$$\begin{aligned} d(\alpha \wedge \beta) &= \frac{1}{k!l!}\frac{\partial}{\partial\kappa^\lambda}(\alpha_{\mu_1\cdots\mu_k}\beta_{\nu_1\cdots\nu_l}) d\kappa^\lambda \wedge d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k} \wedge d\kappa^{\nu_1} \wedge \cdots \wedge d\kappa^{\nu_l} \\ &= \frac{1}{k!l!}\left(\frac{\partial}{\partial\kappa^\lambda}\alpha_{\mu_1\cdots\mu_k}\right)\beta_{\nu_1\cdots\nu_l} d\kappa^\lambda \wedge d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k} \wedge d\kappa^{\nu_1} \wedge \cdots \wedge d\kappa^{\nu_l} \\ &+ (-1)^k \frac{1}{k!l!}\alpha_{\mu_1\cdots\mu_k}\left(\frac{\partial}{\partial\kappa^\lambda}\beta_{\nu_1\cdots\nu_l}\right) d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k} \wedge d\kappa^\lambda \wedge d\kappa^{\nu_1} \wedge \cdots \wedge d\kappa^{\nu_l} \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \end{aligned} \tag{2.12}$$

Or, using compact notation again:

$$\begin{aligned} d(\alpha \wedge \beta) &= \frac{1}{k!l!}\partial_\lambda(\alpha_K \beta_L) d\kappa^\lambda \wedge d\kappa^{KL} \\ &= \frac{1}{k!l!}(\partial_\lambda \alpha_K)\beta_L d\kappa^{\lambda K} \wedge d\kappa^L + (-1)^k \frac{1}{k!l!}\alpha_K(\partial_\lambda \beta_L) d\kappa^K \wedge d\kappa^{\lambda L} = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \end{aligned}$$

When $M = M_1 \times M_2$, and restricting to $\omega \in \Omega^{k+l}(M)$ which can be factorized according to $\omega = \alpha \wedge \beta$ for any $\alpha \in \Omega^k(M_1)$ and $\beta \in \Omega^l(M_2)$, the exterior derivative d on M depends on the exterior derivatives d_i on M_i. We see that $d\alpha = d_1\alpha$ and $d\beta = d_2\beta$, thus, when we write $d(\alpha \wedge \beta) \in \Omega^{k+l+1}(M)$ as in (2.12), we obtain

$$d\omega = d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta = d_1\alpha \wedge \beta + (-1)^k \alpha \wedge d_2\beta. \tag{2.13}$$

An important property of the d-operator is that it is nilpotent: $d^2 = 0$. This motivates us to study a cohomology theory of differential forms.

De Rham cohomology. The spaces of smooth k-forms are in general infinite dimensional. However, it is possible to construct finite dimensional spaces from them (at least when M is compact) which are especially important when studying topological properties of M. Let ω be a k-form. Then it is called *closed* if $d\omega = 0$, and it is called *exact* if there exists a $(k-1)$-form α such that $\omega = d\alpha$. It is easy to show that any exact form is also closed. We can write this in formal language. A closed k-form is an element of

$$\mathrm{Ker}(d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)),$$

and an exact k-form is an element of

$$\mathrm{Im}(d : \Omega^{k-1}(M) \longrightarrow \Omega^k(M)).$$

We should note that the spaces of closed and exact forms are closed under addition. Observe that any exact form is also a closed form, and as such the space of exact k-forms is a subgroup of the space of closed k-forms, and they are both subgroups of $\Omega^k(M)$. Because all groups mentioned here are abelian, any subgroup of another group is automatically a normal subgroup, meaning that we can define quotients of them without any problems. Thus, the space of exact k-forms is a normal subgroup of the space of closed k-forms, which in turn is a normal subgroup of the space of k-forms. Now we are ready to define cohomology. The k-th *de Rham cohomology group* of M is defined as

$$H^k_{\mathrm{dR}}(M) = H^k_{\mathrm{dR}}(M;\mathbb{R}) := \mathrm{Ker}(d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)) / \mathrm{Im}(d : \Omega^{k-1}(M) \longrightarrow \Omega^k(M)),$$

or, just in words, the space of closed k-forms modulo the space of exact k-forms. To explain this further, we should define an equivalence relation between closed k-forms. Let ω and ω' be closed k-forms. Then they are equivalent if there exists a $(k-1)$-form α such that $\omega' = \omega + d\alpha$. In this case we say that ω and ω' are *cohomologous*. Each equivalence class $[\omega]$ is called a *cohomology class*, and all elements of $H^k_{\mathrm{dR}}(M)$ are written like that. An important remark is that any $H^k_{\mathrm{dR}}(M)$ is independent of the used atlas, and as such the cohomology groups can be interpreted as objects describing global properties of M. As these cohomology groups are torsion-free, we can simply write

$$H^k_{\mathrm{dR}}(M) \simeq \mathbb{R}^{b^k},$$

for some integers $b^k = \dim(H^k_{\mathrm{dR}}(M))$.

The philosophy behind cohomology is that any exact form is automatically closed, so they aren't really interesting. A physicist is temped to describe this as *gauge theory*. Any closed form represents a cohomology class, but as any exact form is cohomologous to the zero form, it would be represented by $[0]$. Thus, precisely those closed forms which are not exact represent non-trivial cohomology classes.

It is not that straightforward to find out what these de Rham cohomology groups look like for an arbitrary smooth manifold, however there are some procedures for at least finding out their dimensions. In most cases knowledge about these dimensions is sufficient. First of all, in this thesis I am mainly interested in compact manifolds without boundary, which implies that the dimensions of all the cohomology groups are finite, and exactly equal the Betti numbers:

$$b^k = \dim(H^k_{\mathrm{dR}}(M)) = \mathrm{rank}(H_k(M;\mathbb{Z})) = b_k.$$

This is called *de Rham's theorem*. Especially when M is a contractible space, de Rham's theorem implies that, for all $k > 0$, any closed k-form on M is also exact. This result can be used to assert that any closed form, defined on any arbitrary compact M, is *locally exact*. When we restrict a closed form ω to any contractible patch $U \subset M$, written as ω_U, we directly see that ω_U is also a closed form. As U is contractible, all of its cohomology groups (except the 0-th) are trivial, thus any closed form on U, in particular ω_U, is exact. This means that in any case ω itself is locally exact.

Another important theorem, which holds for any compact oriented manifold without boundary and which says that $b^k = b^{m-k}$ for any $0 \leq k \leq m$, is called the *Poincaré duality*.

There is another interesting theorem. When M and N are two smooth manifolds which are diffeomorphic, their de Rham cohomology groups are isomorphic:

$$H^k_{\mathrm{dR}}(M) \simeq H^k_{\mathrm{dR}}(N)(\forall k).$$

De Rham cohomology: Some examples. There are a few simple spaces though to begin with. For example, $H^0_{\mathrm{dR}}(S^m) \simeq H^m_{\mathrm{dR}}(S^m) \simeq \mathbb{R}$, and all others only contain the trivial class. Thus, $b_0(S^m) = b_m(S^m) = 1$ and $b_k(S^m) = 0$ otherwise. Another interesting and very essential space to study is the *m-torus* $T^m \simeq$

$S^1 \times \cdots \times S^1$. We know that $T^m \simeq \mathbb{R}^m/\mathbb{Z}^m$, thus when we use coordinates x^μ inherited from $\mathbb{R}^m$, with periodic identifications, we have dx^μ as a proper basis at any point for the cotangent space. A very important remark is that all dx^μ are globally defined, and so are arbitrary wedge products of them. Then all $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$ are global closed forms themselves. The functions x^μ are only correctly defined locally, thus the dx^μ are not exact. The higher order wedge products of dx^μ are also not exact. They are all independent of each other, in the sense that there exist no exact forms that transform one into the other. This should be a proper set of forms to make cohomology classes of. There are exactly $\binom{m}{k}$ of these, so $b_k(T^m) = \dim(H^k_{\mathrm{dR}}(T^m)) = \binom{m}{k}$.

The Künneth formula applied to Betti numbers. When M is a more complicated space, there are ways to find its Betti numbers. We should try to imagine we can construct M by gluing together a sequence of simpler spaces. Spaces of which we already know the cohomology groups and the corresponding Betti numbers. The Mayer-Vietoris sequence, which is a long exact sequence, is one method to construct H^k. A manifold can be constructed by using simpler building blocks of which the H^k are already known. The Mayer-Vietoris sequence can then be used to construct the H^k of the final manifold.

Another trick is the *Künneth formula*, of which we only discuss its effects on Betti numbers here. For any pair of compact smooth oriented manifolds M_1, M_2 of which we know the Betti numbers, we can easily compute the Betti numbers of $M := M_1 \times M_2$, which is another compact smooth oriented manifold. The Künneth formula reads as follows:

$$b_k(M) = b_k(M_1 \times M_2) = \sum_{i+j=k} b_i(M_1) b_j(M_2). \tag{2.14}$$

This is an alternative argument implying validity of $\chi(M) = \chi(M_1)\chi(M_2)$ discussed earlier, see formula (2.2). As a trivial example: we know that the Betti numbers of the circle S^1 and the torus $T^2 = S^1 \times S^1$ are

$$(b_0(S^1), b_1(S^1), b_2(S^1)) = (1,1,0) \quad , \quad (b_0(T^2), b_1(T^2), b_2(T^2)) = (1,2,1).$$

Indeed this satisfies (2.14):

$$\begin{aligned} b_0(T^2) &= b_0(S^1)b_0(S^1), \\ b_1(T^2) &= b_0(S^1)b_1(S^1) + b_1(S^1)b_0(S^1), \\ b_2(T^2) &= b_0(S^1)b_2(S^1) + b_1(S^1)b_1(S^1) + b_2(S^1)b_0(S^1). \end{aligned}$$

Metrics. A *metric* on a smooth manifold M is a globally defined tensor field g of type $(0,2)$. Thus, $g_p : T_pM \otimes T_pM \to \mathbb{R}$ for any $p \in M$, or $g_p \in T_p^*M \otimes T_p^*M$. Dropping the subscript p it is written with respect to a coordinate κ^μ as $g = g_{\mu\nu} d\kappa^\mu \otimes d\kappa^\nu$. We define

$$\gamma = \gamma_{(\kappa)}(g) = \det(g_{\mu\nu}). \tag{2.15}$$

We see that this is not really a scalar, as it still depends on the used coordinate. Additional properties of g are that it is symmetric and non-degenerate. The first property means that $g_p(V,W) = g_p(W,V)$ for any pair $V, W \in T_pM$, or, in components, that $g_{\mu\nu} = g_{\nu\mu}$. The second property means that $\gamma \neq 0$ for any $p \in M$. This automatically implies that g has a certain global signature. When γ is positive (or negative), it can not cross zero, thus it should remain positive (or negative) for any other $q \in M$. We will say that a metric has Riemannian signature if $\gamma > 0$, and that it has Lorentzian signature if $\gamma < 0$. However we will only encounter metrics with all eigenvalues positive, which we will call Riemannian metrics, or with all eigenvalues except one positive, which we will call Lorentzian metrics. The standard Lorentzian metric on a flat space is the well-known Minkowski metric. The metric g, being non-degenerate, also has an inverse denoted by g^{-1}, which at any p should be regarded as an element of $T_pM \otimes T_pM$, which is the space dual to $T_p^*M \otimes T_p^*M$. Thus, g^{-1} is a smooth symmetric non-degenerate tensor field of type $(2,0)$, and it is written as $g^{-1} = g^{\mu\nu} \frac{\partial}{\partial \kappa^\mu} \otimes \frac{\partial}{\partial \kappa^\nu}$. It is inverse to g in the sense that $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$, and $\det(g^{\mu\nu}) = \gamma^{-1}$. Here δ^μ_λ, being the Kronecker-δ, are the components of the identity map on the tangent space, which is thus an isotrope tensor field (of type $(1,1)$). The metric and its inverse define natural maps between the spaces of tensor fields of different types. In terms of components, they can be used to change lower indices to upper indices and vice versa. Using multi-indices, we write

$$g_{I,J} := g_{i_1j_1}g_{i_2j_2}\cdots g_{i_kj_k} \quad , \quad g^{I,J} := g^{i_1j_1}g^{i_2j_2}\cdots g^{i_kj_k},$$

where $|I| = |J| = k$. One example of raising (and lowering) indices is $T^K{}_L = g^{K,M}T_{ML}$. For future purposes, we define the following pseudo-scalar:

$$G := G_{(\kappa)} = \sqrt{|\gamma_{(\kappa)}|} = \sqrt{|\det(g_{\mu\nu})|}. \tag{2.16}$$

When g_i are metrics on M_i, $i \in \{1,2\}$, we can trivially construct a metric g on $M := M_1 \times M_2$ from g_1 and g_2. Using coordinates on M, again induced from the coordinates on M_i, we can write the components of g as a block-diagonal matrix, where the i-th block contains the components of g_i. In this case $\gamma = \gamma_1\gamma_2$ thus also $G = G_1G_2$. We will also write $g = g_1 + g_2$.

Riemannian manifolds. A *Riemannian manifold* is a pair (M,g) where g is a Riemannian metric. A *Lorentzian manifold* is a pair (M,g) where g is a Lorentzian metric. A Lorentzian manifold is an example of a pseudo-Riemannian manifold. The metric defines a nice connection ∇ on the tangent bundle, the cotangent bundle and tensor products of them. This connection is called the *Levi-Civita connection*, and for any vector field V it assigns a (k,l)-tensor field $\nabla_V T$ to any (k,l)-tensor field T. The action of ∇_V on T is called a *covariant derivative*, and any tensor field T is called *covariantly constant* if for all V its covariant derivative is zero. The Levi-Civita connection is defined such that the metric itself is covariantly constant, thus it is a *metric connection*. With respect to a coordinate, this ∇_V satisfies

$$\nabla_V = \nabla_{V^\mu\partial_\mu} = V^\mu\nabla_{\partial_\mu} \equiv V^\mu\nabla_\mu.$$

So we only need to look at the definition of ∇_μ. The action of ∇_V on a scalar function f is simply $\nabla_V f = V[f]$. In other words, $\nabla_\mu f = \partial_\mu f$. On basis vectors and covectors the connection acts as follows:

$$\nabla_\mu\partial_\nu = \Gamma^\lambda{}_{\nu\mu}\partial_\lambda \quad , \quad \nabla_\mu d\kappa^\nu = -\Gamma^\nu{}_{\mu\lambda}d\kappa^\lambda.$$

The symbols $\Gamma^\lambda{}_{\mu\nu}$ are defined by

$$\Gamma^\lambda{}_{\mu\nu} := \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

and are called the *Christoffel symbols* of the Levi-Civita connection. They are symmetric in the indices μ,ν. They can be derived from the fact that g is covariantly constant. Now we know how ∇_μ acts on the basis vectors and covectors, the Leibnitz rule

$$\nabla_V(fW) = (\nabla_V f)W + f\nabla_V W$$

tells us that for any vector V and for any covector ω we have

$$\begin{aligned}
\nabla_\mu V &= \nabla_\mu(V^\nu\partial_\nu) = (\partial_\mu V^\nu)\partial_\nu + V^\nu\nabla_\mu\partial_\nu = (\partial_\mu V^\nu)\partial_\nu + V^\nu\Gamma^\lambda{}_{\nu\mu}\partial_\lambda = (\partial_\mu V^\lambda + \Gamma^\lambda{}_{\mu\nu}V^\nu)\partial_\lambda,\\
\nabla_\mu\omega &= \nabla_\mu(\omega_\nu d\kappa^\nu) = (\partial_\mu\omega_\nu)d\kappa^\nu + \omega_\nu\nabla_\mu d\kappa^\nu = (\partial_\mu\omega_\nu)d\kappa^\nu - \omega_\nu\Gamma^\nu{}_{\mu\lambda}d\kappa^\lambda = (\partial_\mu\omega_\lambda - \Gamma^\nu{}_{\mu\lambda}\omega_\nu)d\kappa^\lambda.
\end{aligned}$$

Or, in components,

$$(\nabla_\mu V)^\lambda = \partial_\mu V^\lambda + \Gamma^\lambda{}_{\mu\nu}V^\nu \quad , \quad (\nabla_\mu\omega)_\lambda = \partial_\mu\omega_\lambda - \Gamma^\nu{}_{\mu\lambda}\omega_\nu.$$

These rules can be generalized to any smooth (k,l)-tensor field T, with components $T^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_l}$. The Leibnitz rule also linearly extends to tensor products, which yields

$$\begin{aligned}
(\nabla_\mu T)^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_l} = \partial_\mu T^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_l} &+ \Gamma^{\nu_1}{}_{\mu\lambda}T^{\lambda\nu_2\cdots\nu_k}{}_{\rho_1\cdots\rho_l} + \cdots + \Gamma^{\nu_k}{}_{\mu\lambda}T^{\nu_1\cdots\nu_{k-1}\lambda}{}_{\rho_1\cdots\rho_l}\\
&- \Gamma^\lambda{}_{\mu\rho_1}T^{\nu_1\cdots\nu_k}{}_{\lambda\rho_2\cdots\rho_l} - \cdots - \Gamma^\lambda{}_{\mu\rho_l}T^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_{l-1}\lambda}.
\end{aligned}$$

Any such smooth (k,l)-tensor field T gives us a smooth $(k,l+1)$-tensor field, with components

$$(\nabla_\mu T)^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_l} = \nabla_\mu T^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_l}.$$

(From now on we can ignore the brackets.)

Metrics and submanifolds. Let M be a manifold with a (Riemannian or Lorentzian) metric g, let $N \subset M$ be a smooth submanifold of M and let $\iota : N \to M$ be the canonical (smooth) embedding with pullback map $\iota^* : T^*_{\iota(p)}M \to T^*_p N$. Then we can pull back the metric from M to N.

If g is a Riemannian metric, then any N can be made into a Riemannian submanifold of M. Then $h := \iota^* g$ defines a Riemannian metric on N. We say h is an *induced metric*. If g is not a Riemannian metric, but if h has a conserved signature, then h also defines a metric on N.

Let κ be a coordinate on N, around a point $p \in N$, and let λ be a coordinate on M, around $\iota(p) \in M$. Then (2.7) implies the following relation between the components of h and g:

$$h_{(\kappa)\mu\nu} = \frac{\partial\lambda^\alpha}{\partial\kappa^\mu}\frac{\partial\lambda^\beta}{\partial\kappa^\nu} g_{(\lambda)\alpha\beta} = \frac{\partial(\lambda\circ\iota\circ\kappa^{-1})^\alpha}{\partial\kappa^\mu}\frac{\partial(\lambda\circ\iota\circ\kappa^{-1})^\beta}{\partial\kappa^\nu} g_{(\lambda)\alpha\beta}. \tag{2.17}$$

(Of course, in case of a canonical embedding, the map ι can be ignored and removed again from the last expression.)

Covariantly constant differential forms. A differential form $\omega \in \Omega^k(M)$ is called covariantly constant if for all of its components $\omega_{\nu_1,\cdots,\nu_k}$ we can write

$$\nabla_\mu \omega_{\nu_1,\cdots,\nu_k} = 0.$$

If ω is covariantly constant, then it is easy to prove that ω is also automatically closed. On the other hand, if ω is an arbitrary closed form, then it is not necessarily covariantly constant.

Vielbeins and Riemann normal coordinates. In a pointwise manner, a *vielbein* is just a basis for T_pM, written as e_a, with respect to which the component matrix of the metric is diagonal with unit entries, except a minus sign in front of the first entry in case of a Lorentzian metric. Thus, with respect to this basis e_a, the metric has Kronecker (or Minkowski) form at p. This basis e_a should satisfy $g_p(e_a, e_b) = \delta_{ab}$ (or η_{ab}), or:

$$g_p(e_a, e_b) = (g_p)_{\mu\nu} d\kappa^\mu \otimes d\kappa^\nu (e_a{}^\alpha \frac{\partial}{\partial\kappa^\alpha}, e_b{}^\beta \frac{\partial}{\partial\kappa^\beta}) = g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \delta_{ab} \quad \text{or} \quad \eta_{ab}.$$

This $e_a{}^\mu$ defines a local orthonormal frame in T_pM, and we assume that its determinant is (strictly) positive: $e := \det(e_a{}^\mu) > 0$. (Here we assume that the coordinate κ we started with, is related to an already chosen and fixed orientation, and that this orientation will be induced on $e_a{}^\mu$.)

We should note that this is a pointwise definition, as it is a transformation from $(\partial/\partial\kappa^\mu)|_p$ to another basis of T_pM. As this is only a basis transformation at the point p, this is not a transformation induced by a coordinate transformation. That is why we call the new basis *non-coordinate*. However, it is often possible to induce a lift of this basis transformation which defines a coordinate transformation. This lift is far from unique, but can be chosen in a trivial way. In this sense there is a coordinate with respect to which g is Kronecker (or Minkowski) at p. The question arises whether it is possible to redefine any coordinate of the patch U such that g is Kronecker (or Minkowski) in *all* $p \in U$. We will see that this is only possible when the dimension of M is 1. When $\dim(M) = 2$ it is only possible up to a (positive) scalar factor.

An important remark should be made. The basis e_a is not unique. We can do a rotation of the e_a, which really is a rotation with respect to the metric. Thus $e_a \mapsto e'_a = \Lambda_{ab} e_b$. This rotation should be orientation preserving. In case of a Riemannian metric this matrix Λ should be an 'element' of SO(m), and in case of a Lorentzian metric this matrix should be an 'element' of SO$(1, m-1)$. An extension of the concept of a lift of the vielbein is the notion of *Riemann normal coordinates*. A coordinate κ is called a Riemann-normal coordinate with respect to p if the metric with respect to this coordinate is Kronecker (or Minkowski), thus if $g_{\mu\nu} = \delta_{\mu\nu}$ (or $\eta_{\mu\nu}$) at p, and if $\partial_\lambda g_{\mu\nu} = 0$ at p. For any $p \in M$ we can find such a coordinate.

In a global manner, a vielbein should be regarded as a smooth section of the frame bundle. At least this means that also objects like $\partial_\mu {e_a}^\nu$ are well defined. The notion of vielbeins will, for example, be used in the context of "*light-cone gauge quantization*" (as mentioned in [5]). Then the concept of *spin connection* is also mentioned, and the vielbein postulate is introduced. The vielbein postulate states that also the vielbein is covariantly constant. (We already know that the metric itself is covariantly constant.)

The Hodge star operator. On an oriented, not necessarily Riemannian, manifold (M, g) of dimension m, the *Hodge star operator* $*$ is a linear map

$$* : \Omega^k(M) \longrightarrow \Omega^{m-k}(M),$$

and on the basis of $\Omega^k(M)_p$ it is defined as

$$*(d\kappa^{\mu_1} \wedge \cdots \wedge d\kappa^{\mu_k}) = \frac{1}{(m-k)!} G \epsilon^{\mu_1 \cdots \mu_k}{}_{\nu_{k+1} \cdots \nu_m} d\kappa^{\nu_{k+1}} \wedge \cdots \wedge d\kappa^{\nu_m},$$

or, using again compact notation:

$$*(d\kappa^I) = \frac{1}{(m-k)!} G \epsilon^I{}_J d\kappa^J,$$

with $|I| = k$, $|J| = m - k$ and G as defined in (2.16). Here ϵ, with only lower indices, is the permutation symbol as usual, thus we have $\epsilon_I \simeq \epsilon_{i_1 \cdots i_k} \in \{-1, 0, 1\}$, and $\epsilon^K{}_L = g^{K,M} \epsilon_{ML}$. Thus, on an arbitrary k-form, the Hodge star operator acts like

$$*\omega = \frac{1}{k!} \omega_I * d\kappa^I = \frac{1}{k!(m-k)!} \omega_I G \epsilon^I{}_J d\kappa^J = \frac{1}{k!(m-k)!} \epsilon_{IJ} G \omega^I d\kappa^J. \tag{2.18}$$

If we apply $*$ two times, then we obtain:

$$* * \omega = \frac{1}{k!(m-k)!} G \omega_I \epsilon^I{}_J * d\kappa^J = \frac{1}{k!k!(m-k)!} G^2 \omega_I \epsilon^I{}_J \epsilon^J{}_K d\kappa^K.$$

If we use

$$\begin{aligned} \epsilon^I{}_J \epsilon^J{}_K &= \epsilon^{IJ} \epsilon_{JK} = \gamma^{-1} \epsilon_{IJ} \epsilon_{JK} = (-1)^{|J||K|} \gamma^{-1} \epsilon_{IJ} \epsilon_{KJ} \Rightarrow \\ \omega_I \epsilon^I{}_J \epsilon^J{}_K &= (-1)^{|J||K|} \gamma^{-1} \omega_I \epsilon_{IJ} \epsilon_{KJ} = (-1)^{k(m-k)} k!(m-k)! \gamma^{-1} \omega_K, \end{aligned}$$

where γ is defined as in (2.15), then we obtain

$$* * \omega = \frac{1}{k!} (-1)^{k(m-k)} \gamma^{-1} G^2 \omega_K d\kappa^K = \frac{1}{k!} \operatorname{sgn}(\gamma) (-1)^{k(m-k)} \omega_K d\kappa^K = (-1)^{k(m-k)+\sigma} \omega, \tag{2.19}$$

where $\operatorname{sgn}(\gamma) = (-1)^\sigma$. Thus, $\sigma = 0$ for positive γ, thus for Riemannian metrics, and $\sigma = 1$ for negative γ, thus for Lorentzian metrics. We see that $*$ defines an isomorphism between $\Omega^k(M)$ and $\Omega^{m-k}(M)$, and this isomorphism is even pointwise.

The canonical volume form. The *canonical volume form*, a nowhere vanishing m-form written as $*1$, is defined by

$$*1 = \frac{1}{m!} G \epsilon_{\nu_1 \cdots \nu_m} d\kappa^{\nu_1} \wedge \cdots \wedge d\kappa^{\nu_m} = G d\kappa^1 \wedge \cdots \wedge d\kappa^m \tag{2.20}$$

and gives a sensible meaning to the integration over M of any smooth function $f : M \to \mathbb{R}$. Note that we used here that $\epsilon_K d\kappa^K$ (no summation) is invariant under permutations of K. As there are $m!$ permutations we indeed exactly obtain the equality (2.20). Now we can define the *integral* over M of f, denoted by $I(f)$ as follows:

$$I(f) := \int_M *f = \int_M f * 1 = \int_M fGd\kappa^1 \wedge \cdots \wedge d\kappa^m.$$

If $f = 1$, then this simply defines the volume of the manifold:

$$\mathrm{Vol}_M := I(1) = \int_M *1.$$

If M is a compact manifold, then its volume Vol_M is finite, and if g is also Riemannian, then Vol_M is also positive.

An inner product induced by the Hodge star operator. The Hodge star operator induces an inner product on the spaces of k-forms. For $\alpha, \beta \in \Omega^k(M)$ we define

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge *\beta. \tag{2.21}$$

Using (2.18), we see that

$$\alpha \wedge *\beta \quad = \quad \frac{1}{k!k!(m-k)!} G\alpha_K \beta^L \epsilon_{LM} d\kappa^K \wedge d\kappa^M = \frac{1}{k!} * (\alpha_K \beta^K) = \frac{1}{k!} * (\beta_K \alpha^K) = \beta \wedge *\alpha.$$

Thus, the inner product (2.21) is symmetric:

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta = \int_M \beta \wedge *\alpha = \langle \beta, \alpha \rangle.$$

If, beside being orientable, (M, g) is also Riemannian the inner product (2.21) is positive and non-degenerate.

The Hodge star operator and direct products of manifolds. Let $M = M_1 \times M_2$, with $m = \dim(M) = \dim(M_1) + \dim(M_2) = m_1 + m_2$, and let the metrics g_i on M_i induce a metric g on M. Then $*_i$ are the Hodge star operators on M_i, induced by g_i. Then, for any $\omega = \alpha \wedge \beta$, with $\alpha \in \Omega^k(M_1)$ and $\beta \in \Omega^l(M_2)$, we can induce $*$ on M acting on this ω, by $*_i$. In this case, using (2.10) and (2.18), we write

$$\begin{aligned}
*\omega \quad &= \quad *(\alpha \wedge \beta) = \frac{1}{k!l!} \alpha_K \beta_L * (d\kappa^K \wedge d\kappa^L) = \frac{1}{k!l!(m-k-l)!} G\alpha^K \beta^L \epsilon_{KLP} d\kappa^P \\
&= \quad \frac{1}{k!l!(m_1-k)!(m_2-l)!} G\alpha^K \beta^L \epsilon_{KLMN} d\kappa^M \wedge d\kappa^N \\
&= \quad (-1)^{|L||M|} \frac{1}{k!l!(m_1-k)!(m_2-l)!} G\alpha^K \beta^L \epsilon_{KMLN} d\kappa^M \wedge d\kappa^N \\
&= \quad (-1)^{l(m_1-k)} \frac{1}{k!l!(m_1-k)!(m_2-l)!} G_1 G_2 \alpha^K \beta^L \epsilon_{KM} \epsilon_{LN} d\kappa^M \wedge d\kappa^N \\
&= \quad (-1)^{l(m_1-k)} \frac{1}{k!(m_1-k)!} G_1 \alpha_K \epsilon^K{}_M d\kappa^M \wedge \frac{1}{l!(m_2-l)!} G_2 \beta_L \epsilon^L{}_N d\kappa^N \\
&= \quad (-1)^{(m_1-k)l} *_1 \alpha \wedge *_2 \beta.
\end{aligned} \tag{2.22}$$

Note that here K, M are indices tangent to M_1 only, and L, N are indices tangent to M_2 only, and as such we see that the multi-indices K, M and L, N are always disjoint, thus we are indeed allowed to write $\epsilon_{KMLN} = \epsilon_{KM}\epsilon_{LN}$, at least when contracting with the (also antisymmetric) basis $d\kappa^M \wedge d\kappa^N$. One subtlety of (2.22) is the reordering of the multi-index P to obtain the multi-indices M, N such that M is purely tangent to M_1 and N is purely tangent to M_2. This reordering ($\epsilon_{KLP} \to \epsilon_{KLMN}$) in the first line of (2.22) gives an extra combinatorial factor of $(m-k-l)!/((m_1-k)!(m_2-l)!)$.

From now on, when discussing any $M = M_1 \times M_2$, we automatically assume g on M to be induced by g_i on M_i. In this case (2.22) is always valid for any such $\omega = \alpha \wedge \beta$.

The exterior co-derivative. The exterior derivative is a linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$. For any oriented smooth manifold M of dimension m which carries a metric g, the Hodge star operator can be used to construct an adjoint of the exterior derivative with respect to the inner product (2.21). We will call this adjoint the *exterior co-derivative*. For any fixed k it is defined by

$$d^\star : \Omega^k(M) \longrightarrow \Omega^{k-1}(M) \quad , \quad d^\star = (-1)^{m(k+1)+1-\sigma} * d*,$$

where σ again depends on whether M is a Riemannian or a Lorentzian manifold. This simplifies when m is even. Then $d^\star = -(-1)^\sigma * d*$. This operator is defined to satisfy $\langle \alpha, d\beta \rangle = \langle d^\star \alpha, \beta \rangle$ in case M is also compact and has no boundary. (We can use Stokes' theorem to prove this.) Another property is that $(d^\star)^2 = 0$, which is implied by $d^2 = 0$. This motivates to look at cohomology corresponding to $d^\star$. A k-form ω is called *co-closed* if $d^\star\omega = 0$, and it is called *co-exact* if there exists a $(k+1)$-form α such that $\omega = d^\star\alpha$.

Let again $M = M_1 \times M_2$, $\alpha \in \Omega^k(M_1)$ and $\beta \in \Omega^l(M_2)$. Then we can use (2.13) and (2.22) to derive how $d^\star$, acting on $\omega = \alpha \wedge \beta$, depends on $d_1^\star$ and $d_2^\star$. In later applications we are only interested in the case that m_1 and m_2 are even (to be precise, $m_1 = 4$ and $m_2 = 6$), and M_1 is a Lorentzian manifold, thus $\sigma_1 = 1$, and M_2 is a Riemannian manifold, thus $\sigma_2 = 0$. In this case also M has even dimension, and it is a Lorentzian manifold. This simplifies some computations, for example $d^\star = *d*$, $d_1^\star = *_1 d_1 *_1$ and $d_2^\star = - *_2 d_2 *_2$, and (2.19) simplifies to $*_1^2\alpha = (-1)^{k+1}\alpha$ and $*_2^2\beta = (-1)^l\beta$. Now we can derive a relation between $d^\star$, $d_1^\star$ and $d_2^\star$:

$$\begin{aligned}
d*(\alpha\wedge\beta) &= d((-1)^{(m_1-k)l} *_1 \alpha \wedge *_2\beta) = (-1)^{kl} d(*_1\alpha \wedge *_2\beta) \\
&= (-1)^{kl}(d_1 *_1 \alpha \wedge *_2\beta + (-1)^{m_1-k} *_1 \alpha \wedge d_2 *_2 \beta) \\
&= (-1)^{kl} d_1 *_1 \alpha \wedge *_2\beta + (-1)^{k(l-1)} *_1 \alpha \wedge d_2 *_2 \beta \Rightarrow \\
d^\star\omega &= *d*(\alpha\wedge\beta) = (-1)^{kl} * (d_1 *_1 \alpha \wedge *_2\beta) + (-1)^{k(l-1)} * (*_1\alpha \wedge d_2 *_2 \beta) \\
&= (-1)^{kl+(k-1)(m_2-l)} *_1 d_1 *_1 \alpha \wedge *_2^2\beta + (-1)^{k(l-1)+k(m_2-l+1)} *_1^2 \alpha \wedge *_2 d_2 *_2 \beta) \\
&= (-1)^{kl-(k-1)l+l} d_1^\star\alpha \wedge \beta + (-1)^{k(l-1)+k(1-l)+k+2} \alpha \wedge d_2^\star\beta) \\
&= d_1^\star\alpha\wedge\beta + (-1)^k \alpha \wedge d_2^\star\beta.
\end{aligned} \tag{2.23}$$

The Laplace operator. For any oriented smooth manifold M with a metric g it is possible to define a second order differential operator Δ called the Laplace-Beltrami operator, the Hodge-de Rham Laplacian, the Hodge-Laplace operator or simply the *Laplace operator*. For any fixed k this operator is written in terms of the exterior derivate and co-derivative:

$$\Delta : \Omega^k(M) \longrightarrow \Omega^k(M) \quad , \quad \Delta := (d + d^\star)^2 = dd^\star + d^\star d. \tag{2.24}$$

A special property of the Laplace operator. Let again $M = M_1 \times M_2$, then the Laplace operators Δ_i, derived from g_i,d_i and $*_i$, induce a Laplace operator Δ on M. (The metric g on M is also canonically derived from the already given metrics g_1 on M_1 and g_2 on M_2: using simple notation we write $g = g_1 + g_2$.)

For future purposes, we would like to examine how the Laplace operator acts on any product of the form $\omega = \alpha\wedge\beta$, where $\alpha \in \Omega^k(M_1)$ and $\beta \in \Omega^l(M_2)$. We would like to use the results of (2.12) and (2.23) to work this out. In later applications, we are only interested in manifolds M_i of even dimension. To be precise, we will study a Lorentzian manifold M_1 of dimension 4 and a Riemannian manifold M_2 of dimension 6. The product manifold $M = M_1 \times M_2$ will thus be a Lorentzian manifold of dimension 10. In this case the Laplace operators Δ, Δ_1 and Δ_2 are related as follows:

$$\begin{aligned}
\Delta\omega &= \Delta(\alpha\wedge\beta) \\
&= dd^\star(\alpha\wedge\beta) + d^\star d(\alpha\wedge\beta) = d(d_1^\star\alpha\wedge\beta + (-1)^k\alpha\wedge d_2^\star\beta) + d^\star(d_1\alpha\wedge\beta + (-1)^k\alpha\wedge d_2\beta) \\
&= d_1 d_1^\star\alpha\wedge\beta + (-1)^{k-1} d_1^\star\alpha\wedge d_2\beta + (-1)^k d_1\alpha\wedge d_2^\star\beta + (-1)^{2k}\alpha\wedge d_2 d_2^\star\beta \\
&\quad + d_1^\star d_1\alpha\wedge\beta + (-1)^{k+1} d_1\alpha\wedge d_2^\star\beta + (-1)^k d_1^\star\alpha\wedge d_2\beta + (-1)^{2k}\alpha\wedge d_2^\star d_2\beta \\
&= (d_1 d_1^\star + d_1^\star d_1)\alpha\wedge\beta + \alpha\wedge(d_2 d_2^\star + d_2^\star d_2)\beta \\
&= \Delta_1\alpha\wedge\beta + \alpha\wedge\Delta_2\beta.
\end{aligned}$$

As Δ_1 does not act on β and Δ_2 does not act on α, we write $\Delta\omega = \Delta_1\omega + \Delta_2\omega$, or just

$$\Delta = \Delta_1 + \Delta_2. \tag{2.25}$$

This relation (2.25) also holds in general, for any arbitrary $\omega \in \Omega^m(M)$. For example if $m = 0$, then locally, at some open contractible patch, we can always write $\omega(x, y) = \sum_j \alpha_j(x)\beta_j(y)$, where $x \in M_1$ and $y \in M_2$. (See Sobolev arguments.) This is possible at any point $(x, y) \in M$, so indeed $\Delta\omega = \Delta_1\omega + \Delta_2\omega$ also holds, globally, in this case.

Harmonic forms. A k-form ω is called *harmonic* if $\Delta\omega = 0$. In Chapter 3 we will further discuss this kind of k-forms, when we will introduce massless superstring theory. For any compact Riemannian manifold we can say that a k-form ω is harmonic if and only if it is closed and co-closed at the same time:

$$\Delta\omega = 0 \quad \Leftrightarrow \quad d\omega = d^\star\omega = 0. \tag{2.26}$$

(This is easy to prove.)

According to *Hodge's theorem* the space of harmonic k-forms defined on a compact orientable Riemannian manifold M, is isomorphic to $H^m_{\mathrm{dR}}(M)$.

2.3 Complex Manifolds

Complex manifolds. We can easily generalize the notion of a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ to multivalued functions. A map $F : \mathbb{C}^n \to \mathbb{C}^d$, written as $F = (f^1, \cdots, f^d)$, is called holomorphic if each f^m $(1 \leq m \leq d)$ is. There is a canonical identification between $\mathbb{R}^{2d}$ and $\mathbb{C}^d$ by writing $z^m = x^m + iy^m$ for all m, where $z = (z^1, \cdots, z^d) \in \mathbb{C}^d$ and $(x, y) = (x^1, \cdots, x^d, y^1, \cdots, y^d) \in \mathbb{R}^{2d}$.

A *complex manifold* of complex dimension d is a smooth manifold M of real dimension $2d$, where its atlas $\mathcal{A} = \{(U_n, \kappa_n)\}$ satisfies a special property: every smooth coordinate transformation, written as $\psi_{mn} : \kappa_m(U_m) \to \kappa_n(U_n)$, can be written as a *holomorphic* coordinate transformation $\Psi_{mn} : K_m(U_m) \to K_n(U_n)$. This is by canonical identification of $\kappa_m(U_m) \subset \mathbb{R}^{2d}$ and $K_m(U_m) \subset \mathbb{C}^d$. The atlas $\mathcal{A}' := \{(U_n, K_n)\}$ will be called a *holomorphic atlas*.

Every complex manifold is a smooth manifold, but not every smooth manifold can be described as a complex manifold. Any smooth manifold of even dimension has an atlas with smooth coordinate transformations which can be represented by complex maps, but these are not necessarily holomorphic. But, sometimes it is possible to find a subatlas $\mathcal{A}'$ of $\mathcal{A}$ which can be described as a holomorphic atlas. In this case it is often possible to find many inequivalent subatlases, each being holomorphic, but coordinate transformations between these subatlases cannot be represented by holomorphic maps. This is related to a special property called *complex structure*, which will be explained later.

Holomorphic maps. For any pair of complex manifolds M and N, with $m = \dim_{\mathbb{C}}(M)$ and $n = \dim_{\mathbb{C}}(N)$, a map $f : M \to N$ is called *holomorphic* if for any $p \in M$ and for any two charts κ around p and λ around $f(p)$ the map $\lambda \circ f \circ \kappa^{-1} : \mathbb{C}^m \to C^n$ is holomorphic. This property is independent of the choices of coordinates around $p \in M$ and $f(p) \in N$.

We say M is *biholomorphic* to N if an invertible holomorphic map $f : M \to N$ exists. Then f itself is called a *biholomorphism*. Such an f is always a diffeomorphism, and being holomorpic it also induces a biholomorphism $f^{-1} : N \to M$. (Then also $m = n$.) In this case the complex structures of M and N can be identified: $(M, J_M) \simeq (N, J_N)$.

Holomorphic functions and the maximum modulus principle. A *holomorphic function* on M is simply a holomorphic map between M and $\mathbb{C}$. If $f : M \to \mathbb{C}$ is holomorphic, then the *maximum modulus principle* tells us the following: if f has an absolute maximum at a point in some connected open subset U of M, then f must be constant on U. If M is compact, then f *must* reach an absolute maximum. This implies that any holomorphic function $f : M \to \mathbb{C}$, with a compact domain M, must be a constant function.

The complexified tangent space. We would like to define smooth complex functions on any complex manifold M. For any pair $f, g : M \to \mathbb{R}$ we can define a complex function $h := f + ig : M \to \mathbb{C}$, and any complex function can be decomposed like this. Some of these functions can be holomorphic. An ordinary holomorphic complex function $f(z)$ satisfies $\partial f/\partial \bar{z} = 0$. Similarly we would like a holomorphic function $h : M \to \mathbb{C}$ to satisfy $V[h] = 0$ for any anti-holomorphic vector field V. However, these V do not lie in TM, but rather in its complexified cousin $TM^{\mathbb{C}}$. This is a natural straightforward extension, and any $Z \in T_pM^{\mathbb{C}}$ can be written as $X + iY$, for some $X, Y \in T_pM$. When tangent spaces are complexified, we also need to complexify cotangent spaces. For any pair $\alpha, \beta \in T_p^*M$ we have an $\omega := \alpha + i\beta \in T_p^*M^{\mathbb{C}}$.

