Topological Quantum Field Theories and Frobenius Structure

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Abstract

In this thesis I would like to discuss the idea of n-dimensional topological quantum field theories, or nD-TQFTs. Any nD-TQFT can be described as a symmetric monoidal functor from **nCob**, the category of n-dimensional cobordisms, to $\mathbf{Vect}_{\Bbbk}$, the category of finite-dimensional vector spaces over some ground field $\Bbbk$. An nD-TQFT is also known as a linear representation of **nCob**. Especially the examples of **2Cob** and **1Cob** can be interesting.

I will discuss the fact that **2Cob** is a free symmetric monoidal category on a commutative Frobenius object. This means that **2Cob** carries Frobenius structure. The image of **2Cob** under any 2D-TQFT will automatically carry the same structure, and specifically the image of the circle, a basic object in **2Cob**, will be a commutative Frobenius algebra. A Frobenius algebra itself can be regarded as a Frobenius object in $\mathbf{Vect}_{\Bbbk}$. The conclusion of this part will be that the collection of 2D-TQFTs itself is again a category, and that this category is equivalent to the category of commutative Frobenius algebras.

I will also discuss the fact that **1Cob** is a free symmetric monoidal category on a dualizable object. The image of the positively oriented point will be a dualizable vector space, which can be regarded as a dualizable object in $\mathbf{Vect}_{\Bbbk}$. The conclusion of this part will be that the category of 1D-TQFTs is equivalent to the category of dualizable vector spaces.

I will mainly follow the book by J. Kock ([7]), be it in a rather different order, but I will also discuss some other articles, namely those of M. Atiyah ([4]) and of C. Blanchet and M. Turaev ([8]). I will discuss three different ways of defining topological quantum field theories, as mentioned in [7], [4] and [8], and I will try to compare them.

I will also discuss adjacent research topics. Chapters 3 and 9 can be viewed as reports on independent research I did. In Chapter 3 I will discuss the topic of porting over monoidal structure from an arbitrary monoidal category to a skeleton, not yet provided with structure. There I will try to use a general, or universal approach. In Chapter 9 I will discuss the category **1Cob** and its skeleton **1cob**. There I will mainly focus on the graded disjoint union and its behaviour.

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1 Introduction

The central questions this thesis will answer are the following:

- How are the three different definitions of topological quantum field theories, as mentioned in [7], [4] and [8], related? The answer can be found in Section 10.1, 10.2 and 10.3.
- What can we say about the category of 1-dimensional topological quantum field theories, and how is this category related to the category of dualizable vector spaces? The answer can be found in Section 10.5. The category of dualizable objects in an arbitrary symmetric monoidal category will be introduced in Section 2.4.
- What can we say about porting over the symmetric monoidal structure from **1Cob** to the skeleton **1cob**? The answer can be found in Chapter 9.
- What can we say in general about porting over the monoidal structure from an arbitrary monoidal category to one of its skeletons? The answer can be found in Chapter 3.
- How does a topological quantum field theory $\mathcal{A}$ with source category **1Cob** induce a topological quantum field theory $\mathcal{A}'$ with source category **1cob**? The answer can also be found in Section 10.5, where we introduce $\mathbf{SVect}_{\Bbbk}$, the category of signed vector spaces.

About Chapter 1. In this chapter (see Section 1.1 and 1.2) I would like to give a somewhat more physical introduction to the topic of topological quantum field theories. The next chapters will be about definitions from category theory, dualizable objects, Frobenius objects, Frobenius algebras, the theory of cobordisms and finally topological quantum field theories.

About Chapter 2,3,4 and 5. Chapter 2 will mainly be about categories. A category can be regarded as a collection of elements, including connections with direction, between elements. The elements will be renamed as *objects*, and the connections will be renamed as *arrows*. Arrows can be composed, or *horizontally composed*. We will introduce the idea of skeletons of a category. A skeleton of a category can be regarded as a minimal subset of the collection of objects and arrows, still carrying the same structure of the category itself. We also say that a skeleton is a minimal full subcategory of the main category. In many cases we have many possibilities for choosing a skeleton.

We can add more structure to many categories, for example monoidal structure and symmetric structure. We will call the result a symmetric monoidal category. In such a category, objects and arrows can also be *vertically composed*. A functor is a map between categories and a symmetric monoidal functor is a map between symmetric monoidal categories.

As a key example, we will shortly introduce $\mathbf{Vect}_{\Bbbk}$, the category of vector spaces over some ground field $\Bbbk$. Its objects are vector spaces, and its arrows are linear maps. The horizontal composition of arrows will be the ordinary composition of linear maps, and the vertical composition of objects (or arrows) will be a matter of taking tensor products of vector spaces (or linear maps).

In Section 2.4 we will introduce *dualizable objects* in the general setting. (In Chapter 9 and in Section 10.5 we will introduce dualizable objects in different specific settings.)

After introducing the concept of symmetric monoidal categories, we can directly discuss the technical details of porting over monoidal structure from a (symmetric) monoidal category to a skeleton, see Chapter 3. Then we will discuss the details of (commutative) Frobenius objects in a (symmetric) monoidal category, see Chapter 4. In Section 4.1 we will introduce Frobenius objects, which are objects satisfying some extra rules. Then we will also introduce Frobenius structure, which applies to categories generated by a single object. In Section 4.2 we will introduce Frobenius objects in $\mathbf{Vect}_{\Bbbk}$, or *Frobenius algebras*, as a special example, and in Section 4.3 we will introduce $\mathbf{cFA}_{\Bbbk}$, the category of commutative Frobenius algebras. Then Chapter 5 will be a self-contained short introduction of Morse functions. We will use *special* Morse functions in the next chapters, for splitting up cobordisms.

About Chapter 6,7,8 and 9. Chapter 6 will be about cobordisms in general, and Chapter 7 will be about how to define (symmetric monoidal) categories of cobordisms. A cobordism in dimension n is an oriented smooth manifold with a closed oriented boundary of dimension $n-1$. We can say that each cobordism has an initial boundary, or in-boundary, and a final boundary, or out-boundary. Thus a cobordism (or, to be more precise, a cobordism *class*) can be regarded as an arrow between two objects. The name of the symmetric monoidal category in question is **nCob**. The horizontal composition of these arrows is a matter of connecting the out-boundary of one cobordism to the in-boundary of another cobordism, and the vertical composition is a matter of taking disjoint unions of the objects or arrows in question. We can also do horizontal *decomposition*, or just a *splitting*, of a cobordism. Then we can split up cobordisms into smaller ones, and for this we can use special Morse functions, as introduced in Chapter 5. In Chapter 6 a procedure for splitting up cobordisms will be introduced, and in Chapter 7 a procedure for gluing them will be presented.

We will also discuss two special examples of cobordism categories. In Chapter 8 we will discuss **2Cob** and in Chapter 9 we will discuss **1Cob**. There we will also discuss skeletons of these categories, and we will try to turn them into symmetric monoidal categories also. Generators and relations of these skeletons will also be presented.

About Chapter 10. Chapter 10 will be about topological quantum field theories. A topological quantum field theory (we have more of them) can be regarded as a map between a collection of cobordisms and the collection of vector spaces and linear maps. Using more formal language: an nD-TQFT is a symmetric monoidal functor between the symmetric monoidal category of cobordisms of dimension n, or **nCob**, and the symmetric monoidal category of vector spaces and linear maps, or $\mathbf{Vect}_{\Bbbk}$. This is mainly the definition of [7], see Section 10.1. In Section 10.2 and 10.3 we will compare this definition with alternative definitions, as mentioned in [4] and [8]. These three definitions look very different at first sight, and we will discuss how they are related.

About Section 10.4. Here we will discuss 2-dimensional topological quantum field theories. We can start with the source category **2Cob** of any 2D-TQFT, but we could as well start with **2cob** (or *small* **2Cob**) instead, which is a skeleton of **2Cob**, generated by the standard oriented (unit) circle. At least Kock [7] uses this approach in his Section 3.3. We can say that a 2D-TQFT mapping from **2cob** instead of **2Cob** is much easier to describe, but it still carries the most relevant information.

The circle can be regarded as a commutative Frobenius object in **2cob**, as already explained in Chapter 8. Then we can say that this skeleton **2cob** (thus not **2Cob** itself!) is a free symmetric monoidal category on a commutative Frobenius object, or a free symmetric monoidal category carrying *Frobenius structure*. These notions will already be introduced in Section 4.1. Any 2D-TQFT will map the circle to a commutative Frobenius algebra. These algebras will already be introduced in Section 4.2.

In this section we will also mention that the collection of 2D-TQFTs can again be turned into a category. Then the category of 2D-TQFTs with source category **2Cob** will be equivalent to $\mathbf{cFA}_{\Bbbk}$, and the category of 2D-TQFTs with source category **2cob** will be isomorphic to $\mathbf{cFA}_{\Bbbk}$.

About Section 10.5. Here we will discuss 1-dimensional topological quantum field theories. The original idea was to directly start with the source category **1cob**, a skeleton of **1Cob**, of any 1D-TQFT, just as we did in Section 10.4, when we started with **2cob** as the source category of any 2D-TQFT. However, we can as well start with the source category $\mathbf{1Cob}'$, which can be regarded as a minimal full symmetric monoidal subcategory of **1Cob**. We will present both $\mathbf{1Cob}'$ and **1cob** as a source category of 1D-TQFT's.

The objects p_+ and p_-, which are positively and negatively oriented points, can be regarded as the objects generating both **1cob** and $\mathbf{1Cob}'$. These points can be regarded as dual objects of each other. We can also say that p_+ (or p_-) is a *dualizable object*. Then we can say that both $\mathbf{1Cob}'$ and **1cob** (thus, again, not **1Cob** itself!) are free symmetric monoidal categories on a dualizable object. Now any 1D-TQFT will map this p_+ to some vector space V, and p_- to some W (which is isomorphic to V^*, the canonical dual of V). Then V and W will be dualizable objects in $\mathbf{Vect}_{\Bbbk}$, or *dualizable vector spaces*.

We will mention that also the collection of 1D-TQFTs can again be turned into a category. Then the category of 1D-TQFTs with source category **1Cob** will be equivalent to $\mathbf{DVS}_{\Bbbk}$, the category of dualizable vector spaces over $\Bbbk$. The categories of 1D-TQFTs with source category **1cob** or $\mathbf{1Cob}'$ will both be isomorphic to $\mathbf{DVS}_{\Bbbk}$.

More about Chapter 3 and 9. It was my choice to study 1D-TQFTs with source category **1cob**. For me this raised the question of how to carry over the symmetric monoidal structure correctly from **1Cob** to **1cob**. The ordinary disjoint union, an operation on **1Cob**, will not work correctly on **1cob**, so the monoidal structure cannot be exactly copied from **1Cob** to **1cob**. We can say **1cob** is a minimal full subcategory of **1Cob**, but not a symmetric monoidal one. Thus a slightly different approach is needed then, and this is why some questions arose to me. Chapter 3 and a large part of Chapter 9 are about the independent research I did.

Chapter 9 will be about **1Cob** and its skeleton **1cob**. We mainly need the graded disjoint union, as **1cob** is not closed under the operation of taking ordinary disjoint unions. Mainly without consulting literature, I did some independent research for Chapter 9 to find an explicit method of checking if porting over the symmetric monoidal structure from **1Cob** to **1cob** will work out correctly. I used an explicitly defined projection functor $P : \mathbf{1Cob} \to \mathbf{1cob}$, and made a special choice for this P. This P was meant for porting over the monoidal structure, but the symmetric structure of **1Cob** was already needed for this P. Another question arose: does it really depend on this explicit choice of definition of P, whether or not the monoidal structure can be successfully ported over from **1Cob** to **1cob**? Chapter 3 was meant for finding this out.

Chapter 3 will have its own topic, but this topic is still adjacent to the main topic of this thesis. If we know that $\mathcal{C}$ is a monoidal category, and if $\mathcal{C}'$ is a skeleton of $\mathcal{C}$, not yet provided with structure, then we can choose a *projection* functor $P : \mathcal{C} \to \mathcal{C}'$, and check if this functor can help us porting over the structure. I also did some independent research to find a formal, universal approach for finding out if this functor P will really do the job, also mainly without consulting literature. The result was meant to find an alternative way for porting over the monoidal structure from **1Cob** to **1cob**, and this way fairly differs from the explicit method, discussed in Chapter 9. For example, the universal approach does not need strict monoidal structure, explicit definitions or any symmetric structure. This chapter assures that it does *not* depend on the explicit behaviour of P or on the specific properties of **1Cob**.

About Chapter 11. Chapter 11 will be about the conclusions and retrospective views of this thesis. There will be a summary of the answers to the central questions.

1.1 The physical background

Classical Mechanics. In classical mechanics, physicists study moving objects in a specified setting, and the main problem is to specify and to solve the *equations of motion*. These equations of motion are formulated as boundary value problems or differential equations. The standard procedure to find a solution of a boundary value problem often consists of three stages. In the first stage one needs to perform a series of algebraic manipulations. In the second stage one needs to compute a series of integrals, which yields a so-called 'general solution'. In the third and final stage, one should insert initial values or *boundary values*, which could for example be a specified position and velocity of an object, given at a certain time. This third stage reflects the fact that classical experiments which look different at first sight, can be described by the same underlying theory. For example, an apple falling from a tree and the moon orbiting the earth are described by the same equation. The difference lies in the boundary values.

When studying objects which are accelerated under the influence of a force, one first needs to specify a differential equation. Usually this equation involves a function which depends on time, and this equation is valid in the general sense. Whatever the initial location and speed of the object is, it should always be possible to derive the time evolution of the object's motion by this equation. After performing the first two stages of solving this equation, one must insert the boundary values. In physics these boundary values are often referred to as *degrees of freedom*.

In the example mentioned above it was clear that there is just one degree of freedom. We needed to specify two boundary values, which were the initial location and velocity, but conventionally we speak of just *one* degree of freedom. This reflects the fact that the equations of motion correspond to a second order differential equation. This means that the space of solutions can be described as a manifold of dimension two, at least when we are dealing with a linear equation. To find a solution, one only needs to specify initial conditions, which are values for the function in question and for its time derivative, or, using physical words, the position and velocity of an object at a given moment in time. For a combined system, consisting of a finite number of moving objects, also possibly interacting with each other, we say that the system has just as many degrees of freedom as the number of objects involved. In the theory of classical mechanics any such combined system and its mechanics are described by a corresponding boundary value problem.

To go one step further, we can consider the boundary value problem of a chain of many classically interacting objects. Imagine a series of objects tied to each other with springs. There are now many degrees of freedom. In theory we can describe the dynamics of a violin string by performing a limiting procedure on such a model with a finite number of degrees of freedom. In this procedure we keep the length of the chain constant, but the distance between all the links goes to zero, while the number of links goes to infinity. This is why we should call this a model with an infinite number of degrees of freedom.

Classical Field Theory. Classical field theory describes physical systems with an infinite number of degrees of freedom. Mathematically these are described by partial differential equations, in contrast to a single object or a finite group of objects, which are described by ordinary differential equations and a finite number of degrees of freedom. The aim of classical field theory is, for example, to solve the equations of motion for a violin string, or a vibrating membrane. The ripples in a pond can also be described by a classical field theory, and here the interesting part starts. Imagine a piece of wood floating in the middle of a pond, which at first is totally in rest. After you throw a stone in the water, you will see a series of waves propagating from the point where the stone hit the water. At a certain moment the waves will reach the piece of wood, which will start to move. In this scenario we thus see a kind of interaction from a distance between two objects.

There exist many simplified models for classical field theories, described by linear homogeneous partial differential equations which have exact solutions. In reality though, classical field theories tend to be inhomogeneous and non-linear. Very few of these problems are exactly solvable; most of them can only approximately be solved by using classical perturbation theory.

Quantum Mechanics. In the early 20th century the theory of *Quantum Mechanics* was born. It can be regarded as an extension of the already fully developed theory of classical mechanics. At the time this theory of quantum mechanics was fully developed, the procedure for quantizing any classical boundary value problem was generally applicable. Quantum mechanics is centered around the famous *Schrödinger equation*, which is a linear partial differential equation of the following form:

$$i\hbar\frac{\partial}{\partial t}\psi(r,t) = \hat{H}\psi(r,t) \tag{1.1}$$

The variables r and t are space and time variables respectively. The operator $\hat{H}$, called the *Hamiltonian*, is a linear partial differential operator which contains derivatives up to second order with respect to space variables only. Any solution of this equation is specified by a *wave function*, or *state function*, which is complex valued. We note that equation (1.1) is of first order with respect to t, thus we could say that it directly dictates the time evolution of a wave function. The boundary values, needed to solve (1.1) completely, are specified by a *time-independent wave function*, which is a wave function at a fixed moment in time, and as such only depends on space coordinates. Every smooth and square-integrable function should be valid. If a time-independent wave function $\psi_i(r)$ is known at a specific time t_i, we can easily generate a solution of (1.1) as follows:

$$\psi(r,t) := \exp(-\frac{i}{\hbar}(t-t_i)\hat{H})\psi_i(r). \tag{1.2}$$

Thus the result will be a time-dependent function. We thus see that an initial time-independent wave function must be specified in order to solve the Schrödinger equation completely.

In more mathematical terms, we are dealing with a Hilbert space $\mathcal{H}$ of possible time-independent wave functions. This Hilbert space is characterized by the specific boundary value problem belonging to the classical model considered. We will use vector notation for elements of $\mathcal{H}$, thus any $\psi(r)$ will be written as $|\psi(r)\rangle$. Then there is a time evolution operator $T(t_f, t_i) : \mathcal{H} \to \mathcal{H}$, which is linear. This operator is equivalent to the exponent as written in (1.2). If $|\psi_i(r)\rangle$ is a boundary value of the theory, then we can find a specific solution $|\psi(r,t)\rangle$ of (1.1). We should note that these time-dependent state vectors also generate a Hilbert space, but this is not the same space as $\mathcal{H}$. Restricting this $|\psi(r,t)\rangle$ to any other constant time t_f, it will satisfy:

$$|\psi(r,t)\rangle_{t=t_f} := |\psi_f(r)\rangle := T(t_f, t_i)|\psi_i(r)\rangle.$$

Note that, in most traditional applications, we assume the model itself will not change in time, thus we will use the same space $\mathcal{H}$ of possible time-independent wave functions for all moments in time.

In quantum mechanics we are often interested in a statistical quantity called the *transition amplitude*. It is defined as the probability of a quantum mechanical system prepared in a certain state $|\psi_i(r)\rangle$ at time t_i to be found in state $|\phi_f(r)\rangle$ at another time t_f. Therefore we first need to apply time evolution to the initial state $|\psi_i(r)\rangle$, and then we can compute the inner product of the resulting state and $|\phi_f(r)\rangle$. This yields $\langle\phi_f(r)|T(t_f, t_i)|\psi_i(r)\rangle$, where $\langle\cdot|\cdot\rangle$ is just the inner product as defined on $\mathcal{H}$. Taking the square of its absolute value will give us a probability number, which we will call the transition amplitude. Traditionally, the transition amplitude is computed by first solving the Schrödinger equation and then computing its corresponding time evolution operator.

Quantum Field Theory. In a later stage of the development of quantum mechanics, new techniques were invented to compute transition amplitudes and other quantities of statistical importance belonging to the quantum model corresponding to a classical *field* theory, which we will call a *quantum field theory*. Here *classical* just means *not quantum*, thus also models obeying Einstein's theory of special relativity are allowed. Especially the theory of electromagnetism was one of the first to be described in this new setting of a quantum field theory. The new techniques of quantum field theory differ from ordinary quantum mechanics, based on the Schrödinger equation, and a lot more brute-force computations are needed, for example the *Feynman path integral*.

1.2 Topological Quantum Field Theories

A topological quantum field theory is a quantum field theory with the additional property that transition amplitudes do not depend on the geometry of the underlying space-time, only on its topology. Thus when a space-time for example smoothly warps or contracts, the transition amplitudes do not change. We could also define a metric on the underlying space, but the model will not depend on this metric. We should note that if this underlying space-time is just Minkowski space, which is the background space-time $\mathbb{R}^4$ equipped with the Minkowski metric of special relativity, then there is really nothing interesting going on. The Minkowski space is a contractible space, thus it has only trivial topological properties.

It gets more interesting when the background space-time has non-trivial topology, especially when not every time slice has the same topology also. This type of spaces especially shows up in for example *string theory*. In closed superstring theory, we consider compact oriented manifolds of dimension two, embedded in a Minkowski space of dimension 10, and the embedding map can be regarded as a field theory. Usually, these manifolds are described as Riemann surfaces, or to be more precise, these M are the *image* of some embedding $\Sigma \to \mathbb{R}^{10}$, where Σ is a Riemann surface. Let $T(t_0)$ be the manifold of constant time t_0, then $T(t_0)$ is a contractible submanifold of $\mathbb{R}^{10}$ of dimension 9. Then we can define the intersection $X(t_0) := M \cap T(t_0)$.

In many cases $X(t_0)$ will be a compact manifold of dimension one. This $X(t_0)$ has some number $N(t_0)$ of connected components $X(t_0)_j$, and each component carries its own space $\mathcal{H}(X(t_0)_j)$ of possible quantum states. The total space $\mathcal{H}(X(t_0))$ of quantum states possible on $X(t_0)$ is the tensor product of these $\mathcal{H}(X(t_0)_j)$. However, we should take care when $X(t_0)$ is *not* a proper manifold. Any compact M without

boundary has intersections of this type which are not proper submanifolds. We assume that it is always possible to find a Morse function, defined on M, such that for all t_0 we can interpret $X(t_0)$ as a level set of this Morse function. In case this $X(t_0)$ is not a proper manifold, we are dealing with a critical point of this Morse function, lying in $X(t_0)$.

We should note that in many quantum field theories the metric is a dynamical field. This means that making any final conclusions about the model in use, we also need to consider all possible metrics, thus we need to perform an extra path integral over this dynamical variable. In a topological quantum field theory we do not desire transition amplitudes or any other quantity depending on a metric. According to [5], to make sure this will indeed not happen, we first need to realize that a background metric is needed when constructing the theory itself, so one needs to choose a background metric. So at first sight the complete Hilbert space $\mathcal{H}_C$ will include quantum states whose amplitudes will depend on the metric. However, any TQFT admits a symmetry, a BRST-like operator $Q : \mathcal{H}_C \to \mathcal{H}_C$, which is nilpotent. The Hilbert space of physical states is defined as the Q-cohomology group:

$$\mathcal{H}_{phys} := \mathrm{Ker}(Q)/\mathrm{Im}(Q).$$

This will project out all quantum states whose amplitudes depend on the metric, or, to be even more precise, it divides out all metric-dependence. This means that it does not matter which specific background metric we start with, and varying this metric also does not have any effect. Thus a path integral over all possible metrics is not needed now, thus many computations can be simplified.

In quantum mechanics it is common use to make a difference between three types of models. Let $\mathcal{H}$ be the Hilbert space corresponding to the model, then we consider the following three types:

- $\mathcal{H}$ is infinite-dimensional with an uncountable basis.
- $\mathcal{H}$ is infinite-dimensional with a countable basis.
- $\mathcal{H}$ is finite-dimensional, thus consequently it carries a finite basis, which is countable by definition.

A model of a free particle propagating in $\mathbb{R}^3$ with constant velocity is an example of a model of the first type. A model of an oscillating particle, trapped in a potential, is an example of a model of the second type. A model of a particle only carrying spin is an example of a model of the last type, as there is only a finite number of spin-states. Especially models of the last type are interesting in the context of TQFTs. The classical axioms of a TQFT, described in [4], apply to finite-dimensional Hilbert spaces by definition, and in this case these axioms are exact. According to [4] it is also possible to apply the theory of TQFTs to infinite-dimensional Hilbert spaces, but then we must make proper redefinitions of the axioms, making them less exact. For simplicity we will ignore the first two possible types from now on, and only consider finite-dimensional Hilbert spaces.

2 Monoids, Categories and Functors

I would like to give an introduction of some basic concepts in category theory, without entering too much into detail. For the reader who is interested in a more detailed introduction, especially written for the subjects discussed further, I would like to suggest reading [7]. In this book one can find a lot of figures which might help to get more intuition.

The development of the concept of categories is a successful attempt to describe the grammar of many mathematical structures. The structure of Frobenius algebras and of cobordisms, and the things we would like to do with them, can be nicely summarized in terms of categorical language. Just like words can be decomposed into letters, we can decompose any 2-dimensional cobordism into elementary pieces. We will observe that some of these 'words', built up as different sequences of 'letters', will in fact be equal as cobordisms. This yields relations, in fact identities, between words or parts of words, which might look different at first sight.

As a standard example, we will shortly introduce the symmetric monoidal category of vector spaces near the end of Section 2.3. Then we will also introduce the category of dualizable objects in Section 2.4. More examples of categories, also symmetric monoidal ones, will be studied in more detail in later chapters.

2.1 Monoids

A monoid is a set M, together with a binary operation $\mu : M \times M \to M$, written like multiplication $\mu : (a, b) \mapsto a \cdot b$. This multiplication is associative and has a neutral element $e \in M$, satisfying $e \cdot a = a \cdot e = a$ for all $a \in M$. This e is unique.

A more formal description of a monoid can be introduced. Let M be a set. Let M^m denote the m-fold Cartesian product of M, written as $M^m = M \times \cdots \times M$. We define the *singleton set* $1 := M^0$. The singleton set satisfies the property $1 \times M = M = M \times 1$. In what follows $\mathrm{Id}_M : M \to M$ will mean the identity function.

A *monoid* is a set M together with two maps

$$\mu : M \times M \to M \quad , \quad \eta : 1 \to M,$$

where μ satisfies the *associativity relation*

$$\mu \circ (\mu \times \mathrm{Id}_M) = \mu \circ (\mathrm{Id}_M \times \mu)$$

and η satisfies

$$\mu \circ (\mathrm{Id}_M \times \eta) = \pi_{(M,1)} \quad , \quad \mu \circ (\eta \times \mathrm{Id}_M) = \pi_{(1,M)}.$$

Here $\pi_{(M,1)}$ and $\pi_{(1,M)}$ are the canonical projection maps

$$\pi_{(M,1)} : M \times 1 \to M \quad , \quad \pi_{(1,M)} : 1 \times M \to M.$$

We can express these properties using the following commuting diagrams:

$$\begin{tikzcd}
 & M \times M \times M \arrow[dl, "\mu \times \mathrm{Id}_M"'] \arrow[dr, "\mathrm{Id}_M \times \mu"] & \\
M \times M \arrow[dr, "\mu"'] & & M \times M \arrow[dl, "\mu"] \\
 & M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times 1 \arrow[r, "\mathrm{Id}_M \times \eta"] \arrow[dr, "\pi_{(M,1)}"'] & M \times M \arrow[d, "\mu"] & 1 \times M \arrow[l, "\eta \times \mathrm{Id}_M"'] \arrow[dl, "\pi_{(1,M)}"] \\
 & M &
\end{tikzcd}
\tag{2.1}$$

We note that associativity implies that μ induces uniquely defined maps $\mu^{(m)} : M^m \to M$. Any monoid is written as a triple (M, μ, η), or alternatively $(M, \cdot, e)$.

When M and M' are two monoids, a *monoid homomorphism* $\phi : M \to M'$ is a function for which the following diagrams commute:

$$\begin{tikzcd} M \times M \arrow[r, "\phi\times\phi"] \arrow[d, "\mu"'] & M' \times M' \arrow[d, "\mu'"] \\ M \arrow[r, "\phi"'] & M' \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "\phi"] & & M' \\ & 1 \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

With this formal description of a monoid we can later define the notion of a monoidal category.

2.2 Categories

Categories: The main definition. A *category* $\mathcal{C}$ is given by a collection $\mathcal{C}_0$ of *objects* and a collection $\mathcal{C}_1$ of morphisms, or *arrows*. These collections are equipped with the following structure:

- Each arrow has a *domain* and a *codomain*, which are objects. If X is the domain of the arrow f, and Y its codomain, alternatively written as $X = \mathrm{dom}(f)$ and $Y = \mathrm{cod}(f)$, then we write $f : X \to Y$.
- For any two arrows $f : X \to Y$ and $g : Y \to Z$, another (unique) arrow $gf : X \to Z$ exists. We say gf is the *composition* of f and g.
- The composition is *associative*, that is, given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, then $h(gf) = (hg)f$. Thus we are allowed to write $hgf : X \to W$.
- For every object X there is an *identity arrow* $\mathrm{Id}_X : X \to X$ which satisfies $f\,\mathrm{Id}_X = f$ and $\mathrm{Id}_X\, g = g$ for all $f : X \to Y$ and $g : Y \to X$. For all X this identity arrow is unique.

We remark that the arrows always point from an object to an object, thus it defines some sense of direction. There is one subtlety though. When an arrow $f : X \to Y$ exists, there is no reason to assume another arrow $g : Y \to X$ exists. For any pair $X, Y \in \mathcal{C}_0$ we define $\mathcal{C}(X, Y)$ as the collection of arrows with domain X and codomain Y.

Example of a category. As an example of a category, considering monoids as objects and monoid homomorphisms as arrows, we can define the category of monoids, written as **Mon**.

Subcategories. Let $\mathcal{C}$ and $\mathcal{D}$ both be categories, then $\mathcal{D}$ is a subcategory of $\mathcal{C}$ if the following hold:

- Any object in $\mathcal{D}$ is also an object in $\mathcal{C}$.
- Any arrow in $\mathcal{D}$ is also an arrow in $\mathcal{C}$.
- The composition of arrows with respect to $\mathcal{C}$ and with respect to $\mathcal{D}$ is the same.
- The identity arrows with respect to $\mathcal{C}$ and with respect to $\mathcal{D}$ are the same.

If for every pair of objects X and Y in $\mathcal{D}$ we have $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)$, then we call $\mathcal{D}$ a *full* subcategory of $\mathcal{C}$.

Isomorphisms, isomorphism classes and skeletons. An arrow $f : X \to Y$ is called an *isomorphism* if there exists another (unique) arrow $g : Y \to X$ such that $gf = \mathrm{Id}_X$ and $fg = \mathrm{Id}_Y$. Then we say X and Y are *isomorphic*, and we will write $g = f^{-1}$. To any object X an *isomorphism class* ι_X is associated, which contains all objects isomorphic to X. For any Y isomorphic to X we have $\iota_Y = \iota_X$.

A *skeleton* of a category $\mathcal{C}$ is a category $\mathcal{D}$ satisfying the following:

- Every object in $\mathcal{D}$ is an object in $\mathcal{C}$.
- For every pair of objects X and Y in $\mathcal{D}$, the arrows in $\mathcal{D}$ are precisely the arrows in $\mathcal{C}$, thus $\mathcal{D}(X,Y) = \mathcal{C}(X,Y)$.
- For every object X in $\mathcal{D}$, its identity arrow with respect to $\mathcal{D}$ coincides with its identity with respect to $\mathcal{C}$.
- The composition law of arrows in $\mathcal{D}$ equals the composition law of arrows in $\mathcal{C}$, when restricted to arrows in $\mathcal{D}$.
- Every object in $\mathcal{C}$ is isomorphic to some object in $\mathcal{D}$.
- There exists no isomorphism between any pair of distinct objects in $\mathcal{D}$.

This means that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ and that it contains exactly one object from each distinct isomorphism class of $\mathcal{C}$, and that the only isomorphisms in $\mathcal{D}$ are arrows, not necessarily identities, from an object to itself.