Real and complex coordinates. For any complex manifold M, with $m = \dim_{\mathbb{C}}(M)$ and coordinates z^μ, we can always represent the coordinates by real numbers by the identity $z^\mu = x^\mu + iy^\mu$. Then the complex differentials, which are a basis for $T_p^*M^{\mathbb{C}}$, become

$$dz^\mu = dx^\mu + idy^\mu \quad , \quad d\bar{z}^\mu = dx^\mu - idy^\mu. \tag{2.27}$$

The basis for $T_pM^{\mathbb{C}}$, dual to the basis of complex differentials, becomes

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2}(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}) \quad , \quad \frac{\partial}{\partial \bar{z}^\mu} = \frac{1}{2}(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}). \tag{2.28}$$

(Note that $\dim_{\mathbb{R}}(T_pM^{\mathbb{C}}) = 4m$.)

Another useful notation is used for the spaces of vectors and covectors which have purely holomorphic or anti-holomorphic indices. For the tangent space we define

$$T_pM^+ := \mathrm{span}\{\frac{\partial}{\partial z^\mu}\} \quad , \quad T_pM^- := \mathrm{span}\{\frac{\partial}{\partial \bar{z}^\mu}\},$$

and we write $T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-$. We do the same for the cotangent space $T_p^*M^{\mathbb{C}} = T_p^*M^+ \oplus T_p^*M^-$, with

$$T_p^*M^+ := \mathrm{span}\{dz^\mu\} \quad , \quad T_p^*M^- := \mathrm{span}\{d\bar{z}^\mu\}.$$

Holomorphic and anti-holomorphic vector fields. Any *holomorphic vector field* V can be written as

$$V = V^\mu(z)\frac{\partial}{\partial z^\mu},$$

and any *anti-holomorphic vector field* W can be written as

$$W = W^{\bar{\mu}}(\bar{z})\frac{\partial}{\partial \bar{z}^\mu}.$$

We see that the components $V^\mu(z)$ and $W^{\bar{\mu}}(\bar{z})$ are holomorphic and anti-holomorphic functions respectively.

This can be related to holomorphic and anti-holomorphic functions. A complex function $f : M \to \mathbb{C}$ is holomorphic if and only if for any anti-holomorphic vector field W it satisfies $W[f] = 0$. Similarly, it is anti-holomorphic if and only if for any holomorphic vector field V it satisfies $V[f] = 0$.

Complex manifolds and orientation. For a complex manifold we know that all of its coordinate transformations Ψ_{mn} are holomorphic. This implies that the corresponding tangent maps $(\Psi_{mn})_*$ are all linear holomorphic maps, thus they have a positive determinant when representing them as real matrices. We conclude that any complex manifold is orientable, which also means that a non-orientable manifold cannot be described as a complex manifold. Sometimes we even say a complex manifold is already *oriented*, as $\mathbb{C}^d$ carries a standard orientation and the image of any chart is contained in $\mathbb{C}^d$.

Complex structure. Let J be a globally defined smooth tensor field of type $(1,1)$ on a smooth manifold M, thus $J \in TM \otimes T^*M$. Then J can also be reformulated as a linear map $J : TM \to TM$. It is a tensor field, thus at every point $p \in M$ it defines a map $J_p : T_pM \to T_pM$. Then J is called an *almost complex structure* on M if it satisfies $J^2 = -\operatorname{Id}_{TM}$, or pointwise, for any p and for any $V \in T_pM$, $J_p(J_p(V)) = -V$. As we can always construct a not necessarily smooth tensor field J which satisfies the last identity in a pointwise manner, we must indeed restrict to smooth tensor fields to be able to do something sensible. Another subtlety is, when J is smooth and satisfies $J^2 = -\operatorname{Id}$ on some coordinate chart, that it will not necessarily extend globally. It turns out that not every smooth manifold can be equipped with an almost complex structure, but when a smooth manifold permits an almost complex structure, it is called an *almost complex manifold*.

However, we will only be dealing with complex manifolds from now on, thus there is a unique almost complex structure corresponding to the holomorphic atlas, and from now on we will just call it the *complex structure*. The complex structure J is a real tensor field which can be extended to $TM^{\mathbb{C}}$. Expressed in holomorphic coordinates it can be defined as

$$J := i\frac{\partial}{\partial z^\mu} \otimes dz^\mu - i\frac{\partial}{\partial \bar{z}^\mu} \otimes d\bar{z}^\mu. \tag{2.29}$$

This is an isotrope tensor, thus it will have this appearance with respect to any arbitrary holomorphic coordinate in the same atlas. When we are dealing with a complex manifold M, we are automatically dealing with a pair (M, J).

When a smooth manifold can be described as a complex manifold, the choice of a maximal holomorphic atlas may not be unique. To any choice of a maximal holomorphic atlas, a complex structure J is related. We say that J is an element of the *complex structure moduli space* $\mathcal{M}_C$. As a trivial example, there is only one maximal holomorphic atlas for $\mathbb{C}^N$ (by definition!), thus it carries a unique complex structure. We write $\dim(\mathcal{M}_C(\mathbb{C}^N)) = 0$. Any torus T^2, described as a smooth manifold, allows a (complex) 1-dimensional family of holomorphic atlases, thus

$$\dim(\mathcal{M}_C(T^2)) = \dim_{\mathbb{C}}(\mathcal{M}_C(T^2)) = 1. \tag{2.30}$$

We will discuss this later.

If (M_1, J_1) and (M_2, J_2) are complex manifolds, then $(M, J) := (M_1 \times M_2, J)$ is again a complex manifold, where the complex structure J is canonically derived from the already given complex structures J_1 and J_2: using simple notation we write $J = J_1 + J_2$.

Complex tensor fields and differential forms. When A and B are two real (k,l)-tensors, $A, B \in \mathcal{T}^k_l(M)_p$, then $C := A + iB \in \mathcal{T}^k_l(M)^{\mathbb{C}}_p = \mathcal{T}^k_l(M)_p \otimes \mathbb{C}$ is said to be a complex tensor. Again all complex tensor fields can be decomposed this way. The bases (2.27) and (2.28) can be extended to bases of complex tensors. Officially we should then say that a complex tensor C has type $(k,l;m,n)$, and it is decomposed as

$$C = C^{\mu_1\cdots\mu_k\bar{\nu}_1\cdots\bar{\nu}_l}{}_{\rho_1\cdots\rho_m\bar{\sigma}_1\cdots\bar{\sigma}_n}\frac{\partial}{\partial z^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial z^{\mu_k}} \otimes \frac{\partial}{\partial \bar{z}^{\nu_1}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{z}^{\nu_l}} \otimes dz^{\rho_1} \otimes \cdots \otimes dz^{\rho_m} \otimes d\bar{z}^{\sigma_1} \otimes \cdots \otimes d\bar{z}^{\sigma_n}.$$

This notation has a canonical extension to tensor fields C of type $(k,l;m,n)$. Of course we can reorder the basis in such an expression, for example $dz \otimes d\bar{z} \mapsto d\bar{z} \otimes dz$, and the result should be a different tensor, so to make this complete we should also indicate the ordering of holomorphic and anti-holomorphic indices. Fortunately we are mainly interested in tensors with covector indices only. We will mainly look at complex differential forms of type (k,l), which are complex tensor fields of type $(0,0;k,l)$. The space of these forms is denoted by $\Omega^{k,l}(M)$, which is of course a complex vector space, and we should note that $\Omega^{k,l}(M) \subset \Omega^{k+l}(M)^{\mathbb{C}}$. As differential forms are completely antisymmetrized tensors, we will not care about the ordering of holomorphic and anti-holomorphic indices. We will write the decomposition of any (k,l)-form using an ordering prescription. When $\omega \in \Omega^{k,l}(M)$, we decompose it as

$$\omega = \frac{1}{k!l!}\omega_{\mu_1\cdots\mu_k\bar{\nu}_1\cdots\bar{\nu}_l} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_k} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_l}.$$

Multiplying complex differential forms is a straightforward procedure. For any $\alpha \in \Omega^{k,l}(M)$ and $\beta \in \Omega^{m,n}(M)$ we write $\alpha \wedge \beta \in \Omega^{k+m,l+n}(M)$.

The complex conjugate of any $\omega \in \Omega^{k,l}(M)$ is written as $\bar{\omega} \in \Omega^{l,k}(M)$, and its components are just the complex conjugate of the components of ω. This means that when components are holomorphic functions, then the complex conjugate of these components are anti-holomorphic functions. A form $\alpha \in \Omega^{k,0}(M)$ is called holomorphic if all of its components are holomorphic functions with respect to any coordinate. This means that $\bar{\alpha} \in \Omega^{0,k}(M)$ and that all of its components are anti-holomorphic functions. Such an α is called a *holomorphic k-form.* (Then $\bar{\alpha}$ is called an anti-holomorpic k-form.)

Holomorphic top-forms. For any compact connected complex manifold M, with $m = \dim_{\mathbb{C}}(M)$, a so-called holomorphic *top-form* is a form $\Omega \in \Omega^{m,0}(M)$, with only holomorphic components. It can be written as follows:

$$\Omega = \Omega_{\mu_1 \cdots \mu_m}(z^1, \cdots, z^m) dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_m} \simeq \Omega(z^1, \cdots, z^m) dz^1 \wedge \cdots \wedge dz^m. \tag{2.31}$$

Note that $\Omega(z^1, \cdots, z^m)$ is not a scalar function, as it transforms under coordinate transformations. However, if Ω and Ω' are different top-forms, and if Ω' can be rewritten as $\phi\Omega$, where $\phi : M \to \mathbb{C}$ is a holomorphic scalar function, then the maximum modulus principle tells us that ϕ is a constant. Then Ω' is a constant multiple of Ω. Of course such a function ϕ only exists if Ω and Ω' have the same zero set. This means that the space of holomorphic top-forms which have the same zero set, has complex dimension one. Especially the space of nowhere vanishing holomorphic top-forms has at most complex dimension one.

The holomorphic and anti-holomorphic exterior derivatives. The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ can be trivially extended to a map $d : \Omega^k(M)^{\mathbb{C}} \to \Omega^{k+1}(M)^{\mathbb{C}}$. For any $\omega \in \Omega^{k,l}(M)$, we find

$$d\omega = \frac{1}{k!l!}\left(\frac{\partial}{\partial z^\lambda}\omega_{\mu_1\cdots\mu_k\bar{\nu}_1\cdots\bar{\nu}_l} dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1\cdots\mu_k\bar{\nu}_1\cdots\bar{\nu}_l} d\bar{z}^\lambda\right) \wedge dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_k} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_l}$$

by using relations (2.27) and (2.28). We see that $d\omega \in \Omega^{k+1,l}(M) \oplus \Omega^{k,l+1}$. We write $d\omega = \partial\omega + \bar{\partial}\omega$, where

$$\begin{aligned}
\partial\omega &= \frac{\partial}{\partial z^\lambda}\omega_{\mu_1\cdots\mu_k\bar{\nu}_1\cdots\bar{\nu}_l} dz^\lambda \wedge dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_k} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_l} \\
\bar{\partial}\omega &= \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1\cdots\mu_k\bar{\nu}_1\cdots\bar{\nu}_l} d\bar{z}^\lambda \wedge dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_k} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_l} \\
&= (-1)^k \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1\cdots\mu_k\bar{\nu}_1\cdots\bar{\nu}_l} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_k} \wedge d\bar{z}^\lambda \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_l},
\end{aligned}$$

thus $d = \partial + \bar{\partial}$. We see that ∂ is a map from $\Omega^{k,l}$ to $\Omega^{k+1,l}$, and $\bar{\partial}$ is a map from $\Omega^{k,l}$ to $\Omega^{k,l+1}$. Just like the d-operator, these operators satisfy a product rule, which in fact is equal to that of d. For any $\alpha \in \Omega^k(M)^{\mathbb{C}}$ and $\beta \in \Omega^l(M)^{\mathbb{C}}$ we have

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta \quad , \quad \bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \bar{\partial}\beta.$$

A form $\omega \in \Omega^{k,l}(M)$ is called ∂-closed ($\bar{\partial}$-closed) if $\partial\omega = 0$ ($\bar{\partial}\omega = 0$), and ∂-exact ($\bar{\partial}$-exact) if there exists an $\alpha \in \Omega^{k-1,l}(M)$ (an $\alpha \in \Omega^{k,l-1}(M)$) such that $\omega = \partial\alpha$ ($\omega = \bar{\partial}\alpha$). Again, it is easily shown that $\partial^2 = \bar{\partial}^2 = 0$, thus we can study their cohomology theory. However it is sufficient just to look at one of the two operators, as for any (k,l)-form ω we have $\partial\bar{\omega} = \overline{\bar{\partial}\omega}$. By convention we are only interested in $\bar{\partial}$-cohomology.

Using the identity $0 = d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial = \partial\bar{\partial} + \bar{\partial}\partial$ we see that ∂ and $\bar{\partial}$ anticommute. Now $d(\partial - \bar{\partial}) = (\partial + \bar{\partial})(\partial - \bar{\partial}) = \bar{\partial}\partial - \partial\bar{\partial} = -2\partial\bar{\partial}$, thus we obtain the identity

$$\partial\bar{\partial} = -\frac{1}{2} d(\partial - \bar{\partial}). \tag{2.32}$$

Dolbeault cohomology and Hodge numbers. For any complex manifold M, with $m = \dim_{\mathbb{C}}(M)$, we can study the so-called *Dolbeault cohomology*, or $\bar{\partial}$-cohomology. In fact it resembles de Rham cohomology in many ways, except that now we are not looking at the behaviour of the total exterior derivative d, but only at the behaviour of the anti-holomorphic exterior derivative $\bar{\partial}$. Formally written, a $\bar{\partial}$-closed (k,l)-form is an element of

$$\mathrm{Ker}(\bar{\partial} : \Omega^{k,l}(M) \longrightarrow \Omega^{k,l+1}(M)),$$

and a $\bar{\partial}$-exact (k,l)-form is an element of

$$\mathrm{Im}(\bar{\partial} : \Omega^{k,l-1}(M) \longrightarrow \Omega^{k,l}(M)).$$

Again the space of $\bar{\partial}$-exact (k,l)-forms is a normal subgroup of the space of $\bar{\partial}$-closed (k,l)-forms, and they are both subgroups of $\Omega^{k,l}(M)$. The (k,l)-th *Dolbeault cohomology group* of M is defined as

$$H^{k,l}(M) = H^{k,l}(M;\mathbb{C}) := \mathrm{Ker}(\bar{\partial} : \Omega^{k,l}(M) \longrightarrow \Omega^{k,l+1}(M))/\,\mathrm{Im}(\bar{\partial} : \Omega^{k,l-1}(M) \longrightarrow \Omega^{k,l}(M)).$$

Alternative notations are $H^{k,l}(M) = H^{k,l}_D(M) = H^{k,l}_{\bar{\partial}}(M)$. Let ω and ω' be $\bar{\partial}$-closed (k,l)-forms. Then they are equivalent if there exists a $(k,l-1)$-form α so that $\omega' = \omega + \bar{\partial}\alpha$, and we say that ω and ω' are $\bar{\partial}$-*cohomologous*. In this case we can write

$$H^{k,l}(M) \simeq \mathbb{C}^{h^{k,l}},$$

for some integers $h^{k,l} = h^{k,l}_{\bar{\partial}} := \dim_{\mathbb{C}}(H^{k,l}(M))$. These integers are called the *Hodge numbers* of M, which again are finite when M is compact. Of course we could also look at the ∂-cohomology, defined by

$$H^{k,l}_{\partial}(M) := \mathrm{Ker}(\partial : \Omega^{k,l}(M) \longrightarrow \Omega^{k+1,l}(M))/\,\mathrm{Im}(\partial : \Omega^{k-1,l}(M) \longrightarrow \Omega^{k,l}(M))$$

instead. Then, writing $h^{k,l}_{\partial} = \dim_{\mathbb{C}}(H^{k,l}_{\partial}(M))$, we obtain the identity $h^{k,l}_{\bar{\partial}} = h^{l,k}_{\partial}$, which is generally applicable. We conclude that, in general, we never have more than $(m+1)^2$ independent Hodge numbers for any complex manifold. The *Hodge diamond* of M expresses all Hodge numbers of M:

$$\begin{array}{cccc} h^{0,0} & h^{0,1} & \cdots & h^{0,m} \\ h^{1,0} & h^{1,1} & & h^{1,m} \\ \vdots & & \ddots & \vdots \\ h^{m,0} & h^{m,1} & \cdots & h^{m,m}. \end{array}$$

(A rotation of 45 degrees still misses here!) As an example, we note that for any contractible complex manifold M, for example $\mathbb{C}^n$, we can write $h^{k,l}_M = \delta_{k,0}\delta_{l,0}$.

Some remarks about Hodge numbers. The Hodge numbers are not pure topological invariants, but rather *pseudo-topological*. They depend on topological properties and on properties of the complex structure. When M and N are complex manifolds, and if $\phi : M \to N$ is a diffeomorphism, we know that its Betti numbers are equal. The same is not necessarily valid for the Hodge numbers. However, if ϕ is also holomorphic, it preserves the complex structure, and we say that ϕ is a *biholomorphism*, and it has a unique holomorphic inverse. In this case the Hodge numbers are equal: $h^{k,l}_M = h^{k,l}_N$.

2.4 Examples of Complex Manifolds

The trivial ones. The sets $\mathbb{C}^m$ can be regarded as trivial complex manifolds. For any $n < m$ the set $\mathbb{C}^n$ can be embedded in $\mathbb{C}^m$ so that it is a complex submanifold of $\mathbb{C}^m$. However, these submanifolds are not compact, and in the main subject of this thesis we will mainly be interested in *compact* complex manifolds (without boundary).

The only compact complex manifolds embedded in $\mathbb{C}^m$ (as complex submanifolds) are points, thus they are trivial ones, having complex dimension zero. This is related to the maximum modulus principle, which will of course contradict the statement that the domain U of any non-trivial holomorphic map $f : U \to \mathbb{C}^m$ is compact. In other words, a compact complex manifold cannot be embedded in $\mathbb{C}^m$. Therefore we need other kinds of sets instead, to be described as complex manifolds, which admit more interesting compact complex submanifolds.

Complex projective spaces. Many compact algebraic manifolds, or *hypersurfaces*, can be described (or represented) as a subset of some complex projective space. We define the complex projective space of (complex) dimension N, denoted by $\mathbb{C}P^N$, or in short $\mathbb{P}^N$, as follows:

$$\mathbb{P}^N := \frac{\mathbb{C}^{N+1} - \{(0, \cdots, 0)\}}{\mathbb{C} - \{0\}} = (\mathbb{C}^{N+1})^*/\mathbb{C}^*.$$

Elements of $\mathbb{P}^N$ are denoted by $[Z^0 : Z^1 : \cdots : Z^N]$, where all Z^j are complex. The element $[0 : 0 : \cdots : 0]$ is not allowed. For any $\lambda \in \mathbb{C}^*$ the following identity holds:

$$[Z^0 : Z^1 : \cdots : Z^N] = [\lambda Z^0 : \lambda Z^1 : \cdots : \lambda Z^N].$$

These $\mathbb{P}^N$ are all compact complex manifolds of (complex) dimension N. It is important to realize that these Z^j are not coordinates on some chart of $\mathbb{P}^N$. However, we can restrict to a part of $\mathbb{P}^N$ on which for example Z^0 is non-zero. Then we can write any point on this subset in a unique way:

$$[Z^0 : Z^1 : Z^2 : \cdots : Z^N] = [1 : \frac{Z^1}{Z^0} : \frac{Z^2}{Z^0} : \cdots : \frac{Z^N}{Z^0}] = [1 : W^1 : W^2 : \cdots : W^N].$$

With these W^j, which are called *inhomogeneous coordinates* by convention, it is possible to define one of the charts contained in a minimal holomorphic atlas of $\mathbb{P}^N$. This atlas contains $N + 1$ similar charts in total. It is easy to show that the corresponding chart transition functions are all holomorphic, and it is clear that we are dealing with an orientable manifold. A special case is $\mathbb{P} = \mathbb{P}^1$, called the Riemann sphere. It has complex dimension 1, and we will call it the *complex projective line*.

We should note that the complex structure of $\mathbb{C}^N$ can be regarded as being unique, by definition. Similarly the complex structure of $\mathbb{P}^N$ can also be regarded as being unique, by definition.

One important remark is that the inhomogeneous coordinates W^j are induced by the coordinates Z^j which were defined on $(\mathbb{C}^{N+1})^*$. In this sense, the W^j are unique as they are uniquely related to the Z^j (up to a global scale factor of course). We could though have chosen other coordinates Z'^j, and study coordinate transformations from Z to Z'. (As we are dealing with a holomorphic atlas, we can safely assume that a holomorphic map from Z to Z' exists.) According to [6] we should restrict ourselves to coordinate transformations which are invertible linear maps with complex coefficients. Only these are free of poles and are invariant under the complex rescaling we will apply when inducing coordinates W' on $\mathbb{P}^N$ from Z'.

The (smooth) set of invertible linear holomorphic maps from $\mathbb{C}^k$ to $\mathbb{C}^k$ forms a Lie group called the *general linear group* with complex coefficients, or, in short, $\mathrm{GL}(k, \mathbb{C})$. Elements of $\mathrm{GL}(k, \mathbb{C})$ can be represented as k-by-k matrices with complex coefficients and non-zero determinant. Any of the groups $\mathrm{GL}(k, \mathbb{C})$ induces a collection of holomorphic coordinate transformations on $\mathbb{P}^k$.

We note that $\mathbb{P}^N$ can be embedded into $\mathbb{P}^{N+1}$ in many different ways. As a trivial example, we can write

$$[Z^0 : \cdots : Z^N] \in \mathbb{P}^N \mapsto [Z^0 : \cdots : Z^N : 0] \in \mathbb{P}^{N+1}.$$

Complex projective spaces: A cell complex. Let $\mathbb{P}^1 = X_1 \cup X_2$, with

$$X_1 = \{[0 : 1]\} \quad , \quad X_2 = \{[1 : W^1] | W^1 \in \mathbb{C}\}.$$

We see that X_1 is just a point, and that $X_2 \simeq \mathbb{C}$, thus $C(\mathbb{P}^1) = C_0 \cup C_2$, which is a disjoint union of a 0-cell and a 2-cell. Similarly we can write $\mathbb{P}^2 = Y_1 \cup Y_2 \cup Y_3$, with

$$Y_1 = \{[0 : 0 : 1]\}, \quad Y_2 = \{[0 : 1 : W^2]\}, \quad Y_3 = \{[1 : W^1 : W^2]\}.$$

We see that $Y_3 \simeq \mathbb{C}^2$, thus $C(\mathbb{P}^2) = C_0 \cup C_2 \cup C_4 = C(\mathbb{P}^1) \cup C_4$. In general we can write:

$$C(\mathbb{P}^N) = C(\mathbb{P}^{N-1}) \cup C_{2N} = C_0 \cup C_2 \cup C_4 \cup \cdots \cup C_{2N-2} \cup C_{2N}.$$

As a consequence we can write $\chi(\mathbb{P}^N) = N + 1$.

Complex projective spaces: Betti numbers and Hodge numbers. The Betti numbers of $\mathbb{P}^N$ are expressed as follows:

$$\begin{aligned}
(b_0(\mathbb{P}^1), \cdots, b_2(\mathbb{P}^1)) &= (1, 0, 1) \\
(b_0(\mathbb{P}^2), \cdots, b_4(\mathbb{P}^2)) &= (1, 0, 1, 0, 1) \\
(b_0(\mathbb{P}^3), \cdots, b_6(\mathbb{P}^3)) &= (1, 0, 1, 0, 1, 0, 1) \\
(b_0(\mathbb{P}^4), \cdots, b_8(\mathbb{P}^4)) &= (1, 0, 1, 0, 1, 0, 1, 0, 1).
\end{aligned}$$

(It is a coincidence that the cell complexes described above would also give the same result.)

The Hodge diamond of $\mathbb{P}^N$ is expressed as

$$\begin{pmatrix} h^{0,0}(\mathbb{P}^N) & h^{0,1}(\mathbb{P}^N) & \cdots & h^{0,N}(\mathbb{P}^N) \\ h^{1,0}(\mathbb{P}^N) & h^{1,1}(\mathbb{P}^N) & & h^{1,N}(\mathbb{P}^N) \\ \vdots & & \ddots & \vdots \\ h^{N,0}(\mathbb{P}^N) & h^{N,1}(\mathbb{P}^N) & \cdots & h^{N,N}(\mathbb{P}^N) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{2.33}$$

or just simply by

$$h^{k,l}(\mathbb{P}^N) = \delta_{k,l}.$$

Later we will discuss the relation between the Betti numbers and the Hodge numbers for a specific kind of compact complex manifolds, namely compact Kähler manifolds.

A real projective space embedded in a complex projective space. Note that the set $[Z^0 : \cdots : Z^N]$ with all $Z^j \in \mathbb{R}$ is embedded in $\mathbb{P}^N$. This set is called the *real projective space* of (real) dimension N, and we write $\mathbb{R}\mathrm{P}^N$. Thus, we write $\mathbb{R}\mathrm{P}^N \subset \mathbb{P}^N$. Note that $\mathbb{R}\mathrm{P}^M$ can also be embedded into $\mathbb{P}^N$ if $M < N$. Another way to express $\mathbb{R}\mathrm{P}^N$ is

$$\mathbb{R}P^N = \frac{\mathbb{R}^{N+1} - \{(0, \cdots, 0)\}}{\mathbb{R} - \{0\}} = (\mathbb{R}^{N+1})^*/\mathbb{R}^* \simeq S^N/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ represents the group identifying antipodal points of the N-sphere. If N is odd, then $\mathbb{R}\mathrm{P}^N$ is an orientable manifold, and if N is even it is not.

Hypersurfaces in complex projective spaces. Let $p : \mathbb{C}^{(N+1)*} \mapsto \mathbb{C}$ be a homogeneous holomorphic polynomial of degree n, then we can look at the zero set of p, which is defined as $\tilde{X} := p^{-1}(0) \subset \mathbb{C}^{(N+1)*}$. Homogeneity of p means that for any $\lambda \in \mathbb{C}^*$ we have $p(\lambda Z^j) = \lambda^n p(Z^j)$ for all $Z^j \in \mathbb{C}^{(N+1)*}$. Homogeneity of p ensures that for any point $(Z^0, \cdots, Z^N)$ lying in $\tilde{X}$, the point $(\lambda Z^0, \cdots, \lambda Z^N)$ also lies in $\tilde{X}$, namely

$$p(Z^j) = 0 \Rightarrow p(\lambda Z^j) = \lambda^n p(Z^j) = 0 \qquad (\forall \lambda \in \mathbb{C}^*).$$

Thus the $\mathbb{C}^*$ action, expressed by multiplying by λ, leaves the set $\tilde{X}$ invariant. This ensures that we can consistently define the set $X := \tilde{X}/\mathbb{C}^* \subset \mathbb{P}^N$, and we will call X a compact *hypersurface* (or *hypercurve*) in $\mathbb{P}^N$, with $\dim_{\mathbb{C}}(X) = N - 1$. For any p we can define a holomorphic differential

$$dp := \frac{\partial p}{\partial z^\mu} dz^\mu.$$

If $dp = 0$ has no solution on $\tilde{X}$ we say that p generates a complex manifold. This manifold is the pullback of $\tilde{X}$ to $\mathbb{P}^N$, written as $X := \tilde{X}/\mathbb{C}^* \subset \mathbb{P}^N$. Saying that $dp = 0$ has no solution on $\tilde{X}$ is equivalent to saying that X contains no singular points. As $\mathbb{P}^N$ itself is a compact complex manifold, we see that X is a compact complex submanifold of $\mathbb{P}^N$.

Hypersurfaces in complex projective spaces: Complex structures. We should be careful in making conclusions about the possible complex structures on such a hypersurface. First let X be an arbitrary smooth submanifold of $\mathbb{P}^N$, where $\dim_{\mathbb{R}}(X) = 2N-2$. The complex structure J on $\mathbb{P}^N$ may be unique, but in general there is no canonical way to pull back this J to X decently, as J is a tensor field of mixed type. Then we need to redefine a complex structure on X from the start (if possible at all).

On the other hand, if X is a *hypersurface*, embedded in $\mathbb{P}^N$, then it is already defined with respect to the holomorphic atlas of $\mathbb{P}^N$. Thus, X is already described as a complex submanifold of $\mathbb{P}^N$. This means that we can describe X with a complex structure J_X, canonically induced by J. (Doing this the other way round is not possible: J cannot be induced by J_X.)

If X and X' are different hypersurfaces embedded in $\mathbb{P}^N$, then they still have the same topology, and a homeomorphism $f : X \to X'$ always exists. On the other hand, a biholomorphism $\phi : X \to X'$ does not always exist. If a map, represented by an element of $\mathrm{GL}(N, \mathbb{C})$, inducing a biholomorphism from X to X', exists, then X and X' carry the same complex structure. But, in many cases such a biholomorphism does not exist. Then X and X' carry different complex structures. We conclude that a hypersurface carries new degrees of freedom for complex structures, compared to the embedding space $\mathbb{P}^N$.

Complex submanifolds of a hypersurface. Now we understand that the complex structure J of $\mathbb{P}^N$ canonically induces a complex structure J_X on any hypersurface $X \subset \mathbb{P}^N$, with $\dim_{\mathbb{C}}(X) = N-1$. We can say that J_X is an *induced complex structure*, generated by J.

Now assume $Y \subset X$ is a hypersurface (holomorphically) embedded in X, with $\dim_{\mathbb{C}}(Y) = N-2$. Then we can say that J_Y is an induced complex structure, generated by J_X. In later chapters we will see some examples of Riemann surfaces embedded in a K3 surface, in turn embedded in $\mathbb{P}^3$.

Now we will study some examples of hypersurfaces which will return in later chapters.

Cubics in $\mathbb{P}^2$. A general homogeneous third order polynomial on $\mathbb{C}^{3*}$ can be written as

$$p : \mathbb{C}^{3*} \longrightarrow \mathbb{C} : (Z^0, Z^1, Z^2) \mapsto p_{ijk} Z^i Z^j Z^k,$$

where the coefficients p_{ijk} are assumed to be complex and symmetric in all indices. Most choices of p_{ijk} will give us a non-singular $X \subset \mathbb{P}^2$, thus X will be a complex manifold with $\dim_{\mathbb{C}}(X) = 1$. We will call this X a cubic in $\mathbb{P}^2$. We say that X is a compact *Riemann surface*. (Riemann surfaces will be discussed later.) To be more precise: X is a Riemann surface of genus 1, thus it is a complex *torus*.

As p_{ijk} must be symmetric, there are only $\binom{5}{3} = 10$ independent coefficients. However, we can make a further reduction. The group $\mathrm{GL}(3, \mathbb{C})$, which has complex dimension 9, acts on the Z^j. Dividing out polynomials which are equivalent under this group, results in a reduction from 10 to just 1 independent coefficient. This means that the moduli space of complex structures of all complex tori, embedded in $\mathbb{P}^2$, has (at least) complex dimension one: $\dim_{\mathbb{C}}(\mathcal{M}_C(T^2)) \geq 1$.

Cubics in $\mathbb{P}^2$: Deformations. We can also study deformations of one cubic into another. If we have, for example, the two polynomials

$$p : (Z^0, Z^1, Z^2) \mapsto (Z^0)^3 + (Z^1)^3 + (Z^2)^3 \quad , \quad q : (Z^0, Z^1, Z^2) \mapsto (Z^0)^3 - (Z^1)^3 - (Z^2)^3,$$

then the zero sets of these functions generate hypersurfaces X_p and X_q in $\mathbb{P}^2$. Then we can study their linear interpolation

$$r_s : (Z^0, Z^1, Z^2) \mapsto (Z^0)^3 + (1-2s)((Z^1)^3 + (Z^2)^3),$$

where $s \in [0, 1]$. However, the map $r_{1/2} : (Z^0, Z^1, Z^2) \mapsto (Z^0)^3$, with $dr_{1/2} = 3(Z^0)^2 dZ^0$, gives us a hypersurface with singularities: note that the equations $r_{1/2} = 0$ and $dr_{1/2} = 0$ have exactly the same solution set. This actually means that all points in $X_{1/2} \subset \mathbb{P}^2$, which is the hypersurface generated by $r_{1/2}$, are singularities. It is however still a complex manifold, as $X_{1/2} \simeq \mathbb{P}$. (The equation $r_{1/2}(Z^0, Z^1, Z^2) = dr_{1/2}(Z^0, Z^1, Z^2) = 0$ will be satisfied if $Z^0 = 0$, thus for any point $[0 : Z^1 : Z^2] \in \mathbb{P}^2$.) Even if this $X_{1/2}$ is a complex manifold, we should ignore it in the family of complex tori.

On the other hand, we can redefine r_s and do a polar interpolation instead:

$$r_s : (Z^0, Z^1, Z^2) \mapsto (Z^0)^3 + e^{s\pi i}((Z^1)^3 + (Z^2)^3).$$

Then the zero sets of r_s correctly define a one-parameter family of hypersurfaces, starting at X_p and ending at X_q, proving that X_p and X_q have the same topology. (In general we can always use some arbitrary interpolation, avoiding singularities, to prove this for any pair of hypersurfaces embedded in $\mathbb{P}^N$.)

In this example the complex structure will not change, but in general this method also works for a pair of hypersurfaces with different complex structures: then we can see how the complex structure smoothly changes.

Quartics in $\mathbb{P}^3$. A general homogeneous fourth order polynomial on $\mathbb{C}^{4*}$ can be written as

$$p : \mathbb{C}^{4*} \longrightarrow \mathbb{C} : (Z^0, Z^1, Z^2, Z^3) \mapsto p_{ijkl} Z^i Z^j Z^k Z^l.$$

When $X = p^{-1}(0)/\mathbb{C}^*$ is non-singular, it will be a complex manifold with $\dim_{\mathbb{C}}(X) = 2$. We will call this X a quartic in $\mathbb{P}^3$. (These quartics are examples of K3 surfaces, see later chapters). Now there are $\binom{7}{4} = 35$ independent coefficients, and we are dealing with a reduction by $\mathrm{GL}(4, \mathbb{C})$ which has dimension 16. There are only 19 out of 35 left. This means that the moduli space of complex structures of all quartics, embedded in $\mathbb{P}^3$, has precisely complex dimension 19. (In [6] we can read that for any hypersurface X embedded in $\mathbb{P}^N$, with $N > 2$, all of the complex structures can be found in this way.)

Quintics in $\mathbb{P}^4$. A general homogeneous fifth order polynomial on $\mathbb{C}^{5*}$ can be written as

$$p : \mathbb{C}^{5*} \longrightarrow \mathbb{C} : (Z^0, Z^1, Z^2, Z^3, Z^4) \mapsto p_{ijklm} Z^i Z^j Z^k Z^l Z^m.$$

Again, when $X = p^{-1}(0)/\mathbb{C}^*$ is non-singular, it will be a complex manifold with $\dim_{\mathbb{C}}(X) = 3$. We will call this X a *quintic* in $\mathbb{P}^4$. Now there are $\binom{9}{5} = 126$ independent coefficients, and we are dealing with a reduction by $\mathrm{GL}(5, \mathbb{C})$ which has dimension 25. There are only 101 out of 126 left. This means that the moduli space of complex structures of all quintics, embedded in $\mathbb{P}^4$, has precisely complex dimension 101.

Riemann surfaces. When the complex dimension of a complex manifold M is 1, we call M a *Riemann surface*. From now on we only discuss compact connected Riemann surfaces S. By convention we assume these to have no boundary. Any Riemann surface is orientable, thus any S can be described as a 2-dimensional surface with genus g. Its Betti numbers are $b_0(S) = b_2(S) = 1$ and $b_1(S) = 2g$, thus its Euler number is $\chi(S) = 2 - 2g$. For any 2-dimensional surface with genus 0, thus a sphere S^2, there is a unique maximal holomorphic atlas, which makes it into the complex projective line $\mathbb{P}$. Uniqueness means that any two holomorphic atlases of $\mathbb{P}$ are equal up to a holomorphic transformation. The Riemann-Roch theorem, combined with Chow's theorem, tells us that any Riemann surface of genus 0 can be identified with $\mathbb{P}$, which is a unique identification, and any Riemann surface of genus 1, thus any (complex) torus, can be identified with a cubic in $\mathbb{P}^2$. This means that the moduli space of complex structures of all complex tori has precisely complex dimension one, see (2.30).

Riemann surfaces: Some other examples. Some Riemann surfaces can be described as hypersurfaces embedded in $\mathbb{P}^2$. Let

$$p : \mathbb{C}^{3*} \longrightarrow \mathbb{C} : (X, Y, Z) \mapsto X^k + Y^k + Z^k$$

be a (trivial) homogeneous polynomial on $\mathbb{C}^{3*}$ of order $k > 0$, and let $\tilde{S} := p^{-1}(\{0\})$. Then $S := \tilde{S}/\mathbb{C}^*$ is a non-singular compact Riemann surface, embedded in $\mathbb{P}^2$. Now we can examine a cell complex of this S, and compute its Euler number $\chi(S)$. We start with $\chi = 0$, and we will add some intermediate terms δ_χ:

- We know that $[0 : 0 : 1] \notin S$, so that $\delta_\chi = 0$. Thus we have $\chi \mapsto \chi + \delta_\chi = 0 + 0 = 0$.

- We know that $[0:1:Z] \in S$ if $1+Z^k=0$. This has k points as solutions for Z, so that $\delta_\chi = k$. Thus we have $\chi \mapsto \chi + \delta_\chi = 0 + k = k$.
- We know that $[1:Y:Z] \in S$ if $1+Y^k+Z^k=0$. If $1+Y^k=0$, which has k points as solutions for Y, then this has the trivial solution $Z^k = 0 \Rightarrow Z = 0$. Then we have $\delta_\chi = k$, so that $\chi \mapsto \chi + \delta_\chi = k + k = 2k$. Now, if $1+Y^k \neq 0$, then this has k solutions for Z for any $Y \in \mathbb{C}_{-k}$, where $\mathbb{C}_{-k}$ equals $\mathbb{C}$ with k points removed. Knowing that $\chi(\mathbb{C}_{-k}) = 1-k$, we obtain $\delta_\chi = k(1-k)$, so that $\chi \mapsto \chi + \delta_\chi = 2k + k(1-k) = k(3-k)$.

Thus, the result is that $\chi(S) = k(3-k)$. We also know that $\chi(S) = 2 - 2g_S$. Then we can finally say that a trivial homogeneous polynomial of order k generates a compact Riemann surface of genus $g_S = \frac{1}{2}(k-1)(k-2)$. This gives us the following table:

k	1	2	3	4	5	$\cdots$
g_S	0	0	1	3	6	$\cdots$

(2.34)

2.5 Kähler Manifolds

The complexified metric. When a complex manifold M is equipped with a Riemannian metric g, we can always naturally extend g to a metric acting on elements of the complexified tangent space. For any $Z, W \in T_pM^{\mathbb{C}}$ there exist $X, Y, U, V \in T_pM$ such that $Z = X + iY$ and $W = U + iV$. Then

$$g_p(Z,W) = g_p(X,U) - g_p(Y,V) + i(g_p(X,V) + g_p(Y,U)). \tag{2.35}$$

We can express components of g with respect to complex coordinates:

$$(g_p)_{\mu\nu} = g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) \quad , \quad (g_p)_{\mu\bar{\nu}} = g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}),$$
$$(g_p)_{\bar{\mu}\nu} = g_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}) \quad , \quad (g_p)_{\bar{\mu}\bar{\nu}} = g_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}).$$

We see that also (2.35) is symmetric in its arguments, thus the components satisfy the following identities:

$$g_{\mu\nu} = g_{\nu\mu}, \quad g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}}, \quad g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu}. \tag{2.36}$$

Another property, derived from $g_p(\bar{Z}, \bar{W}) = \overline{g_p(Z,W)}$, is

$$\overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu} \quad , \quad \overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}}. \tag{2.37}$$

Hermitian metrics. A Riemannian metric g is called *Hermitian* if it satisfies

$$g_p(J_pX, J_pY) = g_p(X,Y)$$

for all $p \in M$ and for all $X, Y \in T_pM$. (Then $g_p(J_pX, X) = 0$ for all $X \in T_pM$.) In this case the pair (M, g) is called a *Hermitian manifold*. We should note that when h is an arbitrary Riemannian metric on M, then a Hermitian metric g can be constructed from h. For all $p \in M$ and for all $X, Y \in T_pM$ we write

$$g_p(X,Y) := \frac{1}{2}(h_p(X,Y) + h_p(J_pX, J_pY).$$

To check whether this is indeed a Hermitian metric is an easy exercise. There is a trivial extension of a Hermitian metric to the complexified tangent space. It is easy to check that the components $g_{\mu\nu}$ and $g_{\bar{\mu}\bar{\nu}}$ of g with respect to complex coordinates are zero for any Hermitian metric. Thus, for any Hermitian metric, properties (2.36) and (2.37) can be summarized as

$$g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0, \quad g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu}, \quad \overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu}.$$

The full expansion of g in components will be

$$g_p = (g_p)_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + (g_p)_{\bar{\mu}\nu} d\bar{z}^\mu \otimes dz^\nu = (g_p)_{\mu\bar{\nu}}(dz^\mu \otimes d\bar{z}^\nu + d\bar{z}^\nu \otimes dz^\mu). \tag{2.38}$$

We should note that X and JX are mutually orthogonal with respect to any Hermitian metric.

Hermitian metrics and complex submanifolds. Let M be a complex manifold with a Hermitian metric g, let $N \subset M$ be a complex submanifold of M and let $\iota : N \to M$ be the canonical holomorphic embedding with pullback map $\iota^* : T^*_{\iota(p)}M \to T^*_p N$. Then we can pull back g from M to N, so that $h := \iota^* g$ defines a Hermitian metric on N. Let now z be a complex coordinate on N, around a point $p \in N$, and let w be a complex coordinate on M, around $\iota(p) \in M$. Then (2.17) has an analog, describing a relation between the components of h and g:

$$h_{(z)\mu\bar{\nu}} = \frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu} g_{(w)\alpha\bar{\beta}}. \tag{2.39}$$

The Kähler form. When a metric g and a complex structure J are defined on M, a Kähler form $\omega \in \Omega^{1,1}(M)$ can be constructed. For any pair $V, W \in T_p M$, we define

$$\omega(V, W) := g(JV, W).$$

From (2.38) we then derive that the full expansion of ω in components will be

$$\omega_p = (g_p)_{\mu\bar{\nu}}(i dz^\mu \otimes d\bar{z}^\nu - i d\bar{z}^\nu \otimes dz^\mu) = i(g_p)_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu. \tag{2.40}$$

As g and J are still real tensor fields when restricted to real input vectors, also ω is a real 2-form when restricted to real input vectors.

Kähler manifolds. When the Kähler form ω on a Hermitian manifold M is closed, thus when $d\omega = 0$, we call M a Kähler manifold. We know that $\omega \in \Omega^{1,1}(M)$, thus $d\omega = (\partial + \bar{\partial})\omega = \partial\omega + \bar{\partial}\omega$, where $\partial\omega \in \Omega^{2,1}(M)$ and $\bar{\partial}\omega \in \Omega^{1,2}(M)$, thus $d\omega \in \Omega^{2,1}(M) \oplus \Omega^{1,2}(M)$. This implies that ω, being d-closed, is also ∂- and $\bar{\partial}$-closed: $\partial\omega = \bar{\partial}\omega = 0$. Thus ω is locally ∂-exact and at the same time locally $\bar{\partial}$-exact on any contractible open $U_i \subset M$. This means that there exists a smooth function $K_i : U_i \to \mathbb{C}$ such that $\omega = i\partial\bar{\partial}K$ on U. In components we write

$$\omega = i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu = i \frac{\partial^2 K_i}{\partial z^\mu \partial \bar{z}^\nu} dz^\mu \wedge d\bar{z}^\nu \quad \Leftrightarrow \quad g_{\mu\bar{\nu}} = \frac{\partial^2 K_i}{\partial z^\mu \partial \bar{z}^\nu}.$$

This $K_i = K_i(z^\mu, \bar{z}^\mu)$ is not unique, as we can add functions which are purely holomorphic or anti-holomorphic, let's say a holomorphic function $F(z^\mu)$ and an anti-holomorhic function $\bar{G}(\bar{z}^\mu)$. Then from $K_i' := K_i + F + \bar{G}$ and K_i the same $\omega|_{U_i}$ is obtained. We will call K_i a *Kähler potential*, which is thus not unique. In fact, if K_j is another such function on a different coordinate patch U_j, then nothing guarantees that it is possible that K_i and K_j coincide on $U_i \cap U_j$. This is because K_i and K_j are only *locally* exact. However, it is always possible to find a holomorphic $F_{ij}(z^\mu)$ and an anti-holomorphic $\bar{G}_{ij}(z^\mu)$, both defined on $U_i \cap U_j$, so that the equality $K_j = K_i + F_{ij} + \bar{G}_{ij}$ holds on $U_i \cap U_j$.

The Dolbeault class $[\omega] \in H^{1,1}(M)$ is called the *Kähler class*. A property of a Kähler manifold is that $[\omega]$ is non-trivial.

If (M_1, g_1) and (M_2, g_2) are Kähler manifolds, then $(M, g) := (M_1 \times M_2, g)$ is again a Kähler manifold, where g is canonically derived from g_1 and g_2. The corresponding Kähler forms ω_1 and ω_2 can be used to construct a Kähler form $\omega := \omega_1 + \omega_2$ on M.

The volume of a compact Kähler manifold. If M is a compact Kähler manifold, with $m = \dim_{\mathbb{C}}(M)$, then the Kähler form $\omega \in \Omega^{1,1}(M)$ also defines a volume form $\omega^m \in \Omega^{m,m}(M)$:

$$\omega^m = \omega \wedge \cdots \wedge \omega \quad , \quad \mathrm{Vol}_M := \int_M *1 = \int_M \frac{\omega^m}{m!}. \tag{2.41}$$

Note that $*1$ is the canonical Riemannian volume form. (Knowing that ω is a real 2-form, this also makes ω^m into a real $2m$-form.)

A connection on a Kähler manifold. For a general Hermitian manifold we can define the *Hermitian connection* which, at first instance, differs from the Levi-Civita connection. First of all we write its Christoffel symbols with respect to complex coordinates, thus these have holomorphic and anti-holomorphic indices. It is defined such that a holomorphic vector after parallel transport is still a holomorphic vector. This means that we can restrict ourselves to Christoffel symbols which are zero for all mixed holomorphic and anti-holomorphic indices. The only non-zero Christoffel symbols thus are $\Gamma^{\kappa}{}_{\mu\nu}$ and $\Gamma^{\bar{\kappa}}{}_{\bar{\mu}\bar{\nu}}$. Again we require this connection to be a metric connection, thus the Hermitian metric must be covariantly constant with respect to this connection. We thus require

$$\nabla_{\kappa} g_{\mu\bar{\nu}} = \partial_{\kappa} g_{\mu\bar{\nu}} - \Gamma^{\lambda}{}_{\kappa\mu} g_{\lambda\bar{\nu}} = 0 \quad , \quad \nabla_{\bar{\kappa}} g_{\mu\bar{\nu}} = \partial_{\bar{\kappa}} g_{\mu\bar{\nu}} - \Gamma^{\bar{\lambda}}{}_{\bar{\kappa}\bar{\nu}} g_{\mu\bar{\lambda}} = 0,$$

which implies that

$$\Gamma^{\kappa}{}_{\mu\nu} = g^{\kappa\bar{\lambda}} \partial_{\mu} g_{\nu\bar{\lambda}} \quad , \quad \Gamma^{\bar{\kappa}}{}_{\bar{\mu}\bar{\nu}} = g^{\lambda\bar{\kappa}} \partial_{\bar{\mu}} g_{\lambda\bar{\nu}}. \tag{2.42}$$

For any Hermitian metric, the corresponding Hermitian connection is unique. In general the Christoffel symbols of the Hermitian connecton are not symmetric in the lower indices, which means that the *torsion* of this connection does not vanish. However, we are only interested in Kähler manifolds. On a Kähler manifold we have a Kähler potential locally, thus, by interchanging derivatives, we can identify $\partial_{\mu} g_{\nu\bar{\lambda}} = \partial_{\nu} g_{\mu\bar{\lambda}}$. This implies that, on a Kähler manifold, we are dealing with a torsion-free Hermitian connection, which coincides with the Levi-Civita connection with respect to complex coordinates. The Christoffel symbols of the Levi-Civita connection thus simplify and become (2.42), still satisfying

$$\Gamma^{\kappa}{}_{\mu\nu} = \Gamma^{\kappa}{}_{\nu\mu} \quad , \quad \Gamma^{\bar{\kappa}}{}_{\bar{\mu}\bar{\nu}} = \Gamma^{\bar{\kappa}}{}_{\bar{\nu}\bar{\mu}}.$$

The complex structure J is a trivial example of a tensor field which is covariantly constant with respect to the Hermitian connection defined on any Hermitian manifold. If M is a Kähler manifold, then J is automatically also covariantly constant with respect to the Levi-Civita connection. According to a theorem the converse is also true. If M is a Hermitian manifold and if J is also covariantly constant with respect to the Levi-Civita connection, then M must be a Kähler manifold.

The Ricci form on a Hermitian manifold. Any Hermitian manifold M admits a Ricci form $\mathcal{R}$ which is based on a real 2-form which can be complexified in a trivial way. Representing it with respect to complex coordinates gives us:

$$\mathcal{R} := i\mathcal{R}_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu} \quad , \quad \mathcal{R}_{\mu\bar{\nu}} := -\partial_{\bar{\nu}}(g^{\alpha\bar{\beta}} \partial_{\mu} g_{\alpha\bar{\beta}}) = -\partial_{\bar{\nu}} \Gamma^{\alpha}{}_{\mu\alpha}.$$

We would like to rewrite the components of the Ricci form. For any square matrix A with real, positive and non-zero eigenvalues we have the following identities:

$$\partial_j \log A = A^{-1} \partial_j A \quad , \quad \operatorname{tr} \log A = \log \det A.$$

Now g is a Riemannian metric, when restricted to the real part of the complexified tangent space, thus it satisfies the needed conditions to be able to write:

$$\Gamma^{\alpha}{}_{\mu\alpha} = g^{\alpha\bar{\beta}} \partial_{\mu} g_{\alpha\bar{\beta}} = \operatorname{tr} \partial_{\mu} \log g_{\alpha\bar{\beta}} = \partial_{\mu} \operatorname{tr} \log g_{\alpha\bar{\beta}} = \partial_{\mu} \log \det g_{\alpha\bar{\beta}}. \tag{2.43}$$

Now $\det g_{\alpha\bar{\beta}}$ really means the *complex* determinant of the top-right part of the matrix of the metric coefficients, thus, using (2.15), it satisfies:

$$\det g_{\alpha\bar{\beta}} = \sqrt{\det g_{jk}} = \sqrt{|\det g_{jk}|} = G. \tag{2.44}$$

Thus, the components of the Ricci form become $\mathcal{R}_{\mu\bar{\nu}} = -\partial_{\bar{\nu}} \partial_{\mu} \log G$.

Now we should be careful drawing conclusions too fast, as $\log G$ is not really a scalar function. The function $\log G$ itself depends on the used coordinates, and it will not extend globally on the entire manifold. However, as $g_{\alpha\bar{\beta}}$ transforms as

$$g^{(y)}_{\alpha\bar{\beta}} \mapsto g^{(z)}_{\alpha\bar{\beta}} = \frac{\partial y^\mu}{\partial z^\alpha}\frac{\partial \bar{y}^\nu}{\partial \bar{z}^\beta} g^{(y)}_{\mu\bar{\nu}},$$

under coordinate transformations $y^\mu \mapsto z^\mu$, we see that $G_{(y)} \mapsto G_{(z)} = G_{(y)} \det H \det A$, where H is purely holomorphic and A is purely anti-holomorphic. As a consequence, taking the logarithm of this transformation yields $\log G_{(y)} \mapsto \log G_{(z)} = \log G_{(y)} + \log\det H + \log\det A$. Now $\log\det H$ is again purely holomorphic and $\log\det A$ is again purely anti-holomorphic, thus when we put the derivatives in front we will see that these terms will vanish:

$$\begin{aligned} \mathcal{R}^{(y)}_{\mu\bar{\nu}} = -\partial_{\bar{\nu}}\partial_\mu \log G_{(y)} \mapsto \mathcal{R}^{(z)}_{\mu\bar{\nu}} &= -\partial_{\bar{\nu}}\partial_\mu \log G_{(z)} \\ &= -\partial_{\bar{\nu}}\partial_\mu(\log G_{(y)} + \log\det H + \log\det A) = -\partial_{\bar{\nu}}\partial_\mu \log G_{(y)} = \mathcal{R}^{(y)}_{\mu\bar{\nu}}. \end{aligned}$$

This property suggests that $\mathcal{R}$ can be written as $-\bar{\partial}\partial f$, where f is a local scalar, thus we can represent $\mathcal{R}$ in a coordinate independent manner:

$$\mathcal{R} = -i\bar{\partial}\partial \log G = i\partial\bar{\partial}\log G = -\frac{i}{2}d(\partial - \bar{\partial})\log G. \tag{2.45}$$

The last equality comes from (2.32). We should however be careful. This expression is coordinate independent, but that is still not enough to make $\mathcal{R}$ into a globally exact form. Thus it is important to realize that $\mathcal{R}$ is only locally exact, and (2.45) is a local identity.

We will conclude that $\mathcal{R}$ is closed as it is locally exact, thus it represents a cohomology class. This cohomology class, up to a constant, is called the *first Chern class*, and it is written as $c_1(M) := [\mathcal{R}/2\pi]$. The first Chern class defines an element in $H^2_{\mathrm{dR}}(M;\mathbb{R})$.

The Ricci form on a Kähler manifold. Now, if M is also a (compact) Kähler manifold, then the components of the Ricci form coincide with the components of the Ricci tensor. The Kähler metric is called *Ricci-flat* if the corresponding Ricci tensor vanishes. Thus, if M is a Kähler manifold, then also the Ricci form vanishes. As a consequence also the first Chern class vanishes. Thus, if the metric is Ricci-flat, then the first Chern class vanishes.

The converse implication is more complicated, as the first Chern class only carries topological information. However, there exists a proposition telling us that the first Chern class is invariant under smooth changes of the metric. Thus, if a Ricci-flat metric g exists on M, then the first Chern class $c_1(M,g)$, related to this g, vanishes. However, this $c_1(M,g)$, only carrying topological information, will not change after smoothly changing g to any other metric g'. Thus, in general we have the identity $c_1(M,g) = c_1(M,g') = c_1(M)$, thus if $c_1(M,g) = 0$ then also $c_1(M,g') = 0$. To summarize, if M admits a Ricci-flat metric, then its first Chern class vanishes. The Calabi conjecture, proven by Yau, also claims the converse implication. If the first Chern class of a compact Kähler manifold vanishes, then it admits a Ricci-flat metric.

Many of the manifolds to be studied later are said to *admit* a Ricci-flat metric, but in many cases this metric is not known (exactly).

The complex Hodge star operator. The Hodge star operator $* : \Omega^k(M) \to \Omega^{m-k}(M)$ naturally induces a map $*^{\mathbb{C}} : \Omega^k(M)^{\mathbb{C}} \to \Omega^{m-k}(M)^{\mathbb{C}}$. Now suppose M is a complex manifold, with $m = \dim_{\mathbb{C}}(M)$. Then $* = *^{\mathbb{C}}$ also nicely maps from $\Omega^{k,l}(M)$ to $\Omega^{m-l,m-k}(M)$, as we can read in [18].

Its conjugate operator is $\bar{*} : \Omega^{k,l}(M) \to \Omega^{m-k,m-l}(M)$. For any complex form α it is defined by $\bar{*}\alpha := \overline{*\alpha} = *\bar{\alpha}$. Then $\bar{*}$ can be used to construct an inner product. Let α and β be (k,l)-forms. Then their inner product can be defined as

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge \bar{*}\beta. \tag{2.46}$$

The adjoint operators of ∂ and $\bar{\partial}$. We already know that the total exterior derivative operator d has an adjoint operator $d^\star$. We can do the same for ∂ and $\bar{\partial}$:

$$\partial^\dagger := - * \bar{\partial} * \quad , \quad \bar{\partial}^\dagger := - * \partial * .$$

These operators also satisfy $(\partial^\dagger)^2 = (\bar{\partial}^\dagger)^2 = 0$. These operators can be used to construct other Laplace operators.

Laplace operators on Hermitian manifolds. We already have the usual Laplace operator Δ, as defined in (2.24). On a Hermitian manifold we define other Laplace operators:

$$\begin{aligned} \Delta_\partial &:= (\partial + \partial^\dagger)^2 = \partial\partial^\dagger + \partial^\dagger\partial, \\ \Delta_{\bar{\partial}} &:= (\bar{\partial} + \bar{\partial}^\dagger)^2 = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \end{aligned}$$

If $\alpha \in \Omega^{k,l}(M)$, then it is said to be ∂-*harmonic* if $\Delta_\partial\alpha = 0$, and $\bar{\partial}$-*harmonic* if $\Delta_{\bar{\partial}}\alpha = 0$. If α is ∂-harmonic, then it also satisfies $\partial\alpha = \partial^\dagger\alpha = 0$, and if α is $\bar{\partial}$-harmonic, then it also satisfies $\bar{\partial}\alpha = \bar{\partial}^\dagger\alpha = 0$:

$$\Delta_\partial\alpha = 0 \Leftrightarrow \partial\alpha = \partial^\dagger\alpha = 0 \quad , \quad \Delta_{\bar{\partial}}\alpha = 0 \Leftrightarrow \bar{\partial}\alpha = \bar{\partial}^\dagger\alpha = 0. \tag{2.47}$$

These equivalences, being similar to (2.26), can also be easily proven. Now we can define the group of $\bar{\partial}$-harmonic forms:

$$\mathcal{H}^{k,l}_{\bar{\partial}}(M) := \{\alpha \in \Omega^{k,l}(M) | \Delta_{\bar{\partial}}\alpha = 0\}.$$

These groups turn out to be isomorphic to the Dolbeault-cohomology groups:

$$\mathcal{H}^{k,l}_{\bar{\partial}}(M) \simeq H^{k,l}(M).$$

A proof can be found in [18]. This means that every $\bar{\partial}$-harmonic form represents a $\bar{\partial}$-cohomology class.

Laplace operators on compact Kähler manifolds. Let now M be a compact Kähler manifold, and $m = \dim_{\mathbb{C}}(M)$. Then all Laplace operators are equal:

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{2.48}$$

A proof can be found in [1]. Combining (2.48) and (2.47) will give us the following result:

$$\partial\alpha = \partial^\dagger\alpha = 0 \quad \Leftrightarrow \quad \bar{\partial}\alpha = \bar{\partial}^\dagger\alpha = 0.$$

As a consequence we can say that if α is a holomorphic (or anti-holomorphic) k-form, thus $\alpha \in \Omega^{k,0}(M)$ (or $\alpha \in \Omega^{0,k}(M)$), then it automatically satisfies $\bar{\partial}\alpha = \bar{\partial}^\dagger\alpha = 0$ (or $\partial\alpha = \partial^\dagger\alpha = 0$), thus $\Delta_{\bar{\partial}}\alpha = 0$ (or $\Delta_\partial\alpha = 0$). Then (2.48) implies that $\Delta\alpha = 0$. Thus, with respect to the Kähler metric, any holomorphic (or anti-holomorphic) k-form is harmonic, ∂-harmonic and $\bar{\partial}$-harmonic at the same time. Conversely, $\Delta\alpha = 0$ implies $\bar{\partial}\alpha = 0$ (or $\partial\alpha = 0$), thus every harmonic $(k,0)$-form (or $(0,k)$-form) is a holomorphic (or anti-holomorphic) k-form.

We can say that especially a holomorphic top-form Ω is automatically a harmonic form. We already know that the space of holomorphic top-forms with the same zero set, has complex dimension one, if M is connected. This means that the space of harmonic $(m,0)$-forms with the same zero set, also has complex dimension one. This in turn means that $h^{m,0}(M)$ tells us more about the number of different zero sets (including the empty zero set in case of a nowhere vanishing Ω) any holomorphic top-form can have.

Also note that $\bar{\partial}\Omega = 0$ and Ω cannot be $\bar{\partial}$-exact, thus $[\Omega]$ represents a Dolbeault cohomology class.

Hodge numbers of Kähler manifolds. If M is a compact Kähler manifold of complex dimension m, then its Hodge numbers $h^{k,l} = h^{k,l}(M)$ satisfy the following rule for all k and l:

$$h^{k,l} = h^{l,k} = h^{m-k,m-l} = h^{m-l,n-k}. \tag{2.49}$$

(Relation (2.48) implies that the Hodge numbers satisfy the symmetry property $h^{k,l} = h^{l,k}$, and the Hodge star operator implies that they should satisfy $h^{k,l} = h^{m-k,m-l}$.)

This means that the Hodge diamond of M is symmetric in the horizontal and in the vertical direction. Then the number of independent Hodge numbers $(m+1)^2$ can be reduced to

- $(\frac{1}{2}m+1)^2$ if m is even,
- $\frac{1}{4}(m+1)(m+3)$ if m is odd.

The Hodge decomposition. The Hodge numbers and Betti numbers of any compact Kähler manifold are related:

$$b_j(M) = \sum_{k+l=j} h^{k,l}(M).$$

There are two extra conditions. If $1 \leq j \leq m$, then b_{2j-1} is even, and $b_{2j} \geq 1$. (A proof can be found in [18].) This can be written alternatively:

$$H^j_{\mathrm{dR}}(M)^{\mathbb{C}} = \bigoplus_{k+l=j} H^{k,l}(M).$$

(Note that $b_j(M) = \dim_{\mathbb{R}}(H^j_{\mathrm{dR}}(M)) = \dim_{\mathbb{C}}(H^j_{\mathrm{dR}}(M)^{\mathbb{C}})$.) This is related to the following identity:

$$\Omega^j(M)^{\mathbb{C}} = \bigoplus_{k+l=j} \Omega^{k,l}(M).$$

Hodge numbers of tori. Let $M_n := T^{2n}$ be a torus of complex dimension n. Then its Hodge numbers are:

$$h^{k,l}(M_n) = \binom{n}{k}\binom{n}{l}.$$

The Künneth formula applied to Hodge numbers. Now we will present the effects of the *Künneth formula* on Hodge numbers. For any two compact Kähler manifolds M_1 and M_2, of which we already know the Hodge numbers, we can directly compute the Hodge numbers of $M := M_1 \times M_2$, which is again a compact Kähler manifold. The Künneth formula now reads as follows:

$$h^{k,l}(M) = \sum_{a+c=k} \sum_{b+d=l} h^{a,b}(M_1) h^{c,d}(M_2). \tag{2.50}$$

This formula can be found in [19], and it is in harmony with the Künneth formula applied to Betti numbers, see (2.14). (We also note that (2.50) is related to properties of harmonic forms.)

As a trivial example: we know that the Hodge diamonds of the torus T^2, the 4-torus $T^4 = T^2 \times T^2$ and the 6-torus $T^6 = T^2 \times T^2 \times T^2$ are

$$\begin{pmatrix} h^{0,0}(T^2) & h^{0,1}(T^2) \\ h^{1,0}(T^2) & h^{1,1}(T^2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{2.51}$$

$$\begin{pmatrix} h^{0,0}(T^4) & h^{0,1}(T^4) & h^{0,2}(T^4) \\ h^{1,0}(T^4) & h^{1,1}(T^4) & h^{1,2}(T^4) \\ h^{2,0}(T^4) & h^{2,1}(T^4) & h^{2,2}(T^4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \tag{2.52}$$

$$\begin{pmatrix} h^{0,0}(T^6) & h^{0,1}(T^6) & h^{0,2}(T^6) & h^{0,3}(T^6) \\ h^{1,0}(T^6) & h^{1,1}(T^6) & h^{1,2}(T^6) & h^{1,3}(T^6) \\ h^{2,0}(T^6) & h^{2,1}(T^6) & h^{2,2}(T^6) & h^{2,3}(T^6) \\ h^{3,0}(T^6) & h^{3,1}(T^6) & h^{3,2}(T^6) & h^{3,3}(T^6) \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}. \tag{2.53}$$

It is, again, easy to check that these satisfy (2.50). For example:

$$h^{1,1}(T^4) = h^{0,0}(T^2)h^{1,1}(T^2) + h^{1,0}(T^2)h^{0,1}(T^2) + h^{0,1}(T^2)h^{1,0}(T^2) + h^{1,1}(T^2)h^{0,0}(T^2).$$

2.6 Examples of Kähler Manifolds

Riemann surfaces. Any compact connected Riemann surface S admits a Hermitian metric. As we have $\dim_{\mathbb{C}}(S) = 1$, the corresponding Kähler form $\omega \in \Omega^{1,1}(S)$ is automatically closed at the same time, thus any S is a Kähler manifold.

The Hodge numbers of S are $h^{0,0} = h^{1,1} = 1$ and $h^{1,0} = h^{0,1} = g_S$, where g_S is the genus of S. Then we see that the relation between the Betti numbers and Hodge numbers indeed says

$$b_0(S) = h^{0,0}(S) = 1, \quad b_1(S) = h^{1,0}(S) + h^{0,1}(S) = 2g_S, \quad b_2(S) = h^{1,1}(S) = 1.$$

(We already know that $\chi(S) = b_0(S) - b_1(S) + b_2(S) = 2 - 2g_S$.)

Fubini-Study metric. We already know that $\mathbb{P}^N$ is a compact complex manifold, with *homogeneous* coordinates $[Z^0 : \cdots : Z^j : \cdots : Z^N]$. Then we can choose patch j, with $Z^j \neq 0$, and express a Kähler potential K_j with respect to this coordinate:

$$K_j := \log(\sum_{k=0}^{N} |\frac{Z^k}{Z^j}|^2).$$

This yields a globally defined closed 2-form $\omega = i\partial\bar{\partial}K_j$. (We could have started with any other j, obtaining the same result.) Then $\mathbb{P}^N$, together with this ω, can be regarded as a compact Kähler manifold. The *Fubini-Study metric* g_{FS}, which is unique, can be directly derived from ω.

Any complex submanifold of a Kähler manifold, especially any hypersurface, which is generated by a polynomial, is also a Kähler manifold. (We can always pull back the Kähler metric to the embedded submanifold, using (2.39).) As $\mathbb{P}^N$, equipped with the Fubini-Study metric, is a Kähler manifold, any cubic (torus) embedded in $\mathbb{P}^2$, any quartic (K3 surface) embedded in $\mathbb{P}^3$ and any quintic embedded in $\mathbb{P}^4$ is also a Kähler manifold.

We note that g_{FS} is *not* a Ricci-flat metric, and $\mathbb{P}^N$ does not admit a Ricci-flat metric either. If we pull back g_{FS} to any hypercurve, then also the resulting metric is not Ricci-flat, but in some cases this hypercurve *does* admit a Ricci-flat metric. (See again the cubic, the quartic and the quintic.)

We note that if no metric is defined yet on $\mathbb{P}^N$, then we also cannot say that it has a radius or volume either. But, if a Kähler metric is defined on $\mathbb{P}^N$, then it definitely has a radius and volume. If we use (2.41), applied to $\mathbb{P}^N$, then we can also compute its volume. As we can read in [7] we can indeed say that $\mathbb{P}^N$ has a radius, related to the choice of Kähler class.

2.7 Calabi-Yau Manifolds

Calabi-Yau manifolds. Let M be a compact connected Kähler manifold, and $m = \dim_{\mathbb{C}}(M)$, then M is a *Calabi-Yau manifold*, or Calabi-Yau m-fold, if it satisfies one of the following equivalent properties.

- There exists a *nowhere vanishing* holomorphic top-form $\Omega \in \Omega^{m,0}(M)$.
- There exists a Ricci-flat metric g on M.

(As M is connected, we can say that $h^{0,0}(M) = 1$.) There are in fact still other properties equivalent to the above, for example

- The first Chern class $c_1(M)$ vanishes.
- The holonomy of M is contained in $\mathrm{SU}(m)$.

but I will not use them in this thesis.

Direct products of Calabi-Yau manifolds. Let M_1 and M_2 be Calabi-Yau manifolds. Then nowhere vanishing holomorphic top-forms Ω_1 and Ω_2 exist. Then $\Omega := \Omega_1 \wedge \Omega_2$ defines a nowhere vanishing holomorphic top-form on $M := M_1 \times M_2$. As a consequence we can say that the direct product of any pair of Calabi-Yau manifolds is again a Calabi-Yau manifold.

Some examples of Calabi-Yau manifolds. Any complex torus T^2 describes a Calabi-Yau 1-fold. It even describes the *only* possible Calabi-Yau of complex dimension one. We can write down direct products of the standard flat torus, with $\Omega(T^2) := dz$:

$$\begin{array}{ll} T^4 := T^2 \times T^2, & \Omega(T^4) := dz^1 \wedge dz^2, \\ T^6 := T^2 \times T^2 \times T^2, & \Omega(T^6) := dz^1 \wedge dz^2 \wedge dz^3. \end{array}$$

Then T^4 is a Calabi-Yau 2-fold and T^6 is a Calabi-Yau 3-fold. See (2.51), (2.52) and (2.53) for the Hodge diamonds of these three Calabi-Yau manifolds respectively.

A counterexample. We note that none of the complex projective spaces $\mathbb{P}^N$ is a Calabi-Yau manifold. For any $\mathbb{P}^N$, which is a compact Kähler manifold with $\dim_{\mathbb{C}}(\mathbb{P}^N) = N$, we have $h^{k,l}(\mathbb{P}^N) = \delta_{k,l}$, see (2.33). Thus we have $h^{N,0} = 0$. This means that no holomorphic top-forms exist at all on any $\mathbb{P}^N$. On the other hand, some of the hypersurfaces embedded in a $\mathbb{P}^N$ *are* Calabi-Yau manifolds.

Properties of a holomorphic top-form. A holomorphic top-form Ω existing on a Calabi-Yau manifold M satisfies the following properties:

1. Ω is closed.

2. Ω is harmonic, and as a consequence it defines a Dolbeault-class in $H^{m,0}(M)$.

3. Ω is covariantly constant with respect to the Ricci-flat metric.

We will prove these properties in the specific case we are dealing with a $(3,0)$-form Ω, existing on a Calabi-Yau 3-fold, as in the application in superstring theory we are mainly interested in 3-folds. These properties can be proven in a similar manner in other dimensions.

- As Ω is holomorphic, we see that automatically $\bar{\partial}\Omega = 0$, and as Ω is a top-form, we also have $\partial\Omega = 0$. This means that Ω is closed, as $d\Omega = \partial\Omega + \bar{\partial}\Omega = 0$. This proves the first property.
- The identity (2.48) and its consequences can directly be used to explain why Ω, being holomorphic, is also harmonic. This proves the second property.
- To prove the third property needs more work. However, a slightly similar (though less clear) proof can be found in [4]. First of all, we will expand Ω into components:

  $$\Omega = \frac{1}{3!}\Omega_{\alpha\beta\gamma} dz^\alpha \wedge dz^\beta \wedge dz^\gamma. \tag{2.54}$$

  Here the component functions $\Omega_{\alpha\beta\gamma} = \Omega_{\alpha\beta\gamma}(z^1, z^2, z^3)$ are purely holomorphic. As $\Omega_{\alpha\beta\gamma}$ is purely antisymmetric, we can write it as follows:

  $$\Omega_{\alpha\beta\gamma} = f\epsilon_{\alpha\beta\gamma} \quad , \quad \bar{\Omega}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \bar{f}\epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}}.$$

  Here f is some holomorphic function defined on the coordinate patch. Now, using (2.31), we can simply rewrite Ω as

  $$\Omega = \frac{1}{3!} f\epsilon_{\alpha\beta\gamma} dz^\alpha \wedge dz^\beta \wedge dz^\gamma = f dz^1 \wedge dz^2 \wedge dz^3. \tag{2.55}$$

Then $\Omega \wedge \bar{\Omega}$ is purely imaginary, thus up to an imaginary constant it is a real 6-form and using (2.20) we can write

$$\begin{aligned}\Omega \wedge \bar{\Omega} &= |f|^2 dz^1 \wedge dz^2 \wedge dz^3 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 \\ &\simeq \frac{|f|^2}{G} G d\kappa^1 \wedge d\kappa^2 \wedge d\kappa^3 \wedge d\kappa^4 \wedge d\kappa^5 \wedge d\kappa^6 = \frac{|f|^2}{G} * 1 = *\frac{|f|^2}{G} = *\|\Omega\|^2,\end{aligned} \tag{2.56}$$

where κ is just a real coordinate, and $*$ is related to the real part of the metric. The symbol $\simeq$ used here only means equality up to the imaginary constant. (Note that this identity is in harmony with the definition of the inner product of differential forms. See (2.46) and the identity $\bar{*}\Omega \simeq \bar{\Omega}$, up to the imaginary constant. Then we have $\langle \Omega, \Omega \rangle = \int_M *\|\Omega\|^2$.)

We should note that both $|f|^2$ and G transform as the component of a real 6-form under coordinate transformations, thus $\|\Omega\|^2 = |f|^2/G$ nicely represents a nowhere vanishing globally defined positive real scalar. It should be clear that $\|\Omega\|^2$ satisfies the following identity:

$$\begin{aligned}\|\Omega\|^2 &= \frac{|f|^2}{G} = \frac{1}{3!} f \epsilon_{\alpha\beta\gamma} \bar{f} \frac{1}{G} \epsilon_{\alpha\beta\gamma} = \frac{1}{3!} f \epsilon_{\alpha\beta\gamma} \bar{f} g^{\alpha\bar{\lambda}} g^{\beta\bar{\mu}} g^{\gamma\bar{\nu}} \epsilon_{\bar{\lambda}\bar{\mu}\bar{\nu}} \\ &= \frac{1}{3!} \Omega_{\alpha\beta\gamma} g^{\alpha\bar{\lambda}} g^{\beta\bar{\mu}} g^{\gamma\bar{\nu}} \bar{\Omega}_{\bar{\lambda}\bar{\mu}\bar{\nu}} = \frac{1}{3!} \Omega_{\alpha\beta\gamma} \bar{\Omega}^{\alpha\beta\gamma}.\end{aligned} \tag{2.57}$$

Now writing $G = |f|^2/\|\Omega\|^2$ we can explicitly compute the components $\mathcal{R}_{\alpha\bar{\beta}}$ of the Ricci form:

$$\begin{aligned}\mathcal{R}_{\alpha\bar{\beta}} &= \partial_\alpha \partial_{\bar{\beta}} \log G = \partial_\alpha \partial_{\bar{\beta}} \log \frac{|f|^2}{\|\Omega\|^2} \\ &= \partial_\alpha \partial_{\bar{\beta}} \log |f|^2 - \partial_\alpha \partial_{\bar{\beta}} \log \|\Omega\|^2 = -\partial_\alpha \partial_{\bar{\beta}} \log \|\Omega\|^2.\end{aligned}$$

The last equality comes from the fact that f is purely holomorphic. Then $\log f$ is purely holomorphic and $\log \bar{f}$ is purely anti-holomorphic, thus $\partial_\alpha \partial_{\bar{\beta}} \log |f|^2 = \partial_\alpha \partial_{\bar{\beta}} (\log f + \log \bar{f}) = 0$.

As $\|\Omega\|^2$ is a globally defined scalar, we see that the Ricci form, already being closed, is also (globally) exact. Then the first Chern class $c_1(M) = [\mathcal{R}/2\pi]$ must vanish, thus M admits a Ricci-flat metric. Now suppose the metric g is already Ricci-flat, then the corresponding Ricci form vanishes also, thus $\partial_\alpha \partial_{\bar{\beta}} \log \|\Omega\|^2 = 0$. This implies $\|\Omega\|^2$ itself can be written as a sum of a purely holomorphic and a purely anti-holomorphic (scalar) function, say $\log \|\Omega\|^2 = H + A$. Now, as M is compact, $H : M \to \mathbb{C}$ is holomorphic, and $A : M \to \mathbb{C}$ is anti-holomorphic, the maximum modulus principle tells us that these functions H and A must be constant. Then $\|\Omega\|^2 = e^H e^A$ should also be constant. In general it should not depend on the metric whether a function is constant, but $\|\Omega\|^2$ is constructed from a higher rank tensor field and the metric, thus it *depends* on the metric, and it is only constant if this metric is Ricci-flat. Thus, to conclude, if an Ω exists, M also admits a Ricci-flat metric g, and the *norm* $\|\Omega\|^2$ of Ω is a constant if it is evaluated with respect to this g.

Finally, when $\|\Omega\|^2$ is a constant, we can easily derive that Ω itself is covariantly constant with respect to the Ricci-flat metric g. We directly see that $\nabla_{\bar{\kappa}} \Omega_{\lambda\mu\nu} = 0$, as Ω is purely holomorphic, so to prove this we will only compute $\nabla_\kappa \Omega_{\lambda\mu\nu}$. As $\Omega_{\lambda\mu\nu}$ itself is totally antisymmetric, thus vanishes if λ, μ and ν are not all different, $\nabla_\kappa \Omega_{\lambda\mu\nu}$ will also vanish in that case. So, we only need to check it for λ, μ and ν all different. Thus

$$\begin{aligned}\nabla_\kappa \Omega_{\lambda\mu\nu} &= \partial_\kappa \Omega_{\lambda\mu\nu} - \Gamma^\rho{}_{\kappa\lambda} \Omega_{\rho\mu\nu} - \Gamma^\rho{}_{\kappa\mu} \Omega_{\lambda\rho\nu} - \Gamma^\rho{}_{\kappa\nu} \Omega_{\lambda\mu\rho} \\ &= \partial_\kappa f \epsilon_{\lambda\mu\nu} - f(\Gamma^\rho{}_{\kappa\lambda} \epsilon_{\rho\mu\nu} + \Gamma^\rho{}_{\kappa\mu} \epsilon_{\lambda\rho\nu} + \Gamma^\rho{}_{\kappa\nu} \epsilon_{\lambda\mu\rho}) \\ &= \partial_\kappa f \epsilon_{\lambda\mu\nu} - f(\Gamma^\lambda{}_{\kappa\lambda} \epsilon_{\lambda\mu\nu} + \Gamma^\mu{}_{\kappa\mu} \epsilon_{\lambda\mu\nu} + \Gamma^\nu{}_{\kappa\nu} \epsilon_{\lambda\mu\nu}) \\ &= \partial_\kappa f \epsilon_{\lambda\mu\nu} - f(\Gamma^\lambda{}_{\kappa\lambda} + \Gamma^\mu{}_{\kappa\mu} + \Gamma^\nu{}_{\kappa\nu}) \epsilon_{\lambda\mu\nu} = (\partial_\kappa f - f \Gamma^\rho{}_{\kappa\rho}) \epsilon_{\lambda\mu\nu}.\end{aligned}$$

Note that we did not do any summation over λ, μ and ν as they are no dummy indices. Now using (2.43) and knowing that $\|\Omega\|^2$ is a constant we can write

$$\begin{aligned}\Gamma^\rho{}_{\kappa\rho} &= \partial_\kappa \log G = \partial_\kappa \log \frac{|f|^2}{\|\Omega\|^2} = \partial_\kappa \log |f|^2 - \partial_\kappa \log \|\Omega\|^2 = \partial_\kappa \log |f|^2 \\ &= \partial_\kappa(\log f + \log \bar{f}) = \partial_\kappa \log f = \frac{1}{f}\partial_\kappa f.\end{aligned}$$

Thus

$$\nabla_\kappa \Omega_{\lambda\mu\nu} = (\partial_\kappa f - f\Gamma^\rho{}_{\kappa\rho})\epsilon_{\lambda\mu\nu} = (\partial_\kappa f - \partial_\kappa f)\epsilon_{\lambda\mu\nu} = 0.$$

We conclude that indeed Ω is covariantly constant with respect to the Ricci-flat metric. This proves the third property.

Now we should note that in [10] the authors wrote that since Ω is *covariantly constant*, its norm $\|\Omega\|^2$ is a constant. However, it is exactly the other way round. We can directly prove that $\|\Omega\|^2$ is a constant with respect to a Ricci-flat metric, so that Ω is covariantly constant. In [4] similar things are discussed, but from that article it is very hard to conclude that the author indeed means that it is the other way round, as it was not written that clear, so I assume it was easy for the authors of [10] to make this small mistake, if they were reading [4] for some studying purposes.

Thus, we conclude, $\|\Omega\|^2$ is a constant with respect to a Ricci-flat metric, so that Ω is covariantly constant. In fact we mainly need the properties of $\|\Omega\|^2$, instead of Ω being covariantly constant.

A relation between the Kähler form and the holomorphic top-form. Let M be a Calabi-Yau m-fold with a Ricci-flat metric g, a Kähler form ω and a holomorphic m-form Ω. Then Ω and ω are related as follows:

$$\Omega \wedge \bar{\Omega} \simeq *\|\Omega\|^2 \quad \Rightarrow \quad \Omega \wedge \bar{\Omega} = *c = c * 1 = \frac{c}{m!}\omega^m = \frac{c}{m!}\omega \wedge \cdots \wedge \omega. \tag{2.58}$$

The left equation comes from (2.56), the right equation comes from (2.41) and c is some (complex) constant. (We know that $\|\Omega\|^2$ is a constant.) A similar relation can also be found in, for example, [16].

Strict Calabi-Yau manifolds. We will call a Calabi-Yau manifold M a *strict* Calabi-Yau manifold if its holonomy exactly equals $\mathrm{SU}(m)$. (For example in [13] the notions of strict Calabi-Yau manifolds and the difference between strict and nonstrict Calabi-Yau manifolds are *not* mentioned, but we see that the Calabi-Yau manifolds mentioned there indeed turn out to be the strict ones only. Only the strict ones are interesting in the context of superstring theory.)

As briefly mentioned in [13] we can say that then the Hodge numbers of M satisfy the following restrictions.

- $h^{k,0} = h^{0,k} = 0$ for any $0 < k < m$.
- $h^{m,0} = h^{0,m} = 1$.

If M is simply connected, then its fundamental group is trivial. Then we also know that $h^{1,0}(M) + h^{0,1}(M) = b_1(M) = 0$, thus $h^{1,0}(M) = h^{0,1}(M) = 0$. This purely topological property is equivalent to the fact that the holonomy of M exactly equals $\mathrm{SU}(m)$, if $m \geq 2$. (Note that if $b_1(M) = 0$, then M itself is not necessarily simply connected.)

Then the symmetry rules, see (2.49), and strictness imply that a Calabi-Yau 2-fold only has one independent Hodge number, and that a Calabi-Yau 3-fold only has two.

In the context of compactified superstring theory we are only interested in strict Calabi-Yau 3-folds. Then the corresponding holonomy group is precisely $\mathrm{SU}(3)$, so that it admits precisely one covariantly constant spinor field η of positive chirality.

We know that $h^{m,0}(M) = 1$, which holds for any strict Calabi-Yau m-fold M, implies that *all* of the holomorphic top-forms are nowhere vanishing: a strict Calabi-Yau m-fold does *not* admit any holomorphic top-form with a non-empty zero set. Thus, we can say that a strict Calabi-Yau manifold *only* admits a nowhere vanishing holomorphic top-form.

Note: if M_1 and M_2 are strict Calabi-Yau manifolds, then $M := M_1 \times M_2$ is not necessarily a strict Calabi-Yau manifold.

Some examples of strict Calabi-Yau manifolds. Note that the complex torus T^2, being a Calabi-Yau 1-fold, is automatically strict. The holonomy group SU(1) is trivial: it only contains the identity.

Examples of strict Calabi-Yau 2-folds are *K3 surfaces*. They all have the following Hodge diamond:

$$\begin{pmatrix} h^{0,0}(K3) & h^{0,1}(K3) & h^{0,2}(K3) \\ h^{1,0}(K3) & h^{1,1}(K3) & h^{1,2}(K3) \\ h^{2,0}(K3) & h^{2,1}(K3) & h^{2,2}(K3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

In short, writing $h^{1,1}(K3) = 20$ is sufficient. These K3 surfaces turn out to be the only strict Calabi-Yau 2-folds existing. All of these surfaces are at least diffeomorphic, and many of these surfaces can be represented as a hypersurface embedded in $\mathbb{P}^3$.