A skeleton is not unique, but any two skeletons $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic, in the sense that for both the objects and arrows in $\mathcal{D}$ and $\mathcal{D}'$ a one-to-one relation exists, thus for every object or arrow in $\mathcal{D}$ there is a unique corresponding object or arrow in $\mathcal{D}'$. Thus a skeleton is only unique up to isomorphisms of categories.

We will not claim that for every category there exists a skeleton, but when a skeleton of $\mathcal{C}$ exists, we can say that it describes the essential structure of $\mathcal{C}$.

Example of a category: A groupoid. A category $\mathcal{C}$ is called a *groupoid* if all of its arrows are isomorphisms. Then we see that, for any object X, $\mathcal{C}(X,X)$ can be regarded as a group. As a nice example we could introduce the *fundamental groupoid* Π_X of a topological space X. The objects in Π_X are points in X, and for any two points p and q the arrows from p to q are homotopy classes of paths, lying in X, from p to q. Here we recall that a path from p to q is specified by a continuous map $\gamma : [0,1] \to X$, with $\gamma(0) = p$ and $\gamma(1) = q$. Then $\pi_1(X,p) := \Pi_X(p,p)$ is the fundamental group of X at p. We note that a topological space X needs not be connected. This means that its fundamental groupoid is also not connected. Of course, in an arbitrary category, two distinct objects do not need to be connected by arrows. In general, a groupoid needs not be connected.

Generators and relations. When we have found a skeleton $\mathcal{D}$ of $\mathcal{C}$ it is possible to study its further structure. A *generating set* $G(\mathcal{D})$ is a collection of arrows in $\mathcal{D}$ so that every other arrow in $\mathcal{D}$ can be obtained by composing arrows in $G(\mathcal{D})$. This set is not necessarily unique, but we assume it is minimal. Any arrow in $G(\mathcal{D})$ is called a *generator*. A relation is the equality of two distinct ways of writing a given arrow in terms of these generators. A minimal set $R(\mathcal{D})$ of relations is called *complete* if every other relation can be obtained by combining relations in $R(\mathcal{D})$.

Functors. A functor is a map between categories. Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of operations $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$, so that for each $f : X \to Y$, where $X, Y \in \mathcal{C}_0$ and $f \in \mathcal{C}_1$, we have $F_1(f) : F_0(X) \to F_0(Y)$ and:

- For $f : X \to Y$ and $g : Y \to Z$, we have $F_1(gf) = F_1(g)F_1(f)$.
- $F_1(\mathrm{Id}_X) = \mathrm{Id}_{F_0(X)}$ for each $X \in \mathcal{C}_0$.

A trivial example is the *identity functor* from a category $\mathcal{C}$ to itself. It maps any object or arrow to itself. It is written as $\mathrm{Id}_{\mathcal{C}}$.

Strictly speaking, it is common use to make a difference between *covariant* functors and *contravariant* functors. The definition above is really a covariant functor, as it *preserves* the direction of arrows. A contravariant functor *reverses* the direction of arrows. From now on, when saying some map F is a functor, we will automatically mean a covariant functor.

The Cartesian product of categories. For each pair of categories $\mathcal{C}$ and $\mathcal{D}$, we define its Cartesian product $\mathcal{C} \times \mathcal{D}$ as follows. For any $X \in \mathcal{C}_0$ and $Y \in \mathcal{D}_0$ we have an object (X, Y) of $\mathcal{C} \times \mathcal{D}$. The set of arrows from (X, Y) to (X', Y') is the Cartesian product $\mathcal{C}(X, X') \times \mathcal{D}(Y, Y')$. This defines the Cartesian product of categories completely. The *singleton category*, or the empty product category, which means the product of zero factors, is denoted by $\mathbf{1} = \mathcal{C}^0$. This $\mathbf{1}$ contains only one object and only one arrow, which is the identity arrow of the single object.

The Cartesian product of categories satisfies the rules $(\mathcal{C} \times \mathcal{D}) \times \mathcal{E} = \mathcal{C} \times (\mathcal{D} \times \mathcal{E})$ and $\mathcal{C} \times \mathbf{1} = \mathcal{C} = \mathbf{1} \times \mathcal{C}$. The equality symbols should be interpreted as natural identifications, not as exact equalities. Strictly speaking, the Cartesian product of sets itself is not associative. As a consequence, the Cartesian product of categories is also not associative. However, there are natural identifications, and these are generated by canonical functors. From now on we will not care about this subtlety. We will ignore the brackets and we will use the n-fold Cartesian product of categories. For example, we will write $\mathcal{C}^n = \mathcal{C} \times \cdots \times \mathcal{C}$. Furthermore, we can define the *canonical projection functors* $\pi_{(\mathbf{1},\mathcal{C})}$ and $\pi_{(\mathcal{C},\mathbf{1})}$:

$$\pi_{(\mathbf{1},\mathcal{C})} : \mathbf{1} \times \mathcal{C} \to \mathcal{C} \quad , \quad \pi_{(\mathcal{C},\mathbf{1})} : \mathcal{C} \times \mathbf{1} \to \mathcal{C}.$$

The composition and the Cartesian product of functors. The composition and the Cartesian product of two functors can be defined in a natural way. The composition of a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is written as $GF : \mathcal{C} \to \mathcal{E}$. This composition is associative. For any pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{C}' \to \mathcal{D}'$ their Cartesian product is written as $F \times G : \mathcal{C} \times \mathcal{C}' \to \mathcal{D} \times \mathcal{D}'$.

Isomorphism of categories: The formal definition. We say the categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ exist such that $GF = \mathrm{Id}_{\mathcal{C}}$ and $FG = \mathrm{Id}_{\mathcal{D}}$. As mentioned earlier, any two skeletons $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{C}$ are isomorphic. However, we should realize that the responsible isomorphism is not unique.

A functor from a category to a skeleton. If $\mathcal{C}$ is a category, and if $\mathcal{C}'$ is a skeleton of $\mathcal{C}$, then we can define a projection functor $P : \mathcal{C} \to \mathcal{C}'$. Any object X' in $\mathcal{C}'$ is also an object in $\mathcal{C}$. Let $i_{X'}$ be its isomorphism class in $\mathcal{C}$. Then P will map any object $X \in i_{X'}$ to X', thus we will write $P_0(X) = X'$. This projection functor is not unique. Only its behaviour with respect to objects is unique, but we need to make a choice for its behaviour with respect to arrows, and there does not really exist any canonical choice.

So, how to choose this functor? For any $X' \in \mathcal{C}'_0$ and for any $X \in i_{X'}$ there are isomorphisms from X to X'. In fact it is possible there are many different isomorphisms from X to X', and this is really the cause of a projection functor not being unique. Suppose X and Y are two arbitrary objects in $\mathcal{C}$, and f is an arrow from X to Y. Now define $X' := P_0(X)$ and $Y' := P_0(Y)$, which are objects in $\mathcal{C}'$. Then f can be mapped to an arrow from X' to Y', which is an arrow in $\mathcal{C}'$. If for example $\iota_1 : X \to X'$, $\iota_2 : X \to X'$, $\iota_3 : Y \to Y'$ and $\iota_4 : Y \to Y'$ are all different isomorphisms, then we could define $f_1 := \iota_3 f \iota_1^{-1}$, $f_2 := \iota_3 f \iota_2^{-1}$, $f_3 := \iota_4 f \iota_1^{-1}$ and $f_4 := \iota_4 f \iota_2^{-1}$. All of these should be different arrows in $\mathcal{C}'$ from X' to Y', and all of these can equally well be chosen as the image $P_1(f)$, so none of these should be preferred.

The functor P is completely determined after choosing one of the isomorphisms for every pair of distinct objects from the same isomorphism class: for any $X \in i_{X'}$ one isomorphism $\iota_X : X \to X' = P_0(X)$ is chosen. If $X = X'$ then we will choose $\iota_X = \iota_{X'} := \mathrm{Id}_{X'}$, but for all other $X \neq X'$ such a canonical choice is not possible. If f is an arrow from X to Y in $\mathcal{C}$, then $f' := P_1(f)$ is an arrow from $X' := P_0(X)$ to $Y' := P_0(Y)$, and it is defined by $P_1(f) := \iota_Y f \iota_X^{-1}$. Doing this for all arrows in $\mathcal{C}$ should define P completely. If f is already an arrow in $\mathcal{C}'$, then we write $P_1(f) = f$.

We can introduce a notation convention. Whenever an object carries a $'$-sign, it is automatically assumed to lie in $\mathcal{C}'$. For any object X in $\mathcal{C}$, we will write $X' := P_0(X)$, but it will depend on the context. Sometimes we will only discuss objects in a skeleton. Then we do not automatically assume that X' is the projected object related to some other X.

There is a unique canonical inclusion functor, say $I : \mathcal{C}' \to \mathcal{C}$. Any object or arrow in $\mathcal{C}'$ is always an object or arrow in $\mathcal{C}$, thus I will really do nothing but mapping any object or arrow to itself. This functor satisfies $PI = \mathrm{Id}_{\mathcal{C}'}$, which implies that the action of P is uniquely determined when P is restricted to $\mathcal{C}'$ itself, as it should be. Two exact relations hold:

$$\pi_{(\mathbf{1},\mathcal{C}')} = P\pi_{(\mathbf{1},\mathcal{C})}(\mathrm{Id}_{\mathbf{1}} \times I), \quad \pi_{(\mathcal{C}',\mathbf{1})} = P\pi_{(\mathcal{C},\mathbf{1})}(I \times \mathrm{Id}_{\mathbf{1}}). \tag{2.2}$$

Natural transformations. A *natural transformation* α between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of arrows $\alpha_X : F_0(X) \to G_0(X)$, which are arrows in $\mathcal{D}$, where X indexes all objects in $\mathcal{C}$, such that for any pair $X, Y \in \mathcal{C}_0$ and for any arrow $f \in \mathcal{C}_1, f : X \to Y$ the following diagram commutes in $\mathcal{D}$:

$$\begin{tikzcd} F_0(X) \arrow[r, "\alpha_X"] \arrow[d, "F_1(f)"'] & G_0(X) \arrow[d, "G_1(f)"] \\ F_0(Y) \arrow[r, "\alpha_Y"'] & G_0(Y) \end{tikzcd} \tag{2.3}$$

In this case we say the arrows α_X are *natural*. We will write $\alpha : F \Rightarrow G$, or draw the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \end{tikzcd}$$

For any functor $F : \mathcal{C} \to \mathcal{D}$ there exists an identity natural transformation $\mathrm{Id}_F : F \Rightarrow F$. We will write $(\mathrm{Id}_F)_X = \mathrm{Id}_{F_0(X)}$.

The Cartesian product of natural transformations. The Cartesian product of functors naturally induces a Cartesian product of natural transformations. For any pair of natural transformations $\alpha : F \Rightarrow G$ and $\beta : H \Rightarrow J$ we write $\alpha \times \beta : F \times H \Rightarrow G \times J$. Identity natural transformations are trivially related by $\mathrm{Id}_{F \times G} = \mathrm{Id}_F \times \mathrm{Id}_G$.

Serial composition of natural transformations. Natural transformations can be composed, and can be interpreted as arrows between functors. If $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are natural transformations, then $\beta\alpha : F \Rightarrow H$, defined by $(\beta\alpha)_X := \beta_X \alpha_X$ for all X, is the result of composing them. We can call this *serial composition*. Associativity of composing arrows implies associativity of serial composition of natural transformations. Any pair of categories $\mathcal{C}, \mathcal{D}$ induces a new category $\mathcal{D}^{\mathcal{C}}$, where the objects are functors $F : \mathcal{C} \to \mathcal{D}$ and the arrows are natural transformations between these functors. Any natural transformation $\alpha : F \Rightarrow G$ satisfies $\mathrm{Id}_G\, \alpha = \alpha = \alpha\, \mathrm{Id}_F$.

Parallel composition of natural transformations. Let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$, and let H and J be functors from $\mathcal{D}$ to $\mathcal{E}$. If natural transformations $\alpha : F \Rightarrow G$ and $\beta : H \Rightarrow J$ exist, then these trivially induce another natural transformation $\gamma : HF \Rightarrow JG$, which can be regarded as a *parallel composition* of α and β. We could write $\gamma = \beta * \alpha$ to express this. This can be visualized as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{E} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "HF"] \arrow[r, bend right, "JG"'] \arrow[r, phantom, "\Downarrow \gamma"] & \mathcal{E} \end{tikzcd}$$

So, how to define this parallel composition? For any object X in $\mathcal{C}$ we can define an arrow $\gamma_X : H_0(F_0(X)) \to J_0(G_0(X))$:

$$\gamma_X := J_1(\alpha_X)\beta_{F_0(X)} = \beta_{G_0(X)}H_1(\alpha_X). \tag{2.4}$$

For any arrow $f : X \to Y$ in $\mathcal{C}$ it is easy to show that it is compatible with γ:

$$\gamma_Y H_1(F_1(f)) = \beta_{G_0(Y)}H_1(\alpha_Y)H_1(F_1(f)) = \beta_{G_0(Y)}H_1(\alpha_Y F_1(f)) = \beta_{G_0(Y)}H_1(G_1(f)\alpha_X) = \\ \beta_{G_0(Y)}H_1(G_1(f))H_1(\alpha_X) = J_1(G_1(f))\beta_{G_0(X)}H_1(\alpha_X) = J_1(G_1(f))\gamma_X.$$

Thus we can safely conclude that γ is indeed another natural transformation. Associativity of composing functors implies associativity of parallel composition of natural transformations.

For any pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, and for identity natural transformations Id_F and Id_G, we will write $\mathrm{Id}_{GF} = \mathrm{Id}_G * \mathrm{Id}_F$.

Composing natural transformations and functors. Suppose we have the following situation:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E} \end{tikzcd}$$

Then $\alpha : F \Rightarrow G$ and H induce another natural transformation $\mathrm{Id}_H * \alpha : HF \Rightarrow HG$. Using (2.4) we can write

$$(\mathrm{Id}_H * \alpha)_X = (\mathrm{Id}_H)_{G_0(X)}H_1(\alpha_X) = \mathrm{Id}_{H_0(G_0(X))}\, H_1(\alpha_X) = H_1(\alpha_X).$$

We could call this the *left composition* of a natural transformation and a functor.

We could also have the following situation:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{E} \end{tikzcd}$$

Then F and $\alpha : G \Rightarrow H$ induce another natural transformation $\alpha * \mathrm{Id}_F : GF \Rightarrow HF$. Again, using (2.4) we can write

$$(\alpha * \mathrm{Id}_F)_X = \alpha_{F_0(X)}G_1((\mathrm{Id}_F)_X) = \alpha_{F_0(X)}G_1(\mathrm{Id}_{F_0(X)}) = \alpha_{F_0(X)}\,\mathrm{Id}_{G_0(F_0(X))} = \alpha_{F_0(X)}.$$

We could call this the *right composition* of a natural transformation and a functor.

Combining the previous two diagrams gives us:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{E} \arrow[r, "J"] & \mathcal{F} \end{tikzcd}$$

Then F, α and J induce another natural transformation $\mathrm{Id}_J * \alpha * \mathrm{Id}_F : JGF \Rightarrow JHF$, and we can write

$$(\mathrm{Id}_J * \alpha * \mathrm{Id}_F)_X = J_1((\alpha * \mathrm{Id}_F)_X) = J_1(\alpha_{F_0(X)}).$$

Natural isomorphisms. We say a natural transformation α between functors $F, G : \mathcal{C} \to \mathcal{D}$ is *invertible* if for all objects X in $\mathcal{C}$ the arrows α_X in $\mathcal{D}$ are isomorphisms. In this case we will call all α_X *natural isomorphisms*, as they all commute with the structure as in diagram (2.3).

Equivalence of categories. We say the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and invertible natural transformations $\alpha : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ and $\beta : \mathrm{Id}_{\mathcal{D}} \Rightarrow FG$. It should be clear that an isomorphism of categories is a special case of an equivalence of categories: the natural transformations in question are trivial.

As a key example, when $\mathcal{C}'$ is a skeleton of $\mathcal{C}$, then we can say $\mathcal{C}$ and $\mathcal{C}'$ are equivalent: the functors in question are the canonical injection functor $I : \mathcal{C}' \to \mathcal{C}$ and a projection functor $P : \mathcal{C} \to \mathcal{C}'$. We already know $PI : \mathcal{C}' \to \mathcal{C}'$ equals $\mathrm{Id}_{\mathcal{C}'}$, resulting in the identity natural transformation $\beta : \mathrm{Id}_{\mathcal{C}'} \Rightarrow PI$, and we can indicate a natural transformation $\kappa : \mathrm{Id}_{\mathcal{C}} \Rightarrow IP$. When P is specified, we already choose isomorphisms $\iota_X : X \to X' = P_0(X)$. For any arrow $f : X \to Y$ in $\mathcal{C}$ there is an arrow $P_1(f) : X' \to Y'$, defined by $P_1(f) = \iota_Y f \iota_X^{-1}$. This is also an arrow in $\mathcal{C}$ itself, as I shows. Now we need to find an invertible natural transformation. An obvious choice of natural isomorphisms would be ι_X itself, as we already know $P_1(f)\iota_X = \iota_Y f$ for any f, thus defining $\kappa_X := \iota_X$ for any X defines an invertible natural transformation from $\mathrm{Id}_{\mathcal{C}}$ to IP. In fact this is the main principle behind projection functors from a category to a skeleton: a natural transformation κ *defines* the arrows ι_X needed to specify such a projection functor completely.

As a side note, when P and P' are different projection functors from $\mathcal{C}$ to $\mathcal{C}'$, then also two different invertible natural transformations exist, say κ and κ'. As a consequence, there exists an invertible natural transformation from P to P'.

2.3 Monoidal categories

Strict monoidal categories: Main definition. Despite the fact that we are only interested in monoidal categories, also called *tensor categories*, we will start with the definition of a *strict* monoidal category. A *strict monoidal category* is a category $\mathcal{C}$ together with two functors

$$\mu : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad , \quad \eta : \mathbf{1} \to \mathcal{C}, \tag{2.5}$$

where μ satisfies

$$\mu(\mu \times \mathrm{Id}_{\mathcal{C}}) = \mu(\mathrm{Id}_{\mathcal{C}} \times \mu) \tag{2.6}$$

and η satisfies

$$\mu(\eta \times \mathrm{Id}_{\mathcal{C}}) = \pi_{(\mathbf{1},\mathcal{C})}, \quad \mu(\mathrm{Id}_{\mathcal{C}} \times \eta) = \pi_{(\mathcal{C},\mathbf{1})}. \tag{2.7}$$

As now μ and η are functors, and especially μ defines a binary operation, or some kind of product, we would like to pick a symbol for writing products of objects *and* arrows. For any pair of objects X and Y in $\mathcal{C}$ we will write $\mu_0(X, Y) = \mu(X, Y) = X \Box Y$, and for any pair of arrows f and g in $\mathcal{C}$ we will write $\mu_1(f, g) = \mu(f, g) = f \Box g$, where $\mathrm{dom}(f \Box g) = \mathrm{dom}(f) \Box \mathrm{dom}(g)$ and $\mathrm{cod}(f \Box g) = \mathrm{cod}(f) \Box \mathrm{cod}(g)$. Identity arrows are trivially related by

$$\mathrm{Id}_{X \Box Y} = \mathrm{Id}_X \Box \mathrm{Id}_Y \,. \tag{2.8}$$

We will call this the *vertical composition of arrows*, and the ordinary composition of arrows can be called horizontal composition.

Relation (2.6) is called the *associativity axiom*, and relations (2.7) are called the *unit axioms*. Associativity of μ then implies we can forget the brackets when writing products of any number of objects or arrows, for example $X \Box Y \Box Z$. Using the image 1 of the empty product category $\mathbf{1}$, or $1 = \eta(\mathbf{1})$, any strict monoidal category is now written as a triple $(\mathcal{C}, \mu, \eta)$, or alternatively $(\mathcal{C}, \Box, 1)$. We should make clear that 1 is an object satisfying $X \Box 1 = X = 1 \Box X$ for any object X in $\mathcal{C}$ and $f \Box \mathrm{Id}_1 = f = \mathrm{Id}_1 \Box f$ for any arrow f in $\mathcal{C}$. Of course 1 and Id_1 are the only object and arrow satisfying these properties, which implies that there is only one possible choice of η. We say that $(1, \mathrm{Id}_1)$ is the *neutral object* of $\mathcal{C}$.

Another relation: let f, g, h and j be arrows in $(\mathcal{C}, \Box, 1)$, satisfying $\mathrm{dom}(g) = \mathrm{cod}(f)$ and $\mathrm{dom}(j) = \mathrm{cod}(h)$. Then the arrows gf and jh exist, and we can write the relation $(g \Box j)(f \Box h) = gf \Box jh$. This can be derived from the composition rules of arrows in $\mathcal{C} \times \mathcal{C}$ and the standard functor properties. The arrows

f, g, h and j in $\mathcal{C}$ canonically induce arrows (f,h) and (g,j) in $\mathcal{C}\times\mathcal{C}$. Then $(g\Box j)(f\Box h)=\mu(g,j)\mu(f,h)=\mu((g,j)(f,h))=\mu(gf,jh)=gf\Box jh$. A special example is

$$\mathrm{Id}_A\,\Box gf\Box\,\mathrm{Id}_B=(\mathrm{Id}_A\,\Box g\Box\,\mathrm{Id}_B)(\mathrm{Id}_A\,\Box f\Box\,\mathrm{Id}_B), \tag{2.9}$$

for some arbitrary objects A and B.

Strict monoidal categories: Examples. A monoid is a trivial example of a monoidal category, perhaps justifying similarity in the notions. A (strict) monoidal category which has no arrows other than identity arrows, can be regarded as a monoid.

Another example is the category $\mathcal{C}^{\mathcal{C}}$ of functors from a category $\mathcal{C}$ to itself. The objects are functors $F:\mathcal{C}\to\mathcal{C}$ and the arrows are natural transformations $\alpha:F\Rightarrow G$. Composition of these arrows is defined by serial composition of natural transformations. Any two such functors can be composed, giving us again a functor from $\mathcal{C}$ to itself: for any pair F and G we simply write $G\Box F:=GF$. The identity functor Id exactly satisfies $\mathrm{Id}\,\Box F=F=F\Box\,\mathrm{Id}$ for any other F. Associativity and unit properties of functors implies that the associativity and unit axioms of this category are satisfied. Parallel composition of natural transformations $\alpha:F\Rightarrow G$ and $\beta:H\Rightarrow J$ gives us $\beta\Box\alpha:=\beta*\alpha$. The identity natural transformation $\mathrm{Id}_{\mathrm{Id}}$ exactly satisfies $\mathrm{Id}_{\mathrm{Id}}\,\Box\alpha=\alpha=\alpha\Box\,\mathrm{Id}_{\mathrm{Id}}$ for any other α.

Nonstrict monoidal categories. When a category $\mathcal{C}$ only satisfies the properties of a strict monoidal category in the weak sense, which means that the equalities (2.6) and (2.7) of the functors are replaced by invertible natural transformations, satisfying the so-called *coherence constraints*, then $\mathcal{C}$ is called a nonstrict monoidal category, or just a *monoidal category*. So what does this mean? The associativity axiom will be replaced by the *weak associativity axiom*. Instead of assuming equality of $\mu(\mu\times\mathrm{Id}_{\mathcal{C}})$ and $\mu(\mathrm{Id}_{\mathcal{C}}\times\mu)$, which are functors from $\mathcal{C}\times\mathcal{C}\times\mathcal{C}$ to $\mathcal{C}$, we assume an invertible natural transformation $\alpha:\mu(\mu\times\mathrm{Id}_{\mathcal{C}})\Rightarrow\mu(\mathrm{Id}_{\mathcal{C}}\times\mu)$, called the *associator*, exists. This means that for any triple A,B,C of objects in $\mathcal{C}$ there exists a natural isomorphism $\alpha_{A,B,C}:(A\Box B)\Box C\to A\Box(B\Box C)$. These isomorphisms are compatible with all arrows, as in diagram (2.3). Similarly, the unit axioms can be replaced by the *weak unit axioms*. Then we assume invertible natural transformations $\beta:\mu(\eta\times\mathrm{Id}_{\mathcal{C}})\Rightarrow\pi_{(\mathbf{1},\mathcal{C})}$ and $\gamma:\mu(\mathrm{Id}_{\mathcal{C}}\times\eta)\Rightarrow\pi_{(\mathcal{C},\mathbf{1})}$ exist. Thus for any object A in $\mathcal{C}$ there exist natural isomorphisms $\beta_A:1\Box A\to A$ and $\gamma_A:A\Box 1\to A$.

The newly introduced natural transformations α, β and γ should now satisfy the coherence constraints, which means that for any quadruple of objects A,B,C,D the two diagrams

$$\begin{tikzcd}
((A\Box B)\Box C)\Box D \arrow[rr, "\alpha_{A\Box B,C,D}"] \arrow[dr, "\alpha_{A,B,C}\Box\,\mathrm{Id}_D"'] & & (A\Box B)\Box(C\Box D) \arrow[rr, "\alpha_{A,B,C\Box D}"] & & A\Box(B\Box(C\Box D)) \\
& (A\Box(B\Box C))\Box D \arrow[rr, "\alpha_{A,B\Box C,D}"'] & & A\Box((B\Box C)\Box D) \arrow[ur, "\mathrm{Id}_A\,\Box\alpha_{B,C,D}"'] &
\end{tikzcd} \tag{2.10}$$

and

$$\begin{tikzcd}
(A\Box 1)\Box B \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "\gamma_A\Box\,\mathrm{Id}_B"'] & & A\Box(1\Box B) \arrow[dl, "\mathrm{Id}_A\,\Box\beta_B"] \\
& A\Box B &
\end{tikzcd} \tag{2.11}$$

should commute. Or, just using symbols:

$$\alpha_{A,B,C\Box D}\alpha_{A\Box B,C,D}=(\mathrm{Id}_A\,\Box\alpha_{B,C,D})\alpha_{A,B\Box C,D}(\alpha_{A,B,C}\Box\,\mathrm{Id}_D), \tag{2.12}$$

$$(\mathrm{Id}_A\,\Box\beta_B)\alpha_{A,1,B}=\gamma_A\Box\,\mathrm{Id}_B\,. \tag{2.13}$$

Using these weaker definitions, a (nonstrict) monoidal category is specified by a sextuple $(\mathcal{C}, \mu, \eta, \alpha, \beta, \gamma)$, where α, β and γ are invertible natural transformations satisfying the coherence constraints. Then $\mathcal{C}$ is a strict one if α, β and γ are identity natural transformations, which should not lead to any confusion, as in that case we already have the equality $(A\Box B)\Box C = A\Box(B\Box C)$ for any triple of objects.

We should realize that, even if the empty product category $\mathbf{1}$ itself contains only one object, there could be multiple objects in $\mathcal{C}$ to be chosen as the image 1 of this object under the functor η. Thus we have more freedom to choose η, as long as it satisfies the weak unit axioms, and 1 should satisfy (2.11). Thus η is in general not unique, contrary to the case of a strict monoidal category. However, any two possible choices of 1 are related by an isomorphism. Thus the isomorphism class of 1 contains multiple objects, or, using other words, 1 is only unique up to isomorphisms.

We will see that the monoidal categories, discussed in later chapters, will in fact all be *nonstrict* monoidal categories. However, as can be read in more detail in [6], there is no problem in treating nonstrict monoidal categories as strict ones. When the coherence constraints are satisfied, we can treat all natural isomorphisms $\alpha_{A,B,C}$, β_A and γ_A as if they were equalities, without contradictions. So we will not care if the natural transformations are treated as identities, as in (2.6) and (2.7). Thus from now on we will call a category a monoidal category, regardless of whether it is a strict or a nonstrict one. In practice this means we can ignore the parentheses, as we can assume μ to be associative. For example, we can write $A\Box B\Box C$ again, as in the strict monoidal case, instead of $(A\Box B)\Box C$ or $A\Box(B\Box C)$.

Of course this can be expressed with more subtlety. We can choose to make exact identifications between for example $A_1\Box A_2\Box A_3\Box A_4$ and $((A_1\Box A_2)\Box A_3)\Box A_4$, thus writing all parentheses to the left, and ignore all other possible objects with parentheses at different locations. Then a $\Box$-product of two such objects can be written again in this form, using associators. Stated otherwise, we let the natural transformations α, β and γ be absorbed into μ and η, and write $\bar{\mu}$ and $\bar{\eta}$ instead, which should be functors turning $\mathcal{C}$ into a strict monoidal category. Then we can write $(\mathcal{C}, \bar{\mu}, \bar{\eta})$ instead of $(\mathcal{C}, \mu, \eta, \alpha, \beta, \gamma)$, and we can say that now $\mathcal{C}$ has been *strictified*. However, these categories are not identical, but should be regarded as equivalent, or even *monoidally* equivalent, as explained in [6].

In later chapters, where we will look at specific monoidal categories, we will use the symbols of disjoint union $\amalg$ or the tensor product $\otimes$ instead of $\Box$, and we will realize that the associator related to these is such a trivial thing, we could easily forget about it.

Monoidal functors. The notion of a monoid homomorphism can be extended to monoidal categories. If $\mathcal{C}$ and $\mathcal{C}'$ are strict monoidal categories, then a functor $F : \mathcal{C} \to \mathcal{C}'$ is called a *strict monoidal functor* if it commutes with all the specified structure, as in the following commuting diagrams:

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, "F\times F"] \arrow[d, "\mu"'] & \mathcal{C}'\times\mathcal{C}' \arrow[d, "\mu'"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{C}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] & & \mathcal{C}' \\ & \mathbf{1} \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

There is another way to look at it. The image of a strict monoidal functor acting on a monoidal category is again a monoidal category, which means that a strict monoidal functor preserves monoidal structure of the source category.

As we will be dealing with nonstrict monoidal categories in later chapters, we should make a subtle extension to this definition. The functor in question should also be compatible with the natural transformations α, β and γ, specified in the definition of $\mathcal{C}$, and we will still call it a strict monoidal functor. After strictifying $\mathcal{C}$ and $\mathcal{C}'$ we can again say that the functor in question is an ordinary strict monoidal functor.

There are other types of monoidal functors, but, to keep it simple, we will only make a difference between *strict* and *nonstrict* monoidal functors. From now on we will assume that any *monoidal functor* we discuss is also *strict*, unless stated otherwise. In the later chapters we will also ignore the natural transformations α, β and γ. For more details about nonstrict monoidal categories and monoidal functors, the reader can consult [7].

Symmetric Monoidal Categories. A strict monoidal category $(\mathcal{C}, \Box, 1)$ is called a *symmetric monoidal category* if for any pair of objects X, Y a *twist arrow*

$$\tau_{X,Y} : X \Box Y \to Y \Box X$$

is defined, satisfying the following properties:

1. For any pair of objects X, Y the arrows $\tau_{X,Y}$ are natural, which means that τ can be interpreted as a natural transformation between functors $(X, Y) \mapsto X \Box Y$ and $(X, Y) \mapsto Y \Box X$. We will call this *naturality of the twist arrow*. Thus for any pair of arrows $f : X \to Y$ and $g : X' \to Y'$ the equation

$$(g \Box f)\tau_{X,X'} = \tau_{Y,Y'}(f \Box g) \tag{2.14}$$

is satisfied.

2. For any pair of objects X, Y we have $\tau_{Y,X}\tau_{X,Y} = \mathrm{Id}_{X \Box Y}$, making any twist arrow into an isomorphism.

3. For any triple of objects X, Y, Z the following diagram commutes:

$$\begin{tikzcd} X \Box Y \Box Z \arrow[rr, "\tau_{X,Y \Box Z}"] \arrow[dr, "\tau_{X,Y} \Box \mathrm{Id}_Z"'] & & Y \Box Z \Box X \\ & Y \Box X \Box Z \arrow[ur, "\mathrm{Id}_Y \Box \tau_{X,Z}"'] & \end{tikzcd} \tag{2.15}$$

Note that (2.15) implies that $\tau_{Y \Box Z,X} = \tau^{-1}_{X,Y \Box Z} = (\tau_{Y,X} \Box \mathrm{Id}_Z)(\mathrm{Id}_Y \Box \tau_{Z,X})$, or, after a relabeling of the objects:

$$\tau_{X \Box Y,Z} = (\tau_{X,Z} \Box \mathrm{Id}_Y)(\mathrm{Id}_X \Box \tau_{Y,Z}). \tag{2.16}$$

An important remark to be made here is that this collection of twist arrows itself is not related to properties of a given monoidal category, but rather a structure to be specified. We would thus like to write $(\mathcal{C}, \mu, \eta, \tau)$ to specify any symmetric monoidal category. It is another question though whether it is possible to specify a symmetric structure for any monoidal category.

Now suppose $Y = Z = X$ and $X' = Y' = X \Box X$, and write τ_X instead of $\tau_{X,X}$. Then (2.14) reduces to $(\tau_X \Box \mathrm{Id}_X)\tau_{X,X \Box X} = \tau_{X,X \Box X}(\mathrm{Id}_X \Box \tau_X)$, when applied to $f = \mathrm{Id}_X$ and $g = \tau_X$, and (2.15) reduces to $\tau_{X,X \Box X} = (\mathrm{Id}_X \Box \tau_X)(\tau_X \Box \mathrm{Id}_X)$. Combining these will give

$$(\tau_X \Box \mathrm{Id}_X)(\mathrm{Id}_X \Box \tau_X)(\tau_X \Box \mathrm{Id}_X) = (\mathrm{Id}_X \Box \tau_X)(\tau_X \Box \mathrm{Id}_X)(\mathrm{Id}_X \Box \tau_X). \tag{2.17}$$

This relation should remind us of a well known property of ordinary permutations.

Here is another useful relation:

- **Lemma.** For any X the following identity holds:

$$\tau_{X,1} = \mathrm{Id}_X = \tau_{1,X}. \tag{2.18}$$

Proof. We already know that $X \Box 1 = X = 1 \Box X$, thus $\tau_{X,1}$ and $\tau_{1,X}$ are arrows from X to itself. Inserting $Y = Z = 1$ into (2.15) and using $1 \Box 1 = 1 \Rightarrow \tau_{X,1 \Box 1} = \tau_{X,1}$ gives us:

$$\tau_{X,1} = \tau_{X,1 \Box 1} = (\mathrm{Id}_1 \Box \tau_{X,1})(\tau_{X,1} \Box \mathrm{Id}_1) = \tau_{X,1}\tau_{X,1}.$$

Now combining this with the identity $\tau_{1,X}\tau_{X,1} = \tau_{X,1}\tau_{1,X} = \mathrm{Id}_X$ yields

$$\tau_{X,1} = \tau_{X,1}\,\mathrm{Id}_X = \tau_{X,1}\tau_{X,1}\tau_{1,X} = \tau_{X,1}\tau_{1,X} = \mathrm{Id}_X\,.$$

Thus $\tau_{X,1} = \mathrm{Id}_X$. As $\tau_{1,X}$ is the inverse arrow of $\tau_{X,1}$ this implies that $\tau_{1,X} = \mathrm{Id}_X$.

$\Box$

Note that we first assumed that $\mathcal{C}$ is already a strict monoidal category *before* defining symmetric structure. In general this is not necessary. A nonstrict monoidal category can also be equipped with symmetric structure, but the definitions and properties of a nonstrict symmetric monoidal category are more complex then. To keep it simple we will only consider strict monoidal categories equipped with simplified symmetric structure, or strict symmetric monoidal categories.

Symmetric Monoidal Categories: A key example. If V and W are vector spaces over a ground field $\Bbbk$, then $V \otimes W$ is again a vector space over $\Bbbk$. There are canonical identifications $\Bbbk \otimes V \simeq V \simeq V \otimes \Bbbk$ and a canonical twist map $\tau_{V,W} : V \otimes W \to W \otimes V$. This twist map is linear and maps any $v \otimes w$ to $w \otimes v$. Then we can define the symmetric monoidal category

$$(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau). \tag{2.19}$$

Its objects are simply vector spaces over $\Bbbk$, and its arrows are linear maps between these vector spaces. For any pair of arrows $f : V \to W$ and $f' : V' \to W'$, $f \otimes f' : V \otimes V' \to W \otimes W'$ is another arrow, thus the usual tensor product of linear maps can be used to specify the monoidal structure of arrows.

Symmetric Monoidal Functors. When $\mathcal{C}$ and $\mathcal{C}'$ are symmetric monoidal categories, then a monoidal functor $F : \mathcal{C} \to \mathcal{C}'$ is called a *symmetric monoidal functor* if it preserves the symmetric structure. Let τ and τ' be the twist arrows as defined on $\mathcal{C}$ and $\mathcal{C}'$ respectively, then we have

$$F_1(\tau_{X,Y}) = \tau'_{F_0(X),F_0(Y)}$$

for all $X, Y \in \mathcal{C}_0$. Again, there is another way to look at it. The image of a symmetric monoidal functor acting on a symmetric monoidal category is again a symmetric monoidal category.

Monoidal natural transformations. Let $(\mathcal{C}, \Box, 1)$ and $(\mathcal{C}', \Box', 1')$ be a pair of monoidal categories. If F and G are (strict) monoidal functors from $(\mathcal{C}, \Box, 1)$ to $(\mathcal{C}', \Box', 1')$, then a natural transformation $\alpha : F \Rightarrow G$ is called a *monoidal natural transformation* if for every pair of objects A and B we have

$$\alpha_A \Box' \alpha_B = \alpha_{A \Box B}$$

and if $\alpha_1 = \mathrm{Id}_{1'}$. This will also be the case when we add symmetric structure to $\mathcal{C}$ and $\mathcal{C}'$, and if F and G are symmetric monoidal functors.

If $(\mathcal{C}, \Box, 1, \tau)$ and $(\mathcal{C}', \Box', 1', \tau')$ are symmetric monoidal categories, then we can define the category $\mathbf{SymmMonCat}(\mathcal{C}, \mathcal{C}')$ whose objects are the symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$, and whose arrows are the monoidal natural transformations between such functors. The category $\mathbf{SymmMonCat}(\mathcal{C}, \mathcal{C}')$ is called a *symmetric monoidal functor category*.

2.4 Dualizable objects

In this section we will introduce dualizable objects in the general case. In Chapter 9 we will discuss the category **1Cob** of 1-dimensional cobordisms. At least we can say that all of its objects are dualizable. In Section 10.5 we will also mention the category of dualizable objects in the destination category of any topological quantum field theory.

Main definition. Let $(\mathcal{C}, \Box, 1, \tau)$ be a symmetric monoidal category. An object X in $\mathcal{C}$ is called a *dualizable object* if another object Y exists, together with arrows $\kappa : X \Box Y \to 1$ and $\lambda : 1 \to Y \Box X$, such that the following identities hold:

$$\mathrm{Id}_X = (\kappa \Box \mathrm{Id}_X)(\mathrm{Id}_X \Box \lambda) \quad , \quad \mathrm{Id}_Y = (\mathrm{Id}_Y \Box \kappa)(\lambda \Box \mathrm{Id}_Y). \tag{2.20}$$

We will call these the *zig-zag identities*. We can say that Y is a dual object of X, and that (X, Y, κ, λ) specifies all information of one specific dualizable object X, after making a choice for Y, κ and λ. (In general we cannot say that X has a unique dual object.)

As we are dealing with a symmetric monoidal category, we can also define $\tilde{\kappa} := \kappa\tau_{Y,X}$ and $\tilde{\lambda} := \tau_{Y,X}\lambda$. If we use the naturality of the twist arrows, then we can say that (X, Y, κ, λ) automatically induces another dualizable object $(Y, X, \tilde{\kappa}, \tilde{\lambda})$. Thus, if Y is a dual object of X, then X is also a dual object of Y. Using these $\tilde{\kappa} : Y\Box X \to 1$ and $\tilde{\lambda} : 1 \to X\Box Y$ we can rewrite (2.20) as follows:

$$\begin{aligned}(\kappa\Box\,\mathrm{Id}_X)(\mathrm{Id}_X\,\Box\lambda) = \mathrm{Id}_X = (\mathrm{Id}_X\,\Box\tilde{\kappa})(\tilde{\lambda}\Box\,\mathrm{Id}_X),\\ (\tilde{\kappa}\Box\,\mathrm{Id}_Y)(\mathrm{Id}_Y\,\Box\tilde{\lambda}) = \mathrm{Id}_Y = (\mathrm{Id}_Y\,\Box\kappa)(\lambda\Box\,\mathrm{Id}_Y).\end{aligned} \tag{2.21}$$

These identities can also be expressed by the following commuting diagrams:

$$\begin{tikzcd}
 & X\Box Y\Box X \arrow[dr, "\kappa\Box\,\mathrm{Id}_X"] & \\
X \arrow[ur, "\mathrm{Id}_X\,\Box\lambda"] \arrow[rr, "\mathrm{Id}_X"] \arrow[dr, "\tilde{\lambda}\Box\,\mathrm{Id}_X"'] & & X \\
 & X\Box Y\Box X \arrow[ur, "\mathrm{Id}_X\,\Box\tilde{\kappa}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y\Box X\Box Y \arrow[dr, "\tilde{\kappa}\Box\,\mathrm{Id}_Y"] & \\
Y \arrow[ur, "\mathrm{Id}_Y\,\Box\tilde{\lambda}"] \arrow[rr, "\mathrm{Id}_Y"] \arrow[dr, "\lambda\Box\,\mathrm{Id}_Y"'] & & Y \\
 & Y\Box X\Box Y \arrow[ur, "\mathrm{Id}_Y\,\Box\kappa"'] &
\end{tikzcd}$$

We can say that $(1, 1, \mathrm{Id}_1, \mathrm{Id}_1)$ is the trivial dualizable object in any symmetric monoidal category.

Dualizable homomorphisms. If we have a pair of dualizable objects (X, Y, κ, λ) and $(X', Y', \kappa', \lambda')$ in $\mathcal{C}$ and if $f_X : X \to X'$ and $f_Y : Y \to Y'$ are arrows in $\mathcal{C}$, then we say that $f := (f_X, f_Y)$ is a *dualizable homomorphism* in $\mathcal{C}$ if it is compatible with all the structure as in the following commuting diagram:

$$\begin{tikzcd}
1 \arrow[r, "\tilde{\lambda}"] \arrow[dr, "\tilde{\lambda}'"'] & X\Box Y \arrow[r, "\kappa"] \arrow[d, "f_X\,\Box f_Y"] & 1 \\
 & X'\Box Y' \arrow[ur, "\kappa'"'] &
\end{tikzcd}$$

Of course any pair of identity arrows $(\mathrm{Id}_X, \mathrm{Id}_Y)$ is a dualizable homomorphism, and composing two dualizable homomorphisms gives us another dualizable homomorphism: $(f, g)(h, j) = (fh, gj)$. We can also say that $(\mathrm{Id}_1, \mathrm{Id}_1)$ is the trivial dualizable homomorphism from $(1, 1, \mathrm{Id}_1, \mathrm{Id}_1)$ to itself.

Note that if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories and if $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor, then the image $(F_0(X), F_0(Y), F_1(\kappa), F_1(\lambda))$ of any dualizable object (X, Y, κ, λ) in $\mathcal{C}$, will be a dualizable object in $\mathcal{D}$. Similarly, the image $(F_1(f), F_1(g))$ of any dualizable homomorphism (f, g) in $\mathcal{C}$ will be a dualizable homomorphism in $\mathcal{D}$.

The category of dualizable objects. If $\mathcal{C}$ is a symmetric monoidal category, written as $(\mathcal{C}, \Box, 1, \tau)$, then $\mathbf{DO}(\mathcal{C})$ is the *category of dualizable objects* in $\mathcal{C}$. Its objects are the dualizable objects in $\mathcal{C}$ and its arrows are the dualizable homomorphisms in $\mathcal{C}$. This category $\mathbf{DO}(\mathcal{C})$ can be equipped with monoidal structure. If (X, Y, κ, λ) and $(X', Y', \kappa', \lambda')$ are dualizable objects in $\mathcal{C}$, then we can also turn $X\Box X'$ into a dualizable object. We define the needed arrows as follows:

$$\kappa'' := (\kappa\Box\kappa')(\mathrm{Id}_X\,\Box\tau_{X',Y}\Box\,\mathrm{Id}_{Y'}) \quad , \quad \lambda'' := (\mathrm{Id}_Y\,\Box\tau_{X,Y'}\Box\,\mathrm{Id}_{X'})(\lambda\Box\lambda').$$

Then we will write:

$$(X, Y, \kappa, \lambda)\Box(X', Y', \kappa', \lambda') = (X\Box X', Y\Box Y', \kappa'', \lambda''). \tag{2.22}$$

For any pair of dualizable homomorphisms (f, g) and (h, j) we will write $(f, g)\Box(h, j) := (f\Box h, g\Box j)$, which is again a dualizable homomorphism. Using naturality of the twist arrows it is easy to check that also $X\Box X'$, as defined in (2.22), is a dualizable object. We thus see that $\mathbf{DO}(\mathcal{C})$ is closed under vertical composition, or $\Box$-products. Also note that $((1, 1, \mathrm{Id}_1, \mathrm{Id}_1), (\mathrm{Id}_1, \mathrm{Id}_1))$ can be regarded as the neutral object for taking $\Box$-products. We conclude that $(\mathbf{DO}(\mathcal{C}), \Box, 1)$ can be regarded as a monoidal category.

A key example. Any vector space V of finite dimension (over $\Bbbk$) has a unique canonical dual V^*. Thus V, together with choices for $\kappa : V \otimes V^* \to \Bbbk$ and $\lambda : \Bbbk \to V^* \otimes V$, can be regarded as a dualizable object in $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk, \tau)$. Then we say V is a *dualizable vector space*. Of course we can choose another finite-dimensional vector space W as a dual of V, if an isomorphism exists from W to V^*. Any pair of finite-dimensional vector spaces V and V' induces another finite-dimensional vector space $V \otimes V'$, which is again dualizable.

The category of dualizable vector spaces. Now we can define the category $\mathbf{DVS}_\Bbbk$ of dualizable vector spaces over $\Bbbk$:

$$\mathbf{DVS}_\Bbbk := \mathbf{DO}(\mathbf{Vect}_\Bbbk) = \mathbf{DO}(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk, \tau). \tag{2.23}$$

Note that $\mathbf{DVS}_\Bbbk$ can be regarded as a monoidal category $(\mathbf{DVS}_\Bbbk, \otimes, \Bbbk)$. In Section 10.5 we will discuss a relation between $\mathbf{DVS}_\Bbbk$ and the category of 1-dimensional topological quantum field theories.

3 Skeletons of Monoidal Categories

Note: the results of this chapter are not really needed elsewhere in this thesis. This chapter can be regarded as a side topic, so it is not really a problem if the reader decides to skip this chapter and reads it later. However, the subject of this chapter is still one of the purposes of this thesis.

In later chapters we will study some (symmetric) monoidal categories, and some skeletons of these categories. Then we will also study how the monoidal structure of these categories can induce monoidal structure for these skeletons, and we will do this explicitly. In this chapter we will study how to do it in general, using a formal, universal approach. If $\mathcal{C}$ is a monoidal category, and if $\mathcal{C}'$ is a skeleton of $\mathcal{C}$, then we will present a proposal for deriving monoidal structure for $\mathcal{C}'$ from the monoidal structure of $\mathcal{C}$. For this purpose we will use a projection functor $P : \mathcal{C} \to \mathcal{C}'$. (An arbitrary projection functor can be used!)

The presented techniques are especially meant for nonstrict monoidal categories. In general we can only say with certainty that, after porting over the structure, the skeleton $\mathcal{C}'$ will be a nonstrict, or *not necessarily strict* monoidal category. It does not matter if the monoidal structure of $\mathcal{C}$ is strict or nonstrict.

The main thing to do here is checking if the proposed techniques are correct. We will check this by finding out if $\mathcal{C}$ satisfying the coherence constraints, will imply that also $\mathcal{C}'$, equipped with the newly constructed *candidate* monoidal structure, will satisfy the coherence constraints. The answer will be yes, but to check this we need some elaborate steps.

This will also show that in general we do not need the explicit properties of $\mathcal{C}$, or a special choice for these properties. It is also no problem if $\mathcal{C}$ is nonstrict.

So, how will this apply to the things discussed in later chapters? There are examples in which the operations $\square$ and $\square'$ turn out to be the same. For example in **2Cob**, the category of oriented cobordisms of dimension 2. See Chapter 8. Then we define $\square = \square' := \amalg$. The objects in **2Cob** are manifolds of dimension 1, and for any two objects X and Y, their disjoint union $X \amalg Y$ is again an object. Then the objects in **2cob**, a skeleton of **2Cob**, are just disjoint products of circles with standard orientation. Thus $\amalg$ also defines a product of objects in **2cob**. This means that the monoidal structure of $\mathcal{C}$ and $\mathcal{C}'$ coincides, at least for this choice of skeleton. Indeed, the chosen skeleton is closed under the operation of taking disjoint union, as we can also read in [7]. Then we could easily ignore the subtleties discussed in this chapter.

However, we will also study **1Cob**, the category of oriented cobordisms of dimension 1, with skeleton **1cob**. See Chapter 9. Then we are forced to define $\square$ and $\square'$ differently. However, $\square$ is already chosen to be $\amalg$. The objects in **1Cob** are sets containing a finite number of oriented points, and the arrows are oriented manifolds of dimension 1 between them. Specifying the skeleton **1cob** means that we need to pick objects which can be used as basic building blocks. We will encounter objects which are a finite disjoint union of positively and negatively oriented points, and we would like to see all positively oriented points in the left part of the disjoint product. Now, if we take an ordinary disjoint product of two such objects, then not all positively oriented points will be guaranteed to be on the left. At least,**1cob** is not closed under the operation of taking ordinary disjoint union. However, there are still isomorphisms from such an object to another object which satisfies the desired property. This construction will be used to define such a thing as the *graded disjoint union*, which results in a $\square'$ defined differently. To make sure this $\square'$ satisfies elegant properties, derived from the properties of $\square$, is a topic on its own, and an explanation of a solid theoretical basis for this fits best *here*, in the context of the previous chapter.

A more explicit check of properties will be discussed in Chapter 9. Then we will also add symmetric structure τ' to **1cob**, also ported over from the symmetric structure τ of **1Cob** itself. This means that **1Cob** should already be regarded as a *strict* monoidal category then. We should note that τ is needed then for explicitly checking the porting over of the monoidal structure, *before* porting over the symmetric structure. Then the projection functor P will be related to τ.

In the more general case, as discussed here, we do not need any explicitly defined P, and we also do not need any (simple) symmetric structure, thus at the same time we also do not need the source category to

be strict. At least this chapter shows that the explicit check is not needed for porting over the monoidal structure, and it does not depend on the specific properties of the category in question. The only extra thing to do afterwards might be strictifying the monoidal structure of the skeleton, if desired.

So this chapter shows that it is no coincidence that we can port over the monoidal structure from **1Cob** to **1cob**. The possibility of porting over this structure does *not* depend on the specific properties of **1Cob**.

Formal introduction. If $(\mathcal{C}, \mu, \eta, \alpha, \beta, \gamma)$ is a nonstrict monoidal category, and if $\mathcal{C}'$ is a skeleton of $\mathcal{C}$, then we can induce functors μ' and η' and natural transformations α', β' and γ' from μ, η, α, β and γ, turning $\mathcal{C}'$ also into a nonstrict monoidal category. Thus $(\mathcal{C}', \mu', \eta', \alpha', \beta', \gamma')$ can be derived from $(\mathcal{C}, \mu, \eta, \alpha, \beta, \gamma)$. To do this we can make use of a projection functor $P : \mathcal{C} \to \mathcal{C}'$ and the canonical injection functor $I : \mathcal{C}' \to \mathcal{C}$. These functors will be used for deriving a (nonstrict) monoidal structure for $\mathcal{C}'$ from the (nonstrict) monoidal structure of $\mathcal{C}$.

Let $\mathrm{Id} : \mathcal{C} \to \mathcal{C}$ and $\mathrm{Id}' : \mathcal{C}' \to \mathcal{C}'$ be identity functors, let $\kappa : \mathrm{Id} \Rightarrow IP$ be the natural transformation induced by P and let $\kappa^{-1} : IP \Rightarrow \mathrm{Id}$ be its inverse. Apart from that, note that $\mathrm{Id}' = \mathrm{Id}|_{\mathcal{C}'}$. Now define $\mu' := P\mu(I \times I)$ and $\eta' := P\eta$, expressed by commuting diagrams:

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "\mu"] & \mathcal{C} \arrow[d, "P"] \\ \mathcal{C}' \times \mathcal{C}' \arrow[u, "I \times I"] \arrow[r, "\mu'"] & \mathcal{C}' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & \mathcal{C} \arrow[d, "P"] \\ & \mathcal{C}' \end{tikzcd} \tag{3.1}$$

If X' and Y' are objects in $\mathcal{C}'$, then we write

$$X' \Box' Y' := \mu'_0(X', Y') = (\mu_0(X', Y'))' = (X' \Box Y')' = P_0(X' \Box Y'), \tag{3.2}$$

which is another object in $\mathcal{C}'$. Similarly, if $f' : X' \to Y'$ and $g' : Z' \to W'$ are arrows in $\mathcal{C}'$, then this defines another arrow $f' \Box' g' : X' \Box' Z' \to Y' \Box' W'$, and we will write

$$f' \Box' g' := \mu'_1(f', g') = (\mu_1(f', g'))' = (f' \Box g')' = P_1(f' \Box g') = \kappa_{Y' \Box W'}(f' \Box g')\kappa^{-1}_{X' \Box Z'}. \tag{3.3}$$

Porting over the weak associativity axiom. We can write $\mu'(\mu' \times \mathrm{Id}')$ and $\mu'(\mathrm{Id}' \times \mu')$ in terms of μ, Id, I and P:

$$\begin{aligned} \mu'(\mu' \times \mathrm{Id}') &= P\mu(I \times I)((P\mu(I \times I)) \times \mathrm{Id}') = P\mu(IP\mu(I \times I) \times I\,\mathrm{Id}') = P\mu(IP\mu(I \times I) \times \mathrm{Id}\, I) \\ &= P\mu(IP\mu \times \mathrm{Id})(I \times I \times I) = P\mu(IP \times \mathrm{Id})(\mu \times \mathrm{Id})(I \times I \times I) \\ \mu'(\mathrm{Id}' \times \mu') &= P\mu(\mathrm{Id} \times IP)(\mathrm{Id} \times \mu)(I \times I \times I). \end{aligned}$$

- **Lemma.** Weak associativity of μ will imply weak associativity of μ'.

  Proof. The natural transformations κ and κ^{-1} induce natural transformations $\kappa^{-1} \times \mathrm{Id}_{\mathrm{Id}} : IP \times \mathrm{Id} \Rightarrow \mathrm{Id} \times \mathrm{Id}$ and $\mathrm{Id}_{\mathrm{Id}} \times \kappa : \mathrm{Id} \times \mathrm{Id} \Rightarrow \mathrm{Id} \times IP$. These in turn induce natural transformations:

$$\begin{aligned} \mu'(\mu' \times \mathrm{Id}') &= P\mu(IP \times \mathrm{Id})(\mu \times \mathrm{Id})(I \times I \times I) \\ &\Downarrow \mathrm{Id}_{P\mu} * (\kappa^{-1} \times \mathrm{Id}_{\mathrm{Id}}) * \mathrm{Id}_{(\mu \times \mathrm{Id})(I \times I \times I)} \\ P\mu(\mathrm{Id} \times \mathrm{Id})(\mu \times \mathrm{Id})(I \times I \times I) &= P\mu(\mu \times \mathrm{Id})(I \times I \times I) \\ &\Downarrow \mathrm{Id}_P * \alpha * \mathrm{Id}_{I \times I \times I} \\ P\mu(\mathrm{Id} \times \mu)(I \times I \times I) &= P\mu(\mathrm{Id} \times \mathrm{Id})(\mathrm{Id} \times \mu)(I \times I \times I) \\ &\Downarrow \mathrm{Id}_{P\mu} * (\mathrm{Id}_{\mathrm{Id}} \times \kappa) * \mathrm{Id}_{(\mathrm{Id} \times \mu)(I \times I \times I)} \\ P\mu(\mathrm{Id} \times IP)(\mathrm{Id} \times \mu)(I \times I \times I) &= \mu'(\mathrm{Id}' \times \mu') \end{aligned}$$

We can define

$$\begin{aligned}
\rho &:= \mathrm{Id}_{P\mu} * (\kappa^{-1} \times \mathrm{Id}_{\mathrm{Id}}) * \mathrm{Id}_{(\mu\times\mathrm{Id})(I\times I\times I)}, \\
\overline{\alpha} &:= \mathrm{Id}_{P} * \alpha * \mathrm{Id}_{I\times I\times I}, \\
\sigma &:= \mathrm{Id}_{P\mu} * (\mathrm{Id}_{\mathrm{Id}} \times \kappa) * \mathrm{Id}_{(\mathrm{Id}\times\mu)(I\times I\times I)}\,.
\end{aligned} \tag{3.4}$$

Composing these three natural transformations induces another natural transformation $\alpha' : \mu'(\mu' \times \mathrm{Id}') \Rightarrow \mu'(\mathrm{Id}' \times \mu')$, thus $\alpha' = \sigma\overline{\alpha}\rho$. Attaching object labels yields the following commuting diagram:

$$\begin{tikzcd}
(A'\Box'B')\Box'C' \arrow[r, "\rho_{A',B',C'}"] \arrow[rrr, bend right, "\alpha'_{A',B',C'}"'] & ((A'\Box B')\Box C')' \arrow[r, "\overline{\alpha}_{A',B',C'}"] & (A'\Box(B'\Box C'))' \arrow[r, "\sigma_{A',B',C'}"] & A'\Box'(B'\Box'C')
\end{tikzcd}$$

This α' precisely defines an associator on $\mathcal{C}'$. Thus weak associativity of μ implies weak associativity of μ'.

This can be expressed alternatively:

$$\begin{tikzcd}
 & & \mathcal{C}\times\mathcal{C} \arrow[r, bend left, "IP\times\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}\times\mathrm{Id}"'] \arrow[d, phantom, "\Downarrow \kappa^{-1}\times\mathrm{Id}_{\mathrm{Id}}"] & \mathcal{C}\times\mathcal{C} \arrow[dr, bend left, "\mu"] & \\
\mathcal{C}'\times\mathcal{C}'\times\mathcal{C}' \arrow[r, "I\times I\times I"] & \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[ur, bend left, "\mu\times\mathrm{Id}"] \arrow[dr, bend right, "\mathrm{Id}\times\mu"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & & \mathcal{C} \arrow[r, "P"] & \mathcal{C}' \\
 & & \mathcal{C}\times\mathcal{C} \arrow[r, bend left, "\mathrm{Id}\times\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}\times IP"'] \arrow[d, phantom, "\Downarrow \mathrm{Id}_{\mathrm{Id}}\times\kappa"] & \mathcal{C}\times\mathcal{C} \arrow[ur, bend right, "\mu"'] &
\end{tikzcd} \tag{3.5}$$

.

□

We should note that, even if $\mathcal{C}$ is a *strict* monoidal category, then $\mathcal{C}'$ is still not necessarily a strict monoidal category. If we replace α in diagram (3.5) by an identity natural transformation, then the remaining natural transformations will not exactly cancel, so there is no guarantee that α' is an identity natural transformation.

Porting over the weak unit axioms. We can write $\mu'(\eta' \times \mathrm{Id}')$ and $\mu'(\mathrm{Id}' \times \eta')$ in terms of μ, η, Id, I and P:

$$\begin{aligned}
\mu'(\eta' \times \mathrm{Id}') &= P\mu(IP \times \mathrm{Id})(\eta \times \mathrm{Id})(\mathrm{Id}_{\mathbf{1}} \times I), \\
\mu'(\mathrm{Id}' \times \eta') &= P\mu(\mathrm{Id} \times IP)(\mathrm{Id} \times \eta)(I \times \mathrm{Id}_{\mathbf{1}}).
\end{aligned}$$

- **Lemma.** If η satisfies the weak unit axioms, then η' does also.

  Proof. This can again be expressed alternatively:

$$\begin{tikzcd}
 & & \mathcal{C}\times\mathcal{C} \arrow[r, bend left, "IP\times\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}\times\mathrm{Id}"'] \arrow[d, phantom, "\Downarrow \kappa^{-1}\times\mathrm{Id}_{\mathrm{Id}}"] & \mathcal{C}\times\mathcal{C} \arrow[dr, bend left, "\mu"] & \\
\mathbf{1}\times\mathcal{C}' \arrow[r, "\mathrm{Id}_{\mathbf{1}}\times I"] & \mathbf{1}\times\mathcal{C} \arrow[ur, bend left, "\eta\times\mathrm{Id}"] \arrow[rrr, bend right, "\pi_{(\mathbf{1},\mathcal{C})}"'] \arrow[rr, phantom, "\Downarrow\beta"] & & & \mathcal{C} \arrow[r, "P"] & \mathcal{C}'
\end{tikzcd} \tag{3.6}$$

Using (2.2) we see that composing the functors in the bottom part yields $\pi_{(\mathbf{1},\mathcal{C}')}$. Thus the natural transformation $\beta : \mu(\eta \times \mathrm{Id}) \Rightarrow \pi_{(\mathbf{1},\mathcal{C})}$ induces a natural transformation $\beta' : \mu'(\eta' \times \mathrm{Id}') \Rightarrow \pi_{(\mathbf{1},\mathcal{C}')}$. A similar diagram can be drawn in case of $\mu'(\mathrm{Id}' \times \eta')$, so that the natural transformation $\gamma : \mu(\mathrm{Id} \times \eta) \Rightarrow \pi_{(\mathcal{C},\mathbf{1})}$ induces a natural transformation $\gamma' : \mu'(\mathrm{Id}' \times \eta') \Rightarrow \pi_{(\mathcal{C}',\mathbf{1})}$. Then η satisfying the weak unit axioms indeed implies η' satisfying the weak unit axioms.

□

Again we should note that, even if $\mathcal{C}$ is a *strict* monoidal category, then $\mathcal{C}'$ is not necessarily a strict monoidal category. If we replace β in diagram (3.6) by an identity natural transformation, then the remaining natural transformation is not guaranteed to be an identity natural transformation.