Examples of strict Calabi-Yau 3-folds are *quintics*. (See the example already introduced in Section 2.4: hypersurfaces embedded in $\mathbb{P}^4$.) They all have the following Hodge diamond:

$$\begin{pmatrix} h^{0,0}(Q) & h^{0,1}(Q) & h^{0,2}(Q) & h^{0,3}(Q) \\ h^{1,0}(Q) & h^{1,1}(Q) & h^{1,2}(Q) & h^{1,3}(Q) \\ h^{2,0}(Q) & h^{2,1}(Q) & h^{2,2}(Q) & h^{2,3}(Q) \\ h^{3,0}(Q) & h^{3,1}(Q) & h^{3,2}(Q) & h^{3,3}(Q) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 101 & 0 \\ 0 & 101 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.59}$$

In short, writing $h^{1,1}(Q) = 1$ and $h^{2,1}(Q) = 101$ is sufficient.

Proving that any quintic or K3 surface is indeed a Calabi-Yau manifold is not that difficult. We can easily explicitly define nowhere vanishing holomorphic top-forms, and we will do so in later chapters. A proof that they are *strict* Calabi-Yau manifolds can be found in other literature. (For example, in [11] we can read that the "Lefschetz hyperplane theorem" shows that the fundamental group of any K3 surface X is trivial, thus X is simply connected, so that it is a strict Calabi-Yau 2-fold.)

Strict Calabi-Yau 3-folds and their moduli spaces. Let M be a strict Calabi-Yau 3-fold and let $\mathcal{M}(M)$ be the *collection*, a continuous family, of all Calabi-Yau 3-folds being diffeomorphic to M. Then $\mathcal{M}_C(M)$ is the collection of possible complex structures, each of which being related to an $M' \in \mathcal{M}(M)$, and $\mathcal{M}_K(M)$ is the collection of possible Kähler classes, each of which being related to an $M' \in \mathcal{M}(M)$. Then $\mathcal{M}_C(M)$ and $\mathcal{M}_K(M)$ together give us $\mathcal{M}(M)$:

$$\mathcal{M}(M) \simeq \mathcal{M}_C(M) \times \mathcal{M}_K(M).$$

Thus, $\mathcal{M}_C(M)$ is the *complex structure moduli space* related to M, and $\mathcal{M}_K(M)$ is the *Kähler moduli space* related to M. Strict Calabi-Yau 3-folds satisfy the following relation:

$$h^{2,1}(M) = \dim_{\mathbb{C}}(\mathcal{M}_C(M)) \quad , \quad h^{1,1}(M) = \dim_{\mathbb{R}}(\mathcal{M}_K(M)).$$

This yields $\dim_{\mathbb{R}}(\mathcal{M}(M)) = 2h^{2,1}(M) + h^{1,1}(M)$. (More explanations can, for example, be found in [13].)

2.8 Some remaining notions

Involutions. Let M be a smooth manifold. A smooth map $\iota : M \to M$ is called an *involution* if $\iota^2 = \mathrm{Id}_M$, or $\iota^2(p) = p$ for all $p \in M$. This means that ι should be a diffeomorphism. Its inverse map ι^{-1} equals ι.

Holomorphic involutions. Let M be a complex manifold. A map $\iota : M \to M$ is called a *holomorphic involution* if it is an involution and a biholomorphism at the same time.

Anti-holomorphic involutions. Let M be a complex manifold. An involution $\iota : M \to M$ is called an *anti-holomorphic involution* if it is also an anti-holomorphic map.

Isometries. Let M be a smooth manifold and let g be an arbitrary metric defined on M. A diffeomorphism $f : M \to M$ is called an *isometry* (with respect to g) if for all $p \in M$ it preserves the metric as follows:

$$f^* g_{f(p)} = g_p. \tag{2.60}$$

As an equivalent definition, derived from (2.8), we can say that f is an isometry if for any $p \in M$ and for any pair of vectors V and W in T_pM, the following holds: $g_{f(p)}(f_*V, f_*W) = g_p(V, W)$.

Let now κ be a coordinate at p and let λ be a coordinate at $f(p)$. We know a real metric, at p, is written as $g_p = (g_p)_{\mu\nu} d\kappa^\mu \otimes d\kappa^\nu$. Then $f^* g_{f(p)}$ can be expanded in components:

$$\begin{aligned} f^* g_{f(p)} &= f^*(g_{f(p)}) = f^*((g_{f(p)})_{\alpha\beta} d\lambda^\alpha \otimes d\lambda^\beta) = (g_{f(p)})_{\alpha\beta} f^*(d\lambda^\alpha) \otimes f^*(d\lambda^\beta) \\ &= (g_{f(p)})_{\alpha\beta} \frac{\partial(\lambda \circ f \circ \kappa^{-1})^\alpha}{\partial \kappa^\mu} \frac{\partial(\lambda \circ f \circ \kappa^{-1})^\beta}{\partial \kappa^\nu} d\kappa^\mu \otimes d\kappa^\nu = (g_{f(p)})_{\alpha\beta} \frac{\partial \lambda^\alpha}{\partial \kappa^\mu} \frac{\partial \lambda^\beta}{\partial \kappa^\nu} d\kappa^\mu \otimes d\kappa^\nu. \end{aligned}$$

Then the condition (2.60) can be rewritten in components:

$$\frac{\partial \lambda^\alpha}{\partial \kappa^\mu} \frac{\partial \lambda^\beta}{\partial \kappa^\nu} (g_{f(p)})_{\alpha\beta} = (g_p)_{\mu\nu}.$$

Holomorphic isometries. Let now (M, g) be a Hermitian manifold and let $f : M \to M$ be a biholomorphism. Then f is called a *holomorphic isometry* if for all $p \in M$ it satisfies (2.60). Let now z be a complex coordinate at p and let w be a complex coordinate at $f(p)$. We know a Hermitian metric, at p, is written as $g_p = (g_p)_{\mu\bar{\nu}}(dz^\mu \otimes d\bar{z}^\nu + d\bar{z}^\nu \otimes dz^\mu)$. Knowing that f is holomorphic, we can write

$$\frac{\partial w^\alpha}{\partial z^\mu} dz^\mu = \frac{\partial(w \circ f \circ z^{-1})^\alpha}{\partial z^\mu} dz^\mu \quad , \quad \frac{\partial w^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu = \frac{\partial(w \circ f \circ \bar{z}^{-1})^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu = 0,$$

so that

$$f^*(dw^\alpha) = \frac{\partial w^\alpha}{\partial z^\mu} dz^\mu + \frac{\partial w^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu = \frac{\partial w^\alpha}{\partial z^\mu} dz^\mu \quad , \quad f^*(d\bar{w}^\alpha) = \frac{\partial \bar{w}^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu.$$

Then

$$\begin{aligned} f^* g_{f(p)} &= f^*((g_{f(p)})_{\alpha\bar{\beta}}(dw^\alpha \otimes d\bar{w}^\beta + d\bar{w}^\beta \otimes dw^\alpha)) \\ &= (g_{f(p)})_{\alpha\bar{\beta}}(f^*(dw^\alpha) \otimes f^*(d\bar{w}^\beta) + f^*(d\bar{w}^\beta) \otimes f^*(dw^\alpha)) \\ &= (g_{f(p)})_{\alpha\bar{\beta}}(\frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu} dz^\mu \otimes d\bar{z}^\nu + \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu} \frac{\partial w^\alpha}{\partial z^\mu} d\bar{z}^\nu \otimes dz^\mu) \\ &= (g_{f(p)})_{\alpha\bar{\beta}} \frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu} (dz^\mu \otimes d\bar{z}^\nu + d\bar{z}^\nu \otimes dz^\mu). \end{aligned}$$

Then the condition (2.60), applied to a biholomorphism, can be rewritten in components:

$$\frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu} (g_{f(p)})_{\alpha\bar{\beta}} = (g_p)_{\mu\bar{\nu}}.$$

Anti-holomorphic isometries. Let now (M, g) again be a Hermitian manifold and let $f : M \to M$ be an anti-holomorphic diffeomorphism. Then f is called an *anti-holomorphic isometry* if for all $p \in M$ it satisfies (2.60). Let now z be a complex coordinate at p and let w be a complex coordinate at $f(p)$. Knowing that f is anti-holomorphic, we can write

$$\frac{\partial w^\alpha}{\partial z^\mu} dz^\mu = \frac{\partial (w \circ f \circ z^{-1})^\alpha}{\partial z^\mu} dz^\mu = 0 \quad , \quad \frac{\partial w^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu = \frac{\partial (w \circ f \circ \bar{z}^{-1})^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu,$$

so that

$$f^*(dw^\alpha) = \frac{\partial w^\alpha}{\partial z^\mu} dz^\mu + \frac{\partial w^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu = \frac{\partial w^\alpha}{\partial \bar{z}^\mu} d\bar{z}^\mu \quad , \quad f^*(d\bar{w}^\alpha) = \frac{\partial \bar{w}^\alpha}{\partial z^\mu} dz^\mu.$$

Then

$$\begin{aligned} f^* g_{f(p)} &= (g_{f(p)})_{\alpha\bar{\beta}} \left(\frac{\partial w^\alpha}{\partial \bar{z}^\mu} \frac{\partial \bar{w}^\beta}{\partial z^\nu} d\bar{z}^\mu \otimes dz^\nu + \frac{\partial \bar{w}^\beta}{\partial z^\nu} \frac{\partial w^\alpha}{\partial \bar{z}^\mu} dz^\nu \otimes d\bar{z}^\mu \right) \\ &= (g_{f(p)})_{\alpha\bar{\beta}} \frac{\partial w^\alpha}{\partial \bar{z}^\mu} \frac{\partial \bar{w}^\beta}{\partial z^\nu} (d\bar{z}^\mu \otimes dz^\nu + dz^\nu \otimes d\bar{z}^\mu). \end{aligned}$$

Then the condition (2.60), applied to an anti-holomorphic diffeomorphism, can be rewritten in components:

$$\frac{\partial w^\alpha}{\partial \bar{z}^\nu} \frac{\partial \bar{w}^\beta}{\partial z^\mu} (g_{f(p)})_{\alpha\bar{\beta}} = (g_p)_{\mu\bar{\nu}}. \tag{2.61}$$

3 Type IIA Superstring Theory

In this chapter we will mainly introduce the physics. In Section 3.1 we will start with the basic theory of the classical string, a two-dimensional object embedded in a high-dimensional space-time. It has two coordinates: one space-like coordinate and one time-like coordinate. A string can be regarded as a one-dimensional object of finite size, travelling in time. We can quantize the basic model of a string and add fermionic degrees of freedom. Then we will also give a short explanation of supersymmetry. The quantized theory has more constraints than the classical one, but precisely these constraints are also supposed to explain certain natural phenomena. We will look at a supersymmetric model, defined on 10-dimensional space-time. We will start with the flat Minkowski space-time, so it supports special relativity.

In Section 3.2 we will introduce a background field related to a curved space. Then we will look at massless type IIA superstring theory, or type IIA *supergravity*. We will mainly look at the bosonic part of this theory. Then we will also mention the effect of compactifying the curved space-time, and we will use a Calabi-Yau manifold for this. We can use the topological properties of this Calabi-Yau 3-fold for finding out more about the physical degrees of freedom of the effective theory in 4 dimensions.

3.1 Bosonic and fermionic theory in flat Minkowski space-time

Some basics of string theory: Open and closed strings. A *classical* string is described by a map $X^\mu = X^\mu(\tau, \sigma)$, defined on the so-called *world-sheet*: a string is a one-dimensional object travelling in time, so that it traces a surface. The classical string theory can be described in any arbitrary space-time dimension D, so that $X^\mu(\tau, \sigma) \in \mathbb{R}^D$.

The parameters τ and σ can be interpreted as the world-sheet coordinates. (In general these parameters are *only* correctly defined locally, on contractible patches of the string surface.) Here τ is a free coordinate, while, in case of the *open string*, σ will be restricted to a closed interval $[0, \pi]$, thus $(\tau, \sigma) \in \mathbb{R} \times [0, \pi]$. The *closed string* satisfies the boundary condition $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi)$ (for all σ). Then the endpoints of the interval $[0, \pi]$ will 'meet', so that the collection of points (τ, σ) is diffeomorphic to $\mathbb{R} \times S^1$.

Some basics of string theory: Quantizing. Bosonic string theory can only be properly quantized when the space-time dimension D equals 26. It is not that easy to make such a theory correspond to the physical reality of dimension 4 we are familiar with. Things seem to get easier when fermions are included and when we require the theory to be supersymmetric, as the critical dimension then reduces to $D = 10$. Then there are only 6 dimensions too many. However, to construct such a supersymmetric theory we need to do a number of modifications to the bosonic theory which are not that straightforward. But we will get there after taking some intermediate steps.

Bosonic theory. The bosonic theory starts with the *Polyakov action* on a flat space-time background with Minkowski metric $\eta_{\mu\nu}$:

$$S_{\text{Pol}} = -\frac{T}{2} \int_\Sigma d\tau d\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}.$$

Here Σ is the world-sheet, which is canonically embedded into the target space $\mathcal{T} = \mathbb{R}^{26}$, T is the string tension, $h_{\alpha\beta}$ is the world-sheet metric and $h = -\det(h_{\alpha\beta}) > 0$.

If we are only looking at closed string theory, then we can assume that Σ can be represented by some compact Riemann surface. Then Σ is orientable, so that the Hodge star operator $*$ (as introduced in (2.18)) is well-defined. Then this action can be written in a coordinate-free form. Let $\iota : \Sigma \to \mathcal{T}$ be the embedding. Then we can write

$$S_{\text{Pol}} = -\frac{T}{2} \int_\Sigma *(\iota^* \eta(h^{-1})).$$

Writing the Polyakov action this way, we see that *by definition* it is invariant under reparametrizations, also simply known as coordinate transformations. Thus, to check whether S_{Pol} is reparametrization invariant becomes redundant.

Including fermionic theory. The next step will be to include fermions: we will add a field ψ^μ. This requires a vielbein, or in the two-dimensional case a *zweibein*, $e_a{}^\alpha$, which satisfies $h_{\alpha\beta}e_a{}^\alpha e_b{}^\beta = \eta_{ab}$. (This zweibein is assumed to be covariantly constant.) Using the inverse of this zweibein, $e^a{}_\alpha$, we can write $h_{\alpha\beta} = e^a{}_\alpha \eta_{ab} e^b{}_\beta$, so that we obtain $\det(h_{\alpha\beta}) = -\det^2(e^a{}_\alpha)$, which implies that $\sqrt{-\det(h_{\alpha\beta})} = \det(e^a{}_\alpha)$. The zweibein ensures that the added fermion field ψ^μ transforms correctly as a Majorana spinor in two dimensions. Let now $e := \det(e^a{}_\alpha)$, then the following step is writing an action which includes fermionic degrees of freedom and is supersymmetric:

$$S = -\frac{1}{8\pi}\int_\Sigma d\tau d\sigma e(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu(\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta \psi_\mu)). \tag{3.1}$$

About the added auxiliary fields: here the $\rho^\alpha = e_a{}^\alpha \rho^a$ and the ρ^a are the curved and the flat Dirac matrices in two dimensions, and χ_α is the *gravitino.*

The next step is to divide out by the five local symmetries of this action, which are *local supersymmetry* (we will soon explain what supersymmetry is), *Weyl invariance*, *Super Weyl invariance*, *Local Lorentz symmetry* and *Reparametrizations*. Then we can fix a gauge, called the *superconformal gauge*, in the following way. Reparametrization can be used to find a coordinate with respect to which the metric has the form $h_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$. Because the metric has the gauge freedom of Weyl invariance, we can get rid of the factor $e^{2\phi}$. However, this is only possible locally on some chart of the world-sheet. Then we must solve $h_{\alpha\beta} = \eta_{\alpha\beta} = e^a{}_\alpha \eta_{ab} e^b{}_\beta$ for the zweibein, or using matrix notation: $\eta = e^T\eta e$. This is exactly the requirement that the zweibein is an element of the local Lorentz group $\mathrm{SO}(1,1)$, which is a Lie group of dimension one. So in general the zweibein has four degrees of freedom. After fixing $\eta = e^T\eta e$ still one degree of freedom is left, and we are still free to choose this one. The simplest choice is $e_a{}^\alpha = \delta_a^\alpha$, so that $e = 1$. Another gauge choice is one which makes the gravitino vanish, and we will use this gauge choice.

This gauge choice results in a very simple action, and (3.1) can be rewritten as

$$S = -\frac{1}{8\pi}\int_\Sigma d\tau d\sigma(\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu).$$

This gauge choice is called the *superconformal gauge*, which is freely possible in classical theory. In quantum theory it is possible in critical dimension $D = 10$ only.

Supersymmetry. In theoretical physics there are models describing only bosons, or only fermions. There are also models describing both bosons and fermions. Such a model is called *supersymmetric* if a continuous transformation of the fields is possible, so that we can exchange the fields related to the bosons with the fields related to the fermions, and so that the model itself is invariant. In short, this means that a symmetry exists between the bosons and the fermions. (We could represent it with a continuous rotation of a couple of bosonic and fermionic fields.)

In general a supersymmetry transformation is represented in infinitesimal form. If X represents a boson, if ψ represents a fermion, and if X and ψ together can be transformed into each other, then we write the supersymmetry transformation as follows:

$$(X,\psi) \mapsto (X',\psi') = (X + \delta_\varepsilon X, \psi + \delta_\varepsilon \psi), \quad \delta_\varepsilon X = S(X,\psi,\varepsilon), \quad \delta_\varepsilon \psi = T(X,\psi,\varepsilon). \tag{3.2}$$

Here ε is an infinitesimal Grassmann parameter.

In general we can have a $\mathcal{N} = 1$, $\mathcal{N} = 2$, $\mathcal{N} = 4$ or $\mathcal{N} = 8$ supersymmetry. The value of $\mathcal{N}$ mainly expresses the number of gravitinos which occur in the model. In string theory there are slightly different types of supersymmetry transformations. We have so-called *world-sheet* supersymmetry and so-called *space-time* supersymmetry. (These notions are explained in much detail in [5] and [20].) Later, for example in Section 4.2, we will mainly mention $\mathcal{N} = 2$ space-time supersymmetry, thus, in this case we assume there are two gravitinos.

The Ramond sector and the Neveu-Schwarz sector. We distinguish different kinds of string modes. There are certain boundary conditions for the components ψ_+^μ and ψ_-^μ of the spinor ψ^μ, the fermionic degrees of freedom. In case of the open string model, we have the *Neveu-Schwarz sector* (or NS sector) and the *Ramond sector* (or R sector). In the NS sector ψ^μ are anti-periodic:

$$\psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0) \quad , \quad \psi_+^\mu(\tau,\pi) = -\psi_-^\mu(\tau,\pi).$$

In the R sector ψ^μ are periodic:

$$\psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0) \quad , \quad \psi_+^\mu(\tau,\pi) = \psi_-^\mu(\tau,\pi).$$

In case of the closed string model, we have two independent degrees of freedom. Then we have the *Neveu-Schwarz-Neveu-Schwarz sector* (or NS-NS sector), the *Neveu-Schwarz-Ramond sector* (or NS-R sector), the *Ramond-Neveu-Schwarz sector* (or R-NS sector) and the *Ramond-Ramond sector* (or R-R sector). Then we have these conditions for the left moving and right moving modes separately.

The GSO projection. One of the solutions of the equations of motion of the (quantized) string describes the *tachyon*, a mode carrying an imaginary mass: $M^2 < 0$. In reality we will not occur such a particle, so we need to slightly change the theory so that the tachyon can be ignored. The model of superstrings can be changed so that half of the mathematical solutions of the string equations can be projected out, by applying the so-called GSO projection (named after Gliozzi, Scherk and Olive): only the physical modes will remain. There are multiple reasons to apply the GSO projection to the model. For example, the GSO projection also restores the modular invariance of the model, and, after performing the GSO projection, the model will be space-time supersymmetric.

At least the tachyon mode will be projected out, including other modes. However, in case of the superstring model this will not give any inconsistencies.

The different types of superstring theory. After applying the GSO projection, we can say that there are 5 remaining basic superstring theories, each of which having its own constraints: *type I superstring theory*, *type IIA superstring theory*, *type IIB superstring theory*, $SO(32)$ *heterotic string theory* and $E_8 \times E_8$ *heterotic string theory*. For example, type I theory describes both open and closed strings. Type IIA and IIB only describe closed strings. As we are dealing with independent left moving and right moving modes in the closed string model, we can say that the GSO projection is performed for each of these two modes separately. We can say that there are two different choices of these combined GSO projections, and that is why there are two different types of models, i.e. type IIA and IIB. If we choose opposite projections, then we are dealing with type IIA theory, and if we choose the same projections, then we are dealing with type IIB theory.

Both type IIA and IIB theory are $\mathcal{N} = 2$ supersymmetric models. The NS-NS sector and the R-R sector contain bosonic degrees of freedom (space-time bosons), and the NS-R sector and the R-NS sector contain fermionic degrees of freedom. From now on we will mainly discuss type IIA theory, and we will mainly study the bosonic part of the theory, thus we will ignore the NS-R sector and the R-NS sector. (To be even more precise: we will only explicitly study the R-R sector.)

Type IIA superstring theory is the theory with two conserved supercharges of opposite chirality, and this theory is called *nonchiral* because it is left-right symmetric. In the next sections we will discuss what to do with the 6 superfluous dimensions. In fact there are many different solutions for getting rid of them, but we will only discuss the method of compactification, using Calabi-Yau manifolds.

Massless modes of type IIA superstring theory. The only massless bosonic modes in the NS-NS sector of type IIA superstring theory, are related to the fundamental representation of the symmetry group $\mathrm{SO}(8) \times \mathrm{SO}(8)$, which is the 2-tensor representation of $\mathrm{SO}(8)$, the 'little group' related to $\mathrm{SO}(1,9)$. We can split this representation into three parts:

- A scalar field of dimension $\mathbf{1}$, also called the *dilaton*.

- An antisymmetric 2-tensor field of dimension **28**, also called the *axion*.
- A traceless symmetric 2-tensor field of dimension **35**, also called the *graviton*.

The only massless bosonic modes in the R-R sector of type IIA superstring theory, are related to a product of a left chiral spinor and a right chiral spinor, or $\mathbf{8_L} \times \mathbf{8_R}$. (This is a tensor product of a pair of Majorana-Weyl spinors of opposite chirality.) We can split this representation into two parts:

- A vector state of dimension **8**, also written as $\langle \psi_L | \Gamma^j | \psi_R \rangle$.
- An antisymmetric 3-tensor field of dimension **56**, also written as $\langle \psi_L | \Gamma^j \Gamma^k \Gamma^l | \psi_R \rangle$.

This is an even product of fermionic degrees of freedom, so the total product indeed represents bosonic degrees of freedom.

3.2 Superstrings in Background Fields

There are different ways for dealing with the 6 extra dimensions. One of the methods is introducing a curved compact space K, assumed to be very small, so that $\mathcal{T} := M := M_4 \times K$, is the background of interest, where $M_4 = \mathbb{R}^4$. (To be more precise: to any point $p \in M_4$ a space K_p can be added. As long as all of these K_p share the same topology, and if K_p varies smoothly, it will describe a consistent theory. In this thesis we will restrict ourselves to the simplified toy model of an K_p being constant.)

In reality a Calabi-Yau 3-fold is not found yet. We assume it is much smaller than subatomic scale. It is, unforunately, far too small to detect one using the present day measuring devices.

This space K will be a strict Calabi-Yau manifold, with $\dim_{\mathbb{C}}(K) = 3$. Thus it is a simply connected compact complex manifold, also being a Kähler manifold and which also admits a Ricci-flat metric, so that a nowhere vanishing holomorphic 3-form exists. However, until now we only introduced type IIA superstring theory on a flat space-time. Now we can discuss the effects of introducing a curved background space-time. In this section we will mainly follow the book [5], and later on also the books [17] and [20].

On M_4 we define the trivial flat Minkowski metric $g_4 = \eta$, and on K we define the metric g_K. These metrics induce a new metric $g = g_4 + g_K$, defined on M. (When looking at the components of these metrics, with respect to the direct product coordinates, we will see that g is a block-diagonal matrix, where the two blocks contain the components of g_4 and g_K.)

Recall that type IIA superstring theory has $\mathcal{N} = 2$ supersymmetry. A Calabi-Yau 3-fold also supports $\mathcal{N} = 2$ supersymmetry as well, at least if it is a *strict* one. A nonstrict Calabi-Yau 3-fold does *not* support $\mathcal{N} = 2$ supersymmetry.

The effects of a background space-time metric in the bosonic case $D = 26$. Until now we only mentioned the flat Minkowski metric $\eta_{\mu\nu}$ in the presented models. If we use $\eta_{\mu\nu}$ in the model, then the model is Weyl invariant. If we replace $\eta_{\mu\nu}$ (written in the bosonic action) by an arbitrary metric $g_{\mu\nu}$ close to $\eta_{\mu\nu}$, defined on the background space-time, then we are not sure yet if the model is still Weyl invariant. (We want the model to satisfy Weyl invariance, so that the model is renormalizable. Not in every case the Weyl invariance will hold, thus we need to impose a constraint on the metric.)

If we now write $g_{\mu\nu} = g_{\mu\nu}(X^\rho) = \eta_{\mu\nu} + f_{\mu\nu}(X^\rho)$, then $f_{\mu\nu}(X^\rho)$ is a wave function representing a *graviton*. If we derive a world-sheet path integral from the altered bosonic action, then we can write down a perturbation of this action, mainly expressed in terms of f. This model then describes the interaction of strings with an external graviton.

Now we can add other, more general fields to this action, in case of oriented bosonic strings. We already have

$$S_1 = -\frac{1}{4\pi\alpha'} \int_\Sigma d\tau d\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X^\rho),$$

where α' is the Regge slope, directly related to the string tension. Then we can add

$$S_2 = -\frac{1}{4\pi\alpha'} \int_\Sigma d\tau d\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X^\rho).$$

Here we used $\epsilon^{12} = 1$, and $B_{\mu\nu}$ is the background antisymmetric tensor field. Finally, we can add

$$S_3 = \frac{1}{4\pi} \int_\Sigma d\tau d\sigma \sqrt{h} \Phi(X^\rho) R^{(2)}.$$

Here $R^{(2)}$ is the world-sheet Ricci scalar, related to $h_{\alpha\beta}$, and Φ is the background value of the dilaton. The total action $S = S_1 + S_2 + S_3$ describes a *nonlinear sigma model*. Note that if we choose Φ a constant, then S_3 represents a purely topological term. Also the terms S_2 and S_3 are invariant under reparametrizations of the string world-sheet Σ, so this S is the most general action (for X^μ) that is invariant under reparametrizations of Σ. We still need to introduce constraints to make this model consistent.

The condition for Weyl invariance. If we perform the lowest non-trivial approximation in α', then the condition for Weyl invariance to hold for the action S is that the fields $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ satisfy the following constraints:

$$\begin{aligned} R_{\mu\nu} + \frac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} - 2 D_\mu D_\nu \Phi &= 0, \\ D_\rho H^\rho{}_{\mu\nu} - 2 H^\rho{}_{\mu\nu} D_\rho \Phi &= 0, \\ 4 D_\mu \Phi D^\mu \Phi - 4 D_\mu D^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} &= 0. \end{aligned} \tag{3.3}$$

Here $H_{\mu\nu\rho} := \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$, thus H, being a 3-form, expresses the exterior derivative of B, being a 2-form: $H = dB$. In addition, R is the Ricci scalar related to the space-time metric $g_{\mu\nu}$.

These constraints (3.3) must have a physical interpretation. They are the Euler-Lagrange equations corresponding to the following action, living in the 26-dimensional target space:

$$S_{26} = -\frac{1}{2\kappa^2} \int d^{26}x \sqrt{g} e^{-2\Phi} (R - 4 D_\mu \Phi D^\mu \Phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}).$$

(Here κ is Einstein's gravitational constant.) This action is called a *gravitational action*, and it describes the long-wavelength limit of the model of interactions of the massless modes of the (closed) bosonic string.

Massless theory and supergravity. We can do a similar thing with the model of type IIA superstrings in 10 dimensions. The *bosonic* part of the low-energy action can be written as

$$S_{IIA} := S_{IIA,1} + S_{IIA,2} + S_{IIA,3}, \tag{3.4}$$

where

$$\begin{aligned} S_{IIA,1} &:= \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{g} e^{-2\Phi} (R + 4 D_\mu \Phi D^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}), \\ S_{IIA,2} &:= -\frac{1}{8\kappa_0^2} \int (G^{(2)} \wedge *G^{(2)} + \frac{1}{12} G^{(4)} \wedge *G^{(4)}), \\ S_{IIA,3} &:= -\frac{1}{4\kappa_0^2} \int B \wedge dC^{(3)} \wedge dC^{(3)}. \end{aligned}$$

Here $g_{\mu\nu}$ is (again) the metric, B is the NS-NS 2-form, $H = dB$ is its field strength and Φ is the dilaton. We also have the R-R field strengths $G^{(2)} := dC^{(1)}$ and $G^{(4)} := dC^{(3)} + H \wedge C^{(1)}$, where $C^{(1)}$ and $C^{(3)}$ are the R-R potentials. (Here we follow the notation as in [17]. Note that the sign convention is slightly different here.)

Zero modes of the Laplace operator: Massless fields in ten dimensions. Now we can work out some relations between some topological properties of the compactified space K, harmonic forms and the effective model in four dimensions. First we will start with a simple model, in ten dimensions, and work out the desired properties.

Let M be an arbitrary smooth manifold with a metric g. The Laplace operator Δ, as defined in (2.24), has zero modes: a k-form $\omega \in \Omega^k(M)$ is called a *zero mode* of Δ if it satisfies $\Delta\omega = 0$, thus if it is a harmonic k-form. If g is a Riemannian metric, then any zero mode ω also satisfies (2.26):

$$\Delta\omega = 0 \quad \Leftrightarrow \quad d\omega = d^\star\omega = 0. \tag{3.5}$$

This does *not* hold (in general) if g is a Lorentzian metric.

Let now B be a k-form representing a physical theory defined on $M = M_4 \times K$, and let $C := dB$ be a $(k+1)$-form. Then B represents a potential and C its field strength. Then we can define an analog of the Maxwell action:

$$S_0 := \langle C, C\rangle = \int_M C \wedge *C = \frac{1}{(k+1)!}\int_M d\kappa^1 \wedge \cdots \wedge d\kappa^{10}\sqrt{g}g^{\mu_1\nu_1}\cdots g^{\mu_{k+1}\nu_{k+1}}C_{\mu_1\cdots\mu_{k+1}}C_{\nu_1\cdots\nu_{k+1}}.$$

We can use the variational principle to minimize the action S_0. We solve $\delta S_0 = 0$ as follows:

$$d^\star dB = 0. \tag{3.6}$$

A gauge choice is still possible: we can write

$$d^\star B = 0. \tag{3.7}$$

If we combine (3.6) and (3.7), then we know that B also satisfies $\Delta B = 0$, where Δ is the Laplace operator defined on M.

Zero modes of the Laplace operator: Massless fields in four dimensions. Now we can determine the behaviour of B on the manifolds M_4 and K separately. Assume B can be written as $B = \alpha \wedge \beta$, where α is an l-form defined on M_4 and β is an m-form defined on K, so that $0 \le l \le 4$, $0 \le m \le 6$ and $l + m = k$. Assume β already satisfies $\Delta_K\beta = 0$, so that (3.5) implies that also $d_K\beta = d_K^\star\beta = 0$.

We know that a K is a Riemannian manifold, and that M_4 and M are Lorentzian manifolds. Then we can apply (2.25) to B:

$$0 = \Delta B = (\Delta_4 + \Delta_K)(\alpha \wedge \beta) = \Delta_4\alpha \wedge \beta + \alpha \wedge \Delta_K\beta = \Delta_4\alpha \wedge \beta.$$

This in turn implies that $\Delta_4\alpha = 0$. Now note that Δ_4 is a Laplace operator defined on a Lorentzian manifold, so that we cannot simply conclude that $d_4\alpha = d_4^\star\alpha = 0$. We will do some alternative steps. If we now apply (2.23), combined with (3.7) and $d_K^\star\beta = 0$, then we can conclude that

$$0 = d^\star B = d_4^\star\alpha \wedge \beta + (-1)^l\alpha \wedge d_K^\star\beta = d_4^\star\alpha \wedge \beta \quad \Rightarrow \quad d_4^\star\alpha = 0.$$

This, combined with $\Delta_4\alpha = 0$, implies that $d_4^\star d_4\alpha = 0$:

$$d_4^\star\alpha = \Delta_4\alpha = 0 \quad \Rightarrow \quad d_4^\star d_4\alpha = 0.$$

This means that α describes a massless field in four dimensions. Let $c := d\alpha$. Then the action S_0 can be reduced to an effective action in four dimensions (up to some constant):

$$S^{(4)} \simeq \int_{M_4} d\kappa^1 \wedge \cdots \wedge d\kappa^4\sqrt{g_{(4)}}g_{(4)}^{\mu_1\nu_1}\cdots g_{(4)}^{\mu_{l+1}\nu_{l+1}}c_{\mu_1\cdots\mu_{l+1}}c_{\nu_1\cdots\nu_{l+1}}. \tag{3.8}$$

The number of linearly independent massless l-forms, arising in four dimensions and related to $\Omega^k(M)$, is the Betti number $b_{k-l}(K) = b_m(K)$.

Note that linear combinations of solutions of type $\alpha \wedge \beta$ are also solutions:

$$B = \sum_j \alpha_j \wedge \beta_j.$$

Low-energy effective action. We can do a similar thing when we replace the simple action S_0 by the action S_{IIA}, introduced in (3.4). We can use the variational principle to minimize the action S_{IIA} with respect to the NS-NS fluctuations of the fields g, B and Φ, and the R-R fluctuations of the fields $C^{(1)}$ and $C^{(3)}$, see [15]. Then similarly it is possible to split up $M = M_4 \times K$ and construct a low-energy effective action $S_{IIA}^{(4)}$ in four dimensions ($D = 4$), just like we constructed (3.8), using harmonic forms. (We will not explicitly write down this $S_{IIA}^{(4)}$.)

Supermultiplets, vector multiplets and hypermultiplets. The degrees of freedom of the massless fields corresponding to the effective action in four dimensions, are called *supermultiplets*. In general, a supermultiplet is a group representation of a supersymmetry algebra, defined in the supersymmetric theory we are studying. Then it is represented by a collection of particles, which is structured as a collection of 'superpartners'.

The supermultiplets we will mention in the effective type IIA theory, thus in the low-energy effective action in $D = 4$ and in $\mathcal{N} = 2$ supersymmetry, contain *vector multiplets* and *hypermultiplets*. See also [20]: "*The metric perturbations and other scalar zero modes lead to moduli fields that belong to* $\mathcal{N} = 2$ *supermultiplets*". A vector multiplet and a hypermultiplet are the only massless supermultiplets that contain scalar fields, in this case of $D = 4$, $\mathcal{N} = 2$. In both cases there are 4 bosonic and 4 fermionic degrees of freedom. In case of the vector multiplet we are dealing with a supermultiplet with maximal helicity 1. This multiplet contains one vector, two gauginos and two scalars. In case of the hypermultiplet we are dealing with a supermultiplet with maximal helicity 1/2. This multiplet contains two spin 1/2 fields and four scalars.

Especially the numbers of linearly independent vector multiplets and hypermultiplets are determined by the Hodge numbers of the Calabi-Yau manifold K. Each harmonic $(1, 1)$-form, defined on K, can be represented by a vector and 2 scalars, which coincide with the *bosonic part* of a vector multiplet. Each harmonic $(2, 1)$-form, defined on K, can be represented by 4 scalars, which coincide with the *bosonic part* of a hypermultiplet. According to [15], in case of the bosonic part of the effective type IIA theory in $D = 4$, there are

- $h^{1,1}$ vector multiplets,
- $h^{2,1} + 1$ hypermultiplets.

Now $\mathcal{M}^{1,1}(K)$ is the moduli space of vector multiplets, and $\mathcal{M}^{2,1}(K)$ is the moduli space of hypermultiplets. These moduli spaces are parametrized by the scalar fields contained in these multiplets. Each vector multiplet contains 2 real scalar fields, and each hypermultiplet contains 4 real scalar fields. Thus, $\dim_{\mathbb{R}}(\mathcal{M}^{1,1}(K)) = 2h^{1,1}$ and $\dim_{\mathbb{R}}(\mathcal{M}^{2,1}(K)) = 4(h^{2,1} + 1)$.

Each of these moduli spaces is a manifold, where $\mathcal{M}^{1,1}(K)$ is also called a *special-Kähler manifold*, and where $\mathcal{M}^{2,1}(K)$ is also called a *quaternionic Kähler manifold*. The total moduli space is simply a direct product of these two: $\mathcal{M}^{1,1}(K) \times \mathcal{M}^{2,1}(K)$.

4 Wrapping Euclidean D-Branes and Instantons

In this chapter we will shortly introduce p-branes, D-branes, Euclidean D-branes and instanton solutions, and then we will especially pay much attention to Euclidean D2-brane instantons, or *membrane instantons*. These are extended objects to be added to the model of massless superstring theory, and these are non-perturbative objects. These objects will give an adjustment to the effective model: in type IIA superstring theory the moduli space of hypermultiplets can be non-perturbatively corrected by membrane instantons wrapping special Lagrangian submanifolds, or SLags, embedded in the Calabi-Yau manifold in question. In the next chapters we will give a mathematical introduction of some SLags, related to a few different Calabi-Yau manifolds, but first we will give the physical motivation. In Section 4.1 we will mainly follow the book by Becker, Becker and Schwarz [20]. We will only assume that the 10-dimensional space-time M is compactified by $M = M_4 \times K$, where $M_4 = \mathbb{R}^4$ (with Minkowski metric) and K is a Calabi-Yau manifold of complex dimension 3. In Section 4.2 we will mainly follow the article by Becker, Becker and Strominger [10]. There we will discuss membrane instantons in much detail. The central point will be to explain why exactly the SLags are selected for analyzing membrane instantons. Finally we will shortly discuss the non-perturbative corrections of the low-energy effective action, and why we restrict to the hypermultiplets and ignore the vector multiplets.

4.1 Basics

p-branes and D-branes. A p-brane is an extended object in string theory. Conventionally it has p spatial dimensions and one time dimension, thus it has $p + 1$ dimensions in total, and it is an object with tension T_p. We say that a p-brane appears in superstring theory as a non-perturbative excitation. The action of a p-brane is written as $S_p := -T_p V$, where V is the volume of the $(p+1)$-dimensional manifold, corresponding to the brane. This manifold is also called the *world volume* of the p-brane. The classical equations of motion *extremize* this action.

A p-brane is completely defined by a map $\iota : \Sigma \to \mathcal{T}$, where Σ is the *domain space* of the p-brane, and where $\iota(\Sigma) \subset \mathcal{T}$ is the *target space* of the p-brane, embedded in the 10-dimensional space-time. The manifold $\mathcal{T}$ itself has a Minkowski metric, however, in certain subsets of $\mathcal{T}$ this metric can partially have a Euclidean signature. The subset $\iota(\Sigma)$ can *either* have a Minkowski metric, *or* a Euclidean metric. We can pull back this metric to the domain space Σ itself.

A (classical) string itself is a special example of a p-brane, if $p = 1$. (In this case S_p coincides with the Nambu-Goto action of the string.) Other examples are D-branes, or Dp-branes, where D stands for *Dirichlet*. Usually these are specific p-branes upon which an open string, satisfying the Dirichlet boundary conditions, can end. We should note that this contact is only with respect to the spatial direction. A D-brane can be regarded as a boundary object for an open string, and a D-brane itself has no boundary. The tension of the D-branes contained in type IIA (and IIB) superstring theory, is proportional to $1/g_s$, the inverse of the string coupling constant.

p-branes and differential forms. In type IIA (and IIB) theory several antisymmetric tensor fields appear, coming from the effective action in 10 dimensions of supergravity. These tensor fields are gauge fields and are often represented by differential forms. Note that, in general, these k-forms represent certain 'charges', and we say a k-dimensional submanifold of 10-dimensional space-time can carry this charge. (Note that we can always canonically integrate over any k-form if we take a k-manifold as its integration domain.)

Any k-form, regarded as a gauge field, can couple to a p-brane. We say an electric and a magnetic coupling is possible. A k-form can couple electrically to a $(k-1)$-brane, and can couple magnetically to a $(7-k)$-brane. Then we can say that an n-brane is related to a $(6-n)$-brane, being its 'magnetic dual'.

We explain this as follows: a k-form gauge field is related to a $(k+1)$-form field strength, which has another field strength as a Hodge dual, represented by a $(9-k)$-form. Now this dual field strength is related to an $(8-k)$-form gauge field, which is in turn indeed related to a $(7-k)$-brane.

D-branes and differential forms. Especially in the R-R sector of the bosonic part of the type IIA action several specific differential forms appear. The term $S_{IIA,2}$ in the supergravity action (3.4) contains the terms $G^{(2)} \wedge *G^{(2)}$ and $G^{(4)} \wedge *G^{(4)}$, where $G^{(2)}$ and $G^{(4)}$ are the R-R field strengths related to the R-R potentials $C^{(1)}$ and $C^{(3)}$ (respectively). Now we say that this theory should contain *stable* branes that carry these charges $C^{(1)}$ and $C^{(3)}$. Then at least a D0- and a D2-brane are related to $S_{IIA,2}$.

Now note that $G^{(2)}$ is a 2-form, $G^{(4)}$ is a 4-form, $*G^{(4)}$ is a 6-form and $*G^{(2)}$ is an 8-form. We can as well write $G^{(6)} = *G^{(4)}$ and $G^{(8)} = *G^{(2)}$. (These are *no* new field strengths.) Now $G^{(6)}$ and $G^{(8)}$ are the R-R field strengths related to the R-R potentials $C^{(5)}$ and $C^{(7)}$ (respectively). Then we can say that $C^{(5)}$ and $C^{(7)}$ are related to the magnetic duals of the D2- and D0-branes. The D0-brane indeed has a D6-brane as a magnetic dual, and the D2-brane has a D4-brane as a magnetic dual. Indeed, in total we have a D0-, D2-, D4- and a D6-brane. (Note that we will ignore a D8-brane, related to a 10-form field strength. This field is nondynamical, as it is a constant maximal form, so it does not represent a physical degree of freedom.)

These Dp-branes (with $p = 0, 2, 4, 6$), turning out to be stable, are sometimes called *half-BPS D-Branes*, because they preserve half of the supersymmetries. There are also Dp-branes with p being odd, and these turn out to be unstable, because they do not carry conserved charges. (In type IIB theory we are dealing with odd D-branes being stable and even ones being unstable.)

Euclidean D-branes. A Dp-brane, defined by $\iota : \Sigma \to \mathcal{T}$, is usually interpreted as a world volume with p spatial coordinates and one time coordinate. Then the metric defined on Σ has a Minkowski signature, so that we are dealing with degenerate metrics. Thus, if we look at the solutions of the equations of motion of a D-brane, derived from its action, we can say that this is why it is possible to describe D-branes as objects with an infinite time domain, but still a finite action, just like in case of the ordinary string.

Now we can introduce the idea of Euclidean D-branes. In this case Euclidean metrics are defined on Σ and $\iota(\Sigma)$. Only the internal coordinates itself, of the Dp-brane, will be 'Euclideanized' (before extremizing the action). Now we will use $(p+1)$ spatial coordinates and *no* time coordinate. Note that the action $S_{IIA,2}$ does not give any restriction of the kind of manifold we will use for a D-brane, thus $S_{IIA,2}$ also supports Euclidean D-branes.

At least if the Euclidean Dp-brane has finite volume, then we will not encounter any serious problems. Anyway, not much effort is needed if we restrict to D-branes entirely embedded in the Calabi-Yau manifold. Then a D-brane is already compact, so that its volume is indeed finite. (Many non-Euclidean D-branes are not compact.) Then the target space of the D-brane, embedded in the Calabi-Yau space, is called a *cycle*.

In general, if some supersymmetry is preserved when a Euclidean p-brane can wrap a $(p+1)$-cycle, then its target space is called a *supersymmetric cycle*.

Note that minimizing the volume of (for example) a Euclidean D-brane embedded in the Lorentzian manifold $\mathbb{R}^4$ will only give us a trivial solution, a point, as $\mathbb{R}^4$ is a contractible space. If we minimize the volume of a Euclidean D-brane, entirely embedded in the Calabi-Yau 3-fold, then we have a chance we end up with a non-trivial manifold. This is why Euclidean D-branes are mainly studied embedded in the Calabi-Yau 3-fold, when studying instantons in string theory.

Instantons in type II superstring theory. Assume we will only look at Dp-branes entirely embedded in the Calabi-Yau 3-fold K from now on. (Thus K contains the target space C of a Dp-brane.) Such D-branes are perfectly allowed in the R-R sector. This means that we will only look at $p = -1, 0, 1, 2, 3, 4, 5$. In type II theory, if a Euclidean Dp-brane wraps around a topologically non-trivial $(p+1)$-dimensional submanifold C of K, or around a $(p+1)$-*cycle*, then its domain space Σ has a finite but non-zero volume. Solving the equations of motion of the D-brane makes sure that C is a manifold with minimized volume. The map $m : \Sigma \to C \subset K$ is smooth and surjective.

Note that in type IIA theory only for p even the Dp-branes are stable. (The Dp-branes for p odd are stable in type IIB theory.) Now note that the Euclidean D0- and D4-branes are all trivial, as K is a *strict* Calabi-Yau 3-fold. Then $b_1(K) = b_5(K) = 0$, so that any 1-cycle or 5-cycle is topologically equivalent to a point (at least, if K is simply connected). So only 3-cycles remain interesting.

Now, in type IIA theory, the hypermultiplets are only related to the odd cohomology groups $H^1_{\mathrm{dR}}(K)$, $H^3_{\mathrm{dR}}(K)$ and $H^5_{\mathrm{dR}}(K)$. However, we already know that

$$b_1(K) = b_5(K) = 0 \Rightarrow \quad \dim_{\mathbb{R}}(H^1_{\mathrm{dR}}(K)) = b_1(K) = 0, \quad \dim_{\mathbb{R}}(H^5_{\mathrm{dR}}(K)) = b_5(K) = 0,$$

thus the hypermultiplets only arise from $H^3_{\mathrm{dR}}(K)$. This is in harmony with that we are only interested in the 3-cycles.

Instantons in type IIA superstring theory: Membrane instantons. As $b_3(K) > 0$ we can say we have at least one topologically non-trivial 3-cycle L, embedded in K. This L describes the image of a *membrane instanton*, which is one particular kind of all the instantons occurring in type IIA theory. This membrane instanton is an instanton describing a non-perturbative correction in g_s. (For example, fundamental-string instantons describe a non-perturbative correction in α'.)

Before this 3-cycle L plays the role of a membrane instanton, we require it to admit supersymmetry. So, besides it has a minimal volume, we need it to satisfy extra conditions. The pullback of the Kähler form on L must vanish, and the pullback of the holomorphic top-form must be a constant complex multiple of a real 3-form. We will call these conditions the *SLag conditions*, and if L satisfies these conditions, then we call L a *special Lagrangian submanifold* of K, or a *SLag*. In the next section we will work out much more details about these membrane instantons, and how the SLag conditions are caused by the supersymmetry conditions.

If L indeed describes a membrane instanton, then we should note that it is entirely embedded in the Calabi-Yau 3-fold K, *at one instant*: it only exists at one moment in time. In massless effective theory in four dimensions a membrane instanton is even just a single point.

Any SLag L is a compact oriented manifold, with $\dim_{\mathbb{R}}(L) = 3$. Then its Betti numbers satisfy the Poincaré duality, thus $b_j(L) = b_{3-j}(L)$, so that

$$\chi(L) = b_0(L) - b_1(L) + b_2(L) - b_3(L) = b_0(L) - b_1(L) + b_1(L) - b_0(L) = 0. \tag{4.1}$$

Note that L can have multiple connected components, say k. Then $b_0(L) = k$, $b_1(L) = l$, $b_2(L) = l$ and $b_3(L) = k$.

4.2 Membrane instantons

Now we will specifically mention membrane instantons, related to type IIA theory, as mentioned in [10]. We will also use a few related results mentioned in [20]. The SLags themselves, including the SLag conditions, will be explained in much detail in Section 6.1, but here we will mainly explain how the consequence of the supersymmetry conditions (or 'BPS conditions') are the SLag conditions. The supersymmetry conditions mean that the theory is invariant under the supersymmetry transformations. To explicitly work out some relations, we actually do it with respect to the supergravity model in 11 dimensions. We do this because then we can do it with a simpler notation, compared to the 10-dimensional type IIA model. According to [10] we do this in preparation of the analysis of the type IIA theory. After compactifying, we can work out the effect of the membrane instantons with respect to $S^{(4)}_{IIA}$, the low-energy effective type IIA action in 4 dimensions. Then we say there are corrections to the hypermultiplets.

The action of a membrane instanton. Later we will continue with membrane instantons embedded in the 10-dimensional space-time, related to type IIA theory, but first we will start with 11-dimensional space-time. In [10], the authors give a detailed explanation and they start with S_{11}, the bosonic part of the Euclidean action of 11-dimensional supergravity. This action is a lot simpler to work with for the moment, instead of the action S_{IIA}. In S_{11}, a 4-form dC will be gauged to zero, which is needed for supersymmetric compactification. This gives a 3-dimensional effective action S_3 on large enough scales, simplifying the model severely. Then the first term of the (effective) action reduces to the volume of the membrane, and the equations of motion of X, the coordinates of the membrane, require extremization of the volume. This already looks similar to the first step needed for the SLag conditions. We can do something similar with respect to the 10-dimensional model of type IIA theory. Then we would need more intermediate computations.

To work out explicitly, we will start with the bosonic part of the Euclidean action of 11-dimensional supergravity:

$$S_{11} := \frac{1}{2\pi^2\ell^9}\int d^{11}x\sqrt{g}(-R+\frac{1}{48}(dC)^2)+\frac{i}{12\pi^2\ell^2}\int C\wedge dC\wedge dC.$$

Here C is a 3-form potential, and ℓ is the 11-dimensional Planck length. Now this action looks somewhat simpler to work with for the moment. Then we have the following action for the membrane instanton:

$$\begin{aligned} S_3 \quad := \quad & \frac{1}{\ell^3}\int d^3\sigma\sqrt{h}(2h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N g_{MN}-2-i\bar{\Theta}\Gamma^\alpha\nabla_\alpha\Theta \\ + \quad & \frac{i}{3!}\epsilon^{\alpha\beta\gamma}\partial_\alpha X^M\partial_\beta X^N\partial_\gamma X^P C_{MNP}+\cdots). \end{aligned} \tag{4.2}$$

Here σ are the coordinates of the world volume, $h_{\alpha\beta}$, with $\alpha,\beta=1,2,3$, is a world volume metric (with Euclidean signature), $X^M(\sigma)$, with $M,N,P=1,\cdots,11$, is the coordinate field of the membrane and Θ is an 11-dimensional Dirac spinor. Then X represents a bosonic field of the membrane, and Θ represents a fermionic field of the membrane. (Note that also the spin connection should be integrated in the covariant derivative ∇_α.)

Note that Γ^α are the induced gamma matrices. We write $\Gamma_\alpha := \partial_\alpha X^M\Gamma_M$, where Γ_M are the 11-dimensional Euclidean space-time gamma matrices. The induced gamma matrices satisfy the rule

$$\{\Gamma_\alpha,\Gamma_\beta\}=2h_{\alpha\beta}.$$

Note that $\partial_\alpha X^M\partial_\beta X^N\partial_\gamma X^P C_{MNP}$ are the components of the pullback of the 3-form C, induced on the membrane, and that $\epsilon^{\alpha\beta\gamma}=\det(h^{-1})\epsilon_{\alpha\beta\gamma}$, as usual.

We note that in (4.2) certain terms are missing: of all the terms containing powers of the fermi fields only the most important ones, those of leading order, are explicitly written down. (We also redefine the physical units so that we can write $\ell=1$.)

Only the first term of (4.2) is a part containing the world volume metric $h_{\alpha\beta}$. There we see that this metric has well known equations of motion, and its solution is that we turn it into the induced metric, the metric we pull back from 11 dimensions:

$$h_{\alpha\beta}=\partial_\alpha X^M\partial_\beta X^N g_{MN}.$$

Then the first term of the action (4.2) will be equal to the volume of the membrane, which is a quantity with respect to the 11-dimensional space-time, and its metric g, as usual. From now on we will assume that $dC=0$, as according to [10] this is "*required for supersymmetric compactifications*". We can also say that "*the X equation of motion requires extremization of the volume*".

The action of a membrane instanton: Fermionic symmetries. Now there are two kinds of fermionic symmetries, or *supersymmetry transformations*, acting on X and on Θ, a global one and a local one. Before introducing these symmetries we will first define some extra auxiliary symbols. We have 11-dimensional spinors ε and κ, where ε is constant and anticommuting, so that we can use it for a global transformation, or a global supersymmetry, and where κ is a variable spinor which depends on the coordinate σ of the membrane, so that we can use it for a local transformation. We also have a couple of projection operators:

$$P_\pm := \frac{1}{2}(1\pm\frac{i}{3!}\epsilon^{\alpha\beta\gamma}\partial_\alpha X^M\partial_\beta X^N\partial_\gamma X^P\Gamma_{MNP}).$$

Here Γ_{MNP} equals the totally anticommuting operator (as usual):

$$\Gamma_{MNP}=\Gamma_{[M}\Gamma_N\Gamma_{P]}=\frac{1}{3!}(\Gamma_M\Gamma_N\Gamma_P+\Gamma_N\Gamma_P\Gamma_M+\Gamma_P\Gamma_M\Gamma_N-\Gamma_M\Gamma_P\Gamma_N-\Gamma_N\Gamma_M\Gamma_P-\Gamma_P\Gamma_N\Gamma_M).$$

These operators obey the following three properties:

$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0, \quad P_+ + P_- = \mathbf{1}.$$

Now we have infinitesimal variations of the membrane fields, according to these auxiliary spinors, and the projection operator. The global fermionic symmetry acts as follows:

$$\delta_\varepsilon X^M = i\bar{\varepsilon}\Gamma^M\Theta \quad , \quad \delta_\varepsilon\Theta = \varepsilon.$$

And the local fermionic symmetry acts as follows:

$$\delta_\kappa X^M = 2i\bar{\Theta}\Gamma^M P_+\kappa(\sigma) \quad , \quad \delta_\kappa\Theta = 2P_+\kappa(\sigma).$$

(See (3.2) for the formal definition of supersymmetry transformations.) Now the membrane theory is invariant under these κ-transformations, so that it should describe an equivalent solution after such a transformation.

In general, any arbitrary bosonic state of the membrane, described by $X^M(\sigma)$, will not be invariant under the supersymmetries generated by ε, at least if $\delta_\kappa X^M = \delta_\kappa\Theta = 0$. Then we say that ε generates a broken supersymmetry. However, in specific cases we can choose a spinor $\kappa(\sigma)$ compensating the transformation by ε. Then we say that ε together with $\kappa(\sigma)$ generate an unbroken supersymmetry. For such a $\kappa(\sigma)$, we assume that

$$\delta_\varepsilon\Theta + \delta_\kappa\Theta = \varepsilon + 2P_+\kappa(\sigma) = 0 \quad \Rightarrow \quad 2P_+\kappa(\sigma) = -\varepsilon. \tag{4.3}$$

As a result we then automatically have

$$\delta_\varepsilon X^M + \delta_\kappa X^M = i\bar{\varepsilon}\Gamma^M\Theta + 2i\bar{\Theta}\Gamma^M P_+\kappa(\sigma) = i(\bar{\varepsilon}\Gamma^M\Theta - \bar{\Theta}\Gamma^M\varepsilon) = 0.$$

Now we can insert the projection operator P_- in (4.3):

$$P_-(\varepsilon + 2P_+\kappa(\sigma)) = P_-(0) = 0 \quad \Rightarrow \quad P_-\varepsilon + 2P_-P_+\kappa(\sigma)) = P_-\varepsilon + 0 = P_-\varepsilon = 0.$$

Thus, writing that $\delta_\varepsilon\Theta + \delta_\kappa\Theta = 0$ implies that $P_-\varepsilon = 0$. In other words, at least we require $P_-\varepsilon$ to be zero, so that it might be possible to compensate ε. Before we apply this constraint, we will work out some more details about a part of the target space: a Calabi-Yau 3-fold.

Spinors in the Calabi-Yau 3-fold. Let K be an arbitrary strict Calabi-Yau 3-fold, with a Ricci-flat metric, with a Kähler form ω, as written in (2.40), and a holomorphic top-form Ω, as written in (2.54). Let z^μ and $\bar{z}^\mu$ be the complex coordinates, defined on K, with $\mu = 1, 2, 3$. We know that on a strict Calabi-Yau 3-fold we have $\mathcal{N} = 2$ supersymmetry, which is equivalent to that there are 8 covariantly constant spinors, defined on this K. (These spinors automatically define 8 covariantly constant spinors on the total target space.)

We will restrict to membranes, described by (4.2), embedded in this Calabi-Yau space, and thus located at a single point in the remaining dimensions. This means that in the first 5 flat dimensions we have coordinates $X^j(\sigma)$, with $j = 1, 2, 3, 4, 5$, so that $\partial_\alpha X^j = 0$. Then we can also restrict to the 6 dimensions of the Calabi-Yau space when we express (4.2). Now we will also express the other 6 coordinates X^M with respect to the complex coordinates z^μ and $\bar{z}^\mu$.

Now we can look at two of these covariantly constant 6-dimensional spinors, ε_+ and ε_-, which have opposite chirality, where ε_+ has positive chirality and ε_- has negative chirality. They satisfy $\varepsilon_+ = (\varepsilon_-)^*$ and we use an imaginary representation of the gamma matrices (when restricting to the Calabi-Yau space), see the γ-symbols. We choose the following normalization:

$$\gamma_{\mu\nu\rho}\varepsilon_+ = \Omega_{\mu\nu\rho}\varepsilon_-, \quad \gamma_{\bar{\mu}\nu\rho}\varepsilon_+ = i(\omega_{\bar{\mu}\nu}\gamma_\rho - \omega_{\bar{\mu}\rho}\gamma_\nu)\varepsilon_+, \quad \gamma_{\bar{\mu}}\varepsilon_+ = 0. \tag{4.4}$$

Here we applied a relation between Ω and ω, as mentioned in (2.58), which only holds in case of a Ricci-flat metric, and we made a specific choice, for the complex constant c, and this choice is also mentioned in [16]:

$$i\Omega \wedge \bar{\Omega} = \frac{4}{3}\omega \wedge \omega \wedge \omega.$$

(If we make another choice for c, then an extra constant factor, in front of $\Omega_{\mu\nu\rho}$, should be added to (4.4).) Anyway, why do we already restrict to a Ricci-flat metric? We will finally apply this to SLags, and these are only defined with respect to a Ricci-flat metric.

Note that (4.4) automatically relates expressions like $\gamma_{\mu\bar{\nu}\rho}\varepsilon_+$ and $\gamma_{\mu\nu\bar{\rho}}\varepsilon_+$ to $\gamma_{\bar{\mu}\nu\rho}\varepsilon_+$ (so we have three of these), and all the other expressions, like $\gamma_{\bar{\mu}\bar{\nu}\rho}\varepsilon_+$ or $\gamma_{\bar{\mu}\bar{\nu}\bar{\rho}}\varepsilon_+$, are chosen to be zero. We have 3 real indices, and we can make 8 combinations of complex indices, with j holomorphic indices and $3-j$ anti-holomorphic indices. Only 4 of these combinations are non-zero, and 3 of the 4 are in fact equal, so we have in fact 2 non-zero combinations: one with 3 holomorphic indices, and one with 2 holomorphic indices and 1 anti-holomorphic index, but this last one has a factor 3. For ε_- we do it similarly: we only have non-zero expressions $\gamma_{\bar{\mu}\bar{\nu}\bar{\rho}}\varepsilon_-$ and $\gamma_{\mu\bar{\nu}\bar{\rho}}\varepsilon_-$.

Now we define

$$\varepsilon_\theta := e^{i\theta}\varepsilon_+ + e^{-i\theta}\varepsilon_- \quad , \quad \varepsilon := \lambda \otimes \varepsilon_\theta,$$

where λ is a 5-dimensional spinor, defined on the first 5 flat dimensions. Now we can express the relation $P_-\varepsilon = 0$ in terms of these complex coordinates and the spinors ε_+ and ε_-. If $P_-\varepsilon = P_-(\lambda\otimes\varepsilon_\theta) = 0$, then we can restrict to $P_-\varepsilon_\theta = 0$, because $\partial_\alpha X^j = 0$ (with $j = 1,2,3,4,5$), and this $P_-\varepsilon_\theta$ contains the derivatives with respect to z^μ and $\bar{z}^\mu$. Now we introduce some auxiliary symbols:

$$P^{\mu\nu\rho} := \tfrac{i}{3!}\epsilon^{\alpha\beta\gamma}\partial_\alpha z^\mu \partial_\beta z^\nu \partial_\gamma z^\rho \quad , \quad P^{\bar{\mu}\nu\rho} := \tfrac{i}{3!}\epsilon^{\alpha\beta\gamma}\partial_\alpha \bar{z}^\mu \partial_\beta z^\nu \partial_\gamma z^\rho \quad , \quad \cdots \quad .$$

Then, applying

$$P^{\bar{\mu}\nu\rho}\gamma_{\bar{\mu}\nu\rho}\varepsilon_+ = iP^{\bar{\mu}\nu\rho}(\omega_{\bar{\mu}\nu}\gamma_\rho - \omega_{\bar{\mu}\rho}\gamma_\nu)\varepsilon_+ = 2iP^{\bar{\mu}\nu\rho}\omega_{\bar{\mu}\nu}\gamma_\rho\varepsilon_+$$

where the second equality is caused by the anti-symmetry of $P^{\bar{\mu}\nu\rho}$ (when exchanging ν and ρ), we can write

$$\begin{aligned}
P_-\varepsilon_\theta &= P_-(e^{i\theta}\varepsilon_+ + e^{-i\theta}\varepsilon_-) = e^{i\theta}P_-\varepsilon_+ + e^{-i\theta}P_-\varepsilon_- \\
&= \frac{1}{2}e^{i\theta}(\varepsilon_+ - P^{\mu\nu\rho}\gamma_{\mu\nu\rho}\varepsilon_+ - 3P^{\bar{\mu}\nu\rho}\gamma_{\bar{\mu}\nu\rho}\varepsilon_+) + \frac{1}{2}e^{-i\theta}(\varepsilon_- - P^{\bar{\mu}\bar{\nu}\bar{\rho}}\gamma_{\bar{\mu}\bar{\nu}\bar{\rho}}\varepsilon_- - 3P^{\mu\bar{\nu}\bar{\rho}}\gamma_{\mu\bar{\nu}\bar{\rho}}\varepsilon_-) \\
&= \frac{1}{2}e^{i\theta}(\varepsilon_+ - P^{\mu\nu\rho}\Omega_{\mu\nu\rho}\varepsilon_- - 6iP^{\bar{\mu}\nu\rho}\omega_{\bar{\mu}\nu}\gamma_\rho\varepsilon_+) + \frac{1}{2}e^{-i\theta}(\varepsilon_- - P^{\bar{\mu}\bar{\nu}\bar{\rho}}\bar{\Omega}_{\bar{\mu}\bar{\nu}\bar{\rho}}\varepsilon_+ + 6iP^{\mu\bar{\nu}\bar{\rho}}\omega_{\mu\bar{\nu}}\gamma_{\bar{\rho}}\varepsilon_-) \\
&= \frac{1}{2}((e^{i\theta} - e^{-i\theta}P^{\bar{\mu}\bar{\nu}\bar{\rho}}\bar{\Omega}_{\bar{\mu}\bar{\nu}\bar{\rho}} - 6ie^{i\theta}P^{\bar{\mu}\nu\rho}\omega_{\bar{\mu}\nu}\gamma_\rho)\varepsilon_+ + (e^{-i\theta} - e^{i\theta}P^{\mu\nu\rho}\Omega_{\mu\nu\rho} + 6ie^{-i\theta}P^{\mu\bar{\nu}\bar{\rho}}\omega_{\mu\bar{\nu}}\gamma_{\bar{\rho}})\varepsilon_-) \\
&= 0
\end{aligned}$$

(Here we used that $\overline{\omega_{\bar{\mu}\nu}} = \overline{ig_{\bar{\mu}\nu}} = -ig_{\mu\bar{\nu}} = -\omega_{\mu\bar{\nu}}$, applying (2.37).) Thus,

$$\begin{aligned}
(e^{-i\theta} - e^{i\theta}P^{\mu\nu\rho}\Omega_{\mu\nu\rho} + 6ie^{-i\theta}P^{\mu\bar{\nu}\bar{\rho}}\omega_{\mu\bar{\nu}}\gamma_{\bar{\rho}})\varepsilon_- + \text{c. c.} &= 0 \Rightarrow \\
e^{i\theta}P^{\mu\nu\rho}\Omega_{\mu\nu\rho}\varepsilon_- - e^{-i\theta}\varepsilon_- - 6ie^{-i\theta}P^{\mu\bar{\nu}\bar{\rho}}\omega_{\mu\bar{\nu}}\gamma_{\bar{\rho}}\varepsilon_- + \text{c. c.} &= 0,
\end{aligned}$$

so that

$$\frac{i}{3!}e^{i\theta}\epsilon^{\alpha\beta\gamma}\partial_\alpha z^\mu\partial_\beta z^\nu\partial_\gamma z^\rho\Omega_{\mu\nu\rho}\varepsilon_- - e^{-i\theta}\varepsilon_- + e^{-i\theta}\epsilon^{\alpha\beta\gamma}\partial_\alpha z^\mu\partial_\beta\bar{z}^\nu\partial_\gamma\bar{z}^\rho\omega_{\mu\bar{\nu}}\gamma_{\bar{\rho}}\varepsilon_- + \text{c. c.} = 0$$

Now the spinors ε_-, $\gamma_{\bar{\rho}}\varepsilon_-$, ε_+ and $\gamma_\rho\varepsilon_+$ are all linearly independent. Then the first term together with the second term, the third term and the complex conjugate term will all vanish on their own:

$$\frac{i}{3!}e^{i\theta}\epsilon^{\alpha\beta\gamma}\partial_\alpha z^\mu\partial_\beta z^\nu\partial_\gamma z^\rho\Omega_{\mu\nu\rho}\varepsilon_- - e^{-i\theta}\varepsilon_- = 0 \quad , \quad e^{-i\theta}\epsilon^{\alpha\beta\gamma}\partial_\alpha z^\mu\partial_\beta\bar{z}^\nu\partial_\gamma\bar{z}^\rho\omega_{\mu\bar{\nu}}\gamma_{\bar{\rho}}\varepsilon_- = 0.$$

We can rewrite these equations:

$$\epsilon^{\alpha\beta\gamma}\partial_\alpha z^\mu\partial_\beta z^\nu\partial_\gamma z^\rho\Omega_{\mu\nu\rho} = -3!ie^{-2i\theta} \quad , \quad \epsilon^{\alpha\beta\gamma}\partial_\alpha z^\mu\partial_\beta\bar{z}^\nu\partial_\gamma\bar{z}^\rho\omega_{\mu\bar{\nu}} = 0.$$

This pair of equations is equivalent to the following pair of equations:

$$\partial_\alpha z^\mu\partial_\beta z^\nu\partial_\gamma z^\rho\Omega_{\mu\nu\rho} = -ie^{-2i\theta}\epsilon_{\alpha\beta\gamma} = e^{-i\varphi}\epsilon_{\alpha\beta\gamma}, \tag{4.5}$$

$$\partial_\alpha z^\mu\partial_\beta\bar{z}^\nu\omega_{\mu\bar{\nu}} - \partial_\beta z^\mu\partial_\alpha\bar{z}^\nu\omega_{\mu\bar{\nu}} = 0. \tag{4.6}$$

(The angle φ is related to θ as follows: $\varphi = 2\theta + \pi/2$.)

Spinors in the Calabi-Yau 3-fold: The result. We see that (4.5) is equivalent to the statement that the set of components of the pullback of Ω on the membrane equals a complex constant (a phase factor), and we claim that (4.6) is equivalent to the statement that $\iota^*\omega$, the pullback of ω, vanishes. In Section 6.1 the reader can find more details related to the pullback of ω. There, in (6.1), we will indeed see that (4.6) means that $\iota^*\omega$ vanishes. Then we can conclude that (4.5) and (4.6) are equal to the SLag conditions. Thus, we can finally conclude that the supersymmetry conditions are equivalent the SLag conditions. In other words, (4.5) and (4.6) imply that the bosonic membrane field $X^M(\sigma)$ is supersymmetric.

Global corrections to massless effective type IIA superstring theory. If we look at the total theory, then we need to keep track of *all possible* corrections by instantons. We will, for example, also need to keep track of all corrections by world-sheet instantons and NS5-branes. In case of the membrane instantons, non-perturbative objects from the R-R sector, we can say that there are corrections to the geometry of the hypermultiplet moduli space and (thus) to the low-energy effective action for the hypermultiplets, and these corrections will contain contributions of order e^{-1/g_s}.

In fact these non-perturbative corrections will modify the geometry of type IIA superstring theories in four dimensions, with $\mathcal{N} = 2$ space-time supersymmetry. In this four dimensional massless superstring theory (or supergravity), there are vector multiplets and hypermultiplets, which are specific types of supermultiplets, as introduced at the end of Section 3.2. However, we recall that supersymmetry imposes that these two multiplets cannot have any interaction in case of the massless low-energy effective action.

Now note that, in case of type IIA theory, the dilaton lives in a hypermultiplet; it is *not* present in a vector multiplet. According to [10], this means that non-perturbative corrections to the vector multiplet moduli space are not possible. Thus, this means that we can ignore the vector multiplets if we are studying the non-perturbative corrections. We only have non-perturbative corrections to the geometry of the hypermultiplet moduli space. This is why only hypermultiplets are mentioned in the context of membrane instantons.

5 An example of a Calabi-Yau threefold: The Fermat Quintic

In this chapter we will introduce the Fermat quintic, a hypersurface embedded in the complex projective space $\mathbb{P}^4$ and an example of a Calabi-Yau 3-fold. In Section 5.1 we will mainly give a definition and a specific choice of splitting up this manifold. We will build a cell complex of the quintic. In Section 5.2 we will shortly introduce how to construct the mirror quintic, a manifold with exchanged Hodge numbers compared to the original manifold. Then it will be clear why we made the specific choice of splitting up the quintic in Section 5.1.

5.1 Definition

The Fermat quintic in $\mathbb{P}^4$. In this chapter we are mainly interested in the fifth order Fermat polynomial, which corresponds to the special choice of coefficients f_{ijklm}:

$$f(Z^j) = (Z^0)^5 + (Z^1)^5 + (Z^2)^5 + (Z^3)^5 + (Z^4)^5. \tag{5.1}$$

The corresponding hypercurve $Q := f^{-1}(0)/\mathbb{C}^*$, embedded in $\mathbb{P}^4$, is called the *Fermat quintic*: our space of interest.

An atlas of the Fermat quintic. The standard minimal holomorphic atlas of $\mathbb{P}^4$ contains 5 patches. If $Z^0 \neq 0$, then for example the first patch contains points like $[1 : a : b : c : d]$, where $(a, b, c, d) \in \mathbb{C}^4$. Then we can construct a minimal holomorphic atlas of Q, the Fermat quintic.

We know that there is a constraint on a, b, c and d: $1 + a^5 + b^5 + c^5 + d^5 = 0$. This means that we can, for example, express d in terms of the first three coordinates a, b and c:

$$d = d(a, b, c) := -\alpha^k \sqrt[5]{1 + a^5 + b^5 + c^5}, \tag{5.2}$$

where $\alpha = e^{2\pi i/5}$ and $k \in \{0, 1, 2, 3, 4\}$. We could say that this generates 5 patches. (To make sure this consistently describes patches, we could make a restriction: $1 + a^5 + b^5 + c^5 \neq 0 \Leftrightarrow d \neq 0$. Also note that these patches could as well be described as subsets of the collection of points of the following form: $[d' : a' : b' : c' : 1]$, with $d' \neq 0$.) We could as well express, for example, a in terms of b, c and d, in a similar fashion. There are 4 of these possible restrictions of the coordinates. We could say that a minimal holomorphic atlas of $5 \cdot 5 \cdot 4/2 = 5 \cdot \binom{5}{2} = 50$ patches can be constructed this way, completely describing the Fermat quintic.

A holomorphic top-form. Let $[1 : u : v : w : x(u, v, w)]$ be a chart of Q. Then we can define a holomorphic top-form as follows:

$$\Omega := \frac{du \wedge dv \wedge dw}{x^4}. \tag{5.3}$$

(See [10] for this definition.) Now we can check if this is indeed a correct definition of a top-form. Let $[1 : u(v, w, x) : v : w : x]$ be another chart of Q. Then we can check how Ω transforms. Then $u^5(v, w, x) = -(1 + v^5 + w^5 + x^5)$ and $du^5 = 5u^4 du$, so that

$$du = \frac{1}{5u^4} du^5 = -\frac{1}{5u^4}(dv^5 + dw^5 + dx^5) = -\frac{1}{u^4}(v^4 dv + w^4 dw + x^4 dx).$$

Then we can rewrite Ω:

$$\Omega = \frac{du \wedge dv \wedge dw}{x^4} = -\frac{(v^4 dv + w^4 dw + x^4 dx) \wedge dv \wedge dw}{x^4 u^4} = -\frac{x^4 dx \wedge dv \wedge dw}{x^4 u^4} = \frac{dv \wedge dx \wedge dw}{u^4}.$$

Let now $[U : 1 : V : W : X(U, V, W)]$ be again another chart of Q. If $U \neq 0$, then we have the following coordinate transformation:

$$[U : 1 : V : W : X(U, V, W)] = [1 : \frac{1}{U} : \frac{V}{U} : \frac{W}{U} : \frac{X(U, V, W)}{U}] = [1 : u : v : w : x(u, v, w)].$$

Thus we have: $u = 1/U$, $v = V/U$, $w = W/U$ and $x = X/U$. Then du, dv and dw will also transform:

$$du = d(\tfrac{1}{U}) = -\tfrac{1}{U^2}dU, \quad dv = d(\tfrac{V}{U}) = \tfrac{1}{U}dV - \tfrac{V}{U^2}dU, \quad dw = d(\tfrac{W}{U}) = \tfrac{1}{U}dW - \tfrac{W}{U^2}dU.$$

Then Ω will transform as follows:

$$\begin{aligned} \Omega &= \frac{1}{x^4}du \wedge dv \wedge dw = \frac{U^4}{X^4}(-\frac{1}{U^2}dU) \wedge (\frac{1}{U}dV - \frac{V}{U^2}dU) \wedge (\frac{1}{U}dW - \frac{W}{U^2}dU) \\ &= \frac{U^4}{X^4}(-\frac{1}{U^2}dU) \wedge (\frac{1}{U}dV) \wedge (\frac{1}{U}dW) = \frac{1}{X^4}dU \wedge dW \wedge dV. \end{aligned}$$

We conclude that the explicit form of Ω will be preserved after applying any (similar) coordinate transformation, thus we could say that Ω indeed is a correctly (and also globally) defined holomorphic top-form. If we examine the behaviour of Ω on any patch of Q, we can also say that it is nowhere vanishing. (We know that the maximum modulus principle implies that any other holomorphic top-form Ω' should be a constant multiple of Ω.)

Thus Q, when equipped with this Ω, defines a (strict) Calabi-Yau 3-fold, so that also a Ricci-flat metric exists on Q. (However, this Ricci-flat metric is not (exactly) known.)

We note that the explicit form of Ω, described by (5.3), also contains a division, but this description only holds on a certain (open) patch of Q, where $x \neq 0$, so there is no question of a singularity. For $x = 0$ we use another patch.

Splitting up the Fermat quintic. Using (2.1) and (2.2) we can compute the Euler number of the Fermat quintic Q. The only thing needed is an arbitrary cell complex for Q. We can split up $\mathbb{P}^4$ into smaller parts, and determine which points in these parts are lying in Q. The cell complex we will introduce here will have its applications later.

Splitting up the Fermat quintic: The types of points in $\mathbb{P}^4$. Any point in $\mathbb{P}^4$ is of the form $[Z^0 : Z^1 : Z^2 : Z^3 : Z^4]$, where all of the Z^j are complex. Some of them could be zero. We can split up $\mathbb{P}^4$ into subsets on which a fixed number of Z^j is non-zero, and the others are zero. The reason for this choice will become clear later. It is clear that this gives us 5 types of points, where a point of type n has exactly n non-zero components. Using the inhomogeneous coordinate representation, we can list the 5 types of points:

Type 1: $[1:0:0:0:0], [0:1:0:0:0], \cdots, [0:0:0:0:1]$,

Type 2: $[1:a:0:0:0], [1:0:a:0:0], \cdots, [0:0:0:1:a]$,

Type 3: $[1:a:b:0:0], [1:a:0:b:0], \cdots, [0:0:1:a:b]$,

Type 4: $[1:a:b:c:0], [1:a:b:0:c], \cdots, [0:1:a:b:c]$,

Type 5: $[1:a:b:c:d]$.

(Here $a, b, c, d \in \mathbb{C}^*$)

Of course we can do permutations of Z^j for any type of point. Officially we should define subtypes of points. Any two subtypes of 'type n' points can be related by permutation of the coefficients. It is easy to see that the set of 'type 1' and 'type 4' points admit 5 permutations, thus they can be split into 5 subtypes. The 'type 2' and 'type 3' points admit 10 permutations. Type 5 admits only 1 permutation, namely the trivial permutation. In total there are $31 = 2^5 - 1$ subtypes, which correspond to mutually disjoint subsets of $\mathbb{P}^4$. From now on we will ignore the subtypes, as they all behave in the same way. A point of a certain type is automatically of the first Subtype, i.e. the left one in the table above.

Splitting up the Fermat quintic: The types of points in Q. Now we would like to know which points, of any type, lie in Q. We immediately see that there are no 'type 1' points lying in Q. For example $[1:0:0:0:0]$ corresponds to $Z^0 \neq 0$ and $Z^1 = Z^2 = Z^3 = Z^4 = 0$, but then the Fermat polynomial is reduced to $(Z^0)^5$, which has $Z^0 = 0$ as the zero set, which yields a contradiction.

The 'type 2' points in Q are of the form $(Z^0)^5 + (Z^1)^5 = 0$, where Z^0 and Z^1 are both non-zero. In terms of inhomogeneous coordinates this becomes $1 + a^5 = 0$, with $a \in \mathbb{C}^*$. We immediately see that this equation has 5 solutions. Let Q_2 be the corresponding solution set. Then $\chi(Q_2) = 5$. As there are 10 subtypes of type 2, this results in 50 points of type 2 in total, which are lying in Q. In terms of cells this corresponds to 50 0-cells, resulting in a contribution $\chi_2 = 10\chi(Q_2) = 50$ to the total Euler number of the Fermat quintic.

The 'type 3' points in Q are of the form $1 + a^5 + b^5 = 0$. For $1 + a^5 = 0$, which has 5 solutions, we also have $b = 0$, which is not wanted. Thus, to ensure that $b \neq 0$ we need to use an additional constraint: $1 + a^5 \neq 0$. The solution set of this constraint is: $a \in \mathbb{C}^* - Q_2$. For any of these a we know that $b \neq 0$, so that we have 5 distinct solutions for b. (Note that $\chi(\mathbb{C}^*) = 0$ so that $\chi(\mathbb{C}^* - Q_2) = -\chi(Q_2) = -5$.) Let now Q_3 be the solution set of $1 + a^5 + b^5 = 0$ with $a, b \in \mathbb{C}^*$. Then $\chi(Q_3) = -5\chi(Q_2) = -25$. As there are 10 subtypes of type 3, this results in a contribution $\chi_3 = 10\chi(Q_3) = -250$ to the total Euler number of the Fermat quintic.

The 'type 4' points in Q are of the form $1 + a^5 + b^5 + c^5 = 0$. For $1 + a^5 + b^5 = 0$, which has Q_3 as its solution set, we also have $c = 0$, which is not wanted. Thus, to ensure that $c \neq 0$ we need to use an additional constraint: $(a, b) \in (\mathbb{C}^*)^2 - Q_3$. (Then $\chi((\mathbb{C}^*)^2 - Q_3) = -\chi(Q_3) = 25$.) For any such (a, b) we again have 5 solutions for c. Let now Q_4 be the solution set of the equation $1 + a^5 + b^5 + c^5 = 0$ with $a, b, c \in \mathbb{C}^*$. Then $\chi(Q_4) = -5\chi(Q_3) = 125$. As there are 5 subtypes of type 4, this results in a contribution $\chi_4 = 5\chi(Q_4) = 625$ to the total Euler number of the Fermat quintic.

The 'type 5' points in Q are of the form $1 + a^5 + b^5 + c^5 + d^5 = 0$. We again need to use an additional constraint: $(a, b, c) \in (\mathbb{C}^*)^3 - Q_4$. For any such (a, b, c) we again have 5 solutions for d. Let now Q_5 be the solution set of the equation $1 + a^5 + b^5 + c^5 + d^5 = 0$ with $a, b, c, d \in \mathbb{C}^*$. Then $\chi(Q_5) = -5\chi(Q_4) = -625$. As there is only one subtype of type 5, this results in a contribution $\chi_5 = \chi(Q_4) = -625$ to the total Euler number of the Fermat quintic.

As a result we have:

$$\begin{aligned} \chi(Q) &= \chi_2 + \chi_3 + \chi_4 + \chi_5 = 10\chi(Q_2) + 10\chi(Q_3) + 5\chi(Q_4) + \chi(Q_5) \\ &= 50 - 250 + 625 - 625 = -200. \end{aligned} \tag{5.4}$$

This is in harmony with the Hodge numbers, of any quintic, we already know. See (2.59). In the next section we will apply this specific choice of splitting up the Fermat quintic Q.

We note that any of the sets similar to Q_2 meets 3 sets similar to Q_3, and any of the sets similar to Q_3 meets 3 sets similar to Q_2. For example $[1:a:0:0:0]$ meets $[1:a:b:0:0]$, $[1:a:0:b:0]$ and $[1:a:0:0:b]$, and $[1:a:b:0:0]$ meets $[1:a:0:0:0]$, $[1:0:a:0:0]$ and $[0:1:a:0:0]$.

5.2 Mirror Symmetry and the Mirror Quintic

Mirror symmetry. If M is a strict Calabi-Yau 3-fold, then, according to a lemma, another strict Calabi-Yau 3-fold W exists, which satisfies

$$h^{1,1}(W) = h^{2,1}(M) \quad , \quad h^{2,1}(W) = h^{1,1}(M). \tag{5.5}$$

In general we can say: $h^{k,l}(W) = h^{3-k,l}(M)$ and $\chi(W) = -\chi(M)$. This is called *mirror symmetry*. Then we say W is the mirror of M, or (M, W) is a mirror pair of Calabi-Yau 3-folds. (Not *every* arbitrary W satisfying (5.5) is the manifold we are looking for. The specific properties of M and W are also involved, and W is the unique manifold related to M.)

If we examine superstring theory, compactified on these two manifolds M and W, then we will conclude that we are dealing with identical effective field theories. In the context of type IIA and type IIB superstring theory, we see that the following statements are closely related:

- Type IIA compactified on M is dual to type IIB compactified on W.
- Type IIB compactified on M is dual to type IIA compactified on W.

However, in this thesis we will restrict to type IIA theory only, but we will shortly have a look at mirror symmetry, just to examine some properties of M and W to check which one will be the 'easiest'. If we study non-perturbative massless theory, then, in case of type IIA theory, we will study supersymmetric 3-cycles, embedded in the Calabi-Yau 3-fold M or W. There are $b_3(M)$ of independent homology 3-cycles lying in M. If $b_3(W) < b_3(M)$, then maybe it is easier to study type IIA theory compactified on W instead of M.