Porting over the coherence constraints (Step 1). Now we would like to know, when μ and η satisfy the coherence constraints, whether μ' and η' also satisfy the coherence constraints. To find this out we will start with rewriting some already known symbols. Knowing the definitions of α', ρ, $\overline{\alpha}$ and σ, as in (3.4), we will write, for any triple of objects A', B' and C' in $\mathcal{C}'$, the following:

$$\begin{aligned}
\rho_{A',B',C'} &= (\mathrm{Id}_{P\mu} * (\kappa^{-1} \times \mathrm{Id}_{\mathrm{Id}}) * \mathrm{Id}_{(\mu\times\mathrm{Id})(I\times I\times I)})_{A',B',C'} \\
&= (P\mu)_1(\kappa^{-1} \times \mathrm{Id}_{\mathrm{Id}})_{((\mu\times\mathrm{Id})(I\times I\times I))_0(A',B',C')} \\
&= (P\mu)_1(\kappa^{-1} \times \mathrm{Id}_{\mathrm{Id}})_{A'\Box B',C'} \\
&= (P\mu)_1(\kappa^{-1}_{A'\Box B'}, \mathrm{Id}_{C'}) = P_1(\kappa^{-1}_{A'\Box B'} \Box \mathrm{Id}_{C'}) \\
&= \kappa_{(A'\Box B')\Box C'}(\kappa^{-1}_{A'\Box B'} \Box \mathrm{Id}_{C'})\kappa^{-1}_{(A'\Box B')'\Box C'} \\
\overline{\alpha}_{A',B',C'} &= (\mathrm{Id}_P * \alpha * \mathrm{Id}_{I\times I\times I})_{A',B',C'} \\
&= P_1(\alpha_{(I\times I\times I)_0(A',B',C')}) = P_1(\alpha_{A',B',C'}) \\
&= \kappa_{A'\Box(B'\Box C')}\alpha_{A',B',C'}\kappa^{-1}_{(A'\Box B')\Box C'} \\
\sigma_{A',B',C'} &= (\mathrm{Id}_{P\mu} * (\mathrm{Id}_{\mathrm{Id}} \times \kappa) * \mathrm{Id}_{(\mathrm{Id}\times\mu)(I\times I\times I)})_{A',B',C'} \\
&= (P\mu)_1(\mathrm{Id}_{\mathrm{Id}} \times \kappa)_{A',B'\Box C'} = (P\mu)_1(\mathrm{Id}_{A'}, \kappa_{B'\Box C'}) \\
&= P_1(\mathrm{Id}_{A'} \Box \kappa_{B'\Box C'}) = \kappa_{A'\Box(B'\Box C')'}(\mathrm{Id}_{A'} \Box \kappa_{B'\Box C'})\kappa^{-1}_{A'\Box(B'\Box C')}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\alpha'_{A',B',C'} &= (\sigma\overline{\alpha}\rho)_{A',B',C'} = \sigma_{A',B',C'}\overline{\alpha}_{A',B',C'}\rho_{A',B',C'} \\
&= \kappa_{A'\Box(B'\Box C')'}(\mathrm{Id}_{A'} \Box \kappa_{B'\Box C'})\alpha_{A',B',C'}(\kappa^{-1}_{A'\Box B'} \Box \mathrm{Id}_{C'})\kappa^{-1}_{(A'\Box B')'\Box C'}
\end{aligned} \tag{3.7}$$

which is an arrow from $(A'\Box' B')\Box' C'$ to $A'\Box'(B'\Box' C')$. Relation (3.7) can also be explained using the following diagram, which commutes by definition:

$$\begin{tikzcd}
(A'\Box' B')\Box' C' = ((A'\Box B')'\Box C')' \arrow[r, "\kappa^{-1}_{(A'\Box B')'\Box C'}"] \arrow[d, "\alpha'_{A',B',C'}"'] & (A'\Box B')'\Box C' \arrow[r, "\kappa^{-1}_{A'\Box B'} \Box \mathrm{Id}_{C'}"] & (A'\Box B')\Box C' \arrow[d, "\alpha_{A',B',C'}"] \\
A'\Box'(B'\Box' C') = (A'\Box(B'\Box C')')' & A'\Box(B'\Box C')' \arrow[l, "\kappa_{A'\Box(B'\Box C')'}"'] & A'\Box(B'\Box C') \arrow[l, "\mathrm{Id}_{A'} \Box \kappa_{B'\Box C'}"']
\end{tikzcd}$$

Then we can use that, in general, the natural isomorphisms $\alpha_{A,B,C}$ and $\alpha_{A',B',C'}$ are, by definition, related by

$$\alpha_{A,B,C} = (\kappa_A^{-1}\Box(\kappa_B^{-1}\Box\kappa_C^{-1}))\alpha_{A',B',C'}((\kappa_A\Box\kappa_B)\Box\kappa_C), \tag{3.8}$$

as for any triple of objects A, B and C in $\mathcal{C}$, the functors $\mu(\mu\times\mathrm{Id})$ and $\mu(\mathrm{Id}\times\mu)$ map the arrow $(\kappa_A, \kappa_B, \kappa_C) : (A, B, C) \to (A', B', C')$ to $(\kappa_A\Box\kappa_B)\Box\kappa_C$ and $\kappa_A\Box(\kappa_B\Box\kappa_C)$ respectively. As an example, if A is an object

in $\mathcal{C}$ and if B' and C' already are objects in $\mathcal{C}'$, then $\kappa_{B'} = \mathrm{Id}_{B'}$ and $\kappa_{C'} = \mathrm{Id}_{C'}$, thus

$$\begin{aligned}
\alpha_{A,B',C'} &= (\kappa_A^{-1}\Box(\kappa_{B'}^{-1}\Box\kappa_{C'}^{-1}))\alpha_{A',B',C'}((\kappa_A\Box\kappa_{B'})\Box\kappa_{C'}) = \\
&= (\kappa_A^{-1}\Box(\mathrm{Id}_{B'}\Box\mathrm{Id}_{C'}))\alpha_{A',B',C'}((\kappa_A\Box\mathrm{Id}_{B'})\Box\mathrm{Id}_{C'}) \\
&= (\kappa_A^{-1}\Box\mathrm{Id}_{B'\Box C'})\alpha_{A',B',C'}((\kappa_A\Box\mathrm{Id}_{B'})\Box\mathrm{Id}_{C'}).
\end{aligned}$$

Of course we could write, for example, $\alpha_{A'\Box B',C',D'}$ in a similar fashion, and we will encouter similar relations in the following definitions, see (3.9), (3.10) and(3.11).

Porting over the coherence constraints (Step 2). Now define the following natural isomorphisms:

$$\begin{aligned}
i_1 &: ((A'\Box B')\Box C')\Box D' \to ((A'\Box' B')\Box' C')\Box' D', \\
i_2 &: (A'\Box B')\Box(C'\Box D') \to (A'\Box' B')\Box'(C'\Box' D'), \\
i_3 &: A'\Box(B'\Box(C'\Box D')) \to A'\Box'(B'\Box'(C'\Box' D')), \\
i_4 &: (A'\Box(B'\Box C'))\Box D' \to (A'\Box'(B'\Box' C'))\Box' D', \\
i_5 &: A'\Box((B'\Box C')\Box D') \to A'\Box'((B'\Box' C')\Box' D'),
\end{aligned}$$

$$\begin{aligned}
i_1 &:= \kappa_{((A'\Box B')'\Box C')'\Box D'}(\kappa_{(A'\Box B')'\Box C'}\Box\mathrm{Id}_{D'})(\kappa_{A'\Box B'}\Box\mathrm{Id}_{C'})\Box\mathrm{Id}_{D'}), \\
i_2 &:= \kappa_{(A'\Box B')'\Box(C'\Box D')'}(\mathrm{Id}_{(A'\Box B')'}\Box\kappa_{C'\Box D'})(\kappa_{A'\Box B'}\Box\mathrm{Id}_{C'\Box D'}) \\
&= \kappa_{(A'\Box B')'\Box(C'\Box D')'}(\kappa_{A'\Box B'}\Box\mathrm{Id}_{(C'\Box D')'})(\mathrm{Id}_{A'\Box B'}\Box\kappa_{C'\Box D'}) \\
&= \kappa_{(A'\Box B')'\Box(C'\Box D')'}(\kappa_{A'\Box B'}\Box\kappa_{C'\Box D'}), \qquad (3.9) \\
i_3 &:= \kappa_{A'\Box(B'\Box(C'\Box D')')'}(\mathrm{Id}_{A'}\Box\kappa_{B'\Box(C'\Box D')'})(\mathrm{Id}_{A'}\Box(\mathrm{Id}_{B'}\Box\kappa_{C'\Box D'})), \\
i_4 &:= \kappa_{(A'\Box(B'\Box C')')'\Box D'}(\kappa_{A'\Box(B'\Box C')'}\Box\mathrm{Id}_{D'})((\mathrm{Id}_{A'}\Box\kappa_{B'\Box C'})\Box\mathrm{Id}_{D'}), \\
i_5 &:= \kappa_{A'\Box((B'\Box C')'\Box D')'}(\mathrm{Id}_{A'}\Box\kappa_{(B'\Box C')'\Box D'})(\mathrm{Id}_{A'}\Box(\kappa_{B'\Box C'}\Box\mathrm{Id}_{D'})).
\end{aligned}$$

Then we can write out all natural isomorphisms α', appearing in diagram (2.10) applied to $\mathcal{C}'$, in terms of the five isomorphisms defined in (3.9) and the natural isomorphisms α which already satisfy (2.10):

$$\begin{aligned}
\alpha'_{A'\Box' B',C',D'} &= i_2\alpha_{A'\Box B',C',D'}i_1^{-1}, \\
\alpha'_{A',B',C'\Box' D'} &= i_3\alpha_{A',B',C'\Box D'}i_2^{-1}, \\
\alpha'_{A',B',C'}\Box'\mathrm{Id}_{D'} &= i_4(\alpha_{A',B',C'}\Box\mathrm{Id}_{D'})i_1^{-1}, \qquad (3.10) \\
\alpha'_{A',B'\Box' C',D'} &= i_5\alpha_{A',B'\Box C',D'}i_4^{-1}, \\
\mathrm{Id}_{A'}\Box'\alpha'_{B',C',D'} &= i_3(\mathrm{Id}_{A'}\Box\alpha_{B',C',D'})i_5^{-1}.
\end{aligned}$$

Then (2.12) implies

$$\begin{aligned}
\alpha'_{A',B',C'\Box' D'}\alpha'_{A'\Box' B',C',D'} &= i_3\alpha_{A',B',C'\Box D'}i_2^{-1}i_2\alpha_{A'\Box B',C',D'}i_1^{-1} = i_3\alpha_{A',B',C'\Box D'}\alpha_{A'\Box B',C',D'}i_1^{-1} \\
&= i_3(\mathrm{Id}_{A'}\Box\alpha_{B',C',D'})\alpha_{A',B'\Box C',D'}(\alpha_{A',B',C'}\Box\mathrm{Id}_{D'})i_1^{-1} \\
&= i_3(\mathrm{Id}_{A'}\Box\alpha_{B',C',D'})i_5^{-1}i_5\alpha_{A',B'\Box C',D'}i_4^{-1}i_4(\alpha_{A',B',C'}\Box\mathrm{Id}_{D'})i_1^{-1} \\
&= (\mathrm{Id}_{A'}\Box'\alpha'_{B',C',D'})\alpha'_{A',B'\Box' C',D'}(\alpha'_{A',B',C'}\Box'\mathrm{Id}_{D'}). \qquad (3.11)
\end{aligned}$$

In other words, if α satisfies its coherence constraints, then α' does also.

Now, what about β' and γ'? Just like we worked out α' in terms of α and other natural isomorphisms, we can work out β' and γ' from the already given structure. Without showing all the steps we can write for any object A' in $\mathcal{C}'$:

$$\begin{aligned}
\beta'_{A'} &= \beta_{A'}(\kappa_1^{-1}\Box\mathrm{Id}_{A'})\kappa_{1'\Box A'}^{-1} \\
\gamma'_{A'} &= \gamma_{A'}(\mathrm{Id}_{A'}\Box\kappa_1^{-1})\kappa_{A'\Box 1'}^{-1}
\end{aligned}$$

Then, similarly to (3.9), (3.10) and (3.11), we can show that (2.13) implies

$$(\mathrm{Id}_{A'} \,\square'\, \beta'_{B'})\alpha'_{A',1',B'} \quad = \quad \gamma'_{A'} \,\square'\, \mathrm{Id}_{B'}$$

In other words, if β and γ satisfy their coherence constraint, then β' and γ' do also.

One subtlety here is that the unit objects $1 \in \mathcal{C}_0$ and $1' \in \mathcal{C}'_0$, related by $1' = P_0(1)$, are not necessarily the same. We already know that in a nonstrict monoidal category there is only a weak axiom saying that 1 is only unique up to isomorphisms. Knowing that the isomorphism class ι_1 can contain multiple objects, we conclude there is freedom to choose 1 and $1'$ differently. However, in later chapters we will mainly discuss specific nonstrict monoidal categories with only one possible unit object, which can also be used as the unit object of the chosen skeleton, thus in that case we simply write $1' = 1$.

About the functor P itself. Assuming that $\mathcal{C}$ and $\mathcal{C}'$ are not necessarily strict or strictified yet, we can have a closer look at the projection functor $P : \mathcal{C} \to \mathcal{C}'$. A priori we can only say this is a functor of a not yet specified type. After deriving the monoidal structure for $\mathcal{C}'$ from $\mathcal{C}$ we could say it *is* a monoidal functor. However, it is not necessarily a strict one, but we could say it is at least a nonstrict monoidal functor. To be more precise, P is at least a *strong monoidal functor*, and it is not difficult to check this. Of course this is no surprise, as P was *needed* first to derive the (not necessarily strict) monoidal structure for $\mathcal{C}'$ from $\mathcal{C}$. Or, to say differently, only afterwards we can say that P is a strong monoidal functor from $\mathcal{C}$ to another category, $\mathcal{C}'$, which obtained its monoidal structure from $\mathcal{C}$ and P.

A trivial example. First assume $(\mathcal{C}, \mu, \eta)$ is already a strict monoidal category, and let $\mathcal{C}'$ be a skeleton of $\mathcal{C}$. Also assume $\mathcal{C}'$ already contains 1, which is the unit object of $\mathcal{C}$. Then we know that, as usual, μ and η satisfy (2.6) and (2.7). This is equivalent to saying that for any triple of objects and for any triple of arrows, for convenience both denoted by (A, B, C), we can write $(A\square B)\square C = A\square(B\square C)$ and $1\square A = A = A\square 1$ (where 1 is either the unit object itself or the unit arrow Id_1). Now, if for any pair $X', Y' \in \mathcal{C}'_0$ we again have $\mu_0(X', Y') = X'\square Y' \in \mathcal{C}'_0$, and if for any pair $f', g' \in \mathcal{C}'_1$ we again have $\mu_1(f', g') = f'\square g' \in \mathcal{C}'_1$, then we can choose a projection functor $P : \mathcal{C} \to \mathcal{C}'$ satisfying $P_0(X'\square Y') = X'\square Y'$ and $P_1(f'\square g') = f'\square g'$ for all objects X' and Y' and arrows f' and g' in $\mathcal{C}'$. We can already say that P satisfies $1' := P_0(1) = 1$. This means that we can define

$$\mu' := P\mu(I \times I) = \mu|_{\mathcal{C}'} \tag{3.12}$$

which is equivalent to saying that $\square' := \square$. Then it is easily proven that also μ' and η' satisfy (2.6) and (2.7) on $\mathcal{C}'$. Thus we can say $(\mathcal{C}', \square, 1)$ is a skeleton of $(\mathcal{C}, \square, 1)$, and it is also a strict monoidal category.

A summary. Now we can summarize all previous elaborate steps. If $\mathcal{C} = (\mathcal{C}, \mu, \eta, \alpha, \beta, \gamma)$ is a nonstrict monoidal category, then a skeleton $\mathcal{C}'$ can also be turned into a nonstrict monoidal category by porting over the structure. Then we can write $\mathcal{C}' = (\mathcal{C}', \mu', \eta', \alpha', \beta', \gamma')$. Even if $\mathcal{C}$ *is* strict, then $\mathcal{C}'$ is still not guaranteed to become strict. However, if $\mathcal{C}$ satisfies the coherence constraints, then $\mathcal{C}'$ does also. Then we can treat $\mathcal{C}'$ as a strict monoidal category.

Generators and relations for a skeleton of a monoidal category. Now we know that a skeleton $\mathcal{C}'$ of a monoidal category $\mathcal{C}$ can also be described as a monoidal category, we can talk about its generators and relations. As now arrows in a skeleton can be composed horizontally *and* vertically, by using $\square'$, we can try to find a generating set $G(\mathcal{C}')$ of arrows in $\mathcal{C}'$ so that every other arrow can be obtained by composing arrows in $G(\mathcal{C}')$ both horizontally *and* vertically. In many cases this generating set should be smaller, compared to when we are only allowed to split up arrows horizontally to find a set of generators.

4 Categories with Frobenius structure and Frobenius algebras

Now we know what a symmetric monoidal category is, we can introduce Frobenius structure. In Section 4.1 we will first introduce monoids, comonoids and (commutative) Frobenius objects. Then we will introduce the details of (commutative) Frobenius objects in a (symmetric) monoidal category. After this we can say what a free (symmetric) monoidal category on a (commutative) Frobenius object is. Finally we will introduce Frobenius structure, which applies to categories generated by a single object. In Section 4.2 we will introduce Frobenius objects in $\mathbf{Vect}_{\Bbbk}$, also known as *Frobenius algebras*. There are some equivalent definitions for Frobenius algebras. In the rest of this thesis we will mainly focus on the fourth definition. In Section 4.3 we will introduce $\mathbf{cFA}_{\Bbbk}$, the category of commutative Frobenius algebras.

4.1 Frobenius structure

Monoids. A *monoid* in a (strict) monoidal category $(\mathcal{C}, \Box, 1)$ is an object X together with two arrows

$$\mu_X : X \Box X \to X \quad , \quad \eta_X : 1 \to X$$

satisfying *associativity* and *unit axioms*, thus μ_X and η_X satisfy

$$\mu_X(\mu_X \Box \operatorname{Id}_X) = \mu_X(\operatorname{Id}_X \Box \mu_X), \quad \mu_X(\eta_X \Box \operatorname{Id}_X) = \operatorname{Id}_X, \quad \mu_X(\operatorname{Id}_X \Box \eta_X) = \operatorname{Id}_X . \tag{4.1}$$

The arrow μ_X is called *multiplication*, and the arrow η_X is called *unit*. In a strict monoidal category, we have the identity $X \Box 1 = X = 1 \Box X$ for any object X. This means that any arrow with domain or codomain X is also an arrow with domain or codomain $X \Box 1$ or $1 \Box X$. That is why for example $\eta_X \Box \operatorname{Id}_X : 1 \times X \to X \times X$ can be regarded as an arrow with domain X. As a side note, we should not confuse a monoid in a monoidal category with an ordinary monoid, however, an ordinary monoid is an example of a monoid in the specific monoidal category $(\mathbf{Set}, \times, 1)$.

- **Lemma.** For any arrow μ_X the associated arrow η_X is unique.

  Proof. Suppose η_X and η'_X satisfy the monoid relations, then

  $$\eta'_X = \operatorname{Id}_X \eta'_X = \mu_X(\operatorname{Id}_X \Box \eta_X)\eta'_X = \mu_X(\eta'_X \Box \eta_X) = \mu_X(\eta'_X \Box \operatorname{Id}_X)\eta_X = \operatorname{Id}_X \eta_X = \eta_X.$$

  Thus η_X is unique.

$\Box$

Comonoids. A *comonoid* in a (strict) monoidal category $(\mathcal{C}, \Box, 1)$ is an object X together with two arrows

$$\delta_X : X \to X \Box X \quad , \quad \epsilon_X : X \to 1$$

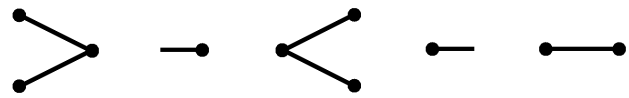
satisfying *coassociativity* and *counit axioms*, thus δ_X and ϵ_X satisfy

$$(\delta_X \Box \operatorname{Id}_X)\delta_X = (\operatorname{Id}_X \Box \delta_X)\delta_X, \quad (\epsilon_X \Box \operatorname{Id}_X)\delta_X = \operatorname{Id}_X, \quad (\operatorname{Id}_X \Box \epsilon_X)\delta_X = \operatorname{Id}_X . \tag{4.2}$$

The arrow δ_X is called *comultiplication*, and the arrow ϵ_X is called *counit*. Thus relations (4.2) for δ and ϵ are similar to relations (4.1), except that now the direction of the arrows is reversed. Similarly to the previous lemma we can say that for any arrow δ_X the associated arrow ϵ_X is unique. There exist categories with objects being a monoid and a comonoid at the same time.

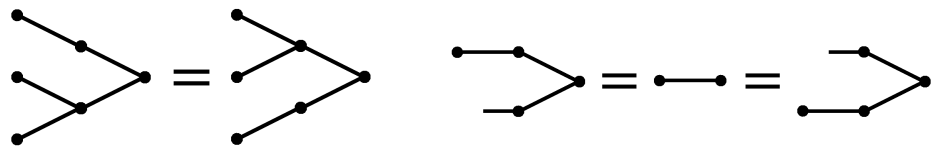
Commutative monoids and cocommutative comonoids. Let X be a monoid in a symmetric monoidal category $(\mathcal{C}, \Box, 1, \tau)$ and let $\tau_X := \tau_{X,X} : X \Box X \to X \Box X$ be the twist arrow related to X. Then X is called *commutative* if it satisfies $\mu_X \tau_X = \mu_X$. If X is a comonoid in a symmetric monoidal category $(\mathcal{C}, \Box, 1, \tau)$, then it is called *cocommutative* if it satisfies $\tau_X \delta_X = \delta_X$.

Graphical representation of monoid and comonoid relations. The monoid and comonoid relations can be graphically represented. By convention, the domain of an arrow is drawn on the left and the codomain of an arrow is drawn on the right. The notation order of two composed arrows is from right to left, but in the diagrams we will now introduce, this order is vice versa. The following diagrams graphically represent the five basic arrows involved in monoid and comonoid relations:



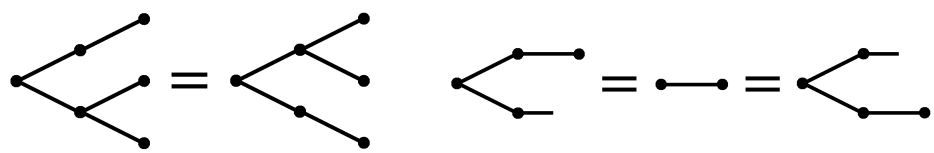
μ_X η_X δ_X ϵ_X Id_X

The following diagrams graphically explain the monoid relations:



$$\mu_X(\mu_X \Box \mathrm{Id}_X) = \mu_X(\mathrm{Id}_X \Box \mu_X) \qquad \mu_X(\eta_X \Box \mathrm{Id}_X) = \mathrm{Id}_X = \mu_X(\mathrm{Id}_X \Box \eta_X) \tag{4.3}$$

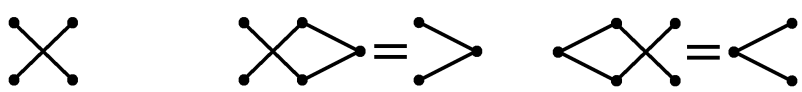
The following diagrams graphically explain the comonoid relations:



$$(\delta_X \Box \mathrm{Id}_X)\delta_X = (\mathrm{Id}_X \Box \delta_X)\delta_X \qquad (\epsilon_X \Box \mathrm{Id}_X)\delta_X = \mathrm{Id}_X = (\mathrm{Id}_X \Box \epsilon_X)\delta_X \tag{4.4}$$

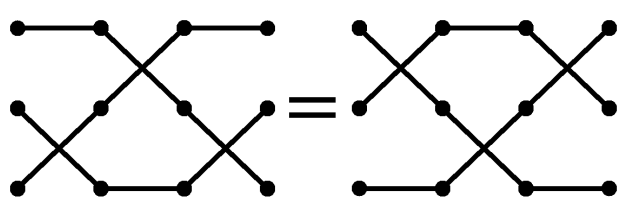
In these diagrams, vertical composition of arrows is ordered from down to up.

Graphical representation of some relations concerning the twist arrows. The following diagrams graphically represent the twist arrow of an object in a symmetric monoidal category and the relations a commutative monoid and a cocommutative comonoid need to satisfy:



τ_X $\qquad$ $\mu_X\tau_X = \mu_X$ $\qquad$ $\tau_X\delta_X = \delta_X$ (4.5)

And these diagrams graphically represent relation (2.17):



$$(\tau_X \Box \mathrm{Id}_X)(\mathrm{Id}_X \Box \tau_X)(\tau_X \Box \mathrm{Id}_X) = (\mathrm{Id}_X \Box \tau_X)(\tau_X \Box \mathrm{Id}_X)(\mathrm{Id}_X \Box \tau_X) \tag{4.6}$$

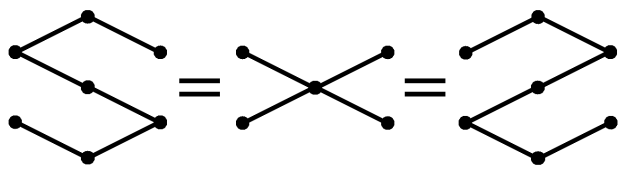
Frobenius objects. A *Frobenius object* in a monoidal category $(\mathcal{C}, \Box, 1)$ is an object X together with four arrows

$$\mu_X : X \Box X \to X \quad , \quad \eta_X : 1 \to X \quad , \quad \delta_X : X \to X \Box X \quad , \quad \epsilon_X : X \to 1$$

satisfying the unit and counit axioms

$$\mu_X(\eta_X \Box \operatorname{Id}_X) = \operatorname{Id}_X = \mu_X(\operatorname{Id}_X \Box \eta_X) \quad , \quad (\epsilon_X \Box \operatorname{Id}_X)\delta_X = \operatorname{Id}_X = (\operatorname{Id}_X \Box \epsilon_X)\delta_X$$

and the *Frobenius relation*



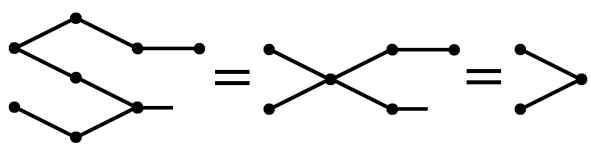
(4.7)

$$(\mu_X \Box \operatorname{Id}_X)(\operatorname{Id}_X \Box \delta_X) = \delta_X \mu_X = (\operatorname{Id}_X \Box \mu_X)(\delta_X \Box \operatorname{Id}_X)$$

Note that we should write $X = (X, \mu_X, \eta_X, \delta_X, \epsilon_X)$ to completely specify a Frobenius object.

- **Lemma.** A Frobenius object satisfies associativity and coassociativity axioms.

  Proof. Composing the arrows in (4.7) with $\epsilon_X \Box \operatorname{Id}_X$ and applying the counit properties yield the following relation:



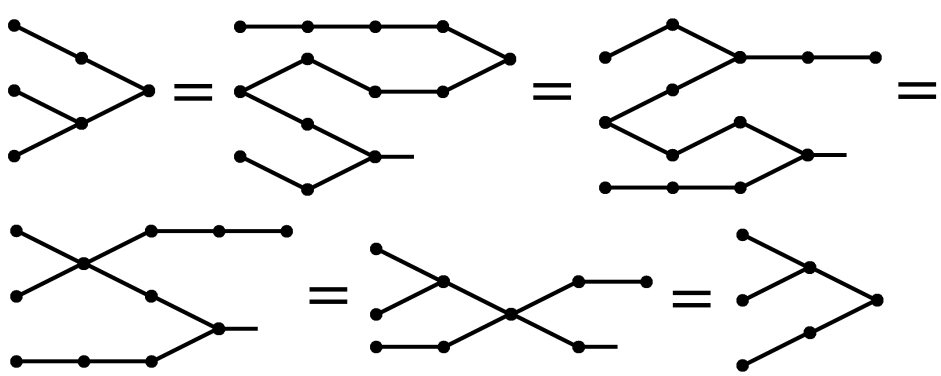
(4.8)

$$(\epsilon_X \Box \operatorname{Id}_X)(\mu_X \Box \operatorname{Id}_X)(\operatorname{Id}_X \Box \delta_X) = (\epsilon_X \Box \operatorname{Id}_X)\delta_X \mu_X = \mu_X$$

As relations like (2.9) show, we can easily insert or remove identity arrows. Inserting (4.8) into $\mu_X(\mu_X \Box \operatorname{Id}_X)$ yields

$$\begin{aligned} \mu_X(\mu_X \Box \operatorname{Id}_X) &= \mu_X((\epsilon_X \Box \operatorname{Id}_X)(\mu_X \Box \operatorname{Id}_X)(\operatorname{Id}_X \Box \delta_X) \Box \operatorname{Id}_X) \\ &= \mu_X(\epsilon_X \Box \operatorname{Id}_X \Box \operatorname{Id}_X)(\mu_X \Box \operatorname{Id}_X \Box \operatorname{Id}_X)(\operatorname{Id}_X \Box \delta_X \Box \operatorname{Id}_X). \end{aligned}$$

Now applying the Frobenius relation multiple times we obtain the following relation, graphically explained by



This finally shows that $\mu_X(\mu_X \Box \mathrm{Id}_X) = \mu_X(\mathrm{Id}_X \Box \mu_X)$, turning μ_X into an associative multiplication arrow.

Of course we can repeat this for the comultiplication: we can easily replace ϵ_X by η_X and interchange μ_X and δ_X. After inverting the direction of composing arrows, we find a coassociative comultiplication δ_X.

□

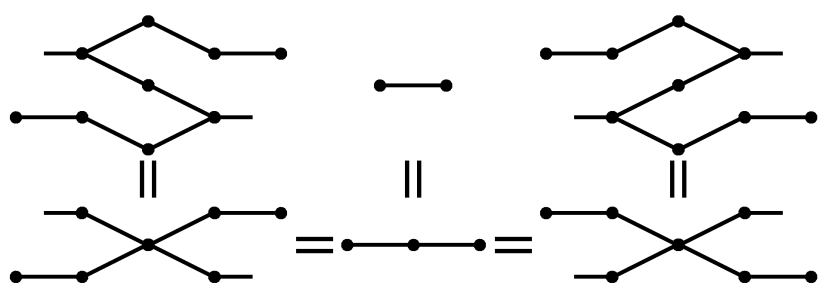
Thus we can safely conclude that a Frobenius object is a monoid and a comonoid at the same time.

- **Lemma.** A Frobenius object satisfies the following relation:

$$\begin{aligned}(\epsilon_X \Box \mathrm{Id}_X)(\mu_X \Box \mathrm{Id}_X)(\mathrm{Id}_X \Box \delta_X)(\mathrm{Id}_X \Box \eta_X) &= \mathrm{Id}_X = \\ (\mathrm{Id}_X \Box \epsilon_X)(\mathrm{Id}_X \Box \mu_X)(\delta_X \Box \mathrm{Id}_X)(\eta_X \Box \mathrm{Id}_X).\end{aligned} \tag{4.9}$$

This relation is called the *snake relation*.

Proof. We can use the Frobenius relation and the unit and counit relations to prove this. We can do this graphically:



(4.10)

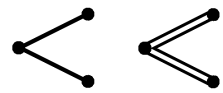
The top part of this diagram just represents relation (4.9).

□

Free monoidal categories on a Frobenius object and Frobenius structure. Let $(\mathcal{C}, \Box, 1)$ be a monoidal category and let X be a Frobenius object in $\mathcal{C}$. Then we call $\mathcal{C}$ a *free monoidal category on a Frobenius object* if all other objects can be written as $\mathbf{n} := X^n = X \Box \cdots \Box X$, and all arrows can be written as serial and parallel composition of the four basic arrows μ_X, η_X, δ_X and ϵ_X. If $\mathcal{C}$ is such a category, then we say $\mathcal{C}$ carries *Frobenius structure*.

Commutative and cocommutative Frobenius objects. Let $(\mathcal{C}, \Box, 1, \tau)$ be a symmetric monoidal category and let X be a Frobenius object in $\mathcal{C}$. Then we call X a *commutative Frobenius object* if it is commutative as a monoid, and it is called a *cocommutative Frobenius object* if it is cocommutative as a comonoid. We will claim that X is cocommutative if and only if X is commutative. To prove this we need some intermediate steps, represented as lemmas.

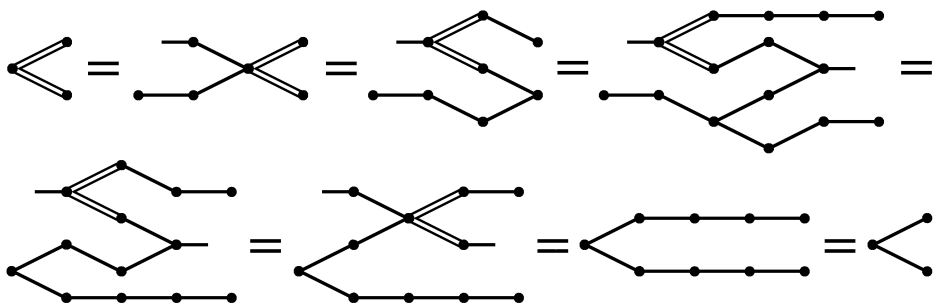
- **Lemma.** If we keep μ_X, η_X and ϵ_X fixed, and if the counit and Frobenius relations are satisfied for both δ_X and δ'_X, graphically represented by



δ_X $\qquad$ δ'_X

then δ_X and δ'_X must be the same arrow.

Proof. We will prove this graphically:

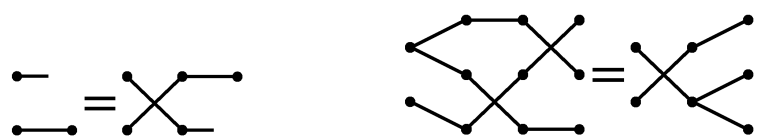


(4.11)

The third equality in this diagram is based on the equality $\mu_X = (\mathrm{Id}_X \,\square\, \epsilon_X)(\mathrm{Id}_X \,\square\, \mu_X)(\delta_X \,\square\, \mathrm{Id}_X)$, which is similar to (4.8). So we conclude that, if μ_X, η_X and ϵ_X are already chosen, then δ_X is unique.

$\square$

- **Lemma.** We have the following two equalities:



$\mathrm{Id}_X \,\square\, \epsilon_X = (\epsilon_X \,\square\, \mathrm{Id}_X)\tau_X$ $\qquad$ $(\mathrm{Id}_X \,\square\, \tau_X)(\tau_X \,\square\, \mathrm{Id}_X)(\mathrm{Id}_X \,\square\, \delta_X) = (\delta_X \,\square\, \mathrm{Id}_X)\tau_X$ (4.12)

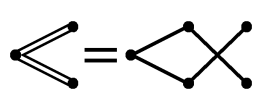
Proof. Naturality of the twist arrow implies the following two commuting diagrams

$$\begin{tikzcd} X\square X \arrow[r, "\tau_X"] \arrow[d, "\mathrm{Id}_X \,\square\, \epsilon_X"'] & X\square X \arrow[d, "\epsilon_X \,\square\, \mathrm{Id}_X"] \\ X\square 1 \arrow[r, "\tau_{X,1}"'] & 1\square X \end{tikzcd} \qquad \begin{tikzcd} X\square X \arrow[r, "\tau_X"] \arrow[d, "\mathrm{Id}_X \,\square\, \delta_X"'] & X\square X \arrow[d, "\delta_X \,\square\, \mathrm{Id}_X"] \\ X\square X\square X \arrow[r, "\tau_{X,X\square X}"'] & X\square X\square X \end{tikzcd}$$

The arrow $\tau_{X,1}$ in left diagram equals Id_X, as proven in (2.18), and we already know that $\tau_{X,X\square X} = (\mathrm{Id}_X \,\square\, \tau_X)(\tau_X \,\square\, \mathrm{Id}_X)$.

$\square$

Now suppose δ_X already satisfies the counit and Frobenius relations, and define $\delta'_X := \tau_X \delta_X$, graphically represented by



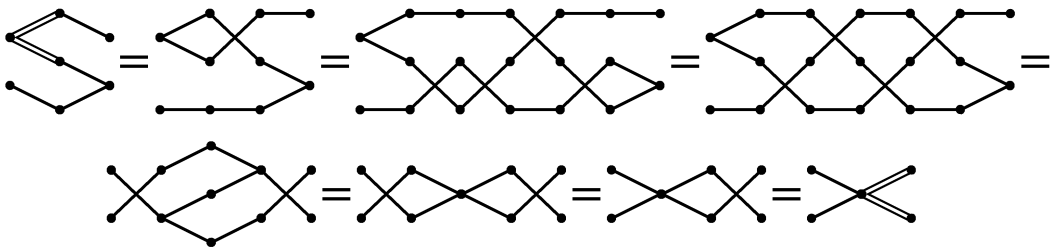
Then we would like to know whether this δ'_X also satisfies the counit and Frobenius relations.

- **Lemma.** If δ_X already satisfies the counit and Frobenius relations, then δ'_X also satisfies the counit and Frobenius relations.

  Proof. If δ_X satisfies the counit relation, then the left diagram of (4.12) implies

$$\begin{aligned}(\epsilon_X \Box \operatorname{Id}_X)\delta'_X &= (\epsilon_X \Box \operatorname{Id}_X)\tau_X\delta_X = (\operatorname{Id}_X \Box \epsilon_X)\delta_X = \\ \operatorname{Id}_X &= (\epsilon_X \Box \operatorname{Id}_X)\delta_X = (\operatorname{Id}_X \Box \epsilon_X)\tau_X\delta_X = (\operatorname{Id}_X \Box \epsilon_X)\delta'_X.\end{aligned}$$

  Thus δ'_X also satisfies the counit relation. Now, if δ_X also satisfies the Frobenius relation, then the right diagram of (4.12) and commutativity of μ_X imply $(\mu_X \Box \operatorname{Id}_X)(\operatorname{Id}_X \Box \delta'_X) = \delta'_X\mu_X$, graphically explained by



  The second equality comes from the identities $(\tau_X \Box \operatorname{Id}_X)(\tau_X \Box \operatorname{Id}_X) = \operatorname{Id}_{X\Box X\Box X}$ and $\mu_X\tau_X = \mu_X$. The third equality comes from applying (4.6). The fourth equality comes from the right diagram of (4.12) and its dual, i.e. turning the diagram 180 degrees. The other steps should be trivial. In a similar way we can show that $\delta'_X\mu_X = (\operatorname{Id}_X \Box \mu_X)(\delta'_X \Box \operatorname{Id}_X)$. Then we see that δ'_X also satisfies the counit and Frobenius relations.

□

To conclude, we see that, in case of $\delta'_X = \tau_X\delta_X$, the conditions for identity (4.11) are satisfied, thus $\delta'_X = \delta_X$. In other words, commutativity of μ_X implies cocommutativity of δ_X. Now we can discuss the final lemma.

- **Lemma.** X is a cocommutative Frobenius object if and only if X is a commutative Frobenius object.

  Proof. If X is a commutative Frobenius object, then μ_X is commutative, as an arrow. Then δ_X is cocommutative, turning X into a cocommutative Frobenius object. The reverse statement being true can be derived from all previous steps when reversing the composition of all arrows and interchanging the roles played by (μ_X, η_X) and (δ_X, ϵ_X).

□

A trivial example of a (commutative) Frobenius object. In any monoidal category $(\mathcal{C}, \Box, 1)$, the object 1, together with the trivial arrow Id_1, can be interpreted as a Frobenius object. As we know, $1 = 1\Box 1$, thus we can make a trivial choice:

$$\mu_1 = \eta_1 = \delta_1 = \epsilon_1 = \operatorname{Id}_1 .$$

It is easy to check that this Frobenius object is also commutative.

Free symmetric monoidal categories on a commutative Frobenius object. Let $(\mathcal{C}, \Box, 1, \tau)$ be a symmetric monoidal category and let X be a commutative Frobenius object in $\mathcal{C}$. Then we call $\mathcal{C}$ a *free symmetric monoidal category on a commutative Frobenius object* if all other objects can be written as $\mathbf{n} := X^n = X\Box\cdots\Box X$, and all arrows can be written as serial and parallel composition of the five basic arrows μ_X, η_X, δ_X, ϵ_X and τ_X. Of course this is a special case of a monoidal category on a Frobenius object, thus $\mathcal{C}$ also carries Frobenius structure. In this case the Frobenius structure is commutative.

Frobenius homomorphisms. Let X and Y be Frobenius objects in a monoidal category $\mathcal{C}$. Then an arrow $f : X \to Y$ is called a *Frobenius homomorphism* in $\mathcal{C}$ if it is compatible with all the structure as in the following commuting diagrams:

$$\begin{tikzcd} X\Box X \arrow[r, "\mu_X"] \arrow[d, "f\Box f"'] & X \arrow[r, "\delta_X"] \arrow[d, "f"] & X\Box X \arrow[d, "f\Box f"] \\ Y\Box Y \arrow[r, "\mu_Y"'] & Y \arrow[r, "\delta_Y"'] & Y\Box Y \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\eta_X"] \arrow[dr, "\eta_Y"'] & X \arrow[r, "\epsilon_X"] \arrow[d, "f"] & 1 \\ & Y \arrow[ur, "\epsilon_Y"'] & \end{tikzcd} \tag{4.13}$$

Of course the identity arrow Id_X is a Frobenius homomorphism, and composing two Frobenius homomorphisms gives another Frobenius homomorphism. If $\mathcal{C}$ is also symmetric, then naturality of the twist arrow implies that a Frobenius homomorphism is automatically compatible with this symmetric structure.

The category of Frobenius objects. If $\mathcal{C}$ is a monoidal category, written as $(\mathcal{C}, \Box, 1)$, then $\mathbf{Frob}(\mathcal{C})$ is the *category of Frobenius objects* in $\mathcal{C}$. Its objects are the Frobenius objects in $\mathcal{C}$ and its arrows are the Frobenius homomorphisms in $\mathcal{C}$. If $\mathcal{C}$ is also symmetric, then $\mathbf{cFrob}(\mathcal{C})$ is the category of commutative Frobenius objects and Frobenius homomorphisms in $\mathcal{C}$.

These two categories $\mathbf{Frob}(\mathcal{C})$ and $\mathbf{cFrob}(\mathcal{C})$ can be equipped with monoidal structure. If $\mathcal{C}$ is a symmetric monoidal category and if $(X, \mu_X, \eta_X, \delta_X, \epsilon_X)$ and $(Y, \mu_Y, \eta_Y, \delta_Y, \epsilon_Y)$ are Frobenius objects in $\mathcal{C}$, then we can also turn $X\Box Y$ into a Frobenius object. We define the needed arrows as follows:

$$\begin{aligned} \mu_{X\Box Y} &:= (\mu_X \Box \mu_Y)(\mathrm{Id}_X \Box \tau_{Y,X} \Box \mathrm{Id}_Y), \\ \eta_{X\Box Y} &:= \eta_X \Box \eta_Y, \\ \delta_{X\Box Y} &:= (\mathrm{Id}_X \Box \tau_{X,Y} \Box \mathrm{Id}_Y)(\delta_X \Box \delta_Y), \\ \epsilon_{X\Box Y} &:= \epsilon_X \Box \epsilon_Y. \end{aligned} \tag{4.14}$$

Using naturality of the twist arrows it is easy to check that associativity of μ_X and μ_Y implies associativity of $\mu_{X\Box Y}$. Similarly coassociativity of δ_X and δ_Y implies coassociativity of $\delta_{X\Box Y}$. Also η_X and η_Y satisfying the unit axioms, implies that $\eta_{X\Box Y}$ satisfies the unit axioms. The same applies to the counit axioms and the Frobenius relation. Thus the four arrows defined in (4.14) turn $X\Box Y$ into a Frobenius object.

If X and Y are commutative Frobenius objects and if we also write

$$\tau_{X\Box Y} := \tau_{X\Box Y, X\Box Y} = (\mathrm{Id}_X \Box \tau_{X,Y} \Box \mathrm{Id}_Y)(\tau_X \Box \tau_Y)(\mathrm{Id}_X \Box \tau_{Y,X} \Box \mathrm{Id}_Y) \tag{4.15}$$

as implied by (2.16), then $X\Box Y$ is again a commutative Frobenius object. Again it is easy to check that commutativity of X and Y implies commutativity of $X\Box Y$.

We thus see that $\mathbf{Frob}(\mathcal{C})$ and $\mathbf{cFrob}(\mathcal{C})$ are closed under vertical composition, or $\Box$-products. As $(1, \mathrm{Id}_1)$ can be regarded as the neutral object for taking $\Box$-products of (commutative) Frobenius objects, we conclude that $(\mathbf{Frob}(\mathcal{C}), \Box, 1)$ and $(\mathbf{cFrob}(\mathcal{C}), \Box, 1)$ can be regarded as monoidal categories. In addition we can assert that a canonical monoidal embedding exists turning $(\mathbf{cFrob}(\mathcal{C}), \Box, 1)$ into a monoidal subcategory of $(\mathbf{Frob}(\mathcal{C}), \Box, 1)$.

We could ask whether these are monoidal subcategories of $\mathcal{C}$ itself. We should be careful however. Of course, for any pair X and Y of objects in $(\mathbf{Frob}(\mathcal{C}), \Box, 1)$, we should write

$$(X, \mu_X, \eta_X, \delta_X, \epsilon_X)\Box(Y, \mu_Y, \eta_Y, \delta_Y, \epsilon_Y) = (X\Box Y, \mu_{X\Box Y}, \eta_{X\Box Y}, \delta_{X\Box Y}, \epsilon_{X\Box Y}).$$

We thus see that the symbol $\Box$ does not exactly have the same meaning, when appearing in $(\mathcal{C}, \Box, 1)$ and in $(\mathbf{Frob}(\mathcal{C}), \Box, 1)$, so the answer to our question is negative. There is even a simpler explanation. The category $\mathbf{Frob}(\mathcal{C})$ is not a subcategory of $(\mathcal{C}, \Box, 1)$, just because more information is needed for specifying any object in $\mathbf{Frob}(\mathcal{C})$). To be more precise, specifying an object in $\mathbf{Frob}(\mathcal{C})$ means specifying an object A

in $(\mathcal{C}, \Box, 1)$ *and* the needed arrows turning A into a Frobenius object. There could be many different ways of turning A into a Frobenius object, so the class of objects in the category $\mathbf{Frob}(\mathcal{C})$ can be much larger than that of $\mathcal{C}$ itself. For example, in case of $\mathcal{C} = \mathbf{Vect}_{\Bbbk}$ every object V admits countlessly many sets of arrows $(\mu_V, \eta_V, \delta_V, \epsilon_V)$ turning V into a Frobenius object.

A subtle remark. If $\mathcal{C}$ is a free symmetric monoidal category on a commutative Frobenius object, and if X is the commutative Frobenius object generating all other objects X^n, then we can say that all the other objects X^n $(n \geq 2)$ are also commutative Frobenius objects. We can simply apply relations (4.14) and (4.15) to these objects. So, saying that $\mathcal{C}$ is a category on *a commutative Frobenius object* does *not* mean that it only contains *one* commutative Frobenius object. It only means that $\mathcal{C}$ contains only one commutative Frobenius object generating all others.

4.2 Frobenius algebras

Algebras. Let $\Bbbk$ be a ground field, for example $\mathbb{R}$ or $\mathbb{C}$, and let A be a vector space over $\Bbbk$. We say that A can be regarded as an abelian group equipped with a $\Bbbk$-action $A \times \Bbbk \to A$. As usual we write $A^* := \mathrm{Hom}(A, \Bbbk)$ for the space dual to A. If A has finite dimension, then A^* is isomorphic to A. If a basis for A is defined, then we can find a dual basis for A^*. There is also a natural isomorphism from A to $A^{**} = \mathrm{Hom}(A^*, \Bbbk)$.

Now let $\mu : A \otimes A \to A$, called a *multiplication map*, and $\eta : \Bbbk \to A$, called a *unit map*, be $\Bbbk$-linear maps. Then A is called a $\Bbbk$-*algebra* if the following diagrams commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\mu \otimes \mathrm{Id}_A"'] \arrow[dr, "\mathrm{Id}_A \otimes \mu"] & \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \Bbbk \arrow[r, "\mathrm{Id}_A \otimes \eta"] \arrow[dr] & A \otimes A \arrow[d, "\mu"] & \Bbbk \otimes A \arrow[l, "\eta \otimes \mathrm{Id}_A"'] \arrow[dl] \\
 & A &
\end{tikzcd}$$

Of course we mean tensor products over $\Bbbk$, or $\otimes = \otimes_{\Bbbk}$. In fact we see that these diagrams are similar to the diagrams in (2.1). Only $\times$ is replaced by $\otimes$ and 1 is replaced by $\Bbbk$. We say a $\Bbbk$-algebra is a monoid in $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$.

Coalgebras. Let $\delta : A \to A \otimes A$, called a *comultiplication*, and $\epsilon : A \to \Bbbk$, called a *counit map*, be $\Bbbk$-linear maps. Then A is called a $\Bbbk$-*coalgebra* if the following diagrams commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A & \\
A \otimes A \arrow[ur, "\delta \otimes \mathrm{Id}_A"] & & A \otimes A \arrow[ul, "\mathrm{Id}_A \otimes \delta"'] \\
 & A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \Bbbk & A \otimes A \arrow[l, "\mathrm{Id}_A \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{Id}_A"] & \Bbbk \otimes A \\
 & A \arrow[ul] \arrow[u, "\delta"] \arrow[ur] &
\end{tikzcd}$$

We say a $\Bbbk$-coalgebra is a comonoid in $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$.

Definitions of Frobenius algebras. The following three definitions, copied from [7], are equivalent:

1. A *Frobenius algebra* is a $\Bbbk$-algebra A of finite dimension, equipped with a linear functional $\epsilon : A \to \Bbbk$ whose nullspace contains no non-trivial left ideals. The functional $\epsilon \in A^*$ is called a *Frobenius form*.

2. A *Frobenius algebra* is a $\Bbbk$-algebra A of finite dimension, equipped with an associative nondegenerate pairing $\beta : A \otimes A \to \Bbbk$. We call this pairing the *Frobenius pairing*.

3. A *Frobenius algebra* is a finite-dimensional $\Bbbk$-algebra A equipped with a left A-isomorphism to its dual. Alternatively (and equivalently) A is equipped with a right A-isomorphism to its dual.

The first definition can also be stated alternatively and equivalently. If, instead of *left ideals*, we write *right ideals* nothing really changes.

Terminology not defined here, but also the elaborate proofs of the equivalences can be found in [7]. The author uses a lot of diagrams to make the reader feel comfortable with these proofs. The definitions will turn out to represent just the same underlying concept, but described from diffferent points of view. We should add that the pairing β, mentioned in the second definition, cannot be nondegenerate if we admit a vector space A of infinite dimension. That is why we would like to restrict to finite dimension.

There is another equivalent definition:

4. A *Frobenius algebra* is a $\Bbbk$-vector space A of finite dimension, equipped with four suitable maps

$$\mu : A \otimes A \to A \quad , \quad \eta : \Bbbk \to A \quad , \quad \delta : A \to A \otimes A \quad , \quad \epsilon : A \to \Bbbk$$

turning A into a $\Bbbk$-algebra and a $\Bbbk$-coalgebra at the same time, and making A satisfy the *Frobenius relation*

$$(\mu \otimes \mathrm{Id}_A)(\mathrm{Id}_A \otimes \delta) = \delta\mu = (\mathrm{Id}_A \otimes \mu)(\delta \otimes \mathrm{Id}_A),$$

which is the analogue of (4.7).

We should note that, in general, a vector space A, or an algebra (A, μ, η), is not necessarily a Frobenius algebra. Even a vector space $(A, \mu, \eta, \delta, \epsilon)$, which is already an algebra and a coalgebra at the same time, is still not necessarily a Frobenius algebra. Only if the four maps $(\mu, \eta, \delta, \epsilon)$ are related correctly, we can say A, equipped with these four maps, is a Frobenius algebra. This only depends on properties of these four maps.

We see that a Frobenius algebra is nothing more than a Frobenius object in $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk)$. The symmetric monoidal category $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk, \tau)$ also admits commutative Frobenius objects. We will call these *commutative Frobenius algebras*. For any Frobenius algebra A there exists a (canonical) twist map $\tau_A : A \otimes A \to A \otimes A$, and A is called a commutative Frobenius algebra if $\mu_A \tau_A = \mu_A$. Whether A can be interpreted as a commutative Frobenius algebra or not depends on the structure.

We should also note that in general there are many different possibilities of turning A into a Frobenius algebra. If $A_1 := (A, \mu, \eta, \delta, \epsilon)$ and if $A_2 := (A, \mu', \eta', \delta', \epsilon')$ are Frobenius algebras, then the two groups of related maps can differ. For example, A_1 can be a commutative Frobenius algebra, while A_2 is *not*.

The snake relation. Any Frobenius object satisfies the snake relation, defined in (4.9). Written in the language of $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk)$ this translates to

$$\begin{aligned} (\epsilon \otimes \mathrm{Id}_A)(\mu \otimes \mathrm{Id}_A)(\mathrm{Id}_A \otimes \delta)(\mathrm{Id}_A \otimes \eta) &= \mathrm{Id}_A = \\ (\mathrm{Id}_A \otimes \epsilon)(\mathrm{Id}_A \otimes \mu)(\delta \otimes \mathrm{Id}_A)(\eta \otimes \mathrm{Id}_A) & \end{aligned} \tag{4.16}$$

for any Frobenius algebra. As it is quite common to use the *pairing* $\beta : A \otimes A \to \Bbbk$, $\beta := \epsilon\mu$, and the *copairing* $\gamma : \Bbbk \to A \otimes A$, $\gamma := \delta\eta$, relation (4.16) can also be written as

$$(\beta \otimes \mathrm{Id}_A)(\mathrm{Id}_A \otimes \gamma) = \mathrm{Id}_A = (\mathrm{Id}_A \otimes \beta)(\gamma \otimes \mathrm{Id}_A). \tag{4.17}$$

However, it depends on the chosen point of view whether these β and γ are defined this way or not. If the maps μ, η, δ and ϵ were already defined, then it suffices to define β and γ this way. But, in other cases, see the second definition, β (and γ) are already defined from the start, and then μ and ϵ are related to δ and η by β and γ.

In the later chapters about cobordisms we will see a relation similar to (4.17), and this relation goes under the name of *snake decomposition*. However, there is still a subtle difference between the snake decomposition and the snake relation.

4.3 The category of Frobenius algebras

Let $\Bbbk$ be a ground field. Then $\mathbf{Vect}_{\Bbbk} = (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau)$, as already defined in (2.19), is a symmetric monoidal category. We know that (commutative) Frobenius algebras are (commutative) Frobenius objects in $\mathbf{Vect}_{\Bbbk}$. We can use (4.14) for showing that taking the tensor product of two Frobenius algebras yields another Frobenius algebra. If $A = (A, \mu_A, \eta_A, \delta_A, \epsilon_A)$ and $B = (B, \mu_B, \eta_B, \delta_B, \epsilon_B)$ are Frobenius algebras, then

$$A \otimes B = (A, \mu_A, \eta_A, \delta_A, \epsilon_A) \otimes (B, \mu_B, \eta_B, \delta_B, \epsilon_B) = (A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B}, \delta_{A\otimes B}, \epsilon_{A\otimes B}) \tag{4.18}$$

with

$$\begin{aligned} \mu_{A\otimes B} &:= (\mu_A \otimes \mu_B)(\mathrm{Id}_A \otimes \tau_{B,A} \otimes \mathrm{Id}_B), \\ \eta_{A\otimes B} &:= \eta_A \otimes \eta_B, \\ \delta_{A\otimes B} &:= (\mathrm{Id}_A \otimes \tau_{A,B} \otimes \mathrm{Id}_B)(\delta_A \otimes \delta_B), \\ \epsilon_{A\otimes B} &:= \epsilon_A \otimes \epsilon_B \end{aligned}$$

is another Frobenius algebra. We can do something similar in case of taking the tensor product of two commutative Frobenius algebras. Then the resulting object will also be a commutative Frobenius algebra.

Now we can define the category of Frobenius algebras over $\Bbbk$,

$$\mathbf{FA}_{\Bbbk} := \mathbf{Frob}(\mathbf{Vect}_{\Bbbk}) = \mathbf{Frob}(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau),$$

and the category of commutative Frobenius algebras over $\Bbbk$,

$$\mathbf{cFA}_{\Bbbk} := \mathbf{cFrob}(\mathbf{Vect}_{\Bbbk}) = \mathbf{cFrob}(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau). \tag{4.19}$$

These can be defined as monoidal categories

$$(\mathbf{FA}_{\Bbbk}, \otimes, \Bbbk) \quad , \quad (\mathbf{cFA}_{\Bbbk}, \otimes, \Bbbk)$$

and there exists a canonical monoidal embedding of $(\mathbf{cFA}_{\Bbbk}, \otimes, \Bbbk)$ into $(\mathbf{FA}_{\Bbbk}, \otimes, \Bbbk)$.

The arrows in $\mathbf{FA}_{\Bbbk}$ (and in $\mathbf{cFA}_{\Bbbk}$), which are Frobenius homomorphisms when regarded as arrows in $\mathbf{Frob}(\mathbf{Vect}_{\Bbbk})$, will be called *Frobenius algebra homomorphisms*. According to a lemma, any Frobenius algebra homomorphism is in fact an isomorphism. (A proof of this lemma can be found in [7].) This turns $\mathbf{FA}_{\Bbbk}$ and $\mathbf{cFA}_{\Bbbk}$ into groupoids. These groupoids should of course contain multiple connected components, as we know that not every pair of Frobenius algebras needs to be connected by an isomorphism.

5 Morse functions

First of all, the notion of Morse functions is very useful for the understanding of the category of cobordisms, to be defined in the next two chapters. However, this chapter should be regarded as self-contained. The notion of Morse functions will be explained, but we will also take a look at the most important property of a Morse function. For any Morse function we can define a Hesse matrix at every point. This is in general no tangent space tensor as it does not transform correctly under coordinate transformations. The components of the Hesse matrix are thus not the components of some tensor field defined on our manifold of interest. However, we are only interested in the behaviour of the Hesse matrix at critical points of the Morse function. We will show that, at critical points, the Hesse matrix *does* transform as a tangent space tensor. In other words, at critical points, the Hesse matrix can be written as a coordinate free object. The fact that also the location of critical points of the Morse function, as well as their property of being nondegenerate, do not depend on the coordinate chosen, is one of the main reasons justifying the usage of Morse functions as topological objects. The reader who is already familiar with Morse theory can skip all of this chapter, but is adviced to read at least the last paragraph. There I will explain the custom notion of *special* Morse functions, which we will explicitly use in the next chapters.

Critical points. Recall that a topological m-manifold M is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^m$. Here, m is the dimension of M, which is finite. M is called a *smooth manifold* if there exists a maximal C^∞ atlas on M. The C^∞-property means that all transition maps in this atlas are smooth. Now, let M be a compact smooth manifold and let $f : M \to \mathbb{R}$ be a smooth map. This means that for every chart (U, κ), the map $f \circ \kappa^{-1} : \kappa(U) \to \mathbb{R}$ is also smooth. We define $I := f(M)$, which is of course a closed and bounded, thus also compact, subset of $\mathbb{R}$. Thus $f : M \to I$ is a smooth surjective map.

For any $p \in M$ we denote the tangent space of M at p by T_pM, and f induces a linear map between tangent spaces, $f_*(p) : T_pM \to T_{f(p)}I$, which is called the *tangent map* at p. A point $p \in M$ is called a *critical point* of f if the induced map $f_*(p)$ is the zero map. Now let us define $M_c(f)$, which is the set of critical points of f.

$$M_c(f) := \{p \in M | f_*(p) = 0\}.$$

Another way to write the tangent map is as follows. If $X \in T_pM$ then we can define $X(f)(p) = f_*(p)(X)$. This can be interpreted as X acting on f at p. Let now $\kappa = \{\kappa^i\}$ be a local coordinate around p and define $f^{(\kappa)} := f \circ \kappa^{-1}$. Then we can write $f^{(\kappa)}(\kappa(p)) = (f \circ \kappa^{-1})(\kappa(p)) = (f \circ \kappa^{-1} \circ \kappa)(p) = f(p)$. We can use this to denote the action of X on f at p as follows:

$$X(f)(p) = \sum_{i=1}^{m} X^i_{(\kappa)} \frac{\partial f^{(\kappa)}}{\partial \kappa^i}(\kappa(p)). \tag{5.1}$$

Now we can express the vector X in terms of the local coordinate κ:

$$X = \sum_{i=1}^{m} X^i_{(\kappa)} \frac{\partial}{\partial \kappa^i}.$$

Let T^*_pM denote the dual space of T_pM, then the local coordinate also defines a basis for T^*_pM. For a covector $\omega \in T^*_pM$, we write

$$\omega = \sum_{i=1}^{m} \omega^{(\kappa)}_i d\kappa^i.$$

It acts on a vector $X \in T_pM$ according to

$$\omega(X) = \sum_{i=1}^{m} \omega^{(\kappa)}_i d\kappa^i (\sum_{j=1}^{m} X^j_{(\kappa)} \frac{\partial}{\partial \kappa^j}) = \sum_{i,j=1}^{m} \omega^{(\kappa)}_i X^j_{(\kappa)} d\kappa^i (\frac{\partial}{\partial \kappa^j}) = \sum_{i,j=1}^{m} \omega^{(\kappa)}_i X^j_{(\kappa)} \delta^i_j = \sum_{i=1}^{m} \omega^{(\kappa)}_i X^i_{(\kappa)}.$$

The *differential* of f at p, denoted by $df(p)$, is an element of T_p^*M. In terms of a local coordinate κ, it can be written as

$$df(p) := \sum_{i=1}^{m} \frac{\partial f^{(\kappa)}}{\partial \kappa^i}(\kappa(p))d\kappa^i.$$

In a critical point p, $f_*(p)$ is the zero map. This means that or all $X \in T_pM$ we have $f_*(p)(X) = 0$, or

$$\forall_X : X(f)(p) = 0 \quad \Longrightarrow \quad \forall_X : \sum_{i=1}^{m} X^i \frac{\partial f^{(\kappa)}}{\partial \kappa^i}(\kappa(p)) = 0.$$

Thus the statement "$f_*(p)$ *is the zero map*" is equivalent to the statement "$df(p) = 0$". Note that the property of a point being a critical point of f, is independent of the used coordinate.