If (M, W) is a mirror pair of Calabi-Yau 3-folds, then we should note that both $h^{1,1}(M) \geq 1$ and $h^{1,1}(W) \geq 1$. On any Calabi-Yau manifold the Kähler form is a non-trivial $(1,1)$-form. Thus, if W is the mirror of M, but if $h^{2,1}(M) = 0$, then $h^{1,1}(W) = 0$, so that W cannot be a Calabi-Yau 3-fold. To conclude, both M and W should satisfy $h^{1,1} \geq 1$ and $h^{2,1} \geq 1$.

The mirror quintic: Introduction. We will discuss the mirror quintic M_Q, as discussed in [7], thus M_Q is the mirror of the quintic Q. It is quite difficult to explicitly describe this M_Q, but we can at least introduce its construction. The first step will be to define a finite group G of holomorphic automorphisms $f : Q \to Q$. As we will work out in detail, this group G is abelian and isomorphic to $\mathbb{Z}_5^3 = (\mathbb{Z}/5\mathbb{Z})^3$, thus it has 125 elements. Then we can define the set $Q' := Q/G$ as a candidate manifold.

However, this set Q' still contains singularities. This means that a smooth atlas of Q' is not possible (yet). If Q'' is the maximal subset of Q' with all of the singularities removed, then a smooth atlas is possible, but then Q'' is not necessarily a compact manifold.

A *blow-up* is needed to replace these singularities by their smooth equivalents: we write $M_Q = b(Q')$, and this will again be a Calabi-Yau 3-fold. We will not mention the details of how to do these 'blow-ups': it is quite complicated to explicitly describe these. In [8], [11] and [17] more can be found in the context of 'orbifolds' and 'blow-ups'.

The group of automorphisms of the quintic: Step 1. Let first G_5 be a group of automorphisms of $\mathbb{C}^5$, and let f_i, $1 \geq i \geq 5$ be its generators. Let $\alpha := e^{2\pi i/5}$, so that $\alpha^5 = 1$, then the generators f_i are defined as the following actions:

$$\begin{aligned}
f_1 &: (z^0, z^1, z^2, z^3, z^4) \mapsto (\alpha z^0, z^1, z^2, z^3, z^4), \\
f_2 &: (z^0, z^1, z^2, z^3, z^4) \mapsto (z^0, \alpha z^1, z^2, z^3, z^4), \\
f_3 &: (z^0, z^1, z^2, z^3, z^4) \mapsto (z^0, z^1, \alpha z^2, z^3, z^4), \\
f_4 &: (z^0, z^1, z^2, z^3, z^4) \mapsto (z^0, z^1, z^2, \alpha z^3, z^4), \\
f_5 &: (z^0, z^1, z^2, z^3, z^4) \mapsto (z^0, z^1, z^2, z^3, \alpha z^4).
\end{aligned}$$

Then we see that each of the f_i are of order 5: $f_i^5 = 1$. Thus, the total group G_5 is isomorphic to $\mathbb{Z}_5^5$. We should note that this group is abelian, so that every subgroup is automatically a normal subgroup. We can use the following notation for any element f of G_5. If f is defined as

$$f : (z^0, z^1, z^2, z^3, z^4) \mapsto (\alpha^j z^0, \alpha^k z^1, \alpha^l z^2, \alpha^m z^3, \alpha^n z^4), \tag{5.6}$$

then we write $f = (j, k, l, m, n) = jf_1 + kf_2 + lf_3 + mf_4 + nf_5$. (We note that for example $j + 5 = j$.) From now on, as G_5 is abelian, if $f = (j, k, l, m, n)$ and $f' = (j', k', l', m', n')$ are elements of G_5, then we can write

$$f + f' = (j + j', k + k', l + l', m + m', n + n')$$

as the composition of these elements.

The group of automorphisms of the quintic: Step 2. This group G_5 can now be used to induce a group G_4 of automorphisms of $\mathbb{P}^4$. For any $(z^0, z^1, z^2, z^3, z^4) \in (C^5)^*$ we have $[Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \in \mathbb{P}^4$. (Here Z^j is just the homogenic version of z^j.)

We should note that the action $(1,1,1,1,1)$ in G_5, written as

$$(1,1,1,1,1) : (z^0, z^1, z^2, z^3, z^4) \mapsto (\alpha z^0, \alpha z^1, \alpha z^2, \alpha z^3, \alpha z^4),$$

induces the trivial action in G_4, as we can always write

$$[\alpha Z^0 : \alpha Z^1 : \alpha Z^2 : \alpha Z^3 : \alpha Z^4] = [Z^0 : Z^1 : Z^2 : Z^3 : Z^4].$$

In fact this means that we can restrict to 4 of the 5 generators. We can ignore generator f_1 and define the following generators of G_4:

$$\begin{array}{lcl} F_2 & : & [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : \alpha Z^1 : Z^2 : Z^3 : Z^4], \\ F_3 & : & [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : Z^1 : \alpha Z^2 : Z^3 : Z^4], \\ F_4 & : & [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : Z^1 : Z^2 : \alpha Z^3 : Z^4], \\ F_5 & : & [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : Z^1 : Z^2 : Z^3 : \alpha Z^4]. \end{array} \tag{5.7}$$

If we now define $F := 4F_2 + 4F_3 + 4F_4 + 4F_5$, then we also have

$$F : [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : \alpha^4 Z^1 : \alpha^4 Z^2 : \alpha^4 Z^3 : \alpha^4 Z^4] = [\alpha Z^0 : Z^1 : Z^2 : Z^3 : Z^4].$$

Thus any action f in G_5 induces an action F in G_4:

$$f = (j, k, l, m, n) \mapsto F = (K, L, M, N) = (k - j, l - j, m - j, n - j).$$

The abelian group G_4 of these actions F is isomorphic to $\mathbb{Z}_5^4$.

The group of automorphisms of the quintic: Step 3. As now $(\alpha^j)^5 = (\alpha^5)^j = 1^j = 1$, we also have $(\alpha^j Z^k)^5 = (\alpha^j)^5 (Z^k)^5 = (Z^k)^5$. This means that for any $F \in G_4$ and for any point p lying in the quintic Q, we also have $F(p) \in Q$. Thus $G_4 \simeq \mathbb{Z}_5^4$ also leaves the quintic invariant.

On the other hand, according to [13] and motivated by [8], we need an extra restriction on G_4, say $G \subset G_4$, as in our application every $F \in G$ must leave the holomorphic 3-form Ω invariant. (Not all elements in G_4 will satisfy.) This restriction will assure us that the resulting Ω' on Q/G will still be a nowhere vanishing holomorphic 3-form, automatically making Q/G into a Calabi-Yau 3-fold. At the same time any of these $F \in G$ should also be a holomorphic *isometry*, so that it makes sense to construct a resulting metric g' on Q/G from the original Kähler metric g defined on Q, and this g' will also be a Kähler metric.

None of the 4 standard generators F_j leaves Ω invariant. On the other hand, if we define the following 3 generators:

$$\begin{array}{lcl} g_1 & := & (1,0,0,4) : [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : \alpha Z^1 : Z^2 : Z^3 : \alpha^4 Z^4], \\ g_2 & := & (0,1,0,4) : [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : Z^1 : \alpha Z^2 : Z^3 : \alpha^4 Z^4], \\ g_3 & := & (0,0,1,4) : [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : Z^1 : Z^2 : \alpha Z^3 : \alpha^4 Z^4], \end{array}$$

then we see that G, generated by these g_j, is the maximal (abelian) subgroup G_4 of actions which still leave Ω invariant. For example g_1 will act on Ω as follows:

$$du \mapsto d(\alpha u) = \alpha du \quad , \quad x^4 \mapsto (\alpha^4 x)^4 = \alpha^{16} x^4 = \alpha x^4.$$

(This corresponds to (5.3), a definition of Ω on the desired patch.) According to [13] any of the actions contained in this group G already is a holomorphic isometry.

We finally obtain the group G, isomorphic to $\mathbb{Z}_5^3$. This result corresponds to [7]. Note that also the following automorphism $g_0 := (4,4,4,3)$ can be constructed, using g_1, g_2 and g_3:

$$g_0 := (4,4,4,3) = (4,4,4,48) = 4(1,1,1,12) = 4g_1 + 4g_2 + 4g_3.$$

This automorphism satisfies

$$g : [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [Z^0 : \alpha^4 Z^1 : \alpha^4 Z^2 : \alpha^4 Z^3 : \alpha^3 Z^4] = [\alpha Z^0 : Z^1 : Z^2 : Z^3 : \alpha^4 Z^4]. \tag{5.8}$$

Similarly we can express automorphisms like, for example, $(1,0,4,0)$ or $(4,1,0,0)$ using g_1, g_2 and g_3.

Fixed point sets of the automorphisms. In Section 5.1 we have introduced a way of splitting up the quintic into subsets containing 'type n' points. In general the effect of G will differ for each point in Q, but for any point lying in one of these special subsets Q_j, the effect of G will be the same. For any Q_j there exists a subgroup G_j of G, so that any $g \in G_j$ sends any $p \in Q_j$ to itself. We then say that p is a *fixed point* of the group G_j. Let now H_j be a group so that $G_j \oplus H_j = G$, then we only need to take the quotient $Q_j/G = Q_j/H_j$. (We should also note that any of the Q_j is *invariant* under the total group G, so that Q_j/G makes sense.) This will make it easy to take the quotient of any of these complete subsets of Q by G.

We note that none of the 'type 4' and 'type 5' points are fixed, as they have at least 4 non-zero components: none of the generators g_1, g_2 or g_3 will map any of these points to itself. Then $\chi(Q_4/G) = \chi(Q_4)/125$ and $\chi(Q_5/G) = \chi(Q_5)/125$. Only points of type 2 and 3 remain.

The example set $Q_2 = \{[1 : a : 0 : 0 : 0]\}$, a set of type 2, contains 5 points: $\chi(Q_2) = 5$. It is fixed by the generators $(0,1,0,4)$ and $(0,0,1,4)$, thus $Q_2/G = Q_2/\mathbb{Z}_5$, so that $\chi(Q_2/G) = \chi(Q_2)/5$. We note that $\{[1 : 0 : a : 0 : 0]\}$, another set of type 2, is similar to Q_2, and it is only fixed by the generators $(1,0,0,4)$ and $(0,0,1,4)$. We also note that $\{[0 : 1 : a : 0 : 0]\}$ is similar to Q_2, and it is only fixed by the generators $(4,4,4,3)$ (see (5.8)) and $(0,0,1,4)$. (We can do similar things in case of all the other 'type 2' points.)

The example set $Q_3 = \{[1 : a : b : 0 : 0]\}$, a set of type 3, is a Riemann surface of genus 6 with 15 points missing: $\chi(Q_3) = -25$. Indeed, if we take the union of this Q_3 and the three sets similar to Q_2,

$$\{[1 : a : 0 : 0 : 0]\}, \quad \{[1 : 0 : a : 0 : 0]\}, \quad \{[0 : 1 : a : 0 : 0]\},$$

then we obtain a Riemann surface of genus 6 (see (2.34)). The set Q_3 is fixed by the generator $(0,0,1,4)$, thus $Q_3/G = Q_3/\mathbb{Z}_5^2$, so that $\chi(Q_3/G) = \chi(Q_3)/25$. (We can do similar things in case of all the other 'type 3' points.)

We note that any subset of Q similar to Q_2 is a set of fixed points of some subgroup of G. Similarly, any subset of Q similar to Q_3 is a fixed curve of some subgroup of G.

Constructing the mirror quintic. Now we know the effect of taking the quotient of any of the chosen subsets of Q by the group G, we know how to construct Q/G itself. As none of the points of type 4 and 5 are fixed, we know that $Q_4/G = G_4/\mathbb{Z}_5^3$ and $Q_5/G = G_5/\mathbb{Z}_5^3$. Then we can compute $\chi(Q/G)$:

$$\begin{aligned} \chi(Q/G) &= 10\chi(Q_2/\mathbb{Z}_5) + 10\chi(Q_3/\mathbb{Z}_5^2) + 5\chi(Q_4/\mathbb{Z}_5^3) + \chi(Q_5/\mathbb{Z}_5^3) \\ &= 10\frac{\chi(Q_2)}{5} + 10\frac{\chi(Q_3)}{25} + \frac{5\chi(Q_4) + \chi(Q_5)}{125} = 10\frac{\chi(Q_2)}{5} + 10\frac{\chi(Q_3)}{25}. \end{aligned} \tag{5.9}$$

This is not the mirror quintic yet, as we still need to do the blow-ups.

We need to insert blow-up factors for the fixed points and fixed curves. From [7] we can sort of derive that the blow-up factor related to any fixed point is 25, and related to any fixed curve is 5. (However, in [7] *no* clear explanation or reference is mentioned.) Then we can compute the Euler number $\chi(b(Q/G))$, derived from (5.9).

$$\begin{aligned} \chi(b(Q/G)) &= 10\chi(b(Q_2/\mathbb{Z}_5)) + 10\chi(b(Q_3/\mathbb{Z}_5^2)) = 10 \cdot 25\frac{\chi(Q_2)}{5} + 10 \cdot 5\frac{\chi(Q_3)}{25} \\ &= 50\chi(Q_2) + 2\chi(Q_3) = 50 \cdot 5 + 2 \cdot (-25) = 250 - 50 = 200 = -\chi(Q). \end{aligned}$$

According to [7] this manifold $M_Q = b(Q/G)$ should indeed be the mirror quintic. The Hodge numbers are reversed then: $h^{1,1}(M_Q) = h^{2,1}(Q) = 101$ and $h^{2,1}(M_Q) = h^{1,1}(Q) = 1$. Thus the Hodge diamond of the mirror quintic is directly related to the Hodge diamond of the quintic itself, expressed in (2.59). This yields the following Betti numbers:

$$b_3(Q) = 2(h^{3,0}(Q) + h^{2,1}(Q)) = 204 \quad , \quad b_3(M_Q) = 2(h^{3,0}(M_Q) + h^{2,1}(M_Q)) = 4.$$

This indeed gives us the inequality $b_3(M_Q) << b_3(Q)$. In theory this should strongly simplify the quest for independent homology 3-cycles. In practice it is still very complicated to do some explicit research in this topic. (We could say that the mirror quintic M_Q is still quite an artificial Calabi-Yau 3-fold.) In Chapter 7 we will not discuss SLags lying in the mirror quintic, only in the quintic itself.

In [13] we can read that, in general, not every manifold W having Hodge numbers reversed, compared to the original manifold M, should be regarded as the *mirror manifold* of M. There are also other technical constraints to make sure that W is the mirror manifold of M. In this specific case however, we are indeed dealing with the mirror manifold M_Q of Q.

Taking the quotient of a general Calabi-Yau 3-fold by a group of automorphisms. We should note that the group G, acting on the quintic, only has fixed points and fixed curves. It has no fixed hypersurfaces. If M is a general Calabi-Yau 3-fold and if G is an arbitrary group of automorphisms from M to itself, then G *can* have fixed hypersurfaces.

In [8] we can read that if $N \subset M$ is a hypersurface, thus if $\dim_{\mathbb{C}}(N) = 2$, then the quotient M/G is *not* singular at N. A blow-up is not needed then. Thus, in general, for investigating all the blow-ups, we only need to study the fixed points and fixed curves anyway.

We can have a similar look at Riemann surfaces. If M is a Riemann surface, then any non-trivial holomorphic automorphism $g : M \to M$ can only have fixed points. If G is the group generated by g, then M/G has no singularities: a coordinate redefinition in a neighbourhood of a fixed point p is sufficient for removing the 'fake' singularity. A coordinate redefinition will not help us in case of fixed points and fixed curves in a Calabi-Yau 3-fold.

6 Special Lagrangian Submanifolds

In this chapter we will introduce special Lagrangian submanifolds, or SLags, mainly in a mathematical sense. The definitions of SLags are already introduced in many other already existing articles, using many different approaches. Here we will mainly follow the definitions of [21]. In Section 6.1 we start with the formal definition of SLags. We will also work out the four related properties, or 'features', in much detail. The most important feature is the second one: we see that a SLag has minimal volume, so that we can directly find a membrane instanton. At the end of Section 6.1 we will discuss the problems we can encounter, when trying to find SLags. In Section 6.2, 6.3 and 6.4 we will introduce basic examples of SLags, in the tori T^2, T^4 and T^6, thus in Calabi-Yau manifolds of complex dimension 1, 2 and 3. In Section 6.5 we will define the Borcea-Voisin construction, or the BV construction. If M_1 and M_2 are strict Calabi-Yau manifolds, of complex dimension m_1 and m_2 respectively, then we can construct a new *strict* Calabi-Yau manifold, of complex dimension $m_1 + m_2$. Then we can use simpler Calabi-Yau manifolds, of lower dimension, to construct a Calabi-Yau 3-fold, which will be interesting in superstring theory. (We will continue in Chapter 8.)

6.1 Definition and Features

Of special interest in the search for membrane instantons are so-called *special Lagrangian submanifolds*, or simply *SLags*, of a Calabi-Yau manifold M on which we would like to compactify our type IIA superstring theory. Before describing what exactly a SLag is, it is important to realize why a physicist is interested in SLags at all. Just in words, if M is a Calabi-Yau 3-fold, thus with real dimension 6, if $L \subset M$ is a submanifold of (according to [16]) real dimension 3 and if this L satisfies certain properties called the *SLag conditions*, then it is a good candidate for wrapping a supersymmetric membrane instanton around it.

Any membrane instanton in a Calabi-Yau 3-fold M can be described as a smooth surjective map

$$m : X \to L \subset M,$$

where X is a compact Euclidean domain space and where L is some SLag in M. Then the map m describes a Euclidean D2-brane. This map m can be injective, but it does not need to be, so it can indeed describe wrapped D2-branes as well. We will also discuss SLags in Calabi-Yau manifolds of other dimensions.

A nice definition of SLags and some properties and examples can be found in [21]. However, I would like to give a detailed description of the SLag conditions in mathematical terms, and tell why these conditions are desired for theoretical physics.

Formal definition. First we should look at the formal definition which is quite short. Let M be a Calabi-Yau m-fold, let g be its Ricci-flat metric, let ω be its Kähler form and let Ω be a holomorphic $(m, 0)$-form. Then we write (M, g, ω, Ω). Now let L be a smooth submanifold of M, and let $\iota : L \to M$ be its canonical embedding. According to the definitions introduced in much detail by [16], this L should satisfy $\dim_{\mathbb{R}}(L) = \dim_{\mathbb{C}}(M)$. Then we can pull back g, ω and Ω onto L. There are three conditions such an L can satisfy:

1) The pullback of ω on L vanishes, or $\iota^*\omega = 0$.

2) The pullback of Ω on L can be written as a real nowhere vanishing m-form times a constant phase factor. In other words, there exists a constant *angle* θ such that $\mathrm{Im}(e^{i\theta}\iota^*\Omega) = 0$.

3) L is compact, so that it has a finite volume.

For any vector $V \in T_pL$ we see that $\iota_* V = V \in T_pM$. The first condition means that for any pair of vectors $V, W \in T_pL$, we have $\iota^*\omega(V, W) = \omega(\iota_*V, \iota_*W) = \omega(V, W) = 0$.

If L only satisfies the first condition, then it is called a *Lagrangian submanifold.* If L also satisfies the second condition, then it is called a *special Lagrangian submanifold.* The third condition is important for the study of instantons. If M is compact and if L is a compact submanifold of M, then its volume is finite.

The instanton action is nothing more than this volume, up to a constant prefactor. For any M there is a possibility to find many L which satisfy the first two conditions, but which are not necessarily compact. These L will have infinite volume, thus an infinite action. These must be excluded when studying the application of SLags in the quest for instantons. From now on we use the substantive *SLag* when L satisfies all three listed conditions.

We should note that any SLag L should be an orientable submanifold of M. This condition is explicitly written in [16]. Only if L is orientable a suitably chosen orientation is possible, so that Ω induces a positive volume form on L.

We should also note that especially the second condition causes a strong constraint. We should recall that if Ω is one of the holomorphic m-forms on M, then any other Ω' holomorphic m-form is just a constant scalar multiple of Ω, thus there is no way to 'deform' Ω. Thus, if $M = (M, J, \omega, \Omega)$ and $M' = (M, J, \omega, \Omega')$ are different Calabi-Yau m-folds, then they contain the same SLags. Thus, to make sure that $\mathrm{Im}(e^{i\theta}\iota^*\Omega)$ vanishes, we are restricted to very special submanifolds of M. To conclude, if it is hard to find SLags with respect to one Ω, then it will not be easier to find SLags with respect to another Ω'.

We could say that if we have an arbitrary L, which is not a SLag yet, then we can deform it, to another manifold L', until it satisfies the SLag conditions. Then this deforming is equivalent to finding an L' with minimized volume. (In some specific cases of M many different results of finding such an L' are possible, after starting with the same arbitrary L.)

The SLag features. Now we will discuss the main features of SLag geometry, shortly introduced in [21] and to be derived from the main SLag conditions. In the first feature *normal spaces* are mentioned. There exist slightly different conceptual definitions of normal spaces. We will use the following definition which fits best in the context of differential geometry. Let L be a submanifold of M. Then for any $p \in L$ we know that also $T_pL \subset T_pM$. The *normal space* at p of L, denoted by N_pL, is the set of all vectors $V \in T_pM$ satisfying $g(V, W) = 0$ for all $W \in T_pL$. This means that T_pL and N_pL are perpendicular with respect to the metric. Now we can write $T_pM = T_pL \oplus N_pL$. In this case $\dim_{\mathbb{R}}(T_pL) = \dim_{\mathbb{R}}(N_pL)$ as $\dim_{\mathbb{R}}(M) = 2\dim_{\mathbb{C}}(M) = 2\dim_{\mathbb{R}}(L)$. In the second feature we will mention a real top-form $\Omega_{L,\theta} := e^{i\theta}\iota^*\Omega$. (It satisfies $\mathrm{Im}(\Omega_{L,\theta}) = 0 \Rightarrow \Omega_{L,\theta} = \mathrm{Re}(\Omega_{L,\theta})$ if the second SLag condition is satisfied.) This $\Omega_{L,\theta}$ is also a nowhere vanishing form, thus it can be used as an alternative volume form.

SLag feature 1. The Kähler form ω yields a pointwise isomorphism m from TL to N^*L, thus from T_pL to N_p^*L for any p.

Proof: A thing to note is that ω is *non-degenerate*, as, by definition, for any pair of vectors $V, W \in T_pM$, we have $\omega(V, W) = g(JV, W)$. The Riemannian metric g is also non-degenerate and the complex structure J is an isomorphism from T_pM to itself.

We know that for any $V \in T_pL$ the map $\omega(V, \cdot)$ maps from T_pM to $\mathbb{R}$, thus $\omega(V, \cdot) \in T_p^*M = T_p^*L \oplus N_p^*L$. Knowing that $\omega(V, W) = 0$ for any $W \in T_pL$ we can say that $\omega(V, \cdot) \in N_p^*L$. Now knowing that ω is non-degenerate we can say that the map $m_p : V \mapsto \omega(V, \cdot)$ is injective: if $\omega(V_1, \cdot) = \omega(V_2, \cdot)$ for some $V_1, V_2 \in T_pL$, thus if $\omega(V_1, W) = \omega(V_2, W)$ for all $W \in N_pL$, then $V_1 = V_2$. As the domain and codomain of m_p have the same dimension, thus $\dim_{\mathbb{R}}(T_pL) = \dim_{\mathbb{R}}(N_p^*L)$, we can say that m_p is also surjective. Then m_p is indeed an isomorphism.

SLag feature 2. The SLag L is 'calibrated' by $\Omega_{L,\theta} = e^{i\theta}\iota^*\Omega$. (See [16] for the definition of 'calibrated submanifolds'.) This means that $\Omega_{L,\theta}$, which is an alternative volume form, is absolutely volume minimizing with respect to the metric $h := \iota^*g$, the pullback of the Ricci-flat metric g. In other words, $e^{i\theta}\iota^*\Omega$ equals the volume form $*1$, defined on L and induced by h, up to some constant c. Thus $\Omega_{L,\theta} = e^{i\theta}\iota^*\Omega = *_{L,h}c$.

Proof: We note that L can be described as a real manifold. Let σ^α be real coordinates at some $p \in L \subset M$ and let z^m be complex coordinates at $p = \iota(p) \in M$. Then we can explicitly pull back the Kähler form to L

and apply $\iota^*\omega = 0$:

$$\begin{aligned}
\iota^*\omega &= \iota^*(ig_{m\bar{n}}dz^m \wedge d\bar{z}^n) = ig_{m\bar{n}}(\iota^*(dz^m) \wedge \iota^*(d\bar{z}^n)) = ig_{m\bar{n}}(\frac{\partial z^m}{\partial\sigma^\alpha}d\sigma^\alpha \wedge \frac{\partial\bar{z}^n}{\partial\sigma^\beta}d\sigma^\beta) \\
&= ig_{m\bar{n}}\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta}(d\sigma^\alpha \otimes d\sigma^\beta - d\sigma^\beta \otimes d\sigma^\alpha) = ig_{m\bar{n}}(\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta}d\sigma^\alpha \otimes d\sigma^\beta - \frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta}d\sigma^\beta \otimes d\sigma^\alpha) \\
&= ig_{m\bar{n}}(\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta}d\sigma^\alpha \otimes d\sigma^\beta - \frac{\partial z^m}{\partial\sigma^\beta}\frac{\partial\bar{z}^n}{\partial\sigma^\alpha}d\sigma^\alpha \otimes d\sigma^\beta) = ig_{m\bar{n}}(\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta} - \frac{\partial z^m}{\partial\sigma^\beta}\frac{\partial\bar{z}^n}{\partial\sigma^\alpha})d\sigma^\alpha \otimes d\sigma^\beta.
\end{aligned}$$

Now, if $\iota^*\omega = 0$, then for all α and β the following holds:

$$ig_{m\bar{n}}(\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta} - \frac{\partial z^m}{\partial\sigma^\beta}\frac{\partial\bar{z}^n}{\partial\sigma^\alpha}) = 0 \quad\Rightarrow\quad \frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta}g_{m\bar{n}} = \frac{\partial z^m}{\partial\sigma^\beta}\frac{\partial\bar{z}^n}{\partial\sigma^\alpha}g_{m\bar{n}}. \tag{6.1}$$

Now we can pull back the metric g to L, thus $h := \iota^*g$. If we use (2.38), then

$$\begin{aligned}
h &= \iota^*g = g_{m\bar{n}}(\iota^*(dz^m) \otimes \iota^*(d\bar{z}^n) + \iota^*(d\bar{z}^n) \otimes \iota^*(dz^m)) = g_{m\bar{n}}\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta}(d\sigma^\alpha \otimes \sigma^\beta + \sigma^\beta \otimes \sigma^\alpha) \\
&= g_{m\bar{n}}(\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta} + \frac{\partial z^m}{\partial\sigma^\beta}\frac{\partial\bar{z}^n}{\partial\sigma^\alpha})d\sigma^\alpha \otimes \sigma^\beta = h_{\alpha\beta}d\sigma^\alpha \otimes d\sigma^\beta.
\end{aligned}$$

Now, if we apply (6.1), then we finally obtain the following identity:

$$h_{\alpha\beta} = 2\frac{\partial z^m}{\partial\sigma^\alpha}\frac{\partial\bar{z}^n}{\partial\sigma^\beta}g_{m\bar{n}} = 2\partial_\alpha z^m\partial_\beta\bar{z}^n g_{m\bar{n}}. \tag{6.2}$$

Let now G be the (complex) determinant of $g_{m\bar{n}}$, see also (2.44) and (2.16). Let h be the *real* determinant of $h_{\alpha\beta}$, or $h := \det(h_{\alpha\beta}) > 0$, and let $\det(\partial z)$ be the determinant of the matrix $\partial_\alpha z^\mu$. Then h can be rewritten as

$$h = 2^m \det(\partial z)\det(\partial\bar{z})G = 2^m|\det(\partial z)|^2 G \quad\Rightarrow\quad \sqrt{h} = 2^{m/2}|\det(\partial z)|\sqrt{G}. \tag{6.3}$$

This defines the canonical volume form

$$*_{L,h}1 := \sqrt{h}d\sigma^1 \cdots d\sigma^m.$$

We can write $\Omega = f dz^1 \wedge \cdots dz^m$, using the notation introduced in (2.31). Then we can explicitly pull back Ω to L:

$$\iota^*\Omega \quad=\quad f(\iota^*(dz^1) \wedge \cdots \iota^*(dz^m)) = f\frac{\partial z^1}{\partial\sigma^{\alpha_1}}\cdots\frac{\partial z^m}{\partial\sigma^{\alpha_m}}d\sigma^{\alpha_1} \wedge \cdots \wedge d\sigma^{\alpha_m} = f\det(\partial z)\epsilon,$$

where $\epsilon = d\sigma^1 \wedge \cdots \wedge d\sigma^m$. We know that both f and $\det(\partial z)$ are nowhere vanishing, thus also $f\det(\partial z) \neq 0$. As we want $\Omega_{L,\theta} = e^{i\theta}f\det(\partial z)\epsilon$ to be real, we can rewrite $f\det(\partial z)$ as

$$f\det(\partial z) = e^{i\varphi}|f\det(\partial z)| = e^{i\varphi}|f||\det(\partial z)|,$$

where either $\varphi = -\theta$ or $\varphi = \pi - \theta$. Then

$$\Omega_{L,\theta} = e^{i\theta}e^{i\varphi}|f||\det(\partial z)|\epsilon = \pm|f||\det(\partial z)|\epsilon = |f||\det(\partial z)|(\pm\epsilon).$$

(As L is orientable, we can choose a proper orientation in which $\Omega_{L,\theta}$, thus $\pm\epsilon$, is positive.) Now knowing that (2.57) implies that $|f| = \sqrt{|f|^2} = \sqrt{\|\Omega\|^2 G}$, $\Omega_{L,\theta}$ can be rewritten as

$$\Omega_{L,\theta} = \pm\sqrt{\|\Omega\|^2 G}|\det(\partial z)|\epsilon = \pm 2^{-m/2}\sqrt{\|\Omega\|^2}\cdot 2^{m/2}|\det(\partial z)|\sqrt{G}\epsilon = \pm 2^{-m/2}\sqrt{\|\Omega\|^2}\sqrt{h}\epsilon = *_{L,h}c,$$

where $c = \pm 2^{-m/2}\sqrt{\|\Omega\|^2}$ is a constant. (We already know that $\|\Omega\|^2$ is a constant with respect to the Ricci-flat metric.) Thus indeed $\Omega_{L,\theta} = *_{L,h}c$, so that L is calibrated by $\Omega_{L,\theta}$.

In [10] we can see a similar (but reversed) proving scheme in the question whether a specific subspace of the Fermat quintic can be regarded as a supersymmetric 3-cycle, thus as a SLag.

SLag feature 3. The local deformation theory of a compact SLag L is directly related to topological properties of L. According to a theorem, see [14] and [16], the deformations of L are in one-to-one correspondence with the harmonic 1-forms on L. In other words, Hodge's theorem tells us that locally the deformation moduli space $\mathcal{M}_D(L)$ of L is a smooth manifold of real dimension $b_1(L)$, thus $\dim_{\mathbb{R}}(\mathcal{M}_D(L)) = b_1(L)$. (We should note that $\mathcal{M}_D(L)$ is the "*connected component of the set of special Lagrangian m-folds containing* L", see [16]; that is what *locally* seems to mean.)

This 'deformation' can be interpreted as a very rigid shifting of the SLag in question. We should also note that there is a constraint: if L is continuously shifted to another SLag $L' \subset M$, then the angle θ should also remain constant and the family of manifolds between L and L' should also be SLags, with respect to the same angle.

A family of SLags which could be smoothly deformed into each other, will be called a *deformation class*, and in general a Calabi-Yau space should carry a whole spectrum of possible deformation classes. This spectrum is closely related to the topological properties of the Calabi-Yau space itself. Each class should at least contain one SLag, but it is not immediately clear whether this SLag is unique. The deformation class of one SLag could contain a whole subfamily of manifolds which are all SLags. To conclude: the dimension of this subfamily is directly related to topological properties of the initial SLag.

In Section 6.2, 6.3 and 6.4 we will discuss some examples of SLag's in tori. Then we will clearly see how the SLags can be deformed.

SLag feature 4. The Kähler form 'calibrates' any holomorphic submanifold N of M. We already know that for any such N we can pull back ω to N, so that N is again a Kähler manifold. If $n = \dim_{\mathbb{C}}(N)$, if $\iota : N \to M$ is the canonical embedding and if $\omega_N := \iota^*\omega$, then $\omega_N^n = \omega_N \wedge \cdots \wedge \omega_N$ again defines a volume form on N, see (2.41).

We should note that none of the SLags can be interpreted as a holomorphic submanifold of M, simply because then $\iota^*\omega = 0$. This especially also means that a SLag L itself *cannot* contain a (part of a) holomorphic submanifold of M. In a pointwise manner, none of the tangent spaces T_pL contains a holomorphic plane.

SLags with multiple connected components. Let L be a SLag in M and let $L = L_1 \cup L_2$, with $L_1 \cap L_2 = \varnothing$. Then each of the sets L_1 and L_2 is also a SLag in M. The other way round, if now L_1 and L_2 are SLags with respect to θ, and if $L_1 \cap L_2 = \varnothing$, then we can construct a new SLag $L = L_1 \cup L_2$ with respect to the same θ. However, in general we cannot combine two arbitrary SLags: if L_j are SLags with respect to θ_j, and if $\theta_1 \neq \theta_2$ (modulo π), then the resulting set $L = L_1 \cup L_2$ cannot be a SLag.

How to find SLags. For a general Calabi-Yau space it is not straightforward to find all possible SLags, but some of them can be trivially constructed, and I would like to discuss these.

In theory, to find any SLag in a Calabi-Yau 3-fold M we could start with an arbitrary compact membrane embedded in M. On this membrane we can define an action which contains a term which is similar to a 3-dimensional Nambu-Goto action, and thus is nothing more than the volume of this membrane. Note however that this action is not invariant under Weyl rescaling like in two dimensions, but that is not needed anyway, because we will not try to quantize the field equations corresponding to the membrane action. We could try to smoothly deform this membrane until it has minimal volume, or even better, until the SLag conditions are satisfied.

Physicists often tend to interpret a deformation as an infinitesimal deformation of a metric, while keeping the actual set the same. Here, the deformation of membranes should really be regarded as a process of deforming a set into another set nearby. However these deformations are still smooth, and could be described by a one-parameter group of local diffeomorphisms. The geometry of the membrane is then induced by the ambient space metric, and in this sense not only the metric changes, but also the set itself. This procedure of deforming however requires a lot of computations in general, so we just start off with choosing a possible candidate and check whether it satisfies the SLag conditions. When we conclude that this membrane is a SLag, it can be regarded as contributing to our study of membrane instantons.

So how will we find possible candidates? We will start with the simplest Calabi-Yau space we could imagine, which is the complex torus, a space of dimension 2, or complex dimension 1. It turns out that all SLags in the torus are known, and can be represented by straight lines wrapping it. (See Section 6.2.) Other relatively simple Calabi-Yau spaces are the *quartic* of complex dimension 2, which is actually a K3 surface, and the *quintic* of complex dimension 3. (See later chapters.) These examples can all be studied by looking at their algebraic properties, and finding some SLags in them is not that complicated. The context of so-called 'algebraic geometry' fits best in this research area.

To make any decent and complete conclusion about the contribution of SLags to the study of membrane instantons, we would actually need knowledge of all SLags a Calabi-Yau space could contain. Unfortunately, we only know the complete spectrum of SLags in case of a few trivial Calabi-Yau spaces.

6.2 SLags in T^2

The complex torus. Fortunately we have an extremely simple example to start with. The complex torus T^2 can be represented as a Riemann surface, which is a complex manifold of (complex) dimension 1. It is also compact and closed, i.e. it has no boundary, and of course it is oriented. It is possible to define a Kähler form ω on T^2 which is related to a Ricci-flat metric g, and a nowhere vanishing holomorphic $(1,0)$-form Ω. If T^2 is equipped with all this, then it is a Calabi-Yau manifold.

Let τ be a non-zero complex number satisfying $\mathrm{Im}(\tau) > 0$. Then the group $G_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}$ defines a lattice. Any complex torus can be identified with the space $\mathbb{C}/G_\tau$ for some τ, using a biholomorphism. The value of τ determines a complex structure, and it is a single coordinate in the moduli space of complex tori. All tori are diffeomorphic, but if two tori have different values of τ then they are not biholomorphic. (We should note that here τ and τ' are said to be *different* if $\tau \neq \tau'$ *and* if no modular transformation exists, transforming τ into τ'.)

Making the complex torus into a Calabi-Yau manifold. The standard complex coordinate $z = x+iy$, defined on $\mathbb{C}$, induces a complex coordinate on T^2. Then $z \sim z+1$ and $z \sim z+\tau$ on T^2. The (trivial) metric, the Kähler form and the holomorphic top-form are very simple:

$$g := \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = dx \otimes dx + dy \otimes dy, \quad \omega := \tfrac{i}{2} dz \wedge d\bar{z} = dx \wedge dy, \quad \Omega := dz = dx + i dy.$$

The complex structure is written in the standard way, see (2.29), and $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. (We see that g is a Ricci-flat Kähler metric. It is even a *flat* metric.) These definitions are valid globally, and these make T^2 into a Calabi-Yau 1-fold.

Candidate SLags in the complex torus. Let L be an arbitrary closed smooth 1-dimensional submanifold of T^2, canonically embedded with the map $\iota : L \to T^2$. It is obvious that its tangent space T_pL is also of dimension 1, thus any two (non-zero) tangent vectors V, W are related by a scalar factor: $W = \lambda V$. This means that

$$\iota^*\omega(V,W) = \omega(\iota_* V, \iota_* W) = \omega(V,W) = \lambda\omega(V,V) = 0.$$

The last identity holds because ω is antisymmetric. Thus, the requirement $\iota^*\omega = 0$ is vacuous: L automatically satisfies the first SLag condition. (This confirms the idea that only the first SLag condition is not sufficient for making sure that L has a minimal volume.)

To make sure that $\Omega_{L,\theta}$, the pullback of Ω on L, is a real multiple of some constant phase factor θ, we need to introduce an additional constraint on L. It is only possible to satisfy the second SLag condition when all tangent spaces T_pL are parallel, which means that then we are restricted to straight lines.

To make sure that L is a closed manifold we only need to restrict to straight lines which wrap an arbitrary integer number around any of the two lattice cycles. Otherwise L will not be compact, making up a dense subset of T^2, which is not what we can use. When L is dense, it has infinite length, but for the study of instantons we need a finite value for the action, which is excluded in this case.

We note that these compact straight lines represent non-trivial 1-cycles embedded in T^2, with a 'minimal volume', at least with respect to the flat standard metric.

Candidate SLags in the complex torus: An explicit definition. The explicit definition of any curve which possibly is a compact SLag in T^2, is given by

$$z(t) = z_0 + vt,$$

where $v \in \mathbb{C}^*$. Then the curve L, with real coordinate t, is closed if there exists a $t_0 \neq 0$ and a pair of integers m and n such that $z(t_0) - z(0) = vt_0 = m + n\tau$. (The case $m = n = 0$ will be ignored.) It should be clear that not for all v we can find such a triple (t_0, m, n) making sure that L is closed. Note that, if L is compact, then v can be rescaled, $v \mapsto vt_0$, so from now on we assume $t_0 = 1$. Then we assume $v = m + n\tau$. Now we can express $\Omega_{L,\theta}$ as follows:

$$\iota^*\Omega = \iota^*(dz) = \frac{\partial z}{\partial t}dt = vdt \quad \Rightarrow \quad \Omega_{L,\theta} = e^{i\theta}vdt.$$

Then L is a candidate SLag if $e^{i\theta}v \in \mathbb{R}$. For any θ a v can be found such that L satisfies the first and the second SLag conditions, but L will not be a closed curve for all θ. Thus L is only *compact* if $v = m + n\tau$. In other words, not for all θ a compact SLag can be found. However, we have countably infinitely many possibilities for choosing $(m, n) \in \mathbb{Z}^2 - \{(0, 0)\}$. We know that we can rewrite $m + n\tau$ as $\rho e^{i\varphi}$. If we define either $\theta = -\varphi$ or $\theta = \pi - \varphi$, then we found a compact SLag. Then we can find countably infinitely many SLags, embedded in the complex torus.

Candidate SLags in the complex torus: An example. We can describe one of the simplest examples of a SLag in T^2 as follows. We already assume that $t_0 = 1$ and we define $z_0 := (0, 0)$ and $(\tau, m, n) := (i, 1, 0)$ so that $v = 1$. Then these numbers define $z(t) := t$ and the SLag $L := \{z(t) \in T^2 | t \in [0, 1)\}$. According to the second SLag condition $e^{i\theta}v = e^{i\theta}$ should be a real number. Then L is a SLag with respect to $\theta = 0$ or $\theta = \pi$.

Deformations of SLags in the complex torus. Any of these SLags in T^2 are compact connected manifolds L with $\dim_{\mathbb{R}}(L) = 1$, thus any L is diffeomorphic to the standard circle S^1. Knowing that $b_1(L) = b_1(S^1) = 1$ and applying SLag feature 3, as discussed in Section 6.1, we may conclude that there is a 1-dimensional moduli space of deformations, or $\dim_{\mathbb{R}}(\mathcal{M}_D(L)) = 1$.

If L is a SLag defined by $z(t) = z_0 + vt$ and if L' is another SLag defined by $z'(t) = z_1 + vt$, and assume they do not overlap (i.e. $z_0 \notin L'$ and $z_1 \notin L$), then we can say that L' can be obtained by deforming L. We immediately see that, for any v, there is a 1-dimensional smooth family of SLags which can be deformed into each other. This is indeed in harmony with $\dim_{\mathbb{R}}(\mathcal{M}_D(L)) = 1$. This family of SLags is even a closed set, and we note that all of the SLags in the same family also have the same 'volume'. We conclude that in total there are countably infinitely many 1-dimensional families of connected SLags, embedded in the complex torus. (Thus, there are uncountably infinitely many SLags in total.)