Let now $\lambda = \{\lambda^i\}$ be another local coordinate around p. Then we can express $X(f)(p)$ with respect to λ.

$$X(f)(p) = \sum_{i=1}^{m} X^i_{(\lambda)} \frac{\partial f^{(\lambda)}}{\partial \lambda^i}(\lambda(p)).$$

Now denote the transition map $\lambda \circ \kappa^{-1}$ by $\lambda^{(\kappa)}$. Then $f^{(\kappa)} = f \circ \kappa^{-1} = f \circ \lambda^{-1} \circ \lambda \circ \kappa^{-1} = f^{(\lambda)} \circ \lambda^{(\kappa)}$, and $\lambda(p) = \lambda^{(\kappa)}(\kappa(p))$. Then, for all i,

$$\frac{\partial f^{(\kappa)}}{\partial \kappa^i}(\kappa(p)) = \frac{\partial f^{(\lambda)} \circ \lambda^{(\kappa)}}{\partial \kappa^i}(\kappa(p)) = \sum_{j=1}^{m} \frac{\partial f^{(\lambda)}}{\partial \lambda^j}(\lambda(p)) \frac{\partial (\lambda^{(\kappa)})^j}{\partial \kappa^i}(\kappa(p)).$$

Substituting this into (5.1) gives:

$$\begin{aligned} X(f)(p) &= \sum_{i=1}^{m} X^i_{(\kappa)} \frac{\partial f^{(\kappa)}}{\partial \kappa^i}(\kappa(p)) = \sum_{i=1}^{m} X^i_{(\kappa)} \sum_{j=1}^{m} \frac{\partial (\lambda^{(\kappa)})^j}{\partial \kappa^i}(\kappa(p)) \frac{\partial f^{(\lambda)}}{\partial \lambda^j}(\lambda(p)) \\ &= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} X^i_{(\kappa)} \frac{\partial (\lambda^{(\kappa)})^j}{\partial \kappa^i}(\kappa(p)) \right) \frac{\partial f^{(\lambda)}}{\partial \lambda^j}(\lambda(p)) = \sum_{j=1}^{m} X^j_{(\lambda)} \frac{\partial f^{(\lambda)}}{\partial \lambda^j}(\lambda(p)). \end{aligned}$$

This means that, under a coordinate transformation from coordinate κ to coordinate λ, the vector components of X transform according to:

$$X^i_{(\kappa)} \to X^i_{(\lambda)} = \sum_{j=1}^{m} X^j_{(\kappa)} \frac{\partial (\lambda^{(\kappa)})^i}{\partial \kappa^j}(\kappa(p)).$$

Or, in compact notation, using Einstein summation convention for repeated dummy indices:

$$X^i_{(\kappa)} \to X^i_{(\lambda)} = \frac{\partial \lambda^i}{\partial \kappa^j} X^j_{(\kappa)}.$$

The Hesse matrix. When $p \in M$ is fixed and a local coordinate κ is chosen, we can define the *Hesse matrix* (related to κ) of f at p by:

$$H^{(\kappa)}_{ij}(p) := \frac{\partial^2 f^{(\kappa)}}{\partial \kappa^i \partial \kappa^j}(\kappa(p)).$$

Note that, since f is smooth, the operators $\partial/\partial\kappa^i$ mutually commute. Therefore $H^{(\kappa)}$ is a symmetric matrix, thus it can be diagonalized by conjugation with an orthogonal matrix. There exists an orthogonal matrix M, satisfying $M^T M = M M^T = I$ (the identity matrix), and a diagonal matrix D with real coefficients, so that $H^{(\kappa)} = M^T D M$.

Now let us derive how $H^{(\kappa)}$ transforms under coordinate transformations.

$$\begin{aligned}
H_{ij}^{(\lambda)}(p) &= \frac{\partial^2 f^{(\lambda)}}{\partial\lambda^i\partial\lambda^j}(\lambda(p)) = \frac{\partial}{\partial\lambda^i}\frac{\partial f^{(\lambda)}}{\partial\lambda^j}(\lambda(p)) \\
&= \frac{\partial}{\partial\lambda^i}\left(\frac{\partial f^{(\kappa)}\circ\kappa^{(\lambda)}}{\partial\lambda^j}(\lambda(p))\right) = \frac{\partial}{\partial\lambda^i}\left(\sum_{l=1}^m \frac{\partial f^{(\kappa)}}{\partial\kappa^l}(\kappa(p))\frac{\partial(\kappa^{(\lambda)})^l}{\partial\lambda^j}(\lambda(p))\right) \\
&= \sum_{l=1}^m \frac{\partial}{\partial\lambda^i}\frac{\partial f^{(\kappa)}}{\partial\kappa^l}(\kappa(p))\frac{\partial(\kappa^{(\lambda)})^l}{\partial\lambda^j}(\lambda(p)) + \sum_{l=1}^m \frac{\partial f^{(\kappa)}}{\partial\kappa^l}(\kappa(p))\frac{\partial}{\partial\lambda^i}\frac{\partial(\kappa^{(\lambda)})^l}{\partial\lambda^j}(\lambda(p)) \\
&= \sum_{l=1}^m\sum_{k=1}^m \frac{\partial^2 f^{(\kappa)}}{\partial\kappa^k\partial\kappa^l}(\kappa(p))\frac{\partial(\kappa^{(\lambda)})^k}{\partial\lambda^i}(\lambda(p))\frac{\partial(\kappa^{(\lambda)})^l}{\partial\lambda^j}(\lambda(p)) + \sum_{l=1}^m \frac{\partial f^{(\kappa)}}{\partial\kappa^l}(\kappa(p))\frac{\partial^2(\kappa^{(\lambda)})^l}{\partial\lambda^i\partial\lambda^j}(\lambda(p)).
\end{aligned}$$

Or, dropping arguments and summation symbols, $H_{ij}^{(\kappa)}$ transforms according to

$$H_{ij}^{(\kappa)} \to H_{ij}^{(\lambda)} = \frac{\partial^2 f^{(\lambda)}}{\partial\lambda^i\partial\lambda^j} = \frac{\partial\kappa^k}{\partial\lambda^i}\frac{\partial\kappa^l}{\partial\lambda^j}\frac{\partial^2 f^{(\kappa)}}{\partial\kappa^k\partial\kappa^l} + \frac{\partial^2\kappa^l}{\partial\lambda^i\partial\lambda^j}\frac{\partial f^{(\kappa)}}{\partial\kappa^l} = \frac{\partial\kappa^k}{\partial\lambda^i}\frac{\partial\kappa^l}{\partial\lambda^j}H_{kl}^{(\kappa)} + \frac{\partial^2\kappa^l}{\partial\lambda^i\partial\lambda^j}\frac{\partial f^{(\kappa)}}{\partial\kappa^l}. \tag{5.2}$$

We see that, under a coordinate transformation, $H^{(\kappa)}(p)$ does not transform as a tangent space tensor. In general it is not a tensor field. However, at a critical point $p \in M_c(f)$, we see that $df(p) = (\partial f^{(\kappa)}/\partial\kappa^i)d\kappa^i = 0$. Thus $\partial f^{(\kappa)}/\partial\kappa^i = 0$ for all i. This means that in such a point the second term in (5.2) vanishes. Thus, only in critical points, the components of H perfectly transform like those of a symmetric tangent space tensor of type $(0,2)$:

$$H_{ij}^{(\kappa)} \to H_{ij}^{(\lambda)} = \frac{\partial\kappa^k}{\partial\lambda^i}\frac{\partial\kappa^l}{\partial\lambda^j}H_{kl}^{(\kappa)}.$$

This means that, in critical points $p \in M_c(f)$, $H(p)$ can be used to define a coordinate-independent object. In this case, when we define $M_{ij} := \frac{\partial\kappa^i}{\partial\lambda^j}$, then

$$H_{ij}^{(\lambda)} = M_{ki}H_{kl}^{(\kappa)}M_{lj} = (M^T)_{ik}H_{kl}^{(\kappa)}M_{lj} \quad\Longrightarrow\quad H^{(\lambda)} = M^T H^{(\kappa)} M.$$

For an arbitrary coordinate λ, we can always find a valid coordinate κ such that M, induced by this coordinate transformation, is orthogonal, and $\det(M) = 1$. This coordinate transformation can be regarded as a lift of the change of basis of T_pM, from λ to a non-coordinate basis. For the corresponding coordinate transformation we can simply choose a rotation around the point $\lambda(p)$. We may allow any rotation, thus we can choose κ such that $H^{(\kappa)}$ is diagonal, containing the eigenvalues of the original matrix $H^{(\lambda)}$.

Nondegenerate critical points. A critical point $p \in M_c(f)$ is called *nondegenerate* if and only if the Hesse matrix $H(p)$ with respect to an arbitrary local coordinate κ, written as

$$H_{ij}^{(\kappa)}(p) := \frac{\partial^2 f^{(\kappa)}}{\partial\kappa^i\partial\kappa^j}(\kappa(p)),$$

is *nonsingular*, which means that $H^{(\kappa)}$ has no zero eigenvalues, thus $\det(H^{(\kappa)}) \neq 0$.

Note that the property of a critical point p being nondegenerate is again independent of the used coordinate. The matrix M corresponding to the coordinate transformation is always nonsingular, thus, if we have $\det(H^{(\kappa)}) \neq 0$ at p, then also

$$\det(H^{(\lambda)}) = \det(M^T H^{(\kappa)} M) = \det(M^T)\det(H^{(\kappa)})\det(M) \neq 0.$$

This means that the property of H being nondegenerate, does not depend on the coordinate chosen.

Morse functions. Let M be a compact smooth manifold of dimension m, and let $f : M \to I \subset \mathbb{R}$ be smooth and surjective, then f is called a *Morse function* if all critical points of f are nondegenerate. According to [3], Morse functions always exist. From now on, let f be a Morse function.

Let $p \in M_c(f)$ and let λ be a local coordinate with respect to which H is diagonal. Then the *index* of f at p, denoted by $n_p(f)$, is the number of negative values on the diagonal of $H^{(\lambda)}$. This is equivalent to defining the index as the number of negative eigenvalues of H (counted with multiplicity) with respect to an arbitrary local coordinate κ. The index $n_p(f)$ has another equivalent definition. It equals the maximum of the set of dimensions of possible linear subspaces of T_pM on which H, regarded as a $(0, 2)$ tangent space tensor, is negative definite. Thus the index is also a coordinate-independent quantity, which is also confirmed by Sylvester's law of intertia. A more detailed explanation of this can be found in [1] and [3].

The index $n_p(f)$ can be any element of $\{0, \cdots, m\}$. If $n_p(f) = 0$, then p is called a (local) minimum of f. If $n_p(f) = m$, then p is called a (local) maximum of f. Otherwise p is called a saddle point of f. We will see that f maps all of its saddle points to values lying in the internal part of I. Suppose M is connected. Then I is also connected, thus it is a closed interval. In this case, if a critical point of f is mapped to a boundary point of I, then we are dealing with a global minimum or maximum. The remaining critical points are all local minima or maxima, or saddlepoints. When M is not connected, the previous still holds for the connected components of M, be it separately.

For a Morse function f, the Hesse matrix behaves like a type $(0, 2)$ tangent space tensor at every $p \in M_c(f)$. It is symmetric and nonsingular, thus, locally, it behaves like a metric with signature. In the context of calculus on flat spaces like $\mathbb{R}^m$, the Hesse matrix tells us in which direction a function changes with the largest speed, and these directions, regarded as unit vectors, should be mutually orthonormal. However, we should be careful when trying to adapt this idea to abstract manifolds, as the components of the Hesse matrix always depend on the coordinate chosen. Besides, we would need to define a metric before we can say whether a tangent vector is a unit vector.

Euler characteristic. When the critical points of any function are nondegenerate, we can easily see that these points are isolated, thus we are dealing with countable sets of critical points. So we will not get into trouble computing sums over all critical points. The number of critical points of any Morse function on a compact manifold is finite. This justifies the following alternative definition of the Euler characteristic.

The Euler characteristic of a smooth compact manifold M, denoted by $\chi(M)$, is closely related to the index of all critical points of any Morse function f on M. It can be defined as

$$\chi(M) = \sum_{p \in M_c(f)} (-1)^{n_p(f)}.$$

What is really remarkable is that it does not matter how the Morse function is exactly defined, as long as it is a Morse function.

Special Morse functions. Before introducing the concept of special Morse functions, we will discuss separate critical points. Let p and q be two distinct critical points of a surjective Morse function $f : M \to I \subset \mathbb{R}$, but with the same critical value, say v, thus $f(p) = f(q) = v$. Then we will call p and q mutually separate if there exists no path in the level set $f^{-1}(v) \subset M$ connecting p and q. We will also call p and q mutually separate if $f(p) \neq f(q)$.

We will call f a *special Morse function* if all of its critical points are mutually separate. This is equivalent to saying that every connected component of any level set $f^{-1}(v)$ contains at most one critical point of f. In the application of Morse functions for the understanding of cobordisms, it is desirable to choose a special Morse function. It will especially help us splitting up a cobordism correctly into disjoint products of generators. We can split up I into closed subsets I_k, each of which containing at most one critical value, lying in its interior (or in ∂I), so that $I_k \cap I_l$ does not contain any critical value, for any $k \neq l$. As a consequence, any critical point of f will lie in the internal part of some $f^{-1}(I_j) \subset M$. We claim without proof that such a special Morse function always exists. In the following chapters we will only consider these special Morse functions.

6 Cobordisms

We assume the reader is already familiar with the concept of smooth structure. A cobordism in dimension n can be interpreted as a smooth interpolation between two compact manifolds of dimension $n-1$, without boundary. These $(n-1)$-manifolds are then precisely the boundary of the cobordism. This interpolation needs not be unique. For an arbitrary pair of $(n-1)$-manifolds a cobordism needs not exist between them, however, if it does, we will call these $(n-1)$-manifolds *cobordant*. We will discuss how to split up a given cobordism into smaller cobordisms, making use of Morse functions. We will discuss some trivial but at the same time important examples of cobordisms, for example the *cylinder*. Though we will introduce the concept of *abstract twist cobordism* in the next chapter, we will introduce the concept of *natural twist cobordism* here, which is not really a universal notion to be found in the literature though. In an early stage we will define oriented cobordisms, which permits a sense of direction. The boundary of an oriented cobordism can be split up into the *in-boundary* and the *out-boundary*. We can say this cobordism is a cobordism from its in-boundary to its out-boundary. This is one of the first steps towards the idea of a category of oriented cobordism classes, discussed in great detail in the next chapter.

6.1 Manifolds with boundary

Half-spaces. For any $n \geq 1$ and for any nonzero linear map $L : \mathbb{R}^n \to \mathbb{R}$ we can define a *half-space* H^n_L by

$$H^n_L := \{x \in \mathbb{R}^n | L(x) \geq 0\}.$$

We should note that any half-space is contractible, as its topology is induced from that of $\mathbb{R}^n$. The boundary of H^n_L is defined by

$$\partial H^n_L := \{x \in \mathbb{R}^n | L(x) = 0\}.$$

We should note that ∂H^n_L is homeomorphic to $\mathbb{R}^{n-1}$. When H^n_L is regarded as a subset of $\mathbb{R}^n$, we can study its topology induced from $\mathbb{R}^n$. Following terminology of [2] we will call this induced topology the *subspace topology*. According to this topology, we will call a subset A of H^n_L open if there exists an open $B \subset \mathbb{R}^n$ such that $A = B \cap H^n_L$. This definition of open subsets of H^n_L allows us to give a straightforward definition of a manifold with boundary.

Manifolds with boundary. A topological space M is said to be a *manifold with boundary* of dimension n if every point $p \in M$ has a neighbourhood U homeomorphic to an open subset of some H^n_L. The L should depend on the coordinate patch considered. Here we should note that we could have a patch $U \subset M$ mapped to the internal part of some H^n_L. Here *internal* means internal with respect to the ordinary topology of $\mathbb{R}^n$. In fact, if we have found all such U, then their plain union will be an ordinary manifold, which we will call $\tilde{M}$. The boundary of M is then defined by $\partial M := M - \tilde{M}$. We note that ∂M itself is locally homeomorphic to ∂H^n_L, and as such it should define a manifold of dimension $n-1$. We note that the boundary itself can be defined as an ordinary manifold, thus we say that it has no boundary itself. In general the identity $\partial\partial M = \varnothing$ holds. In later applications of manifolds with boundary, we will require all coordinate transformations to be smooth, and when a manifold with boundary is mentioned, we really mean a *smooth* manifold with boundary.

We should note that the tangent space T_pM, for any $p \in \partial M$, is still a full vector space. Recall that a tangent vector is just an equivalence class of smooth curves through p going in or coming from the same direction. Knowing one half of a curve is sufficient to know its direction.

Some examples.

- For $n = 1$ we have a trivial example. Let M be a closed interval in $\mathbb{R}$, say $[a, b]$ $(a < b)$. Then the internal part (a, b) is an open interval, which is an ordinary manifold of dimension 1. For any $c \in (a, b)$ we see that $[a, c)$ and $(c, b]$ are homeomorphic to open subsets in a half-space of dimension 1. Thus the set $\{a, b\}$ is the boundary of M, which is an ordinary manifold of dimension 0 and has two connected components.

- For $n = 2$ we have another trivial example. Let M be the Cartesian product of the previous example and the circle. Then we have a closed cylinder. The internal part is $(a, b) \times S^1$, which is an ordinary manifold of dimension 2. The set $\{a, b\} \times S^1 = (\{a\} \times S^1) \cup (\{b\} \times S^1)$ is the boundary of M, which is an ordinary manifold of dimension 1 and has two connected components, namely two circles.

Closed manifolds. A closed manifold M is, by definition, a compact manifold without boundary. The latter only means that it suffices to describe M as an ordinary manifold. We should note that, in case we are only discussing ordinary manifolds, the latter property is automatically satisfied. Only in the context of manifolds with boundary, this definition makes sense. Of course every ordinary manifold M can be described as a manifold with boundary, but in this case $\partial M = \varnothing$. From now on, when mentioning a *manifold*, we will mean an ordinary manifold.

Morse functions on manifolds with boundary: A convention. Let M be a (smooth) manifold with boundary. Recall the notion of a *special* Morse function from the previous chapter. It is a Morse function with all critical points being mutually separate. From now on, when discussing any Morse function on M, we will assume that it is a special Morse function, and that it is constant when restricted to ∂M. Thus we will also assume that ∂M contains no critical points of this Morse function. We will mainly be interested in compact manifolds M with boundary, which means that M itself is a compact space, and that its boundary is a closed manifold. Recall that if f is a Morse function on M, which means that it is a smooth function, then $I := f(M)$ is a compact subset of $\mathbb{R}$.

Later we will discuss the behaviour of the extrema of Morse functions. We will also discuss the application of special Morse functions, and hopefully afterwards it will get more clear why we are interested in special Morse functions in the first place. For now we just start with this formal definition without further explanations.

6.2 Orientation

Orientation of vector spaces. Let V be a real vector space of dimension n, and let $\{e_1, \cdots, e_n\}$ be a basis of V. We should remark that we can consider this basis to be ordered. A matrix can be used for reordering the basisvectors to obtain another basis. In general, an invertible matrix M_{ij} can be used to define a new basis for V. If we define

$$f_i := \sum_{j=1}^{n} M_{ij} e_j$$

then we have found another basis for V. The matrix can thus be regarded as a basis transformation. The determinant of an invertible matrix is always nonzero, and as such it has a signature. We will call M *orientation preserving* when its determinant is positive. Otherwise we will call it *orientation reversing*. We directly see that if M is an even permutation of basis vectors, it is orientation preserving, and if M is an odd permutation, it is orientation reversing.

An *orientation* of V assigns a sign to each ordered basis. This should be compatible with the idea that a basis transformation can preserve or reverse the orientation. In other words, all bases which can be transformed into each other by an orientation preserving transformation will have the same sign.

In the special case where V has dimension 0, there is only one (trivial) basis, namely the empty set. The only basis transformation is the trivial one, which is orientation preserving by definition. Thus we are free to assign a plus or a minus sign to this unique basis, which can be regarded as being equivalent to assigning a sign to V itself. Thus, in this case, the property of a map preserving or reversing the orientation, only depends on properties of the sets involved.

When V is equipped with an orientation, we will call V *oriented*.

Orientation of the Cartesian product of two oriented vector spaces. Let V and W be oriented vector spaces of dimension m and n respectively, then there is a canonical way to assign orientation to their

Cartesian product $V \times W$. Let $\{e_1, \cdots, e_m\}$ and $\{f_1, \cdots, f_n\}$ be positively oriented bases of V and W respectively. Then we would like the basis $\{e_1, \cdots, e_m, f_1, \cdots, f_n\}$ of $V \times W$ to be a positively oriented one. From now on we automatically assume that, when we are dealing with a pair of oriented vector spaces, the canonical orientation will be assigned to their Cartesian product.

Orientation of manifolds. Let M be a smooth manifold, let (U, κ) be some coordinate patch in its atlas and let U itself be connected. Then κ induces an ordered basis of T_pM for any $p \in U$. Then, if for some $p \in U$, an orientation is chosen for the tangent space T_pM, we can extend this to an orientation of T_qM for any other $q \in U$, just by smoothly transporting a local frame, associated to the tangent bundle, from p to q. Then we will say the patch U is oriented. Let (V, λ) be another connected coordinate patch, such that $U \cap V$ is nonempty and connected, then the orientation on U can be extended to V. Then, if M is connected, it is called *orientable* if it is possible to repeat this procedure for any patch in its atlas, so that orientation will agree on any intersection of two patches. Note however that this M will remain orientable if we discard the choice of orientation afterwards. Thus orientability of M does neither depend on the actual atlas chosen, nor on the orientation chosen locally on its patches. If M is orientable, we can define a global orientation on M by making a consistent choice of local orientations, agreeing on all intersections of patches. Then we say M is *oriented*. In this case M admits two global orientations. To summarize, every oriented manifold is orientable. However, if a manifold is orientable it does not need to be oriented yet. Thus the collection of orientable manifolds contains the collection of oriented manifolds, but when a manifold is unoriented it is not necessarily unorientable.

Now suppose M is not connected, but instead has a finite number N of connected components. Then for each component M_j we can figure out whether it is orientable, and if so we could fix an orientation. If all its connected components are orientable, we will say M is orientable, and we can make a choice of orientation for each component M_j. This means there are 2^N possible orientations for M.

From now on we will only be dealing with orientable manifolds with a finite number N of connected components, thus with 2^N possible orientations.

In the special case where the dimension of M is zero, we will just assign a sign to any connected component M_j itself. These M_j are just points, and we are still dealing with 2^N possible orientations.

We should make clear that assigning an orientation to any orientable manifold M is a matter of adding information to the bare manifold. If we want to be precise we would need to write (M, σ), where σ tells us how each connected component of M is oriented. If an orientation σ is assigned to M, we can assign the opposite orientation $\overline{\sigma}$ as well. Thus $(M, \overline{\sigma})$ is the same as (M, σ) with *opposite* orientation. We will drop this σ from our notation from now on, and write $\overline{M}$ instead of $(M, \overline{\sigma})$. We always assume M and $\overline{M}$ carry the same smooth structure.

The empty manifold $\varnothing$ has exactly one orientation. We could say there is only one map from $\varnothing$ to $\{-, +\}$. This also means that we can write $\overline{\varnothing} = \varnothing$.

Orientation of the Cartesian product of two oriented manifolds. Let M and N be oriented manifolds, then there is a canonical way to assign orientation to their Cartesian product $M \times N$. For all $p \in M$ and $q \in N$, the tangent spaces T_pM and T_qN are oriented. Using the canonical orientation of the Cartesian product of two vector spaces, we will assign an orientation to $T_{(p,q)}(M \times N) \simeq T_pM \times T_qN$. This induces a canonical orientation on $M \times N$. From now on we automatically assume that, when we are dealing with a pair of oriented manifolds, the canonical orientation will be assigned to their Cartesian product.

Orientation of manifolds with boundary. Let M be an m-dimensional manifold with boundary and let $\tilde{M}$ be its internal part, which is an ordinary manifold. If $\tilde{M}$ is oriented, it assigns an orientation to every $T_p\tilde{M}$. Now note that this can be extended to any $p \in M$, especially for $p \in \partial M$. Thus orientation of $\tilde{M}$ induces an orientation on M. For any $p \in \partial M$, T_pM has an induced orientation. However, ∂M itself is a manifold of dimension $m - 1$. That is why it is not possible to induce orientation on $T_p\partial M$ without introducing extra information. We need to define an orientation on each connected component of ∂M. We should note that if $\tilde{M}$ is orientable, then M is orientable. Then also ∂M is orientable.

In-boundary and out-boundary. In many cases we will see that ∂M has multiple, say N, connected components, denoted by ∂M_j. Each of them can have its own orientation. Suppose M and ∂M are oriented. Now take j fixed. Let $p \in \partial M_j$ and let $\{e_1, \cdots, e_{m-1}\}$ be a basis of $T_p\partial M_j$, with positive orientation. As $T_p\partial M_j$ is a linear subspace of T_pM, we can try to extend this basis to a basis of T_pM. Let $\{e_1, \cdots, e_{m-1}, f\}$ be a basis of T_pM with positive orientation, with respect to the orientation of M. Then there exists a curve through p which has f as its tangent vector. If this curve points inwards, into M, we will call ∂M_j an *in-boundary*, and if it points outwards, out of M, we will call ∂M_j an *out-boundary*. This way we can relate orientation of M and orientation of the connected components of its boundary. We will call f a positive normal. According to [7] we should note that the normal bundle $TM|_{\partial M_j}/T\partial M_j$ is a trivial vector bundle on ∂M_j, thus for any other point $q \in \partial M_j$ we come to the same conclusion whether ∂M_j is an in-boundary or an out-boundary.

We should note that this relation works in both directions. If we know orientations of M and of each ∂M_j, then we also know which ∂M_j are in-boundaries and which are out-boundaries. On the other hand, if we know an orientation of $\tilde{M}$, thus of M, and if orientation of ∂M_j is not specified yet, but if we already know which ones we want to be in-boundaries or out-boundaries, we can define the orientations of all ∂M_j to agree our desires.

Soon we will encounter manifolds with boundary which for example only have an in-boundary. Its out-boundary is just an empty manifold. We will also encounter manifolds with only an out-boundary.

Orientation preserving diffeomorphisms and orientation reversing diffeomorphisms. Let M and N be connected oriented manifolds of dimension m, and suppose a diffeomorphism $\phi : M \to N$ exists. Then ϕ can either preserve the orientation, or reverse the orientation. For any $p \in M$ we can choose an ordered basis of T_pM, say $\{e_1, \cdots, e_m\}$, with positive orientation. Now note again that these vectors e_j correspond to equivalence classes of curves, which are maps from $\mathbb{R}$ to M. It is possible to define a trivial pushforward such that these curves become maps from $\mathbb{R}$ to N, just by composing with ϕ. This means that there is a canonical way to map the vectors e_j to vectors $f_j \in T_{\phi(p)}N$. The orientation of N tells us whether we are dealing with an ordered basis of $T_{\phi(p)}N$ with positive or negative orientation. If positive, we will call ϕ an *orientation preserving diffeomorphism* or *OPD*, and if negative, we will call ϕ an *orientation reversing diffeomorphism* or *ORD*. We should note that we only need to test whether ϕ is an OPD or an ORD in one point p. Global orientability of M and N can be used to transport our checks from p to any other arbitrary $q \in M$. The result will be the same.

Again, if M and N have multiple connected components, and $\phi : M \to N$ is a diffeomorphism, then it can act independently as an OPD or an ORD on each component. In general, a diffeomorphism $\phi : M \to N$ is an OPD (ORD) if and only if it is an OPD (ORD) on each of the connected components of M. On the other hand, when M is an oriented manifold of dimension $m \geq 2$, but N is not oriented yet, we can canonically define an orientation on N by using the natural pushforward of ordered bases of tangent spaces of M. Then ϕ is automatically an OPD.

A special case will be the (artificial) map

$$I_\varnothing : \varnothing \to \varnothing. \tag{6.1}$$

As $\varnothing$ has only one orientation we could say that $I_\varnothing$ is automatically an OPD. This map is idempotent and can always be inserted as an identity. There are no other OPDs from $\varnothing$ to itself.

Trivial examples of OPDs and ORDs.

- If we have $\phi : M \to M : p \mapsto p$, then ϕ is an OPD.
- If we have $\phi : M \to \overline{M} : p \mapsto p$, then ϕ is an ORD.

These simple examples should make clear that being an OPD or an ORD is not totally an intrinsic property of a diffeomorphism. A diffeomorphism being an OPD or an ORD often also depends on the manifolds involved, and how they are oriented.

This especially applies if we are looking at points, also known as connected oriented 0-manifolds. Let p be an arbitrary single point and let p_+ and p_- be oriented versions of p. Of course the orientation of p_+ (p_-) is positive (negative). Then the maps $\phi_{++} : p_+ \to p_+$ and $\phi_{--} : p_- \to p_-$ are orientation preserving, and the maps $\phi_{+-} : p_+ \to p_-$ and $\phi_{-+} : p_- \to p_+$ are orientation reversing. As any map between points can automatically be regarded as a diffeomorphism, we can say ϕ_{++} and ϕ_{--} are OPDs, and ϕ_{+-} and ϕ_{-+} are ORDs. Thus in this case it does not depend on the intrinsic properties of the map at all whether we are dealing with an OPD or an ORD.

Orientation preserving diffeomorphisms between manifolds with boundary. Let M and N be oriented manifolds with boundary and let $\phi : M \to N$ be a diffeomorphism. Then ϕ is an OPD between manifolds with boundary if it is an OPD when restricted to ∂M and to $\tilde{M}$, thus if $\phi|_{\partial M} : \partial M \to \partial N$ and $\phi|_{\tilde{M}} : \tilde{M} \to \tilde{N}$ are OPDs in terms of ordinary manifolds.

6.3 Oriented cobordisms

Let $m \geq 1$, and let M be a compact oriented manifold with boundary and with dimension m. Then ∂M is a closed oriented manifold of dimension $m-1$. Let $\partial_+ M$ be the in-boundary of M, and let $\partial_- M$ be the out-boundary of M. Then $\partial_+ M \cap \partial_- M = \varnothing$ and $\partial_+ M \cup \partial_- M = \partial M$.

Now let X and Y be closed oriented manifolds of dimension $m-1$. A compact oriented manifold M with boundary is called an *oriented cobordism* from X to Y, if there exist orientation preserving diffeomorphisms $\iota_+ : X \to \partial_+ M$ and $\iota_- : Y \to \partial_- M$. Thus a cobordism M is specified by the following information: $(M, X, Y, \iota_+, \iota_-)$. Note that we could as well say that M is just an oriented cobordism from $\partial_+ M$ to $\partial_- M$. However, the more abstract definition will allow us to study cobordisms from X to itself, and to compose cobordisms. This is one of the first steps making it suitable to study categorical aspects of cobordism theory, which we will do in the next chapter.

We should note that alternative definitions exist for oriented cobordisms. It is only a matter of convention to say that there exist OPDs for both the in-boundary and the out-boundary. We could have made a different choice of convention.

From now on we automatically assume any discussed cobordism to be smooth, compact and oriented, and its boundary to be smooth, closed and oriented, so we will drop the adjectives. We assume any boundary is denoted by a capital symbol from the end of the alphabet, for example X, Y, Z or W. Any cobordism itself is denoted by a capital symbol from the middle of the alphabet, for example M, N or P.

In general there will not always exist a cobordism from any X to any Y, but when a cobordism from X to Y exists, X and Y are said to be *cobordant*. It is also possible to study cobordisms from X to Y in case X or Y (or both) is the empty set.

Reversing the orientation of a cobordism. If M is a cobordism from X to Y, equipped with OPDs $\iota_+ : X \to \partial_+ M$ and $\iota_- : Y \to \partial_- M$, then we can reverse the orientation of M to obtain a cobordism from Y to X. Reversing the orientation of M really means reversing the orientation of its internal part $\tilde{M}$, thus the orientation of the boundary of M will be fixed. Despite this subtlety, we will write $\overline{M}$ for this oriented manifold with boundary. As a consequence, the roles of the in- and out-boundaries will be exchanged, so $\partial_+ \overline{M} = \partial_- M$ and $\partial_- \overline{M} = \partial_+ M$. The result will be that $\overline{M}$ will be a cobordism from Y to X.

Examples of oriented cobordisms. The following examples of oriented cobordisms should really be regarded as key examples.

- Let $\{0\} \in \mathbb{R}$ and $\{1\} \in \mathbb{R}$ be manifolds of dimension zero, and with positive orientation. Let the open interval $(0,1) \subset \mathbb{R}$ be a manifold of dimension one, and assign positive orientation to any tangent vector pointing from 0 to 1. Then the closed interval $[0,1] \subset \mathbb{R}$, constructed with these $(0,1)$, $\{0\}$ and $\{1\}$, is an oriented manifold with boundary. Its in-boundary is $\{0\}$ and its out-boundary is $\{1\}$, thus it is a cobordism from 0 to 1. We will call its orientation the *standard orientation*.

- In general we could start with the closed interval $I_{\alpha\beta} := [0,1]$, where $\alpha, \beta \in \{+,-\}$. Here $\{0\}$ will carry orientation α, and $\{1\}$ will carry orientation β, and we will use the symbols 0_α and 1_β to indicate these oriented points. The interior $(0,1)$ will still carry the same orientation as defined in the previous example. Thus I_{++} is the interval $[0,1]$ with standard orientation. Then 0_+ is its in-boundary, and 1_+ its out-boundary, or $\partial_+ I_{++} = 0_+$ and $\partial_- I_{++} = 1_+$. We should be more careful with I_{+-}, I_{-+} and I_{--}. The oriented point 0_- cannot be an in-boundary of $(0,1)$, and 1_- cannot be an out-boundary. On the contrary, 0_- is an out-boundary and 1_- is an in-boundary. Thus we have

$$\begin{array}{ll} \partial_+ I_{++} = 0_+ & \partial_- I_{++} = 1_+ \\ \partial_+ I_{+-} = 0_+ \cup 1_- & \partial_- I_{+-} = \varnothing \\ \partial_+ I_{-+} = \varnothing & \partial_- I_{-+} = 0_- \cup 1_+ \\ \partial_+ I_{--} = 1_- & \partial_- I_{--} = 0_- \end{array}$$

We can picture these cobordisms as follows:

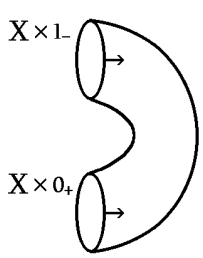
I_{++} $\qquad$ I_{+-} $\qquad$ I_{-+} $\qquad$ I_{--}

In these figures the in-boundaries are on the left and the out-boundaries are on the right. The arrows only indicate the choice of orientation for $(0,1)$.

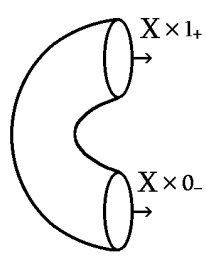
- Let I be the closed interval $[0,1]$ with standard orientation, thus $I := I_{++}$. For any closed oriented X we can construct a trivial cobordism $C_X := X \times I$. Here 'trivial' means that it is induced by the identity map from X to itself. Then, for $\epsilon \in \{+,-\}$, we have $\partial_\epsilon C_X = \partial_\epsilon (X \times I) = X \times \partial_\epsilon I$. Thus the in-boundary is $\partial_+ C_X = X \times \{0\}$, and the out-boundary is $\partial_- C_X = X \times \{1\}$. The maps $X \to \partial_+ C_X$ and $X \to \partial_- C_X$ are trivial OPDs. The cobordism C_X (from X to itself) is called the *cylinder* generated by X.

- Of course we could as well look at the cobordisms $X \times I_{+-}$, $X \times I_{-+}$ and $X \times I_{--}$. In most of the literature these cobordisms are also called cylinders. We should again be careful when indicating the in- and out-boundaries. Again $\partial_\epsilon (X \times I_{\alpha\beta}) = X \times \partial_\epsilon I_{\alpha\beta}$. For example $X \times I_{+-}$ has an empty out-boundary, and its in-boundary is $X \times \partial_+ I_{+-} = X \times (0_+ \cup 1_-)$. Similarly $X \times I_{-+}$ has an empty in-boundary, and its out-boundary is $X \times \partial_- I_{-+} = X \times (0_- \cup 1_+)$.

  We should note that the trivial map from X to $X \times \{p_-\}$ is orientation reversing, for any point p. This can be fixed by replacing X by $\overline{X}$. For example $X \times I_{--}$ can be regarded as a trivial cobordism from $\overline{X}$ to itself. Again we can picture these subtleties:



(6.2)



(6.3)

$X \times I_{+-}$ $\qquad$ $X \times I_{-+}$

The maps from X to $X \times 0_+$ and to $X \times 1_+$, and from $\overline{X}$ to $X \times 0_-$ and to $X \times 1_-$ should all be OPDs.

Cobordisms between diffeomorphic manifolds. For any pair X and Y related by an *orientation preserving* diffeomorphism $\phi : X \to Y$, we can construct a cobordism M from X to Y. We can always start with the cylinder generated by Y. Let $\iota_+ : Y \to \partial_+ C_Y$ and $\iota_- : Y \to \partial_- C_Y$ be the desired OPDs. Then we can precompose ι_+ with ϕ to obtain a cobordism from X to Y. The maps $\iota_+ \circ \phi : X \to \partial_+ C_Y$ and $\iota_- : Y \to \partial_- C_Y$ are the OPDs we can use to describe C_Y as a cobordism from X to Y. Of course there is another way to generate a cobordism between X and Y, based on ϕ. We could as well start with the cylinder generated by X and then precompose ι_- with ϕ^{-1} instead. As will turn out later, this cobordism will be equivalent to M.

Twist cobordisms. Again, let X and Y be related by an OPD, say $\phi : X \to Y$, which has an inverse $\psi : Y \to X$, and assume $X \cap Y = \varnothing$. Now define $\Phi : X \cup Y \to X \cup Y$ as the OPD mapping X to Y, according to ϕ, and Y to X, according to ψ. Thus $\Phi|_X = \phi$ and $\Phi|_Y = \psi$, or $\Phi(X) = Y$ and $\Phi(Y) = X$. Then Φ will generate a non-trivial cobordism from $X \cup Y$ to itself. We can define this cobordism as

$$T_{X,Y} := (X \cup Y) \times I = (X \times I) \cup (Y \times I) = C_X \cup C_Y.$$

Then $\partial_+ T_{X,Y} = \partial_+ C_X \cup \partial_+ C_Y$ and $\partial_- T_{X,Y} = \partial_- C_X \cup \partial_- C_Y$. Now ϕ induces an OPD from X to $\partial_+ C_Y$ and ψ induces an OPD from Y to $\partial_+ C_X$. The OPD from $X \cup Y$ to $\partial_- T_{X,Y}$ is the canonical one.

We will call this $T_{X,Y}$ the *natural twist cobordism*. Later on, when discussing categories of cobordisms, we will also introduce a more abstract twist cobordism which in general is conceptually rather different.

The existence of OPD-generated cobordisms is also interesting when studying OPDs from X to itself. For example in dimension 1, if X has N connected components, which are all diffeomorphic to a circle, then there will always exist diffeomorphisms $\phi : X \to X$ such that for any $p \in X$, lying in some connected component of X, the point $\phi(p)$ lies in some other connected component of X. Thus OPDs from X to itself are not all trivial. Smoothness of ϕ implies that for any connected component X_j we have $\phi(X_j) = X_{\phi_*(j)}$, where ϕ_* is in fact a permutation.

Morse functions, defined on oriented manifolds with boundary. Let M be an oriented manifold with boundary, and let $f : M \to \mathbb{R}$ be a (special) Morse function. From now on we assume that f will reach its global extrema on ∂M: if $\partial_+ M$ is not empty, then we assume that the global minimum of f will occur on $\partial_+ M$, and if $\partial_- M$ is not empty, we assume that the global maximum of f will occur on $\partial_- M$. Note that in these cases the global extrema are not supposed to be related to any critical point of f, but to the fact that f has extrema on the boundary of its domain. This will be different when $\partial_+ M$ or $\partial_- M$ is the empty set, but in every case we can safely assume that none of the critical points of f lie in ∂M.

Splitting cobordisms into smaller ones by using a Morse function. Let M be a cobordism from X to Y, and let $f : M \to I \subset \mathbb{R}$ be a surjective (special) Morse function. Let $\bar{I} := [p, q]$ be the smallest interval containing I, thus $p, q \in I$ and $p \leq r \leq q$ for all $r \in I$. A value a in the interior of I is called a *regular value* if the level set $Z := f^{-1}(a) \subset M$ contains no critical points of f. This implies that Z itself can again be described as a manifold. Now we can split up I into two closed subsets I_1 and I_2 such that $I_1 \cap I_2 = \{a\}$, and such that for all $b \in I_1$ and $c \in I_2$ we have $b \leq a \leq c$. It is clear that $I_1 = I \cap [p, a]$, $I_2 = I \cap [a, q]$ and $I = I_1 \cup I_2$. We will write $I_1 \leq I_2$.

We should note that for any a in the interior of $\bar{I}$, but not lying in I itself, thus $a \in \bar{I} - I$, we can consider intervals I_1 and I_2 as well. As now a is not an element of I, we have $I_1 \cap I_2 = \varnothing$, thus also $Z = \varnothing$.

Defining $M_j := f^{-1}(I_j)$ and assuming that the smooth structure of M is induced on these M_j, we can define new cobordisms. These cobordisms should be regarded as new manifolds with boundary, and as such we need to specify these boundaries. We are free to choose an orientation convention on the new boundary components, and we will do so by doing the following. We will treat Z as the out-boundary of M_1 and at the same time as the in-boundary of M_2. This is indeed a valid choice, as there is no continuous path from a point in the interior of M_1 to a point in the interior of M_2, which will not cross Z. Thus we have

$\partial_+ M_1 = \partial_+ M$, $\partial_- M_1 = Z = \partial_+ M_2$ and $\partial_- M_2 = \partial_- M$. We see that this gives us the following cobordisms: M_1 from X to Z, and M_2 from Z to Y. In case of $a \in \bar{I} - I$ we will find that $Z = \varnothing$. Instead of Z we can also use any other Z' diffeomorphic to Z (by OPD). Then we will say that M, from X to Y, is split up into M_1, from X to Z', followed by M_2, from Z' to Y. Then we will write

$$M = M_2 M_1,$$

and we will call this a *decomposition* of M.

As we know, f is a special Morse function on M, and after splitting up M we see that f restricted to M_1 or M_2 still remains a special Morse function. Thus further splitting up M_1 and M_2 can be done by using the same f. We can repeat this procedure as many times as we desire. We can choose a set of N points a_j lying in $\bar{I} - \partial I$, which are regular points or lying in $\bar{I} - I$, such that $j < k \Leftrightarrow a_j < a_k$ for all j and k. We define $I_j := I \cap [a_{j-1}, a_j]$ and $M_j := f^{-1}(I_j)$, thus $I = \cup_j I_j$ and $M = \cup_j M_j$. Here $a_0 = p$ and $a_{N+1} = q$, thus $f^{-1}(a_0) = \partial_+ M$ and $f^{-1}(a_{N+1}) = \partial_- M$. Again we can write $I_1 \leq I_2 \leq \cdots \leq I_{N+1}$. The orientation of M_j is canonically induced by the orientation of M. Then we can choose a collection of manifolds Z_j diffeomorphic to $M_j \cap M_{j+1} = f^{-1}(a_j)$ (by OPD). This will give us a chain of new cobordisms:

$$X \xrightarrow{M_1} Z_1 \xrightarrow{M_2} Z_2 \xrightarrow{M_3} \cdots \xrightarrow{M_N} Z_N \xrightarrow{M_{N+1}} Y$$

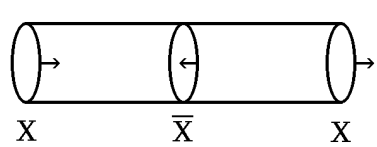
Again note that it is also possible that one of these Z_j is the empty set.

If M is empty it is not possible to define a Morse function on M, and we will not need it either, so we will always assume that M is not empty. If we assume M is connected, then $I = f(M)$ is also connected. On the other hand, if $I = f(M)$ is connected, then M is not necessarily connected.

We will also see that, even if M is a connected cobordism, after splitting it up into parts M_j, these parts need not be connected cobordisms themselves.

Also, if M itself has no boundary, then f can reach its global extrema in arbitrary points, thus we have more freedom in choosing this f, thus also in how to decompose M into smaller pieces. Using this f, we can split up M into a pair of cobordisms: M_1, followed by M_2, where $\partial_+ M_1 = \partial_- M_2 = \varnothing$ and $\partial_- M_1 = \partial_+ M_2 = f^{-1}(a) \neq \varnothing$, for some $a \in I - \partial I$. However, once such a manifold is split up into two pieces, the direction is fixed for splitting it up further, at least when we are dealing with a connected manifold or when we use the same Morse function to continue. We can think of numerous other examples looking more exotic, but we will not discuss these here.

A subtlety in splitting a cylinder. Starting with the cylinder C_X, a trivial cobordism from X to X, we can try to split it up into two smaller cobordisms, each of which is diffeomorphic to the original cylinder. We could for example end up with the cobordisms $M_1 := X \times [0, \frac{1}{2}]$ and $M_2 := X \times [\frac{1}{2}, 1]$. However, the new boundary $M_1 \cap M_2$ in the middle will have an already fixed orientation, according to the splitting rules. Thus it is not possible to split up C_X into smaller cobordisms, by using a copy of X with opposite orientation in the middle, as in this figure:



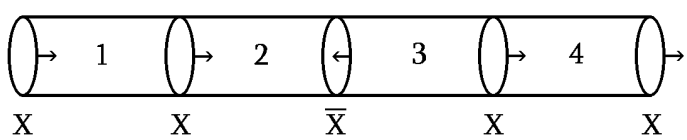
(6.4)

Here X and $\overline{X}$ are submanifolds of this cylinder, up to OPDs, thus there exist OPDs from X to $X \times 0_+$ and $X \times 1_+$, and from $\overline{X}$ to $X \times \frac{1}{2}_-$. This counter-example of splitting a cobordism was shortly introduced in [7], but I would like to elaborate a bit more on this to give more direct and more detailed arguments.

Assume $M_1 := X \times [0_+, \frac{1}{2}_-]$ is the left part and $M_2 := X \times [\frac{1}{2}_-, 1_+]$ is the right part, then $\partial_+ M_1 = \partial M_1 = X \times \{0_+, \frac{1}{2}_-\}$, $\partial_- M_1 = \partial_+ M_2 = \varnothing$ and $\partial_- M_2 = \partial M_2 = X \times \{\frac{1}{2}_-, 1_+\}$. Recall that the arrows, each of which is representing some positive normal, should point inwards in case of an in-boundary, and outwards in case of an out-boundary. The left part is similar to the cylinder $X \times I_{+-}$ (see figure 6.2), having an empty out-boundary, and the right part is similar to the cylinder $X \times I_{-+}$ (see figure 6.3), having an empty in-boundary. Despite the fact that $\partial_- M_1 = \partial_+ M_2$, both being empty, this is not what we are looking for, as something similar to $X \times I_{+-}$ followed by something similar to $X \times I_{-+}$ is not a connected cobordism. Even worse, none of the newly created in or out-boundaries equals the in or out-boundary of the original cobordism. So, what we have here is not a valid decomposition into smaller cobordisms.

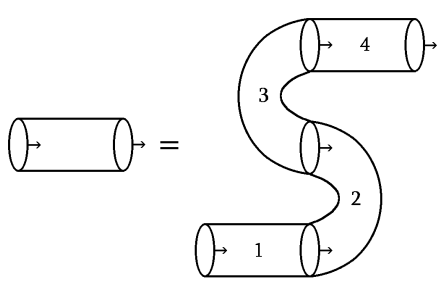
Another argument to reject this type of decomposition comes from looking at Morse functions. A Morse function f responsible for this decomposition, must take on the same value on the in-boundary, on the $\overline{X}$ in the middle and on the out-boundary, however, we would like f to satisfy $f(\partial_+ C_X) < f(X \times \frac{1}{2}_-) < f(\partial_- C_X)$. As both $\partial_+ M$ and $\partial_- M$ are not empty, f should reach its global extrema on the boundary, however, in this case it should also reach a maximum on the interior of M_1, and a minimum on the interior of M_2. This means f will not reach its global extrema on ∂M, unless f is constant, but then it is not a Morse function anymore. In fact, this is why we chose conventions for the Morse function and the newly generated in and out-boundaries in the first place.

The snake decomposition of a cylinder. For any X we can split up the cylinder $C_X = X \times [0_+, 1_+]$ in a special way. It is possible to define a Morse function $f : C_X \to I$ we can use for splitting up this cylinder into smaller cylinders $X \times [0_+, a_+]$, $X \times [a_+, b_-]$, $X \times [b_-, c_+]$ and $X \times [c_+, 1_+]$, for some $0 < a < b < c < 1$, as in the following figure:



This f is chosen such that it takes on the same value $v \in I$ exactly on the three circles, representing $X \times \{a_+, b_-, c_+\}$, in the middle of this figure, thus $f^{-1}(v) = X \times \{b_+, c_-, d_+\}$. In fact this can be interpreted as an extension of the counter-example as depicted in figure (6.4). However, this time it *is* possible to split up this cylinder into two smaller parts, each of which can be described as cobordisms. If we define M_1 as the union of part 1 and 3, and M_2 as the union of part 2 and 4, then we will see that $\partial_+ M_1 = \partial_+ M = X \times 0_+$, $\partial_- M_1 = \partial_+ M_2 = X \times \{a_+, b_-, c_+\}$ and $\partial_- M_2 = \partial_- M = X \times 1_+$. So why is this possible now? This is simple. Note that now f satisfies $f(\partial_+ C_X) < f(X \times \{b_+, c_-, d_+\}) < f(\partial_- C_X)$, thus, contrary to the counter-example, f does not need to satisfy $f(X \times b_+) < f(X \times c_-) < f(X \times d_+))$. We should also realize that now f will not need to take on its global extrema on $f^{-1}(v)$ anymore. The only thing to make sure is that extrema of f on part 2 and 3 lie somewhere between $f(\partial_+ M)$ and $f(\partial_- M)$, and in any case it is possible to find some f satisfying this proposition.

Now we can nicely illustrate this decomposition of C_X into M_1 and M_2:



(6.5)

The left hand side is just the cylinder C_X itself. The right hand side is the cylinder decomposed with respect to f, where M_1 is the left part, and M_2 is the right part. We see that M_1 is nicely followed by M_2, but also that M_1 and M_2 together, despite being disconnected manifolds themselves, will nicely form the cylinder. This decomposition, as illustrated in figure (6.5), of any cylinder is called the *snake decomposition*.

7 Categories of oriented cobordisms

Morse functions are very useful for decomposing cobordisms into smaller ones. However, we also want to study the reverse process of composing two cobordisms, to obtain a new one. This requires some other techniques, and Morse functions will be ignored in this process. In fact this is a totally different context. First we will introduce the concept of cobordism classes. These are much easier to compose and at the same time a lot of superfluous information is ignored. We can already define a category of cobordism classes, however, this category is not the final one yet, as we first need to specify more structure. We will introduce the concept of disjoint union, an operation which will help us finding a monoidal structure. After adding the monoidal structure we will add symmetric structure. Then we will define $(\mathbf{nCob}, \amalg, \varnothing, \tau)$, the symmetric monoidal category of cobordism classes in dimension n. The objects in this category are closed oriented manifolds of dimension $n-1$, and its arrows are cobordism classes based on oriented cobordisms between these manifolds. The cobordisms themselves are manifolds of dimension n.

In the next chapters we will discuss cobordism categories in dimension 2 (see Chapter 8) and 1 (see Chapter 9) and a skeleton of each of these. If we study the skeleton, then we can extract some essential information from the cobordisms and cobordism classes, regarded as arrows. For example the ordering of the connected components of their boundaries. This ordering is again motivated by the concept of disjoint union. However, we should be careful when dealing with this ordering, so we need to discuss some subtleties. For example, if a cobordism M has one or more connected components without boundary, then the possibility of ordering connected components of M will be partially cancelled. See the part about 'closed cobordisms and some properties' at the end of Section 7.3.

7.1 Cobordism classes

An equivalence relation. Let M and M' be oriented cobordisms from X to Y. Then there exist orientation preserving diffeomorphisms (OPDs) $\iota_+ : X \to \partial_+ M$, $\iota_- : Y \to \partial_- M$, $\iota'_+ : X \to \partial_+ M'$ and $\iota'_- : Y \to \partial_- M'$. We will define the following equivalence relation. The cobordisms M and M' are called equivalent if there exists an OPD $\phi : M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M \arrow[dd, "\phi"] & \\
X \arrow[ur, "\iota_+"] \arrow[dr, "\iota'_+"'] & & Y \arrow[ul, "\iota_-"'] \arrow[dl, "\iota'_-"] \\
 & M' &
\end{tikzcd} \tag{7.1}$$

Or, to be more precise, such that ϕ satisfies the following identities:

$$\phi_+ := \phi|_{\partial_+ M} = \iota'_+ \circ \iota_+^{-1} \quad , \quad \phi_- := \phi|_{\partial_- M} = \iota'_- \circ \iota_-^{-1}. \tag{7.2}$$

It is easy to show that this indeed defines an equivalence relation.

The equivalence classes related to this equivalence relation are called *cobordism classes*, and any such cobordism class can be represented by a cobordism. If M and M' are lying in the same cobordism class, then we will write $M \sim M'$, or $[M] = [M']$. (Here $[M]$ *is* a cobordism class.) Dividing out by this equivalence relation will thoroughly reduce the amount of cobordisms to be studied. At least if $M \sim M'$, then we could say that the topological properties of M and M' are equal.

Observe that if X is empty, then the maps ι_+ and ι'_+ will vanish, so that only the right triangle in diagram (7.1) will remain. (Note that we could also say that $\iota_+ = \iota'_+ = \phi_+ = I_\varnothing$. Knowing that $I_\varnothing$, as defined in (6.1), is an idempotent map we conclude that ϕ_+ satisfies (7.2).) Similarly, if Y is empty, then the maps ι_- and ι'_- will vanish, so that only the left triangle in diagram (7.1) will remain. If both X and Y are empty, then only the map ϕ will remain, but this also means that ϕ is not restricted anymore. Then we can simply say that two cobordisms M and M' are equivalent, just if an OPD $\phi : M \to M'$ exists.

How to interpret the equivalence relation generating cobordism classes: A first step. We could start with splitting up a cobordism M into its connected components, say M_j for $1 \leq j \leq N$, where the index should be regarded as just a label. Thus $M = M_1 \cup M_2 \cup \cdots \cup M_N$. Now suppose we are studying an arbitrary OPD $\phi : M \to M'$. Then we could split up M' according to how M was split up. Any diffeomorphism maps connected components to connected components, so we define $M'_j := \phi(M_j)$. This also gives us $\partial_+ M'_j = \phi(\partial_+ M_j)$ and $\partial_- M'_j = \phi(\partial_- M_j)$. Now, if for all j relation (7.2), restricted to ∂M_j instead, holds, then also $M \sim M'$. However, suppose we are examining one ϕ, and the result is negative, thus (7.1) does *not* commute, then it is still too early to draw any conclusions. In some cases it is possible to find *another* ϕ' which will satisfy. This ϕ' can differ from ϕ by a permutation of the labeling of the connected components of M, especially when some of its connected components themselves are diffeomorphic to each other. Then we need to check all other possible permutations until we are really sure. Thus this first method might seem a bit excessive.

However, we will see that it is always possible to make one of the two triangles in (7.1) commute. We can always start with writing $\phi_+ = \iota'_+ \circ \iota_+^{-1}$ by definition, even before we define ϕ itself. Then we can try to find a lift ϕ of ϕ_+, and if this lift exists we can define $\phi_- := \phi|_{\partial_- M}$. If this ϕ_- satisfies $\phi_- = \iota'_- \circ \iota_-^{-1}$, then we are through. Then we immediately can say $M \sim M'$. Otherwise, if one of the steps fails, we immediately know that $M \nsim M'$. We could start this testing method with ϕ_- as well.

We can describe this testing method as a sequence of two steps:

- **Step 1.** Define the OPD $\phi_+ : \partial_+ M \to \partial_+ M'$ by $\phi_+ := \iota'_+ \circ \iota_+^{-1}$. Is it possible to find an OPD $\phi : M \to M'$, which is a lift of ϕ_+? Thus there exists an OPD ϕ such that $\phi|_{\partial_+ M} = \phi_+$? If no, then $M \nsim M'$, end of test. If yes, go to step 2.

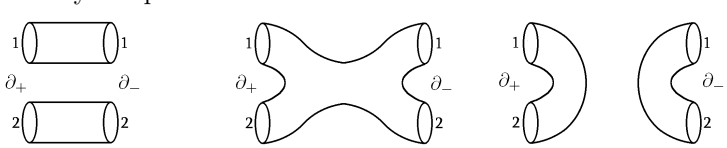
- **Step 2.** Does $\phi_- := \phi|_{\partial_- M}$ satisfy $\phi_- = \iota'_- \circ \iota_-^{-1}$? If no, then $M \nsim M'$, end of test. If yes, then $M \sim M'$ thus (7.1) is satisfied, end of test.

If we found a ϕ so that $M \sim M'$, then we can define $M'_j := \phi(M_j)$, and we will conclude that $M_j \sim M'_j$ for all j individually. We should observe that M_j (and M'_j) are connected sets, but $\partial_+ M_j$ and $\partial_- M_j$ not necessarily. For example, if $\partial_+ M_j$ has multiple connected components for some j, and if an OPD ψ exists from $\partial_+ M_j$ to itself, permuting its connected components, then this action can not be detected by any ϕ satisfying (7.1). I will try to clarify in more detail what this means later.

In [7] we can read about diffeomorphisms *keeping the boundaries fixed*, or *diffeomorphisms rel* ∂. However, $\partial_+ M$ and $\partial_- M$ are not really fixed, but what it means is the following. If ϕ_+ satisfies both steps, so that we find a ϕ satisfying (7.1), then we can pretend that the boundaries are fixed.

How to interpret the equivalence relation generating cobordism classes: Some examples. In the next examples we will look at cobordisms in dimension 2. In all examples we define $X = Y = S^1_1 \cup S^1_2$, a pair of distinct oriented circles, labeled 1 and 2. These circles can be unit circles and can be embedded in any space, but we will not care about that, so they will be regarded as abstract circles. The cobordisms considered, thus M, N, P and Q, will all be cobordisms from X to Y, thus from $S^1_1 \cup S^1_2$ to itself. The used figures should explain themselves for a large part. The labels 1 and 2 attached to the in- and out-boundaries in these figures point out which circle is mapped to which connected component of a boundary by the injections ι.

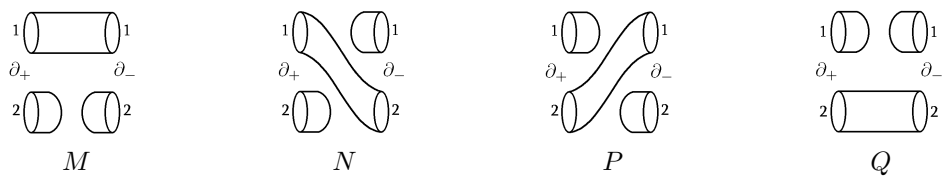
- **Example 1.** The following are examples of cobordisms which are not equivalent according to (7.1). We will only compare M to the other two cobordisms.



M N P

We immediately see that $M \nsim N$, as M has two connected components, and N has only one, so it is not possible to find any diffeomorphism between them. Comparing M to P we see that both have two connected components, each having the form of a cylinder. In fact there exists a diffeomorphism between M and P. However, this diffeomorphism cannot be regarded as a lift of any $\phi_+ : \partial_+ M \to \partial_+ P$. Translating this analysis to the two testing steps, we see that both examples immediately fail the first step. The OPD ϕ_+ is always possible, but in both situations it is impossible to find a proper lift of ϕ_+.

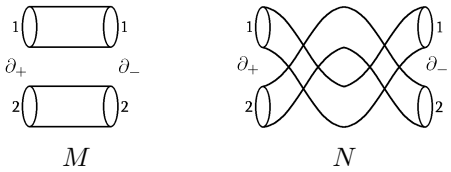
- **Example 2.** The following are examples of cobordisms which are also not equivalent according to (7.1). Their boundaries are not necessarily the same, only up to OPD. We will only compare M to the other three cobordisms.



We see that for any pair of these, there exists an OPD relating the two. Now we can try to find a lift ϕ of ϕ_+. We will see that there exists a lift $\phi_N : M \to N$ of ϕ_+, but no lifts $\phi_P : M \to P$ and $\phi_Q : M \to Q$. Thus we immediately see that P and Q will fail the first step, thus $M \nsim P$ and $M \nsim Q$. We also see that ϕ_N will fail the second step, as M contains a cylinder from S^1_1 to S^1_1, but N contains a cylinder from S^1_1 to S^1_2, thus $\phi_- = \phi_N|_{\partial_- M}$ will not satisfy (7.2).

Seeing it from the other side, we can always define ϕ_- to satisfy (7.2). Then we will see that there exists a lift $\phi_P : M \to P$ of ϕ_-, but no lifts $\phi_N : M \to N$ and $\phi_Q : M \to Q$. Thus we immediately see that N and Q will fail the first step. Now we see that ϕ_P will fail the second step, as $\phi_+ = \phi_P|_{\partial_+ M}$ will not satisfy (7.2).

- **Example 3.** The following is an example of a pair of equivalent cobordisms.



We see that there exists a lift ϕ of ϕ_+, and that $\phi_- = \phi|_{\partial_- M}$ satisfies (7.2).

We see that there are different possible reasons why the first step could fail for a pair M and N of cobordisms from X to Y. For example when there exists no diffeomorphism or no OPD from M to N, as in example 1. In example 2 we see that $\phi_Q : M \to Q$ will always fail the first step, but depending on whether we start the test with ϕ_+ or ϕ_- we see that either $\phi_P : M \to P$ or $\phi_N : M \to N$ will fail the first step, and none of the examples will reach the second step. So the first step is equivalent to testing whether M and N are diffeomorphic at all (by OPD), and whether their in- or out-boundary can be treated as *fixed*.

As a side note, we should realize that in higher dimensions there exist manifolds which can be equipped with different incompatible smooth structures. This means there exists no diffeomorphism from such a manifold to itself, respecting this change of smooth structure. Only a homeomorphism exists. However, we will ignore these subtleties here, as we will mainly concentrate on cobordisms in dimension 1 and 2, in which smooth structure is unique. A detailed analysis can be found in [7], and this analysis applies to cobordisms of any dimension. This analysis is based on the method of gluing topological manifolds. I would like to introduce a slightly different, relatively simple approach, which should be enough in dimension 1 and 2.

Orientation preserving diffeomorphisms and their induced cobordism classes. Let ϕ be an OPD from X to Y. As stated earlier, there are two ways of inducing a cobordism from this ϕ. It is easy to show that these two different cobordisms lie in the same cobordism class. Thus any ϕ uniquely induces cobordism class $[M_\phi]$, a class of cobordisms from X to Y.

Let ϕ and ψ be a pair of OPDs from X to Y. These will generate cobordisms from X to Y, say M_ϕ and M_ψ. This pair of OPDs will be called *smoothly homotopic* if a smooth map $\Phi : X \times [0,1] \to Y$ exists, satisfying $\Phi(p,0) = \phi(p)$ and $\Phi(p,1) = \psi(p)$ for all $p \in X$. A proposition says that $[M_\phi] = [M_\psi]$ if and only if ϕ and ψ are smoothly homotopic, so this proposition can be used to find differences between cobordism classes. The proof of this proposition can be found in [7].