6.3 SLags in T^4

The trivial 4-torus. In this section we will study SLags in the trivial 4-torus T^4. We define $G := \mathbb{Z} \oplus i\mathbb{Z}$ and the trivial 2-torus $T^2 := \mathbb{C}/G$. Then $T^4 := T^2 \times T^2 = \mathbb{C}^2/(G \times G)$. The standard coordinates $(z^1, z^2) = (x^1 + iy^1, x^2 + iy^2)$ of $\mathbb{C}^2$ induce coordinates on T^4. We can define the (trivial flat) metric, the Kähler form and the holomorphic top-form:

$$g := \tfrac{1}{2} \textstyle\sum_{j=1,2} (dz^j \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^j), \quad \omega := \tfrac{i}{2} \textstyle\sum_{j=1,2} dz^j \wedge d\bar{z}^j, \quad \Omega := dz^1 \wedge dz^2.$$

These definitions make T^4 into a Calabi-Yau 2-fold. (Note that T^2 is a strict Calabi-Yau 1-fold but T^4 is not a strict Calabi-Yau 2-fold.)

Candidate SLags in the complex 4-torus. We will study the following candidate SLags. Let α, β, γ and δ be real integer parameters. Then we define

$$y^1(x^1, x^2) = \alpha x^1 + \beta x^2 \quad , \quad y^2(x^1, x^2) = \gamma x^1 + \delta x^2.$$

Then the graph

$$(z^1(x^1, x^2), z^2(x^1, x^2)) = (x^1 + i(\alpha x^1 + \beta x^2), x^2 + i(\gamma x^1 + \delta x^2)) \simeq (x^1, \alpha x^1 + \beta x^2, x^2, \gamma x^1 + \delta x^2) \in \mathbb{R}^4$$

describes a compact 2-dimensional surface L embedded in T^4 and parametrized by the real coordinates x^j.

Candidate SLags in the complex 4-torus: The pullback of the Kähler form. Now we can explicitly write $\iota^*\omega$ with respect to x^j. We know that

$$\partial z^1/\partial x^1 = 1 + i\alpha, \quad \partial z^1/\partial x^2 = i\beta, \quad \partial z^2/\partial x^1 = i\gamma, \quad \partial z^2/\partial x^2 = 1 + i\delta. \tag{6.4}$$

Then

$$\begin{aligned} \iota^*\omega &= \frac{i}{2}\sum_{j=1,2}(\iota^*(dz^j) \wedge \iota^*(d\bar{z}^j)) = \frac{i}{2}\sum_{j=1,2}(\frac{\partial z^j}{\partial x^1}dx^1 + \frac{\partial z^j}{\partial x^2}dx^2) \wedge (\frac{\partial \bar{z}^j}{\partial x^1}dx^1 + \frac{\partial \bar{z}^j}{\partial x^2}dx^2) \\ &= \frac{i}{2}\sum_{j=1,2}(\frac{\partial z^j}{\partial x^1}\frac{\partial \bar{z}^j}{\partial x^2} - \frac{\partial z^j}{\partial x^2}\frac{\partial \bar{z}^j}{\partial x^1})dx^1 \wedge dx^2. \end{aligned}$$

Then (6.4) implies that

$$\begin{aligned} \frac{\partial z^1}{\partial x^1}\frac{\partial \bar{z}^1}{\partial x^2} - \frac{\partial z^1}{\partial x^2}\frac{\partial \bar{z}^1}{\partial x^1} &= (1 + i\alpha)(-i\beta) - i\beta(1 - i\alpha) = -2i\beta, \\ \frac{\partial z^2}{\partial x^1}\frac{\partial \bar{z}^2}{\partial x^2} - \frac{\partial z^2}{\partial x^2}\frac{\partial \bar{z}^2}{\partial x^1} &= (i\gamma)(1 - i\delta) - (-i\gamma)(1 + i\delta) = 2i\gamma, \end{aligned}$$

so that

$$\iota^*\omega = \frac{i}{2}(-2i\beta + 2i\gamma)dx^1 \wedge dx^2 = (\beta - \gamma)dx^1 \wedge dx^2.$$

To conclude, the first SLag condition, $\iota^*\omega = 0$, is satisfied if $\gamma = \beta$. (This means that the matrix corresponding to α, β, γ and δ is symmetric.)

Candidate SLags in the complex 4-torus: The pullback of the holomorphic top-form. Now we can explicitly write $\iota^*\Omega$ with respect to x^j. Then (6.4) implies that

$$\begin{aligned} \iota^*\Omega &= \iota^*(dz^1) \wedge \iota^*(dz^2) = (\frac{\partial z^1}{\partial x^1}dx^1 + \frac{\partial z^1}{\partial x^2}dx^2) \wedge (\frac{\partial z^2}{\partial x^1}dx^1 + \frac{\partial z^2}{\partial x^2}dx^2) \\ &= (\frac{\partial z^1}{\partial x^1}\frac{\partial z^2}{\partial x^2} - \frac{\partial z^1}{\partial x^2}\frac{\partial z^2}{\partial x^1})dx^1 \wedge dx^2 = ((1 + i\alpha)(1 + i\delta) - (i\beta)(i\gamma))dx^1 \wedge dx^2 \\ &= (1 + \beta\gamma - \alpha\delta + i(\alpha + \delta))dx^1 \wedge dx^2. \end{aligned}$$

If we choose for example $\theta = 0$, then $\mathrm{Im}(\Omega_{L,\theta}) = \mathrm{Im}(e^{i\theta}\iota^*\Omega) = \mathrm{Im}(\iota^*\Omega) = (\alpha + \delta)dx^1 \wedge dx^2$. Then the second SLag condition, $\mathrm{Im}(\Omega_{L,\theta}) = 0$, is satisfied if $\delta = -\alpha$.

SLags in the complex 4-torus. To conclude, we found some SLags in T^4, and the SLag conditions imply the constraints $\gamma = \beta$ and $\delta = -\alpha$. Then any of these SLags can be described by

$$z^1(x^1, x^2) = x^1 + i(\alpha x^1 + \beta x^2) \quad , \quad z^2(x^1, x^2) = x^2 + i(\beta x^1 - \alpha x^2). \tag{6.5}$$

We can move such a SLag: we could as well define z^j as

$$z^1(x^1, x^2) = x^1 + i(\alpha x^1 + \beta x^2) + c_1 \quad , \quad z^2(x^1, x^2) = x^2 + i(\beta x^1 - \alpha x^2) + c_2, \tag{6.6}$$

for some arbitrary constants $c_j \in \mathbb{R}$.

Deformations of SLags in the complex 4-torus. Any of these SLags in T^4 are again compact connected manifolds L with $\dim_{\mathbb{R}}(L) = 2$, thus any L is diffeomorphic to the standard torus T^2. Knowing that $b_1(L) = b_1(T^2) = 2$ and applying SLag feature 3, we may conclude that there is a 2-dimensional moduli space of deformations, or $\dim_{\mathbb{R}}(\mathcal{M}_D(L)) = 2$.

For a fixed pair (α, β) we already know that we can smoothly deform the SLag defined by (6.5) to another Slag defined by (6.6). Then the pair of constants $(c_1, c_2) \in \mathbb{R}^2$ indicates the deformation, which is indeed in harmony with $\dim_{\mathbb{R}}(\mathcal{M}_D(L)) = 2$. However, we should note that if we have two different SLags L and L', indicated by different pairs (α, β) and (α', β') respectively, then there is no continuous deformation from L to L' possible. This means that the total family of all SLags, with respect to $\theta = 0$, is a smooth manifold with multiple connected components. In fact this family will have infinitely many connected components, but any of these connected components is still a closed set. This is indeed in harmony with what *local* seems to mean, in the context of *local deformations* of SLags, as described by feature 3.

A counterexample. If we define $L \subset T^4$ as the set of points $z^1 \in T^2$ and $z^2 = 0$, then L is diffeomorphic to all the SLags in T^4 already discussed. Then also $\iota^*(dz^1) = dz^1$ and $\iota^*(dz^2) = 0$, so that

$$\iota^*\omega = \frac{i}{2} dz^1 \wedge d\bar{z}^1 \quad , \quad \iota^*\Omega = dz^1 \wedge 0 = 0.$$

We see that both ω and Ω do not satisfy the SLag conditions: $\iota^*\omega \neq 0$ and we cannot use $\iota^*\Omega$ as a volume form defined on L. Note that the equation $\mathrm{Im}(e^{i\theta}\iota^*\Omega) = 0$ is satisfied for *all* θ, which leads to a contradiction if we interpret L as a SLag. Also note that L is a holomorphic curve embedded in T^4. Thus, we conclude that L is *not* a SLag embedded in T^4.

6.4 SLags in T^6

The trivial 6-torus. In this section we will study SLags in the trivial 6-torus $T^6 := T^2 \times T^2 \times T^2$, where $T^2 := \mathbb{C}/G$ and $G := \mathbb{Z} \oplus i\mathbb{Z}$. The standard coordinates $(z^1, z^2, z^3) = (x^1 + iy^1, x^2 + iy^2, x^3 + iy^3)$ of $\mathbb{C}^3$ induce coordinates on T^6. We can again define the metric, the Kähler form and the holomorphic top-form:

$$g := \tfrac{1}{2} \textstyle\sum_{j=1}^3 (dz^j \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^j), \quad \omega := \tfrac{i}{2} \textstyle\sum_{j=1}^3 dz^j \wedge d\bar{z}^j, \quad \Omega := dz^1 \wedge dz^2 \wedge dz^3.$$

These definitions make T^6 into a Calabi-Yau 3-fold.

Candidate SLags in the complex 6-torus. We will study the following candidate SLags. Let M_{jk} be a real 3-by-3 matrix with integer coefficients. Then we define $y^j(x^1, x^2, x^3) := M_{jk}x^k$, thus

$$\begin{aligned}
y^1(x^1, x^2, x^3) &= M_{11}x^1 + M_{12}x^2 + M_{13}x^3, \\
y^2(x^1, x^2, x^3) &= M_{21}x^1 + M_{22}x^2 + M_{23}x^3, \\
y^3(x^1, x^2, x^3) &= M_{31}x^1 + M_{32}x^2 + M_{33}x^3.
\end{aligned}$$

Then the graph

$$\begin{aligned}
(z^1(x^1, x^2, x^3), z^2(x^1, x^2, x^3), z^3(x^1, x^2, x^3)) &= (x^1 + iM_{1j}x^j, x^2 + iM_{2j}x^j, x^3 + iM_{3j}x^j) \\
&\simeq (x^1, M_{1j}x^j, x^2, M_{2j}x^j, x^3, M_{3j}x^j) \in \mathbb{R}^6
\end{aligned}$$

describes a compact 3-dimensional surface L embedded in T^6 and parametrized by the real coordinates x^j.

Candidate SLags in the complex 6-torus: The pullback of the Kähler form. Now we can explicitly write $\iota^*\omega$ with respect to x^j. We know that

$$z^j = x^j + iy^j = x^j + iM_{jk}x^k \quad \Rightarrow \quad \frac{\partial z^j}{\partial x^k} = \frac{\partial}{\partial x^k}(x^j + iM_{jl}x^l) = \delta_{jk} + iM_{jl}\delta_{kl} = \delta_{jk} + iM_{jk}. \tag{6.7}$$

Then we can pull back the Kähler form (ignoring the summation symbol):

$$\begin{aligned}
\iota^*\omega &= \frac{i}{2}\sum_{j=1}^{3}(\iota^*(dz^j)\wedge\iota^*(d\bar{z}^j)) = \frac{i}{2}(\frac{\partial z^j}{\partial x^k}dx^k)\wedge\frac{\partial \bar{z}^j}{\partial x^l}dx^l) = \frac{i}{2}(\delta_{jk}+iM_{jk})(\delta_{jl}-iM_{jl})dx^k\wedge dx^l \\
&= \frac{i}{2}(\delta_{jk}\delta_{jl}+iM_{jk}\delta_{jl}-i\delta_{jk}M_{jl}+M_{jk}M_{jl})dx^k\wedge dx^l \\
&= \frac{i}{2}(\delta_{kl}dx^k\wedge dx^l+iM_{lk}dx^k\wedge dx^l-iM_{kl}dx^k\wedge dx^l+M_{jk}M_{jl}dx^k\wedge dx^l).
\end{aligned}$$

Now the following coefficients are symmetric: $\delta_{kl}=\delta_{lk}$ and $M_{jk}M_{jl}=M_{jl}M_{jk}$. The 2-form $dx^k\wedge dx^l$ is antisymmetric, thus:

$$\iota^*\omega = \frac{i}{2}(iM_{lk}dx^k\wedge dx^l-iM_{kl}dx^k\wedge dx^l) = \frac{1}{2}(M_{kl}-M_{lk})dx^k\wedge dx^l.$$

If now M_{kl} itself is a symmetric matrix, thus if $M^T=M$, then $\iota^*\omega$ vanishes, so that the first SLag condition is satisfied.

Candidate SLags in the complex 6-torus: The pullback of the holomorphic top-form. Now we can explicitly write $\iota^*\Omega$ with respect to x^j. We define the matrix $D_{jk}:=\partial z^j/\partial x^k=\delta_{jk}+iM_{jk}$, see (6.7). This implies that

$$\begin{aligned}
\iota^*\Omega &= \iota^*(dz^1)\wedge\iota^*(dz^2)\wedge\iota^*(dz^3) = \frac{\partial z^1}{\partial x^j}dx^j\wedge\frac{\partial z^2}{\partial x^k}dx^k\wedge\frac{\partial z^3}{\partial x^l}dx^l = D_{1j}D_{2k}D_{3l}dx^j\wedge dx^k\wedge dx^l \\
&= D_{1j}D_{2k}D_{3l}\epsilon_{jkl}dx^1\wedge dx^2\wedge dx^3 = \det(D)dx^1\wedge dx^2\wedge dx^3.
\end{aligned}$$

If we choose $\theta=0$, then $\mathrm{Im}(\Omega_{L,\theta})=\mathrm{Im}(\iota^*\Omega)=\mathrm{Im}(\det(D))dx^1\wedge dx^2\wedge dx^3$. Then the second SLag condition, $\mathrm{Im}(\Omega_{L,\theta})=0$, is satisfied if $\mathrm{Im}(\det(D))=0$. If now M, being symmetric, and D are explicitly written as

$$M=\begin{pmatrix} M_{11} & M_{12} & M_{13}\\ M_{21} & M_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{pmatrix}=\begin{pmatrix}\alpha & \beta & \gamma\\ \beta & \delta & \epsilon\\ \gamma & \epsilon & \zeta\end{pmatrix} \quad\Rightarrow\quad D=\begin{pmatrix}1+i\alpha & i\beta & i\gamma\\ i\beta & 1+i\delta & i\epsilon\\ i\gamma & i\epsilon & 1+i\zeta\end{pmatrix},$$

then $\mathrm{Re}(\det(D))$ and $\mathrm{Im}(\det(D))$ are explicitly written as

$$\begin{aligned}
\mathrm{Re}(\det(D)) &= 1+\beta^2+\gamma^2+\epsilon^2-\alpha\delta-\alpha\zeta-\delta\zeta,\\
\mathrm{Im}(\det(D)) &= \alpha+\delta+\zeta+\alpha\epsilon^2+\delta\gamma^2+\zeta\beta^2-\alpha\delta\zeta-2\beta\gamma\epsilon.
\end{aligned}$$

Some SLags in the complex 6-torus. To conclude, we found some SLags in T^6, and the SLag conditions imply the constraints $M^T=M$ and $\mathrm{Im}(\det(D))=0$. Any of these SLags can again be moved. We can replace the variables (x^1,x^2,x^3) by $(x^1-x_0^1,x^2-x_0^2,x^3-x_0^3)$ for some arbitrary constants $x_0^j\in\mathbb{R}$.

Deformations of SLags in the complex 6-torus. Any of these SLags in T^6 are again compact connected manifolds L with $\dim_{\mathbb{R}}(L)=3$, thus any L is diffeomorphic to the (real) standard torus $T^3=S^1\times S^1\times S^1$. Knowing that $b_1(L)=b_1(T^3)=3$ and applying SLag feature 3, we may conclude that there is a 3-dimensional moduli space of local deformations, or $\dim_{\mathbb{R}}(\mathcal{M}_D(L))=3$. Any two SLags in the same deformation class must be parallel, and any deformation is indicated by a triple (x_0^1,x_0^2,x_0^3). The total family of all SLags, with respect to $\theta=0$, is again a smooth manifold with infinitely many connected components, each of which being a closed manifold.

A remark in the physical context. The SLags we found in T^6 are not really interesting in the context of representing membrane instantons in type IIA superstring theory, as T^6 is *not* a strict Calabi-Yau 3-fold. Then it does not support $\mathcal{N}=2$ supersymmetry.

6.5 The Borcea-Voisin construction

Motivation. Let M_1 and M_2 be *strict* Calabi-Yau manifolds, with $m_j = \dim_{\mathbb{C}}(M_j)$. In Section 2.7 we already introduced the trivial case of constructing a new Calabi-Yau manifold $M := M_1 \times M_2$, with $m = \dim_{\mathbb{C}}(M) = m_1 + m_2$. However, this resulting manifold M is not necessarily a *strict* Calabi-Yau manifold.

The *Borcea-Voisin construction*, or *BV construction*, introduced in [9] and [12], is an alternative method of constructing a new Calabi-Yau m-fold from M_1 and M_2, and this M will again be a strict one. Then we will shortly introduce the method of constructing SLags in M.

Formal definition. Let $\sigma_j : M_j \to M_j$ be holomorphic involutions, reversing the sign of the respective holomorphic top-forms Ω_j:

$$\sigma_j^*(\Omega_j)_{\sigma_j(p)} = -(\Omega_j)_p,$$

for all $p \in M_j$. Then $\sigma := \sigma_1 \times \sigma_2 : M_1 \times M_2 \to M_1 \times M_2$ is another holomorphic involution.

Note that σ preserves the holomorphic top-form $\Omega := \Omega_1 \wedge \Omega_2$, defined on $M_1 \times M_2$. We can write $\sigma^* = (\sigma_1 \times \sigma_2)^* = \sigma_1^* \times \sigma_2^*$ and $\sigma(p,q) = (\sigma_1(p), \sigma_2(q))$, so that

$$\begin{aligned} \sigma^*\Omega_{\sigma(p,q)} &= \sigma^*((\Omega_1)_{\sigma_1(p)} \wedge (\Omega_2)_{\sigma_2(q)}) = \sigma_1^*(\Omega_1)_{\sigma_1(p)} \wedge \sigma_2^*(\Omega_2)_{\sigma_2(q)} \\ &= (-(\Omega_1)_p) \wedge (-(\Omega_2)_q) = (\Omega_1)_p \wedge (\Omega_2)_q = \Omega_{(p,q)}, \end{aligned}$$

for all $(p,q) \in M_1 \times M_2$.

The involution σ generates a group G isomorphic to $\mathbb{Z}_2$. Then the BV construction is a matter of taking the quotient $M_1 \times M_2/G$ and blowing up the singularities:

$$M := b(M_1 \times M_2/G).$$

Then M is the *BV product* of M_1 and M_2, and this M is a new strict Calabi-Yau manifold. As G preserves Ω, we can say that Ω can be used to represent a holomorphic top-form defined on M, thus we can say that M is again a Calabi-Yau manifold. (Of course this is only valid *after* the blow-up.)

As G does *not* preserve Ω_j (it even reverses them), we can say that some of the Hodge numbers of M will be zero, so that we will end up with a strict Calabi-Yau manifold.

SLags in the BV product of two Calabi-Yau manifolds. Let L_1 be a SLag in M_1, with angle θ_1, and let L_2 be a SLag in M_2, with angle θ_2. If $\iota_j : L_j \to M_j$ are the canonical embeddings of these SLags, then $\mathrm{Im}(e^{i\theta_j}\iota_j^*\Omega_j) = 0$. Then it is possible to construct a SLag in M from L_1 and L_2. Now it is easy to show that $L_1 \times L_2$, with canonical embedding $\iota = \iota_1 \times \iota_2$, is again a SLag in $M_1 \times M_2$, with angle $\theta = \theta_1 + \theta_2$:

$$\begin{aligned} \mathrm{Im}(e^{i(\theta_1+\theta_2)}\iota^*\Omega) &= \mathrm{Im}(e^{i\theta_1}\iota_1^*\Omega_1 \wedge e^{i\theta_2}\iota_2^*\Omega_2) \\ &= \mathrm{Re}(e^{i\theta_1}\iota_1^*\Omega_1) \wedge \mathrm{Im}(e^{i\theta_2}\iota_2^*\Omega_2) + \mathrm{Im}(e^{i\theta_1}\iota_1^*\Omega_1) \wedge \mathrm{Re}(e^{i\theta_2}\iota_2^*\Omega_2) \\ &= \mathrm{Re}(e^{i\theta_1}\iota_1^*\Omega_1) \wedge 0 + 0 \wedge \mathrm{Re}(e^{i\theta_2}\iota_2^*\Omega_2) = 0. \end{aligned}$$

Let now $L := L_1 \times L_2/G$. If L_j are smooth and if $L_1 \times L_2$ does *not* intersect the fixed point set of σ, then also L will be smooth, so that a blow-up is not needed. Then we can directly say that L is a SLag in M. However, if $L_1 \times L_2$ contains some fixed points, then it will be far more difficult, and a blow-up is needed. Then we define $L := b(L_1 \times L_2/G)$.

In general a product of SLags $L_1 \times L_2$ does *not* need to contain any fixed point, but it should at least be invariant under the action of G, thus $\sigma(L_1 \times L_2) = L_1 \times L_2$, which implies that $\sigma_j(L_j) = L_j$. If σ does *not* map the whole set $L_1 \times L_2$ to itself, then the candidate set $L_1 \times L_2/G$ does not make any sense. (Note that [21] does *not* mention this.)

We assume that the holomorphic involution should also be an isometry, thus σ should be a *holomorphic isometric involution*. If g_j are the Ricci-flat metrics defined on M_j, then the metric g, trivially induced by g_1 and g_2, defines a Ricci-flat metric on $M_1 \times M_2$. If σ preserves g, which in fact means that σ_j preserve g_j, then g induces a correctly defined Ricci-flat metric on $M = b(M_1 \times M_2/G)$. Then any candidate SLag L in M, constructed from SLags L_j in M_j, is correctly defined with respect to the Ricci-flat metric g. (Note that [21] does also *not* mention this.)

A remark. Note that the involution sends Ω to $-\Omega$. However, note that if Ω satisfies the SLag conditions, thus if there exists a θ so that $\mathrm{Im}(e^{i\theta}\iota^*\Omega) = 0$, then also $-\Omega$ satisfies the same equation.

7 SLags in the Fermat Quintic

In Chapter 5 we already introduced the Fermat quintic. In this chapter we can also discuss the SLags embedded in the Fermat quintic. We can find one example of a SLag in the quintic in [10], and we discussed this example in Section 7.1. In Section 7.2 we will work out the results of some research. We found 624 copies of the example SLag, given in Section 7.1, so we found 625 SLags in total. These SLags all share the same geometry, and they have the (smooth) topology of $\mathbb{RP}^3$, the real projective space of real dimension 3.

7.1 The Anti-Holomorphic Isometric Involution acting on the Fermat quintic

It is not that simple to find all SLags in the Fermat Quintic Q, defined by (5.1). However, it is possible to construct some specific examples of SLags in Q. These SLags can be found by trying to determine submanifolds of Q which are fixed point sets of certain involutions $D : Q \to Q$. We will mainly discuss the method introduced in [10]. We know that $\dim_{\mathbb{C}}(Q) = 3$, thus any SLag in Q has real dimension 3. To make sure that the fixed point set $L \subset Q$ of D has real dimension 3, we need to use an anti-holomorphic involution. (See Section 2.8 for the definition of anti-holomorphic involutions.) We will also mention that D should be an anti-holomorphic *isometry*. Then D must also satisfy (2.60) and (2.61). Then we say that D is an *anti-holomorphic isometric involution*. In this context L will still be a candidate SLag: we will check if L indeed satisfies the SLag conditions introduced in Section 6.1.

The concept of isometric involutions can be clarified with help of a simple example. The unit sphere S^2 with standard metric induced from its embedding supports a whole family of isometric involutions. Any plane P intersecting S^2 and the origin defines an isometric involution by orthogonal reflection in P of any point $p \in S^2$. The circle $P \cap S^2$ is the set of fixed points of this involution.

We note that in [10] there is no explicit mention of the notion 'SLag'. At the time this article was written, the concept of SLags was still not finished yet. Then we are talking about *supersymmetric 3-cycles* instead, embedded in the Calabi-Yau 3-fold in question.

We should note that there is no Ricci-flat g known for Q (only numerical approximations). However, fortunately the method we will use here does *not* depend on the explicit definition of g.

Formal definition. The anti-holomorphic involution

$$\tilde{\iota} : \mathbb{C}^5 \to \mathbb{C}^5 : (Z^0, Z^1, Z^2, Z^3, Z^4) \mapsto (\bar{Z}^0, \bar{Z}^1, \bar{Z}^2, \bar{Z}^3, \bar{Z}^4)$$

induces an anti-holomorphic involution

$$\iota : \mathbb{P}^4 \to \mathbb{P}^4 : [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [\bar{Z}^0 : \bar{Z}^1 : \bar{Z}^2 : \bar{Z}^3 : \bar{Z}^4].$$

We note that any point $[Z^0 : Z^1 : Z^2 : Z^3 : Z^4]$ in Q satisfies

$$(Z^0)^5 + (Z^1)^5 + (Z^2)^5 + (Z^3)^5 + (Z^4)^5 = 0 = \bar{0} = (\bar{Z}^0)^5 + (\bar{Z}^1)^5 + (\bar{Z}^2)^5 + (\bar{Z}^3)^5 + (\bar{Z}^4)^5,$$

thus

$$[Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \in Q \quad \Leftrightarrow \quad [\bar{Z}^0 : \bar{Z}^1 : \bar{Z}^2 : \bar{Z}^3 : \bar{Z}^4] \in Q.$$

Thus we can safely conclude that ι leaves Q invariant. Let now $D : Q \to Q$ be the anti-holomorphic involution induced by ι. According to [10] this is also an isometry. We will prove this as follows. We will use the same coordinate chart for p and $D(p)$. Now D is an anti-holomorphic isometric involution because it satisfies (2.61), explicitly written as

$$\frac{\partial(Z \circ D \circ \bar{Z}^{-1})^\alpha}{\partial \bar{Z}^\nu} \frac{\partial(\bar{Z} \circ D \circ Z^{-1})^\beta}{\partial Z^\mu} (g_{D(p)})_{\alpha\bar{\beta}} = \delta^\alpha_\nu \delta^\beta_\mu (g_{D(p)})_{\alpha\bar{\beta}} = (g_{D(p)})_{\nu\bar{\mu}} = \overline{(g_p)_{\nu\bar{\mu}}} = (g_p)_{\mu\bar{\nu}}. \tag{7.1}$$

(This holds, for example, on a patch $[1 : Z^1 : Z^2 : Z^3 : Z^4(Z^1, Z^2, Z^3)] \subset Q$, thus then $Z = (Z^1, Z^2, Z^3)$.)

Even if g is the Ricci-flat metric on Q, and if we do not explicitly know its definition, then we can still say that (7.1) implies that D preserves g.

The fixed point set of the involution. Now it is time to study the fixed point set of D. For any $p = [1 : Z^1 : Z^2 : Z^3 : Z^4] \in Q$ we have $D(p) = [1 : \bar{Z}^1 : \bar{Z}^2 : \bar{Z}^3 : \bar{Z}^4]$. Then p is a fixed point of D, thus $D(p) = p$, if p can be expressed in real coordinates: $Z^1, Z^2, Z^3, Z^4 \in \mathbb{R}$. If we describe Z^4 as a function of Z^1, Z^2 and Z^3, then (5.2) implies that

$$Z^4 = Z^4(Z^1, Z^2, Z^3) = -\sqrt[5]{1 + (Z^1)^5 + (Z^2)^5 + (Z^3)^5} \in \mathbb{R}. \tag{7.2}$$

(Note that the extra factor α^k appearing in (5.2) is left out here: only if $k = 0$ we can say that $Z^4 \in \mathbb{R}$.) Thus, we can say that to any triple (Z^1, Z^2, Z^3) exactly one Z^4 is associated.

Let now L be the fixed point set of D. Then L is a real submanifold of Q with $\dim_{\mathbb{R}}(L) = 3$, and (7.2) describes a subset L_1 of L, and this subset is isomorphic to $\mathbb{R}^3$.

What kind of set is the candidate SLag? In total we can split up L into 4 subsets containing certain types of points:

$$\begin{array}{ll} L_1 := \{[1 : Z^1 : Z^2 : Z^3 : Z^4(Z^1, Z^2, Z^3)]\}, & Z^4(Z^1, Z^2, Z^3) = -\sqrt[5]{1 + (Z^1)^5 + (Z^2)^5 + (Z^3)^5}, \\ L_2 := \{[0 : 1 : Z^2 : Z^3 : Z^4(Z^2, Z^3)]\}, & Z^4(Z^2, Z^3) = -\sqrt[5]{1 + (Z^2)^5 + (Z^3)^5}, \\ L_3 := \{[0 : 0 : 1 : Z^3 : Z^4(Z^3)]\}, & Z^4(Z^3) = -\sqrt[5]{1 + (Z^3)^5}, \\ L_4 := \{[0 : 0 : 0 : 1 : Z^4]\}, & Z^4 = -1. \end{array}$$

Note that there are no restrictions on $Z^\mu \in \mathbb{R}$, thus there are diffeomorphisms

$$L_1 \simeq \{[1 : a : b : c]\}, \quad L_2 \simeq \{[0 : 1 : b : c]\}, \quad L_3 \simeq \{[0 : 0 : 1 : c]\}, \quad L_4 \simeq \{[0 : 0 : 0 : 1]\},$$

with $a, b, c \in \mathbb{R}$. Thus, there is a diffeomorphism from L to the real projective space $\mathbb{R}\mathrm{P}^3$. As mentioned in Section 2.4 we can say that $\mathbb{R}\mathrm{P}^3$, thus also L, is an orientable manifold, so that L is possibly a SLag. To conclude, L is an orientable smooth compact connected manifold.

Its Betti numbers are $b_0(\mathbb{R}\mathrm{P}^3) = 1$, $b_1(\mathbb{R}\mathrm{P}^3) = 0$, $b_2(\mathbb{R}\mathrm{P}^3) = 0$ and $b_3(\mathbb{R}\mathrm{P}^3) = 1$, and these indeed satisfy (4.1). These are in fact the same as the Betti numbers of S^3, the 3-sphere. However, there is no diffeomorphism from S^3 to $\mathbb{R}\mathrm{P}^3$. (The torsion subgroups of the homology groups of $\mathbb{R}\mathrm{P}^3$ are still non-trivial.)

On the other hand, an unramified double covering is possible from S^3 to $\mathbb{R}\mathrm{P}^3$. The antipodal points in S^3 can be identified, so that we obtain the points in $\mathbb{R}\mathrm{P}^3$.

Why do we need an isometry? As D is an isometry, it preserves the metric, but what will it do with the Kähler form ω? We know that $D(p) = p$ for all $p \in L$, thus $Z^\mu(p) = \bar{Z}^\mu(p)$, so that $Z^\mu(p) \in \mathbb{R}$. Let now $X^\mu(p)$ be the (real) coordinates of L around p, and let $\iota : L \to Q$ be the canonical embedding. Then $\iota(p) = p$ for all $p \in L$, and $\iota^*(dZ^\mu) = \iota^*(d\bar{Z}^\mu) = dX^\mu$, so that

$$\iota^*\omega_{\iota(p)} = \iota^*\omega_p = \iota^*(i(g_p)_{\mu\bar{\nu}} dZ^\mu \wedge d\bar{Z}^\nu) = i(g_p)_{\mu\bar{\nu}}(\iota^*(dZ^\mu) \wedge \iota^*(d\bar{Z}^\nu)) = i(g_p)_{\mu\bar{\nu}} dX^\mu \wedge dX^\nu.$$

Now (7.1) and $D(p) = p$ imply that $(g_p)_{\mu\bar{\nu}} = (g_{D(p)})_{\nu\bar{\mu}} = (g_p)_{\bar{\mu}\nu}$, thus $(g_p)_{\mu\bar{\nu}} \in \mathbb{R}$, so that

$$\begin{aligned} \iota^*\omega_{\iota(p)} &= i(g_p)_{\mu\bar{\nu}} dX^\mu \wedge dX^\nu = -i(g_p)_{\mu\bar{\nu}} dX^\nu \wedge dX^\mu \\ &= -i(g_p)_{\nu\bar{\mu}} dX^\nu \wedge dX^\mu = -i(g_p)_{\mu\bar{\nu}} dX^\mu \wedge dX^\nu = -\iota^*\omega_{\iota(p)}. \end{aligned}$$

This implies that $\iota^*\omega = 0$. Thus, to conclude, ω satisfies the first SLag condition on L because D is an isometry. (Note that if $\bar{Z}^\mu = Z^\mu$ and if $g_{\mu\bar{\nu}} \in \mathbb{R}$, then the righthand side of (6.1) is automatically satisfied, so that indeed $\iota^*\omega = 0$, thus now we have a reverse implication of (6.1).)

The pullback of the holomorphic top-form. The holomorphic nowhere vanishing top-form Ω on Q is already defined in (5.3). When expressed with respect to Z^μ and $Z^4 = Z^4(Z^1, Z^2, Z^3)$, using (2.55), it can be written as

$$\Omega := \frac{dZ^1 \wedge dZ^2 \wedge dZ^3}{(Z^4)^4} = f dZ^1 \wedge dZ^2 \wedge dZ^3 \quad \Rightarrow \quad f = f(Z^1, Z^2, Z^3) = \frac{1}{(Z^4)^4} = \frac{1}{(Z^4(Z^1, Z^2, Z^3))^4}.$$

(Of course we could have chosen an extra constant complex factor in front of Ω.) On L we have $Z^\mu = \bar{Z}^\mu = X^\mu \in \mathbb{R}$, $\iota^*(dZ^\mu) = dX^\mu$ and $Z^4 = X^4$, so that

$$\iota^*\Omega = \iota^*(f dZ^1 \wedge dZ^2 \wedge dZ^3) = f|_L(\iota^*(dZ^1) \wedge \iota^*(dZ^2) \wedge \iota^*(dZ^3)) = \frac{dX^1 \wedge dX^2 \wedge dX^3}{(X^4)^4}.$$

Then we immediately see that $\iota^*\Omega$ is a real 3-form, thus $\mathrm{Im}(e^{i\theta}\iota^*\Omega) = 0$, for $\theta \in \{0, \pi\}$, so that Ω satisfies the second SLag condition on L. Note that $f = f|_L = 1/(X^4)^4$. If we use a suitable patch with $X^4 \neq 0$, then $(X^4)^4 > 0$ so that

$$f > 0 \quad \Rightarrow \quad f = |f| = \frac{1}{|X^4|^4} = \frac{1}{|Z^4|^4}.$$

Now note that feature 2 (see Section 6.1) implies that $\iota^*\Omega$ is proportional to the volume form defined by h, the pullback of the metric g on L. We can also prove this explicitly, using the same proving scheme used in [10]. Note that (2.56) implies that $\|\Omega\|^2 = |f|^2/G$, where G is the determinant of $g_{\mu\bar{\nu}}$, see (2.44). As g is a Ricci-flat metric we can say that $\|\Omega\|^2$ is a constant, see Section 2.7. If we write $\|\Omega\|^2 = 8c^2$, $c > 0$, then

$$G = \frac{|f|^2}{\|\Omega\|^2} = \frac{|f|^2}{8c^2} = \frac{1}{8c^2|Z^4|^8}.$$

We already know that $\iota^*\omega = 0$, so that (6.1) holds. As a consequence also (6.2) holds with respect to arbitrary real coordinates σ^α of L:

$$h^{(\sigma)}_{\alpha\beta} = 2\frac{\partial Z^\mu}{\partial\sigma^\alpha}\frac{\partial \bar{Z}^\nu}{\partial\sigma^\beta} g_{\mu\bar{\nu}} = 2\partial_\alpha Z^\mu \partial_\beta \bar{Z}^\nu g_{\mu\bar{\nu}}.$$

Then (6.3) implies that

$$h = 8|\det(\partial Z)|^2 G = \frac{8|\det(\partial Z)|^2}{8c^2|Z^4|^8} = (\frac{|\det(\partial Z)|}{c|Z^4|^4})^2 \quad \Rightarrow \quad \sqrt{h} = \frac{|\det(\partial Z)|}{c|Z^4|^4}.$$

Let now $\sigma^\alpha := X^\alpha$. Then

$$\frac{\partial Z^\mu}{\partial\sigma^\alpha} = \frac{\partial}{\partial X^\alpha}(X^\mu + iY^\mu) = \frac{\partial X^\mu}{\partial X^\alpha} = \delta^\mu_\alpha \quad \Rightarrow \quad \det(\partial Z) = \det(\partial\bar{Z}) = |\det(\partial Z)| = 1,$$

so that $\sqrt{h} = 1/c|Z^4|^4 = f/c$, thus

$$\iota^*\Omega = f dX^1 \wedge dX^2 \wedge dX^3 = c\frac{f}{c} dX^1 \wedge dX^2 \wedge dX^3 = c\sqrt{h} dX^1 \wedge dX^2 \wedge dX^3 = c *_{L,h} 1 = *_{L,h} c. \qquad (7.3)$$

An extra phase factor. Note that $\iota^*\Omega$ is already real in this case. In general an extra phase factor will also be there. This is why this extra factor $e^{i\theta}$ should also be added to (7.3).

7.2 Other Anti-Holomorphic Isometric Involutions

A family of involutions. In the previous section we introduced the standard anti-holomorphic involution, acting on the Fermat quintic Q. In total there are 625 similar involutions acting on Q. If $\alpha := e^{2\pi i/5}$, so that $\alpha^5 = 1$, then we can define a family of involutions on $\mathbb{P}^4$, similar to the family of involutions defined in (5.6):

$$D_{j,k,l,m} : [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \mapsto [\bar{Z}^0 : \alpha^{2j}\bar{Z}^1 : \alpha^{2k}\bar{Z}^2 : \alpha^{2l}\bar{Z}^3 : \alpha^{2m}\bar{Z}^4].$$

Then the single involution mentioned in Section 7.1 satisfies $D = D_{0,0,0,0}$. Now note that $(\alpha^{2j}\bar{Z}^1)^5 = (\alpha^5)^{2j}(\bar{Z}^1)^5 = (\bar{Z}^1)^5$. Then any point $[Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \in Q$ satisfies

$$\begin{aligned}(Z^0)^5 + (Z^1)^5 + (Z^2)^5 + (Z^3)^5 + (Z^4)^5 &= 0 = \bar{0} = (\bar{Z}^0)^5 + (\bar{Z}^1)^5 + (\bar{Z}^2)^5 + (\bar{Z}^3)^5 + (\bar{Z}^4)^5 \\ &= (\bar{Z}^0)^5 + (\alpha^{2j}\bar{Z}^1)^5 + (\alpha^{2k}\bar{Z}^2)^5 + (\alpha^{2l}\bar{Z}^3)^5 + (\alpha^{2m}\bar{Z}^4)^5,\end{aligned}$$

thus

$$[Z^0 : Z^1 : Z^2 : Z^3 : Z^4] \in Q \quad \Leftrightarrow \quad [\bar{Z}^0 : \alpha^{2j}\bar{Z}^1 : \alpha^{2k}\bar{Z}^2 : \alpha^{2l}\bar{Z}^3 : \alpha^{2m}\bar{Z}^4] \in Q.$$

Now note that $\overline{\alpha^j} = \alpha^{-j}$, thus any fixed point Z^1 of the action $Z^1 \mapsto \alpha^{2j}\bar{Z}^1$ can be expressed as $Z^1 = \alpha^j X^1$, where $X^1 \in \mathbb{R}$:

$$Z^1 = \alpha^j X^1 \mapsto \alpha^{2j}\overline{\alpha^j X^1} = \alpha^{2j}\alpha^{-j}\bar{X}^1 = \alpha^j X^1 = Z^1.$$

Thus, the fixed points of $D_{j,k,l,m}$ can be expressed as

$$[X^0 : \alpha^j X^1 : \alpha^k X^2 : \alpha^l X^3 : \alpha^m X^4] \quad , \quad X^\mu \in \mathbb{R}.$$

Candidate SLags. Let now $L_{j,k,l,m}$ be the fixed point set, or 'candidate SLag', related to the involution $D_{j,k,l,m}$. Then $L = L_{0,0,0,0}$ is the standard SLag introduced in Section 7.1. We will look if these $L_{j,k,l,m}$ are again fixed point sets of a related isometry, so that we can indeed safely conclude that each is a SLag. First we will do a coordinate transformation, depending on which candidate SLag we are testing.

A family of coordinate transformations. Now note that we can do a coordinate transformation

$$Z^\mu \mapsto W^\mu \quad , \quad [Z^0 : Z^1 : Z^2 : Z^3 : Z^4] = [W^0 : \alpha^j W^1 : \alpha^k W^2 : \alpha^l W^3 : \alpha^m W^4].$$

Then W^μ helps us describing $L_{j,k,l,m}$ as a real manifold: W^μ satisfies $W^\mu = \bar{W}^\mu = X^\mu$ on $L_{j,k,l,m}$. Then Q is still a Fermat quintic with respect to the coordinates W^μ:

$$(Z^0)^5 + (Z^1)^5 + (Z^2)^5 + (Z^3)^5 + (Z^4)^5 = 0 \quad \Leftrightarrow \quad (W^0)^5 + (W^1)^5 + (W^2)^5 + (W^3)^5 + (W^4)^5 = 0.$$

If we now express (7.1) with respect to W^μ, for any $D_{j,k,l,m}$, then we will again see that we are indeed dealing with an anti-holomorphic isometric involution. On $L_{j,k,l,m}$ we again have

$$g_p = (g_p^{(W)})_{\mu\bar{\nu}}(dW^\mu \otimes d\bar{W}^\nu + d\bar{W}^\nu \otimes dW^\mu),$$

with $({g_p}^{(W)})_{\mu\bar{\nu}}$ being real, so that if we pull back the Kähler form, then it vanishes. Thus, $L_{j,k,l,m}$ satisfies the first SLag condition. As the polynomials, with respect to Z^μ and W^μ, are equal, we can say that the candidate SLag $L_{j,k,l,m}$ has exactly the same form as L. They will at least all be diffeomorphic.

Let now $U \subset Q$ (again) be the (standard) patch of inhomogeneous coordinates expressed as

$$[1 : Z^1 : Z^2 : Z^3 : Z^4(Z^1, Z^2, Z^3)] = [1 : \alpha^j W^1 : \alpha^k W^2 : \alpha^l W^3 : \alpha^m W^4(W^1, W^2, W^3)],$$

with respect to the different coordinates Z^μ and W^μ. Then we can say that this describes a finite group G of $5^4 = 625$ coordinate transformations, from the standard coordinate Z^μ to any W^μ. To any involution $D_{j,k,l,m}$ one of these coordinate transformations is related. Now we can also rewrite Ω with respect to W^μ:

$$\begin{aligned}\Omega &= \frac{dZ^1 \wedge dZ^2 \wedge dZ^3}{(Z^4)^4} = \frac{d(\alpha^j W^1) \wedge d(\alpha^k W^2) \wedge d(\alpha^l W^3)}{(\alpha^m W^4)^4} \\ &= \alpha^{j+k+l-4m}\frac{dW^1 \wedge dW^2 \wedge dW^3}{(W^4)^4} = \alpha^{j+k+l+m}\frac{dW^1 \wedge dW^2 \wedge dW^3}{(W^4)^4}.\end{aligned}$$

After pulling back to $L_{j,k,l,m}$ we obtain

$$\iota^*_{(j,k,l,m)}\Omega = \iota^*_{(W)}\Omega = \alpha^{j+k+l+m}\frac{dX^1 \wedge dX^2 \wedge dX^3}{(X^4)^4}.$$

Now we see that Ω has a constant phase with respect to any $L_{j,k,l,m}$, thus any of these candidate SLags satisfies the second SLag condition. As $L_{j,k,l,m}$ is already compact, we simply say that it is a SLag from now on. The coordinate transformations corresponding to $D_{1,0,0,4}$, $D_{0,1,0,4}$, $D_{0,0,1,4}$ and $D_{0,0,0,1}$ can be regarded as 4 generators of the total group G. The first three generators satisfy $j+k+l+m=0$, so that Ω looks the same with respect to both Z^μ and W^μ. Then also its pullbacks $\mathrm{Im}(\iota^*_{(Z)}\Omega)$ and $\mathrm{Im}(\iota^*_{(W)}\Omega)$ vanish with respect to L and $L_{j,k,l,m}$ respectively, thus both are real with respect to the same $\theta=0$. In total there are 125 SLags with respect to this angle, and for any other θ we can do something similar. Now note that there is a finite number of angles, $\theta=2\pi n/5$ (or $\theta=2\pi n/5+\pi$) with $n\in\{0,1,2,3,4\}$, with respect to which there are 125 SLags each. Thus, indeed, there are 625 SLags in total.

A family of holomorphic actions. In Section 5.2 we already introduced a group G_4 of holomorphic actions, isomorphic to $\mathbb{Z}_5^4$, see (5.7). Both the Fermat quintic and $\mathbb{P}^4$ are invariant under the actions of G_4. The 625 elements can be expressed as follows:

$$H_{j,k,l,m}:[Z^0:Z^1:Z^2:Z^3:Z^4]\mapsto[Z^0:\alpha^jZ^1:\alpha^kZ^2:\alpha^lZ^3:\alpha^mZ^4].$$

Then we see that any anti-holomorphic isometric involution $D_{j,k,l,m}$ can be rewritten as

$$D_{j,k,l,m}=H_{2j,2k,2l,2m}D_{0,0,0,0}=H_{j',k',l',m'}D.$$

(Note that there is a 1-to-1 relation between the 625 possible indices (j,k,l,m) of D and (j',k',l',m') of H.) This also means that

$$H_{j',k',l',m'}=H_{j',k',l',m'}DD=D_{j,k,l,m}D,$$

where both $D_{j,k,l,m}$ and D are isometries, so that any $H_{j',k',l',m'}$ is also an isometry itself. Thus, the 625 SLags are all related by isometries, so that they all have the same geometry, the same volume.

We should note that $b_1(L_{j,k,l,m})=b_1(\mathbb{RP}^3)=0$. This means that none of these 625 SLags is a member of a smooth family of SLags: they are all isolated.

A remark in the physical context. The 625 SLags we found in total in the quintic Q are interesting in the context of representing membrane instantons in type IIA superstring theory, as Q is a strict Calabi-Yau 3-fold. Then it does support $\mathcal{N}=2$ supersymmetry. However, we cannot say we found *all* SLags, so we cannot write down a complete correction to the effective theory.

7.3 SLags in the Fermat cubic

The general cubic was already introduced in Section 2.4. The Fermat cubic is a complex manifold C, embedded in $\mathbb{P}^2$ and generated by a homogeneous third order polynomial of Fermat type:

$$f(Z^j)=(Z^0)^3+(Z^1)^3+(Z^2)^3.$$

It is diffeomorphic to the torus, thus it is a Calabi-Yau 1-fold. (It is *not* a flat torus.) To find some SLags in C we can use techniques similar to those used for finding SLags in the Fermat quintic. Here we will only shortly repeat these steps already mentioned in Section 7.1 and 7.2, applied to C. We will again look at a family of anti-holomorphic isometric involutions $D_{j,k}:C\to C$ and their fixed point sets $L_{j,k}\subset C$. These $L_{j,k}$ will again be SLags, and we know that any SLag in C has real dimension 1.

The anti-holomorphic isometric involutions. This time the family of involutions is defined as

$$D_{j,k}:[Z^0:Z^1:Z^2]\mapsto[\bar{Z}^0:\beta^{2j}\bar{Z}^1:\beta^{2k}\bar{Z}^2],$$

where $\beta=e^{2\pi i/3}$, so that $\beta^3=1$. In total we have $3^2=9$ of these $D_{j,k}$. Again we see that any point $[Z^0:Z^1:Z^2]$ in C satisfies

$$(Z^0)^3+(Z^1)^3+(Z^2)^3=0=\bar{0}=(\bar{Z}^0)^3+(\bar{Z}^1)^3+(\bar{Z}^2)^3=(\bar{Z}^0)^3+(\beta^{2j}\bar{Z}^1)^3+(\beta^{2k}\bar{Z}^2)^3,$$

thus any $D_{j,k}$ indeed leaves C invariant. We can use similar arguments as mentioned in Section 7.2 to explain why any $D_{j,k}$ is an isometry. The corresponding holomorphic actions

$$H_{j,k} : [Z^0 : Z^1 : Z^2] \mapsto [Z^0 : \beta^j Z^1 : \beta^k Z^2]$$

are again related to $D_{j,k}$ as

$$D_{j,k} = H_{2j,2k} D_{0,0} = H_{j',k'} D \quad \Rightarrow \quad H_{j',k'} = D_{j,k} D,$$

and these $H_{j',k'}$ are again isometries. (Note again that there is a 1-to-1 relation between the 9 possible indices (j,k) of D and (j',k') of H. These H form an abelian group isomorphic to $\mathbb{Z}_3^2$.) Thus, the 9 SLags are all related by isometries, so that they all have the same geometry, the same volume.

The fixed point sets of these involutions. The points in $L_{j,k}$, the fixed points of $D_{j,k}$, can be expressed as

$$[X^0 : \alpha^j X^1 : \alpha^k X^2] \quad , \quad X^\mu \in \mathbb{R}.$$

For example, we can study $L = L_{0,0}$, the fixed point set of $D = D_{0,0}$. We can split up L into 2 subsets:

$$\begin{array}{ll} L_1 := \{[1 : X^1 : X^2(X^1)]\}, & X^2(X^1) = -\sqrt[3]{1 + (X^1)^3}, \\ L_2 := \{[0 : 1 : X^2]\}, & X^2 = -1. \end{array}$$

(Any $L_{j,k}$ can again be split up into similar subsets.) There are diffeomorphisms $L_1 \simeq \{[1 : a] | a \in \mathbb{R}\}$ and $L_2 \simeq [0 : 1]$, thus there is a diffeomorphism from L to the real projective space $\mathbb{RP}^1$. Now $\mathbb{RP}^1 \simeq S^1/\mathbb{Z}_2 \simeq S^1$, thus, to conclude: L is an orientable smooth compact connected manifold, so that it is indeed a SLag.

The SLags. We again have alternative coordinates W^μ so that $[Z^0 : Z^1 : Z^2] = [W^0 : \beta^j W^1 : \beta^k W^2]$. Then a group G of coordinate transformations exists, from the standard coordinate Z^μ to any W^μ, and this group contains 9 elements. The holomorphic nowhere vanishing top-form on C can be written as

$$\Omega = \frac{dZ^1}{(Z^2)^2} = \beta^{j+k} \frac{dW^1}{(W^2)^2},$$

when expressed with respect to the patch $[1 : Z^1 : Z^2(Z^1)]$. The coordinate transformations corresponding to $D_{1,2}$ and $D_{0,1}$ can be regarded as 2 generators of the total group G. The first generator satisfies $j + k = 0$, so that Ω looks the same with respect to both Z^μ and W^μ. In total there are 3 SLags with respect to this angle, and for any other θ we can do something similar. There are three different angles, $\theta = 2\pi n/3$ with $n \in \{0, 1, 2\}$, with respect to which there are 3 SLags each. To conclude, we have indeed found 9 SLags in total.

We should note that $b_1(L_{j,k}) = b_1(S^1) = 1$. This means that any of these 9 SLags is a member of a 1-dimensional smooth family of SLags.

8 SLags in K3 surfaces

In this chapter we will discuss two different K3 surfaces and SLags in them. First we will discuss the Fermat quartic $F_4 \subset \mathbb{P}^3$, and this is a smooth K3 surface. We can find 24 SLags in F_4, and they are all diffeomorphic to the torus. Then we will discuss a singular K3 surface S_4: this is a blow-up of the manifold $T^4/\mathbb{Z}_2$, which has singularities. We will discuss the BV construction to find SLags in S_4. Finally we will again use the BV construction and try to find SLags in a Calabi-Yau 3-fold $F_4 \times T^2/\mathbb{Z}_2$, the BV product of the Fermat quartic and the flat torus.

We should note that the nonstrict Calabi-Yau 3-fold $F_4 \times T^2$ admits $\mathcal{N} = 4$ supersymmetry, and is surely not simply connected. After performing the BV construction we obtain a strict Calabi-Yau 3-fold $F_4 \times T^2/\mathbb{Z}_2$, which admits $\mathcal{N} = 2$ supersymmetry, which we need.

8.1 SLags in the Fermat quartic

The Fermat quartic in $\mathbb{P}^3$. In this section we are mainly interested in the fourth order Fermat polynomial, which corresponds to the special choice of coefficients f_{ijkl}:

$$f(Z^\mu) = (Z^0)^4 + (Z^1)^4 + (Z^2)^4 + (Z^3)^4. \tag{8.1}$$

The corresponding hypercurve $F_4 := f^{-1}(0)/\mathbb{C}^*$, embedded in $\mathbb{P}^3$, is called the *Fermat quartic*, and this is a specific example of a K3 surface. We will see that there are smooth families of SLags, embedded in F_4, but we will at least find 24 of them. (We will ignore the holomorphic top-form Ω and the phase factors $e^{i\theta}$ for the moment.)

We can use a cell complex of F_4 to compute its Euler number, and we can do this in a similar way as we computed the Euler number of the Fermat quintic, see (5.4). The result is that $\chi(F_4) = 24$, and this indeed corresponds to the Euler number of any K3 surface.

The anti-holomorphic isometric involutions. The family of involutions is defined as

$$D_{j,k,l} : [Z^0 : Z^1 : Z^2 : Z^3] \mapsto [\bar{Z}^0 : \gamma^{2j}\bar{Z}^1 : \gamma^{2k}\bar{Z}^2 : \gamma^{2l}\bar{Z}^3] = [\bar{Z}^0 : i^j\bar{Z}^1 : i^k\bar{Z}^2 : i^l\bar{Z}^3],$$

where $\gamma = e^{\pi i/4}$, so that $\gamma^8 = 1$. In total we have $4^3 = 64$ of these $D_{j,k,l}$. It is easy to check that any $D_{j,k,l}$ leaves F_4 invariant, and that any of these $D_{j,k,l}$ is an anti-holomorphic isometric involution. The corresponding holomorphic actions

$$H_{j,k,l} : [Z^0 : Z^1 : Z^2 : Z^3] \mapsto [Z^0 : i^jZ^1 : i^kZ^2 : i^lZ^3] \tag{8.2}$$

are related to $D_{j,k,l}$ as

$$D_{j,k,l} = H_{j,k,l}D_{0,0,0} = H_{j,k,l}D \quad \Rightarrow \quad H_{j,k,l} = D_{j,k,l}D,$$

thus these $H_{j,k,l}$ are again isometries. These $H_{j,k,l}$ form an abelian group isomorphic to $\mathbb{Z}_4^3$.

The fixed point sets of these involutions. The points in $L_{j,k,l}$, the fixed points of $D_{j,k,l}$, can be expressed as

$$[X^0 : \gamma^jX^1 : \gamma^kX^2 : \gamma^lX^3] \quad , \quad X^\mu \in \mathbb{R}.$$

We should however take care, because there are a few different kinds of these fixed point sets. (Note that, for example, if $X^1 \in \mathbb{R}$, then also $\gamma^4X^1 \in \mathbb{R}$, thus also in this case we will only consider $j, k, l \in \{0, 1, 2, 3\}$.) We will do a few example tests (with respect to the patch $X^0 \neq 0$):

- For example, if $j = k = l = 0$, then we have

$$L_{j,k,l} = L_{0,0,0} = \{[1 : X^1 : X^2 : X^3] \in F_4 | X^\mu \in \mathbb{R}\}.$$

Then we have the following equation:

$$0 = 1 + (Z^1)^4 + (Z^2)^4 + (Z^3)^4 = 1 + (X^1)^4 + (X^2)^4 + (X^3)^4. \tag{8.3}$$

As $X^\mu \in \mathbb{R}$ we have $(X^\mu)^4 \geq 0$, so that $1 + (X^1)^4 + (X^2)^4 + (X^3)^4 \geq 1$. Then (8.3) has no solutions. Thus, the fixed point set $L_{0,0,0}$ (of $D_{0,0,0}$) is empty. Similarly we can prove that the fixed point sets of 7 other involutions, $D_{0,0,2}$, $D_{0,2,0}$, $D_{0,2,2}$, $D_{2,0,0}$, $D_{2,0,2}$, $D_{2,2,0}$ and $D_{2,2,2}$, are also empty.

- For example, if $j = k = 0$ and $l = 1$, then we have

$$L_{j,k,l} = L_{0,0,1} = \{[1 : X^1 : X^2 : \gamma X^3] \in F_4 | X^\mu \in \mathbb{R}\}.$$

Then we have the following equation:

$$0 = 1 + (Z^1)^4 + (Z^2)^4 + (Z^3)^4 = 1 + (X^1)^4 + (X^2)^4 + (\gamma X^3)^4 = 1 + (X^1)^4 + (X^2)^4 - (X^3)^4.$$

This equation has solutions. We have $1+(X^1)^4+(X^2)^4 \geq 1$, so that $(X^3)^4 \geq 1$. Then, for any X^1 and X^2 we have $X^3 = \pm\sqrt[4]{1 + (X^1)^4 + (X^2)^4}$. Thus, for any $(X^1, X^2) \in \mathbb{R}^2$ we have exactly two solutions for X^3: a strictly positive and a strictly negative one. Thus, the total solution set contains two isolated parts, each of which being connected. Now we can split up $L_{0,0,1}$ into 6 subsets:

$$\begin{array}{ll} L_1^\pm := \{[1 : X^1 : X^2 : X^3(X^1, X^2)]\}, & X^3(X^1, X^2) = \pm\sqrt[4]{1 + (X^1)^4 + (X^2)^4}, \\ L_2^\pm := \{[0 : 1 : X^2 : X^3(X^2)]\}, & X^3(X^2) = \pm\sqrt[4]{1 + (X^2)^4}, \\ L_3^\pm := \{[0 : 0 : 1 : X^3]\}, & X^3 = \pm 1. \end{array}$$

There are diffeomorphisms $L_1^+ \simeq L_1^- \simeq \{[1 : a : b] | a, b \in \mathbb{R}\}$, $L_2^+ \simeq L_2^- \simeq \{[0 : 1 : b] | b \in \mathbb{R}\}$ and $L_3^+ \simeq L_3^- \simeq [0 : 0 : 1]$, thus there is a diffeomorphism from L to $2\,\mathbb{R}\mathrm{P}^2$, or two copies of the real projective space $\mathbb{R}\mathrm{P}^2$. Unfortunately $\mathbb{R}\mathrm{P}^2$ is not orientable (see Section 2.4), thus, to conclude: $L_{0,0,1}$ is *not* a SLag. (In total there are 32 of these sets.)

- For example, if $j = 0$ and $k = l = 1$, then we have

$$L_{j,k,l} = L_{0,1,1} = \{[1 : X^1 : \gamma X^2 : \gamma X^3] \in F_4 | X^\mu \in \mathbb{R}\}. \tag{8.4}$$

Then we have the following equation:

$$0 = 1 + (Z^1)^4 + (Z^2)^4 + (Z^3)^4 = 1 + (X^1)^4 + (\gamma X^2)^4 + (\gamma X^3)^4 = 1 + (X^1)^4 - (X^2)^4 - (X^3)^4.$$

This equation has solutions. We have $1 + (X^1)^4 \geq 1$, so that $(X^2)^4 + (X^3)^4 \geq 1$. Then for any $X^1 \in \mathbb{R}$ we have $(X^2)^4 + (X^3)^4 = 1 + (X^1)^4$, and, for each X^1 fixed, the corresponding solution set is diffeomorphic to the circle S^1. Now we can split up $L_{0,1,1}$ into 2 subsets:

$$\begin{array}{ll} L_1 := \{[1 : X^1 : X^2 : X^3]\}, & (X^2)^4 + (X^3)^4 = 1 + (X^1)^4, \\ L_2 := \{[0 : 1 : X^2 : X^3]\}, & (X^2)^4 + (X^3)^4 = 1. \end{array}$$

There are diffeomorphisms $L_1 \simeq \{[1 : a] | a \in \mathbb{R}\} \times S^1$ and $L_2 \simeq [0 : 1] \times S^1$, thus there is a diffeomorphism from $L_{0,1,1}$ to $\mathbb{R}\mathrm{P}^1 \times S^1 \simeq (S^1/\mathbb{Z}_2) \times S^1 \simeq S^1 \times S^1 \simeq T^2$. Thus, to conclude: L is an orientable smooth compact connected manifold, so that it is indeed a SLag. Similarly we can argue that, for example, $L_{1,0,1}$ and $L_{1,1,0}$ are SLags, diffeomorphic to T^2.

Relations between the different SLags. We should note that any of the holomorphic actions

$$H_{2j,2k,2l} : [Z^0 : Z^1 : Z^2 : Z^3] \mapsto [Z^0 : (-1)^j Z^1 : (-1)^k Z^2 : (-1)^l Z^3]$$

will leave all the SLags invariant, so we only need to check the following $2^3 = 8$ holomorphic actions:

$$H_{0,0,0}, \quad H_{0,0,1}, \quad H_{0,1,0}, \quad H_{0,1,1}, \quad H_{1,0,0}, \quad H_{1,0,1}, \quad H_{1,1,0}, \quad H_{1,1,1}.$$

These 8 actions are isometries, and they can be used to make copies of the SLags already found. For example, we find 7 other SLags diffeomorphic to $L_{0,1,1}$, namely $L_{2j,2k+1,2l+1} = H_{j,k,l}(L_{0,1,1})$. These SLags are the fixed point sets of the involutions $D_{2j,2k+1,2l+1} = H_{2j,2k,2l} D_{0,1,1}$. Then these 8 SLags are the following sets:

$$L_{2j,2k+1,2l+1} = \{[1 : i^j X^1 : i^k \gamma X^2 : i^l \gamma X^3]\}.$$

Similarly we can find 7 other SLags diffeomorphic to $L_{1,0,1}$, namely $L_{2j+1,2k,2l+1}$, and 7 other SLags diffeomorphic to $L_{1,1,0}$, namely $L_{2j+1,2k+1,2l}$. Then we have found 24 SLags in total, all being (at least) diffeomorphic.

These three groups of 8 SLags can be related by other holomorphic actions, being isometries and leaving F_4 invariant. For example, we have the holomorphic permutation involution:

$$P : [1 : Z^1 : Z^2 : Z^3] \mapsto [1 : Z^1 : Z^3 : Z^2].$$

Then P can be expressed with respect to another coordinate W^μ, so that $Z^1 = W^1$, $Z^2 = W^2 + W^3$ and $Z^3 = W^2 - W^3$. Then the Fermat polynomial can be expressed with respect to W^μ:

$$f(W^\mu) = 1 + (W^1)^4 + 2(W^2)^4 + 2(W^3)^4 + 12(W^2)^2(W^3)^2.$$

(Note that this is a quartic with the same complex structure: any general linear map will preserve the complex structure, so that we are still dealing with the Fermat quartic.) We can express P with respect to W as $P : (W^1, W^2, W^3) \mapsto (W^1, W^2, -W^3)$. Then it is easy to prove that P can be written as a composition of two simple anti-holomorphic isometric involutions, so that P itself is also an isometry. Similarly we can prove that any other of the holomorphic permutation involutions is an isometry. Then $L_{0,1,1}$, $L_{1,0,1}$ and $L_{1,1,0}$ can be related: there are isometries between the following three sets:

$$\{[1 : X^1 : \gamma X^2 : \gamma X^3]\}, \quad \{[1 : \gamma X^1 : X^2 : \gamma X^3]\}, \quad \{[1 : \gamma X^1 : \gamma X^2 : X^3]\}.$$

In total we have found 24 SLags with the same geometry. Other SLags, like for example

$$\{[\gamma X^0 : \gamma X^1 : X^2 : X^3]\} = \{[X^0 : X^1 : \gamma^7 X^2 : \gamma^7 X^3]\} = \{[X^0 : X^1 : \gamma^3 X^2 : \gamma^3 X^3]\} = L_{0,3,3} \simeq L_{0,1,1},$$

can be identified with one of these 24.

A conclusion. We should note that

$$b_1(L_{2j,2k+1,2l+1}) = b_1(L_{2j+1,2k,2l+1}) = b_1(L_{2j+1,2k+1,2l}) = b_1(T^2) = 2,$$

which means that any of these 24 SLags is a member of a 2-dimensional smooth family of SLags.

8.2 SLags in a singular K3 surface

The BV product of two tori. We can use the BV construction, introduced in Section 6.5, to construct a BV product of the standard flat torus $T^2 := \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ with itself. Let $z \simeq z + 1 \simeq z + i$ be the trivial coordinate on T^2. We will use the standard flat metric and the standard holomorphic top-form $\Omega(T^2) = dz$. This induces a coordinate (z^1, z^2) on $T^4 = T^2 \times T^2$, and a holomorphic top-form $\Omega(T^4) = dz^1 \wedge dz^2$ on T^4. In Section 6.2 we already discussed the (connected) SLags in T^2, so that we can easily construct SLags in T^4: any pair of SLags L_1 and L_2 in T^2 will induce a SLag $L_1 \times L_2$ in T^4.

The BV product of two tori: holomorphic involutions. The holomorphic involutions needed for the BV construction will be

$$\sigma_j : T^2 \to T^2 : z^j \mapsto -z^j \quad \Rightarrow \quad \sigma = \sigma_1 \times \sigma_2 : T^4 \to T^4 : (z^1, z^2) \mapsto (-z^1, -z^2).$$

The maps σ_j indeed map $\Omega_j(T^2) = dz^j$ to $-\Omega_j(T^2) = -dz^j$. This especially holds in a pointwise manner, as $\Omega(T^2)_p = \Omega(T^2)_q$ for all $p, q \in T^2$:

$$\sigma_j^* \Omega_j(T^2)_{\sigma_j(p)} = \sigma_j^*(dz^j) = -dz^j = -\Omega_j(T^2)_p. \tag{8.5}$$

It is easy to prove that these σ_j^* are also holomorphic *isometric* involutions. Then we have $T^4/\{\sigma\} = T^4/\mathbb{Z}_2$ as a candidate BV product of two tori. (Note that we already pointed out in Section 6.5 that such an involution should indeed also be an isometry.)

The BV product of two tori: fixed points and singularities. The involutions σ_j have 4 isolated fixed points each: 0, $1/2$, $i/2$ and $(1+i)/2$. This means that the total involution σ has $4 \cdot 4 = 16$ isolated fixed points. Then the singularities induced by the fixed point set $T^4_f \subset T^4$ of σ cannot be removed by coordinate redefinitions: these points are no 'fake' singularities because $\dim_{\mathbb{C}}(T^4_f) < \dim_{\mathbb{C}}(T^4) - 1$. Then $T^4/\mathbb{Z}_2$ will be a singular surface. Then we need to do a blow-up.

The resulting smooth K3 surface S_4. According to [11] the set $T^4/\mathbb{Z}_2$ will actually be a singular K3 surface, and its blow-up $S_4 := b(T^4/\mathbb{Z}_2)$ will even be a smooth K3 surface.

We can shortly repeat how to do this blow-up, and why S_4 could indeed be a K3 surface. The Betti numbers of T^4 are $(1, 4, 6, 4, 1)$. The 1-forms and 3-forms will be cancelled by the involution σ, so that the Betti numbers of $T^4/\mathbb{Z}_2$ are $(1, 0, 6, 0, 1)$. In [11] we can read that the 16 singular points will be replaced by their smooth equivalents, each of which being diffeomorphic to a sphere S^2. Then these 16 spheres will replace $b_2 = 6$ by $b_2 = 22$. Thus, the Betti numbers of S_4 will finally be $(1, 0, 22, 0, 1)$, and these coincide with the Betti numbers of any smooth K3 surface.

Indeed, if we also look at the Hodge numbers, then we see that the 16 spheres will add 16 real 2-forms, thus 16 will be added to $h^{1,1}(T^4/\mathbb{Z}_2)$:

$$h^{j,k}(T^4) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} \Rightarrow \quad h^{j,k}(T^4/\mathbb{Z}_2) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow \quad h^{j,k}(S_4) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(See (2.52) for the Hodge diamond $h^{j,k}(T^4)$.) The Hodge numbers of S_4 again coincide with the Hodge numbers of any smooth K3 surface.

SLags in S_4. Not all of the *connected* SLags L in T^4, written as $L = L_1 \times L_2$ with L_1 and L_2 being connected SLags in T^2, are invariant under the involution σ. Only the connected SLags L_j in T^2, containing two fixed points will be invariant under σ_j, thus, unfortunately, the BV construction applied to these SLags cannot avoid any of the corresponding singularities. The connected SLags in T^2 are diffeomorphic to a circle, thus the constructed SLags in T^4 are diffeomorphic to T^2. However, any clear article explicitly describing how such a SLag L transforms to a new one $L(S_4)$ in S_4, after the blow-up, is hard to find. Especially any information about the resulting relevant differential forms (Ω and ω), needed to explicitly describe the resulting SLags, is hard to find. All we can write until now is that $L(S_4) = b(L_1 \times L_2/\mathbb{Z}_2)$.

On the other hand, if one of the SLags, say L_1, is *not* connected, then maybe we can avoid the fixed points of σ. For example, if we have a connected SLag $L_0 \subset T^2$ which does *not* contain any of the 4 fixed points, then the involution σ_1 can make a copy of L_0, or $L_0' := \sigma_1(L_0)$, so that $L_0 \cap L_0' = \varnothing$ and both L_0 and L_0' are SLags with respect to the same θ. Now define $L_1 := L_0 \cup L_0'$, a new SLag in T^2 with two connected components. Then L_1 will be diffeomorphic to two copies of the circle, and it will be invariant under the action of σ_1.

Let L_2 still be the same circle as defined earlier. Then $L = L_1 \times L_2$ will again be invariant under the total action of σ. This L will have two connected components, each of which being diffeomorphic to T^2, and L will *not* contain any of the fixed points of σ. Now note that σ will map any point lying in one connected component of L to the other connected component. Then $L/\mathbb{Z}_2$ is correctly defined. Thus, we found a SLag $L(S_4) := L/\mathbb{Z}_2 = L_1 \times L_2/\mathbb{Z}_2$ in S_4, the smooth version of the singular K3 surface, and this SLag is (also) diffeomorphic to T^2. This is in harmony with the result of finding SLags in the Fermat quartic K3 surface, see Section 8.1.

8.3 SLags in the BV product of the Fermat quartic and the flat torus

The direct product of the Fermat quartic and the flat torus. We can compute the Hodge diamond of $F_4 \times T^2$, using the Künneth formula:

$$h^{j,k}(F_4) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad h^{j,k}(T^2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \quad h^{j,k}(F_4 \times T^2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 21 & 21 & 1 \\ 1 & 21 & 21 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Now we see that the resulting space $F_4 \times T^2$ is indeed no strict Calabi-Yau 3-fold, thus we indeed have to perform the BV construction.

A holomorphic involution defined on the Fermat quartic. Let F_4 be the Fermat quartic K3 surface, as described in (8.1). Then we can define a holomorphic involution

$$\sigma_1 : F_4 \to F_4 : [a : b : c : d] \mapsto [a : b : c : -d].$$

(We could choose another σ_1 as well, mapping to $[-a : b : c : d]$, $[a : -b : c : d]$ or $[a : b : -c : d]$, or to a totally different image.)

Note that if L_1 is one of the 24 SLags, and if the point $[a : b : c : d]$ lies in L_1, then also $[a : b : c : -d] \in L_1$. Thus, any of the 24 SLags will be invariant under the involution σ_1.

Now we can define the following nowhere vanishing holomorphic top-form on $[1 : Z^1 : Z^2 : Z^3(Z^1 : Z^2)]$, the standard patch of F_4:

$$\Omega_1 = \Omega(F_4) := \frac{dZ^1 \wedge dZ^2}{(Z^3)^3}.$$

For any point $p \simeq (Z^1, Z^2, Z^3)$ we have $\sigma_1(p) \simeq (Z^1, Z^2, -Z^3)$, when expressing both p and $\sigma_1(p)$ with respect to the same coordinate. Then we have

$$\sigma_1^*(\Omega_1)_{\sigma_1(p)} = \sigma_1^*(\frac{dZ^1 \wedge dZ^2}{(-Z^3)^3}) = -\frac{\sigma_1^*(dZ^1) \wedge \sigma_1^*(dZ^2)}{(Z^3)^3} = -\frac{dZ^1 \wedge dZ^2}{(Z^3)^3} = -(\Omega_1)_p.$$

Now note that σ_1 is one of the holomorphic actions, defined in (8.2): $\sigma_1 = H_{0,0,2}$. This means that σ_1 is a holomorphic *isometric* involution, so that it finally satisfies the needed conditions. (Note that we already pointed out in Section 6.5 that such an involution should indeed also be an isometry.)

Now we can determine the fixed point set of σ_1. Points like $[a : b : c : 1]$, with at least one of the values of a, b and c non-zero, will not be mapped to themselves. Only points like $[a : b : c : 0] \in F_4$ will be mapped to themselves. This generates an interesting subset E_1 of F_4. This subset E_1 can be identified with the Fermat quartic in $\mathbb{P}^2$, and this is a Riemann surface of genus 3, see (2.34). Then $\chi(E_1) = -4$.

SLags in the Fermat quartic and fixed points. Let L_1 be the standard SLag $L_{0,1,1}$ in F_4, see (8.4). Then we can check which of the fixed points, lying in E_1, are also lying in L_1:

$$E_1 \cap L_1 = \{[X^0 : X^1 : \gamma X^2 : 0] \in F_4 | X^\mu \in \mathbb{R}\} \quad \Rightarrow \quad (X^0)^4 + (X^1)^4 = (X^2)^4.$$

For all X^0 and X^1, thus for all $[X^0 : X^1] \in \mathbb{RP}^1 \simeq S^1$ there are precisely two solutions for X^2, namely

$$X^2 = X^2(X^0, X^1) = \pm\sqrt[4]{(X^0)^4 + (X^1)^4}.$$

Thus, $E_1 \cap L_1$ has two connected components, each of which being diffeomorphic to the circle. (Note that for any of the 23 other SLags, say L_1', we can construct an intersection $E_1 \cap L_1'$ in a similar way.)

A holomorphic involution defined on the flat torus. Let $T^2 := \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ be the standard flat torus with $\Omega_2 = \Omega(T^2) = dz$ and with the standard flat metric. Then the holomorphic involution $\sigma_2 : z \mapsto -z$ maps Ω_2 to $-\Omega_2$, as (8.5) explains. We already know that not *every* connected SLag L_2 in T^2 will be invariant under this involution. Only if L_2 contains exactly two points, contained in the set E_2 of 4 isolated fixed points of σ_2, then L_2 will be invariant under σ_2. Thus, $E_2 \cap L_2$ contains two points.

However, if $L_0 \subset T^2$ is again a connected SLag which does *not* contain any of the 4 fixed points, then the involution σ_2 can be used to construct an invariant SLag $L_2 := L_0 \cup \sigma_2(L_0)$, with two connected components, and this SLag does not contain any fixed point of σ_2. Thus, $E_2 \cap L_2 = \varnothing$. (From now on we will ignore the *connected* SLag L_2.)

The total involution defined on the direct product of the Fermat quartic and the flat torus. Now we can define the total involution $\sigma = \sigma_1 \times \sigma_2$ as follows:

$$\sigma : F_4 \times T^2 \to F_4 \times T^2 : ([a : b : c : d], z) \mapsto ([a : b : c : -d], -z).$$

Let now $E := E_1 \times E_2$ be the total fixed point set of σ. Then E is diffeomorphic to 4 copies of the Riemann surface of genus 3. Thus, E, with $\dim_{\mathbb{C}}(E) = 1$, has 4 connected components and

$$\chi(E) = \chi(E_1 \times E_2) = \chi(E_1)\chi(E_2) = -4 \cdot 4 = -16.$$

This fixed point set generates singularities which cannot be removed, because $\dim_{\mathbb{C}}(E) < \dim_{\mathbb{C}}(F_4 \times T^2) - 1$. Then $F_4 \times T^2/\{\sigma\} = F_4 \times T^2/\mathbb{Z}_2$ will be a singular Calabi-Yau 3-fold, so that we need to do a blow-up.

A SLag in the direct product. Knowing that L_1 is a SLag in F_4 and that L_2 is a SLag in T^2, we can define a SLag $L := L_1 \times L_2$ in $F_4 \times T^2$. If L_2 is still the SLag containing no fixed points of σ_2, thus if $L_2 = L_0 \cup \sigma_2(L_0)$, then the intersection of the total fixed point set and L, thus

$$E \cap L = (E_1 \times E_2) \cap (L_1 \times L_2) = (E_1 \cap L_1) \times (E_2 \cap L_2),$$

is a direct product of two circles and the empty set, thus $E \cap L$ is empty. Also knowing that $L_1 \simeq T^2$ and that L_2 is diffeomorphic to two copies of the circle, we may conclude that we have found a SLag L in $F_4 \times T^2$, being diffeomorphic to two copies of $T^2 \times S^1 = T^3$.

A SLag in the BV product. We know that L_j are invariant under the action of σ_j, thus L is also invariant under the action of σ. Then it makes sense to define $L/\mathbb{Z}_2 = L/\{\sigma\}$. This σ will again map any point lying in one connected component of L to the other connected component, so that L will *not* contain any of the fixed points of σ. Now define $M := b(F_4 \times T^2/\mathbb{Z}_2)$, the blow-up of the BV product of the Fermat quartic and the flat torus. Then we can also define a SLag in M:

$$L_M := b(L/\mathbb{Z}_2) = b(L_1 \times L_2/\mathbb{Z}_2).$$

We see that the BV construction applied to these SLags avoids the singularities, so a blow-up is not needed. Thus $L_M \simeq L_1 \times L_2/\mathbb{Z}_2$, and this SLag is diffeomorphic to T^3. (Note, also T^3 satisfies $\chi(T^3) = 0$, see (4.1).)

A remark in the physical context. The SLags we found in the Borcea-Voisin product $M = b(F_4 \times T^2/\mathbb{Z}_2)$ are interesting in the context of representing membrane instantons in type IIA superstring theory, as M is a strict Calabi-Yau 3-fold. Then it does support $\mathcal{N} = 2$ supersymmetry. However, we did not find all SLags, by far, so we cannot write down a complete correction to the effective theory.

9 A summary of the mathematical results

In this chapter we will present the mathematical results of Chapter 6,7 and 8. These chapters, or most of the sections of these chapters each have an independent result, and we will give an overview here. In all of these chapters we have listed results of the quest for SLags in a specific manifold. The SLags listed in Section 7.2, 7.3, 8.1, 8.2 and 8.3 can be regarded as my own results, and the others can be found in the literature.

- In Section 6.2 we started with a manifold T^2 with complex dimension 1. This is the complex torus. Here we described infinitely many SLags. All the connected SLags are diffeomorphic, to the circle (as a real manifold).
- In Section 6.3 we started with a manifold $T^4 = T^2 \times T^2$ with complex dimension 2. Here we also described infinitely many SLags. All the connected SLags we explicitly mentioned are diffeomorphic, to the real torus T^2.
- In Section 6.4 we started with a manifold $T^6 = T^2 \times T^2 \times T^2$ with complex dimension 3. Here we again described infinitely many SLags. All the connected SLags we explicitly mentioned are diffeomorphic, to the real 3-torus $T^3 \simeq T^2 \times S^1$.
- In Section 7.1 we started with the Fermat quintic, a manifold Q with complex dimension 3. Here we just described *one* SLag, and this SLag is diffeomorphic to the real projective space $\mathbb{RP}^3$, with $\dim_{\mathbb{R}}(\mathbb{RP}^3) = 3$.
- In Section 7.2 we continued with the manifold Q of Section 7.1. Here we described 624 other SLags, and these all share the same geometry of the first one from Section 7.1: this means they are at least all diffeomorphic, and the responsible diffeomorphisms are all isometries.
- In Section 7.3 we started with the Fermat cubic, a manifold C with complex dimension 1, and this C is diffeomorphic to T^2. Here we described 9 SLags, and these all share the same geometry. These SLags are all connected manifolds, thus they are all diffeomorphic to the circle, just like the SLags mentioned in Section 6.2.
- In Section 8.1 we started with the Fermat quartic, a K3 manifold F_4 with complex dimension 2. Here we described 24 SLags. These SLags all share the same geometry, and they are diffeomorphic to the real torus T^2.
- In Section 8.2 we started with two manifolds with complex dimension 1 each: they are equal. The Borcea-Voisin product of these manifolds is $S^4 = b(T^2 \times T^2/\mathbb{Z}_2) = b(T^4/\mathbb{Z}_2)$, which has complex dimension 2, and the example SLag mentioned is (also) diffeomorphic to the real torus T^2. (This manifold S^4 is also a K3 manifold, and the manifold with singularities $T^4/\mathbb{Z}_2$ is also called a 'singular' K3 surface.)
- In Section 8.3 we started with two manifolds, one with complex dimension 2 and one with complex dimension 1. The Borcea-Voisin product of these manifolds is $M = b(F_4 \times T^2/\mathbb{Z}_2)$, which has complex dimension 3, and the example SLag mentioned is diffeomorphic to the real 3-torus T^3.

10 Conclusions

This chapter is about the conclusions of this thesis. Here we will present the results of the quest for membrane instantons in type IIA superstring theory.

The SLags found in Section 6.2, 6.3, 7.3, 8.1 and 8.2 are all manifolds of real dimension 1 or 2. They are not embedded in a Calabi-Yau 3-fold, so they cannot directly be related to the quest for membrane instantons, and the compactifying from 10 to 4 dimensions.

The SLags found in Section 6.4 *are* of real dimension 3, so they are indeed embedded in a Calabi-Yau 3-fold, but *not* in a *strict* Calabi-Yau 3-fold, thus these SLags are still not really interesting in the context of representing membrane instantons in type IIA superstring theory. Then it does not support $\mathcal{N} = 2$ supersymmetry, which is required (by definition) in case of type IIA theory.

The following Calabi-Yau 3-folds *are* strict Calabi-Yau 3-folds, thus the corresponding SLags are interesting in the main physical context of membrane instantons, thus of the non-perturbative corrections to massless type IIA superstring theory. The SLags found in Q, the Fermat quintic, represented in Section 7.1 and 7.2, and found in M, the Borcea-Voisin product of the quartic and the torus, represented in Section 8.3, are nice candidate SLags, of real dimension 3, embedded in a strict Calabi-Yau 3-fold, so they can indeed be used to represent the image of membrane instantons. Thus, the SLags embedded in Q, diffeomorphic to $\mathbb{R}\mathrm{P}^3$, represent the images of membrane instantons in Q, and the SLags embedded in M, diffeomorphic to T^3, represent the images of membrane instantons in M. These instantons are then represented by surjective maps

$$m : \Sigma \to L(\simeq \mathbb{R}\mathrm{P}^3) \subset Q \quad \text{or} \quad m : \Sigma \to L(\simeq T^3) \subset M,$$

where $L = m(\Sigma)$ is a SLag, embedded in Q *or* in M. There are many different possibilities for the map m, from different possible domain spaces Σ, and to different possible target spaces L, such that $m(\Sigma) = L$. Then, for any specific L, a specific wrapping number is related to each of these possible m and Σ related to L. In case of Q we found 625 different SLags, thus there are, at least, 625 different (but equivalent) target spaces for membrane instantons.

In both cases, of Q and M, it is still not clear yet if we found all possible SLags, thus we cannot say if we found all possible images of membrane instantons. So, unfortunately, in both cases we cannot write down a full correction to the effective theory.

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