Horizontal composition of cobordism classes. Let M be a cobordism from X to Y and let N be a cobordism from Y to Z. Then we can try to find another cobordism P from X to Z which can be regarded as the horizontal composition of M and N. Of course this P should not be an arbitrary cobordism from X to Z, so we would like to see properties of both M and N returning in P. At first sight we might say we need to find a P which can be split up into two parts using a Morse function, such that one part equals M and the other part equals N. However, it turns out to be highly exceptional for this to be possible, as we would like $\partial_- M$ and $\partial_+ N$ to be the same manifold, and it should be possible to consistently extend the smooth structures of M and N to P, so we need a different approach. We will try to look from the other side.

If P is a cobordism from X to Z, and if it can be split up into two parts, say P_1 followed by P_2, using a Morse function, such that $P_1 \sim M$ is a cobordism from X to Y, and $P_2 \sim N$ is a cobordism from Y to Z, then P can be regarded as a composition of M and N. From now on, assume this is the case. The problem is that this P will not be unique. In other words, it is not possible to consistently compose cobordisms themselves. However, for any other $M' \sim M$ and $N' \sim N$, any such P can also be regarded as a composition of M' and N'. So, to be able to find a unique composition, we will need to divide out by the equivalence relation $\sim$ generating cobordism classes. We can define the operation of horizontal composition of cobordism classes:

$$[P] = [N][M].$$

This means that any $P' \in [P]$ can be split up into P_1', followed by P_2', such that $[P_1'] = [M]$ and $[P_2'] = [N]$.

This approach will also turn the horizontal composition into an associative operation. For any triple of cobordisms, say M from X to Y, N from Y to Z and P from Z to W, there exists a cobordism Q from X to W such that $[Q] = ([P][N])[M] = [P]([N][M])$. We can start with $[Q] = ([P][N])[M]$ and split up Q with respect to a suitable Morse function, into two parts, say Q_1 and R, such that $[Q_1] = [M]$ and $[R] = [P][N]$. Thus $[Q] = [R][M]$. Using the same Morse function we can split up R into Q_2 and Q_3 such that $[Q_2] = [N]$ and $[Q_3] = [P]$. Thus we have split up Q into three parts, Q_1, Q_2 and Q_3. It does not really matter in which order we split it up. We can also split it up into three parts immediately. Now defining $S := Q_1 \cup Q_2$, $\partial_+ S = \partial_+ Q_1$ and $\partial_- S = \partial_- Q_2$, we immediately see that $[S] = [Q_2][Q_1] = [N][M]$, thus $[Q] = [P][S] = [P]([N][M])$. Associativity means that we can ignore parentheses and write $[Q] = [P][N][M]$ from now on.

Now only information about topological properties of the manifolds involved will survive, making the theory suitable for studying topological quantum field theories. There is a reason for calling this composition the *horizontal* composition. There is also a vertical composition. These subtleties will be discussed later.

Invertible cobordisms. Let C_X (C_Y) be the cylinder cobordism from X (Y) to itself. A cobordism M from X to Y is said to be *invertible* if both $[\overline{M}][M] = [C_X]$ and $[M][\overline{M}] = [C_Y]$. The *inverse* of a cobordism cannot unambiguously be indicated. In fact this is another reason to study cobordism classes instead. Then we can say, if M is invertible, then $[\overline{M}]$ is the *unique* inverse of $[M]$.

The cylinder itself is the most trivial example of an invertible cobordism. In fact it is its own inverse, so $[C_X][C_X] = [C_X]$, turning the cylinder into an idempotent cobordism class. It should be clear that if an OPD $\phi : X \to Y$ exists, then the cobordism generated by this ϕ is invertible.

For any manifold M we define $\nu(M)$ as the number of its connected components. Then $\nu(M) = 0$ if M is the empty manifold and $\nu(M) = 1$ if M is connected.

- **Lemma.** If M is an invertible cobordism from X to Y, then $\nu(X) = \nu(Y) = \nu(M)$.

  Proof. First of all, $\nu(C_X) = \nu(X \times I) = \nu(X)$ for any X. Of course we can safely assume $\nu(M) = \nu(\overline{M})$ and $\nu(M') = \nu(M)$ for any $M' \in [M]$, so we can easily define the number of connected components of a cobordism class: $\nu([M]) := \nu(M)$. If M is invertible, then $[\overline{M}][M] = [C_X]$ and $[M][\overline{M}] = [C_Y]$, thus $\nu([\overline{M}][M]) = \nu(X)$ and $\nu([M][\overline{M}]) = \nu(Y)$. From now on assume M is invertible.

  We call M *horizontally connected* if for any $p \in M$ there exist $p_+ \in \partial_+ M$ and $p_- \in \partial_- M$ such that p, p_+ and p_- lie in the same connected component of M. For this to work properly, we assume $\partial_+ M$ and $\partial_- M$ are both not empty. In fact this means that M has no connected component without boundary, or with only an in- or an out-boundary. We already know any cylinder is horizontally connected.

  Splitting up a cylinder into two pieces which can be considered as mirror images of each other will result in a pair of cobordisms which are also horizontally connected, and we can safely assume that the topology of each of the two pieces is similar to the topology of the cylinder itself. This will be the case when splitting up C_X into $C_{X,1} \in [M]$ and $C_{X,2} \in [\overline{M}]$, and C_Y into $C_{Y,1} \in [\overline{M}]$ and $C_{Y,2} \in [M]$. So we may conclude that $\nu(M) = \nu(\overline{M}) = \nu([\overline{M}][M]) = \nu([M][\overline{M}])$, thus $\nu(X) = \nu(M) = \nu(Y)$.

  □

A special case of this lemma is when $\nu(M) = 1$. If M is invertible and connected, then X and Y are also connected.

Composition with a cylinder. Let C_X and C_Y be cylinders.

- **Lemma.** For any cobordism M from X to Y the following identity holds:

$$[C_Y][M] = [M] = [M][C_X]. \tag{7.3}$$

  Proof. We can always find a suitable Morse function f we can use to split up M into three parts, say M_1, M_2 and M_3, assuming f has no critical points on M_1 and M_3. This is only possible when $M_1 \in [C_X]$ and $M_3 \in [C_Y]$, which implies that M and M_2 share the same topological properties. In fact M and M_2 will be diffeomorphic by an OPD satisfying (7.1), so we can write $[M] = [M_3][M_2][M_1] = [C_Y][M][C_X]$. Knowing that any cylinder is idempotent with respect to horizontal composition, we can write $[M] = [C_Y][M][C_X] = [C_Y][C_Y][M][C_X] = [C_Y][M][C_X][C_X]$, thus $[M] = [C_Y][M] = [M][C_X]$.

  □

The category of cobordism classes: A first step. Now it is possible to describe cobordisms and their compositions in terms of category theory. We define **nCob** as the category of oriented cobordisms in dimension n. The objects in **nCob** will be closed oriented manifolds of dimension $n-1$, and the arrows in **nCob** will be cobordism classes. Any arrow can be represented by an oriented manifold of dimension n with boundary. If X and Y are objects, and if M is a cobordism from X to Y, where $\iota_+ : X \to \partial_+ M$ and $\iota_- : Y \to \partial_- M$ are the desired OPDs, then the cobordism class $[M]$ indicates an arrow from X to Y, and we will write $[M] : X \to Y$. We should observe that the maps ι_+ and ι_- form part of the information specifying an arrow.

Indeed, for any pair of cobordisms, say M from X to Y and N from Y to Z, the corresponding arrows $[M]$ and $[N]$ can be composed, and the arrow $[N][M] : X \to Z$ is unique. This composition is associative, as pointed out before. From (7.3) in the preceding paragraph, we can conclude that any cylinder $[C_X] : X \to X$ constitutes an identity arrow, thus that $\mathrm{Id}_X = [C_X]$. Any invertible cobordism M can be interpreted as an isomorphism $[M]$ between two objects.

In this context we can say that the action of reversing the orientation of a cobordism is the effect of a contravariant functor from **nCob** to itself. This functor will map any object to itself, and reverse the direction of any arrow.

We say this is a *first step* because it is possible to indicate more structure. The category **nCob** can be equipped with monoidal structure, and *indicating* this structure is really the right use of language, as no additional structure on **nCob** itself is really needed, only a functor. After indicating monoidal structure we will indicate symmetric structure, turning **nCob** into a symmetric monoidal category. But, before we do so, we first need to introduce the concept of disjoint union.

The category of cobordism classes: Isomorphisms. As stated earlier, if for a pair of closed oriented manifolds X and Y, now considered as objects, an orientation preserving diffeomorphism $\phi : X \to Y$ exists, then we can construct a cobordism M_ϕ, generated by ϕ. The diffeomorphism has an inverse, also being orientation preserving, thus M_ϕ is an invertible cobordism. Then the corresponding cobordism class $[M_\phi]$ also has an inverse arrow, so any cobordism class induced by an OPD between two objects, generates an isomorphism between these two objects. To any OPD ϕ a unique isomorphism $[M_\phi]$ is associated, but it is still possible for two distinct but smoothly homotopic OPDs ϕ and ψ to generate the *same* isomorphism.

7.2 Disjoint unions

The disjoint union of sets. Let $a \neq b$ be two arbitrary points. Then, for any ordered pair of sets (X, Y) we can introduce their (binary) *disjoint union*, written as $X \amalg Y$. We could define it as

$$X \amalg Y := X_a \cup Y_b := X \times a \cup Y \times b.$$

The *universal property* of the disjoint union, which I will not explain in more detail, mainly says that it really does not matter what we chose a and b to be. When we have another such pair of points $a' \neq b'$, we can canonically identify $X_{a'} \cup Y_{b'}$ to $X_a \cup Y_b$. So in fact it does not really matter how a and b were chosen, at least as long as we keep it consistently. As a marginal note, we should realize that writing $X \amalg Y = X \times a \cup Y \times b$ is also a matter of choice. We could as well use $a \times X$ instead of $X \times a$, or something else. From now on we will just write it down in a more abstract way, ignoring a and b, and just write $X \amalg Y = X_1 \cup Y_2$. We can treat these numbers as labels. Later, when X and Y are oriented manifolds, we automatically assume 1 and 2 to be *positively oriented* points, thus $X \amalg Y = X \times 1_+ \cup Y \times 2_+$.

We note that in general $Y \amalg X = Y_1 \cup X_2 \neq X_1 \cup Y_2 = X \amalg Y$, contrary to the *ordinary* union of sets $X \cup Y$, which satisfies $X \cup Y = Y \cup X$ in a natural way, without more structure being specified. Of course the input of the ordinary union could be regarded as an ordered pair of sets, thus $(X, Y) \mapsto X \cup Y$, but as the result for (Y, X) is the same, we say the ordinary union is a symmetric operation, and the disjoint union is not.

However, we can define a canonical isomorphism between $X \amalg Y$ and $Y \amalg X$. For any point $p \in X$ and $q \in Y$ we write p_1 and q_2 as elements of $X \amalg Y$, and p_2 and q_1 as elements of $Y \amalg X$. We also write $\tau_{X,Y} : X \amalg Y \to Y \amalg X$ and $\tau_{Y,X} : Y \amalg X \to X \amalg Y$, which are maps only interchanging the labels of points.

We could try to repeat this for any number of sets, however, the disjoint union is not a priori an associative operation. Writing $(X \amalg Y) \amalg Z = (X_1 \cup Y_2)_1 \cup Z_2 = X_{11} \cup Y_{21} \cup Z_2$ and $X \amalg (Y \amalg Z) = X_1 \cup (Y_1 \cup Z_2)_2 = X_1 \cup Y_{12} \cup Z_{22}$, we see that there is no exact equality. However, there is a unique canonical isomorphism between these two.

The multi-disjoint union of sets. For any n-tuple of sets $X^1, \cdots, X^n$, we can define

$$X^1 \amalg \cdots \amalg X^n := X^1_1 \cup \cdots \cup X^n_n := X^1 \times 1 \cup \cdots \cup X^n \times n.$$

Note that the upper index is just a label to keep track of the sets of interest. We could as well look at sets like, for example, $X^n \amalg \cdots \amalg X^2 \amalg X^1 = X^n_1 \cup \cdots \cup X^2_{n-1} \cup X^1_n$. If the n-tuple is already canonically ordered, then we can easily identify the lower index to the upper one. Comparing this to the previous definition of binary disjoint union, we can write unique canonical isomorphisms $(X \amalg Y) \amalg Z \simeq X \amalg (Y \amalg Z) \simeq X \amalg Y \amalg Z$.

The disjoint union and the multi-disjoint union of closed oriented manifolds. Let (X, Y) be an ordered pair of closed smooth oriented manifolds of dimension n. Then it should be clear that we can copy smooth structure and orientation of X and Y to $X \amalg Y$ and $Y \amalg X$ in a canonical way, turning $X \amalg Y$ and $Y \amalg X$ into smooth oriented manifolds. Doing this for the multi-disjoint union of any ordered n-tuple of such manifolds does not really need any more effort.

A more subtle discussion is based on the idea of splitting up a closed manifold into smaller parts, each again being a closed manifold. In the context of closed smooth manifolds we can easily introduce a concept of irreducibility. If the manifold M is closed and connected, and if we split it up into two parts, each again being a connected set, it is unavoidable that one part is open, thus is not compact, and the other part is compact but has a boundary, thus is not closed. Thus splitting up any closed M, not necessarily being connected, into its connected components gives us a collection of irreducible closed connected manifolds $M^1, \cdots, M^n$. Now the upper index is still nothing more than a label, however, we can order these components as we like, for example using standard ordering $1 < \cdots < n$, and write a canonical isomorphism from M to $M^1 \amalg \cdots \amalg M^n$. Again, another arbitrary ordering is also allowed.

As an example, we could look at a pair of oriented closed connected manifolds X and Y satisfying $X \cap Y = \varnothing$. Then $X \cup Y$ is again an oriented closed manifold and the following diagram commutes:

$$\begin{tikzcd} & X \cup Y \arrow[dl, "\iota_{XY}"'] \arrow[dr, "\iota_{YX}"] & \\ X \amalg Y \arrow[rr, bend left, "\tau_{X,Y}"] & & Y \amalg X \arrow[ll, bend left, "\tau_{Y,X}"] \end{tikzcd} \tag{7.4}$$

Here all maps are canonical orientation preserving diffeomorphisms. The maps ι_{XY} and ι_{YX} are just canonical maps unambiguously attaching a label to any point,

In general, for any manifold M with n connected components, we can make such diagrams based on multi-disjoint unions of n arbitrarily ordered factors. From now on we will ignore the subtle differences between the ordinary disjoint union and the multi-disjoint union.

The disjoint union of compact oriented manifolds with boundary. Let M and N be compact oriented manifolds with boundary. Then $M \amalg N$ is again a compact oriented manifold with boundary. We write $\partial_+(M \amalg N) = \partial_+ M \amalg \partial_+ N$ and $\partial_-(M \amalg N) = \partial_- M \amalg \partial_- N$.

The disjoint product of maps. Let $X = X^1 \amalg X^2$ and $Y = Y^1 \amalg Y^2$, and let $\phi_1 : X^1 \to Y^1$ and $\phi_2 : X^2 \to Y^2$ be OPDs, then these maps canonically induce an OPD $\phi : X \to Y$. Restricted to the disjoint factors X^j, this ϕ will coincide with ϕ_j. We will write $\phi = \phi_1 \amalg \phi_2$, and we will call it the disjoint *product* of the maps ϕ_1 and ϕ_2.

The disjoint product of cobordisms. Let X, X', Y and Y' be closed oriented manifolds of dimension $n-1$. Then $X \amalg X'$ and $Y \amalg Y'$ are again closed manifolds with canonically induced orientation. If M is a cobordism from X to Y and M' from X' to Y', then $M \amalg M'$ is a cobordism from $X \amalg X'$ to $Y \amalg Y'$. This cobordism is again canonically constructed: We can write $\partial_+ M \amalg \partial_+ M' = \partial_+(M \amalg M')$ and $\partial_- M \amalg \partial_- M' = \partial_-(M \amalg M')$. If $\iota_+ : X \to \partial_+ M$, $\iota'_+ : X' \to \partial_+ M'$, $\iota_- : Y \to \partial_- M$ and $\iota'_- : Y' \to \partial_- M'$ are the needed OPDs, then $\iota_+ \amalg \iota'_+ : X \amalg X' \to \partial_+(M \amalg M')$ and $\iota_- \amalg \iota'_- : Y \amalg Y' \to \partial_-(M \amalg M')$ are the OPDs needed to turn $M \amalg M'$ into a cobordism from $X \amalg X'$ to $Y \amalg Y'$.

The disjoint product of cobordism classes. The concept of disjoint products of cobordisms is compatible with the concept of cobordism classes: if M and N are equivalent cobordisms from X to Y, and if M' and N' are equivalent cobordisms from X' to Y', where $\phi : M \to N$ and $\phi' : M' \to N'$ are the OPDs needed to make (7.1) commute in each case, then $\phi \amalg \phi' : M \amalg M' \to N \amalg N'$ is again an OPD satisfying

(7.1). Thus $[M] = [N]$ *and* $[M'] = [N']$ implies $[M \amalg M'] = [N \amalg N']$. The reverse implication is easily shown, just by restricting the maps in question to the disjoint parts, thus

$$\{[M] = [N]\} \wedge \{[M'] = [N']\} \Leftrightarrow \{[M \amalg M'] = [N \amalg N']\}.$$

7.3 The symmetric monoidal category of cobordism classes

Introducing monoidal structure. Knowing **nCob** is a category, we can try to equip it with monoidal structure. The disjoint product structure of cobordisms naturally induces a monoidal structure for the category of cobordism classes. We define a functor $\mu : \mathbf{nCob} \times \mathbf{nCob} \to \mathbf{nCob}$ as introduced in (2.5): for any two objects X and Y we write $\mu_0(X, Y) = X \square Y := X \amalg Y$, and for any two arrows $[M] : X \to Y$ and $[M'] : X' \to Y'$ we write $\mu_1([M], [M']) = [M] \square [M'] := [M \amalg M']$, which is an arrow from $X \amalg X'$ to $Y \amalg Y'$. Instead of $[M] \square [M']$ we will write $[M] \amalg [M']$. We should be careful what $\amalg$ means in this case. It is not literally the disjoint union of $[M]$ and $[M']$, but it should really mean $[M \amalg M']$. We will call this the vertical composition of cobordism classes. Now we can use the $\amalg$-symbol instead of the $\square$-symbol, in case of objects *and* in case of arrows.

Now writing $\mathbf{1} := \mathbf{nCob}^0$, which is the empty product category of **nCob**, we can define a functor $\eta : \mathbf{1} \to \mathbf{nCob}$ as introduced in (2.5). The image of this functor will be the object $1 = \eta(\mathbf{1}) := \varnothing$, and the arrow $\mathrm{Id}_1 = \mathrm{Id}_\varnothing = [C_\varnothing] = [\varnothing]$. (Note that $[C_\varnothing]$ can be regarded as the cylinder generated by the OPD $I_\varnothing$.) This corresponds to the empty cobordism, which of course has empty boundary, thus $[\varnothing] : \varnothing \to \varnothing$. Then there are natural identifications $\varnothing \amalg X = X = X \amalg \varnothing$, for any object X, and $[\varnothing] \amalg [M] = [M] = [M] \amalg [\varnothing]$, for any arrow $[M]$. A special case is when $X = \varnothing$ and $[M] = [\varnothing]$: then we have natural identifications $\varnothing = \varnothing \amalg \varnothing$ and $[\varnothing] = [\varnothing] \amalg [\varnothing]$.

We claim without proof that these functors μ and η satisfy (2.6) and (2.7), be it only in the weak sense, thus only up to invertible natural transformations. We also claim without proof that these invertible natural transformations satisfy the coherence constraints, and we will ignore them from now on. In other words, we claim that the diagrams (2.10) and (2.11) commute in this case. Then $(\mathbf{nCob}, \mu, \eta) = (\mathbf{nCob}, \amalg, \varnothing)$ is a monoidal category, or to be more precise, it is a nonstrict monoidal category. As discussed in Chapter 2 it is possible to treat a nonstrict monoidal category as a strict one, at least when the coherence constraints are satisfied, which is the case here. Finding a canonical identification between repeated ordinary disjoint unions and the multi-disjoint union, can be interpreted as finding a trivial associator making (2.10) and (2.11) commute, after redefining μ and η. This allows us to observe $\amalg$ as an associative operation and to ignore parentheses, thus to write $((W \amalg X) \amalg Y) \amalg Z = W \amalg X \amalg Y \amalg Z$, for example. So from now on we can observe $(\mathbf{nCob}, \amalg, \varnothing)$ as a strict monoidal category, which also means that from now on any natural identification can be regarded as an exact identification.

We should note that an arbitrary closed oriented n-manifold X is not necessarily the disjoint product of smaller manifolds, but, as illustrated in figure (7.4), we can always split up X into connected components $X_1, \cdots, X_N$, and make a choice for ordering them. For any such choice there is a canonical OPD $\phi : X \to X_1 \amalg \cdots \amalg X_N$, and this OPD generates a unique cobordism class $[M_\phi]$, which is an isomorphism.

Twist diffeomorphisms and twist cobordisms: The abstract ones. For any pair X and Y of closed oriented manifolds we can define a canonical diffeomorphism $\tau_{X,Y} : X \amalg Y \to Y \amalg X$. It will only interchange labels: for any $p \in X$ and $q \in Y$ we can write p_1 and q_2 as elements of $X \amalg Y$ and p_2 and q_1 as elements of $Y \amalg X$. Then $\tau_{X,Y}$ will map p_1 to p_2 and q_2 to q_1. We will call this diffeomorphism $\tau_{X,Y}$ the *abstract twist diffeomorphism*. The inverse of $\tau_{X,Y}$ will be denoted by $\tau_{Y,X}$.

The diffeomorphism $\tau_{X,Y}$ induces a cobordism $T_{X,Y}$ called an *abstract twist cobordism*. Its cobordism class $[T_{X,Y}]$ is unique and invertible, and its inverse is written as $[T_{Y,X}]$, thus $[T_{Y,X}][T_{X,Y}] = [C_{X \amalg Y}] = \mathrm{Id}_{X \amalg Y}$ and $[T_{X,Y}][T_{Y,X}] = [C_{Y \amalg X}] = \mathrm{Id}_{Y \amalg X}$.

It should be clear that the abstract twist cobordism is not quite the same thing as the natural twist cobordism, as X and Y do not need to be diffeomorphic themselves, and their intersection does not need to be empty. So, in general, natural twist cobordisms and abstract twist cobordisms are conceptually rather

different. However, any abstract twist cobordism from $X \amalg X$ to itself can also be described as a natural twist cobordism from $X \amalg X$ to itself. Apart from that, monoidal structure was not needed to introduce natural twist cobordisms. The abstract twist cobordism is not really needed to describe a cobordism from X to itself, connecting different connected components, resulting in a natural twist cobordism.

From now on we will not discuss natural twist cobordisms anymore. So from now on, when a twist diffeomorphism or twist cobordism is mentioned, we will assume it is an abstract one.

Now consider two cobordisms M from X to Y and M' from X' to Y'. Then $M \amalg M'$ is a cobordism from $X \amalg X'$ to $Y \amalg Y'$, and $M' \amalg M$ is a cobordism from $X' \amalg X$ to $Y' \amalg Y$. Then $T_{X,X'}$ and $T_{Y,Y'}$ are (abstract) twist cobordisms. We claim without proof that these cobordisms, considered as cobordism classes, satisfy the following:

$$[T_{Y,Y'}][M \amalg M'] = [M' \amalg M][T_{X,X'}].$$

As a consequence, we can write $[M' \amalg M] = [T_{Y,Y'}][M \amalg M'][T_{X',X}]$.

As the diffeomorphisms $\tau_{X,Y}$ will not be discussed anymore from now on, we can reuse the symbol for something else. We will now use the $\tau_{X,Y}$ symbol for the twist arrow, induced by a twist diffeomorphism. So from now on we will write $\tau_{X,Y} := [T_{X,Y}]$.

Twist cobordisms and braiding. We should note that, in this context, specifying a twist cobordism does *not* involve specifying which part twists over or under which other part, and the used figures reflect this. In other words, we will not discuss *braiding*. Determining the degree of braiding is only possible after introducing more rules, and for an arbitrary pair of connected components of a cobordism we need to be sure it is possible at all to regard them as embedded into another space. And, even if the connected components can be regarded as embedded, we still need to specify *in which space* we would like to see them embedded, and we also need a reference point according to which one part twists over or under the other. So, allowing braidings would restrict our framework, but they can be of more use when studying more specific theories. For example in physics, if we would like to study 2-dimensional cobordisms embedded into 4-dimensional spacetime, and the cobordism boundaries will have constant time, then there is a good framework for studying the braiding properties of these cobordisms.

Introducing symmetric monoidal structure. For any X and Y it is possible to construct a twist cobordism $T_{X,Y}$, and its corresponding *unique* cobordism class can be regarded as an arrow $\tau_{X,Y} : X \amalg Y \to Y \amalg X$. We claim without proof that any such arrow satisfies the properties of a *twist arrow*, as discussed in Chapter 2, so all these arrows together can be used to specify symmetric monoidal structure on **nCob**.

The symmetric monoidal category of cobordism classes. From now on we will sometimes drop the square brackets indicating a cobordism class, thus we will write M instead of $[M]$. Hopefully it will be clear from the context whether a cobordism or a cobordism class is mentioned.

We started with the category **nCob**, and turned it into a nonstrict monoidal category $(\mathbf{nCob}, \amalg, \varnothing)$, which can be strictified. Then we can add symmetric structure. The twist cobordism defines a twist arrow for any pair of objects, so the monoidal category $(\mathbf{nCob}, \amalg, \varnothing)$ can be turned into a (strict) symmetric monoidal category $(\mathbf{nCob}, \amalg, \varnothing, \tau)$.

From now on we will sometimes write **nCob** instead of $(\mathbf{nCob}, \amalg, \varnothing, \tau)$. This is a nice starting point for studying two interesting symmetric monoidal categories, namely **1Cob** and **2Cob**, the category of cobordisms in dimension 1 and 2. In this case, also considering that a complete classification of lines and surfaces is known, we do not need to worry about problems with incompatible smooth structures of one and the same manifold. All manifolds of dimension 1 or 2 carry unique smooth structure.

Closed cobordisms and some properties. Let M and N be cobordisms, and let N also be closed, thus $\partial N = \varnothing$. Then N is a cobordism from $\varnothing$ to itself, in which case we will see that the ι-maps, appearing in diagram (7.1), will vanish, so making this diagram commute gets much easier. Then the canonical OPD from $M \amalg N$ to $N \amalg M$, only interchanging labels, is compatible with diagram (7.1), thus it is easy to show that $[M \amalg N] = [N \amalg M]$. We could say that any closed cobordism commutes with all other cobordisms with respect to disjoint union, or vertical composition.

Now let also M be closed, then it is also easy to show that $[M][N] = [N][M]$, thus any two closed cobordisms commute with respect to ordinary (or horizontal) composition.

For any cobordism P in the cobordism class $[M][N]$, an OPD from P to $M \amalg N$ exists, so we can combine the former two properties and obtain the following identity:

$$[M \amalg N] = [N \amalg M] = [M][N] = [N][M]. \tag{7.5}$$

Thus any two closed cobordisms commute with respect to horizontal and vertical composition, and we can even exchange the procedures of horizontal and vertical composition themselves. We will review these properties later when discussing topological quantum field theories.

8 The category of 2-cobordisms

In this chapter we will study the category **2Cob** of cobordism classes of dimension 2. We will also study **2cob**, a skeleton of **2Cob**. It is mainly the skeleton **2cob** we will use later, in Section 10.4, when we will study 2-dimensional topological quantum field theories. Before doing so, it would be nice if we can also turn **2cob** into a symmetric monoidal category. We know that **2Cob** is already a symmetric monoidal category, and after restricting its symmetric monoidal structure to **2cob**, we will see that it still functions as a symmetric monoidal structure. So there is a trivial way of porting over all the structure from **2Cob** to **2cob**. This chapter will also present the properties of **2Cob** and **2cob**. We will study some generators and relations of **2cob**. We have relations like the naturality of the twist, the snake relation, monoid and comonoid relations and the Frobenius relation. The circle S^1 can be regarded as a commutative Frobenius object in **2Cob** and **2cob**. We can say that the circle, also regarded as the basic object of **2cob**, makes **2cob** into a free symmetric monoidal category on a commutative Frobenius object.

2Cob, the category of oriented cobordisms in dimension 2. Any object X in **2Cob** is a closed oriented manifold of dimension 1, and has N_X connected components. Any arrow in **2Cob**, from X to Y, is a class of equivalent cobordisms of dimension 2, from X to Y. Any connected component X_j of X has the topology of a circle, thus we could say that any connected component of a 2-cobordism has the topology of a compact surface of some genus and possibly with some open disks of dimension 2 missing. From now on, if we mention any 1-manifold, we will assume it is closed and oriented. We claim without proof that for any arbitrary pair of objects X and Y there exists at least one cobordism from X to Y.

The oriented circle, a basic object in 2Cob. Let S^1, regarded as a manifold, be the standard unit circle in $\mathbb{R}^2$ with *standard orientation*, and let $\overline{S^1}$ be the same circle with *opposite orientation*. There exists a class of orientation preserving diffeomorphisms from S^1 to $\overline{S^1}$. These two closed oriented manifolds can be regarded as objects in **2Cob**, and any OPD $\phi : S^1 \to \overline{S^1}$ induces a cobordism M_ϕ, which we can call an *orientation reversing cobordism*. The corresponding cobordism class $[M_\phi]$ can be regarded as a unique arrow in **2Cob**, and this arrow is an isomorphism.

Diffeomorphic objects and invertible cobordisms. Now, if we have an arbitrary (connected) object X and if we found a diffeomorphism $\psi : X \to S^1$, then this ψ is either *orientation preserving* or *orientation reversing*. In the latter case ψ is an OPD from X to $\overline{S^1}$, but, as there also exist OPDs ϕ from S^1 to $\overline{S^1}$, there exists an OPD $\phi^{-1} \circ \psi$ from X to S^1. Thus the existence of a diffeomorphism from X to S^1 implies the existence of an *orientation preserving* diffeomorphism from X to S^1. This OPD induces a unique cobordism class, and it again generates an isomorphism in **2Cob**.

Later we will see that this is *not* the case when we are dealing with oriented cobordisms in dimension 1. Then the diffeomorphism we start with must already be orientation preserving. However, as we do not need to restrict to *orientation preserving* diffeomorphisms now, we will see that a skeleton of **2Cob** will be somewhat easier to define.

Another consequence: in general, in the category **nCob**, it is always possible to find a cobordism between $X \amalg \overline{X}$ and $\varnothing$. The cobordisms depicted in figure (6.2) and in figure (6.3), also called cylinders, reflect this. However, in **2Cob** a cobordism between for example $S^1 \amalg S^1$ and $\varnothing$ is also possible. Similar cobordisms are not possible in **1Cob**, as we will discuss later.

Objects diffeomorphic to N copies of the circle. We already know that any connected component X_j of X has the topology of a circle. As we are dealing with smooth manifolds this means that any X_j is diffeomorphic to S^1, which in turn implies that for any X_j there exists an OPD ϕ_j from X_j to S^1. These ϕ_j generate unique cobordism classes $M_j : X_j \to S^1$. Let S^N be the multi-disjoint union of N circles:

$$S^N := \underbrace{S^1 \amalg \cdots \amalg S^1}_{N}. \tag{8.1}$$

Writing $N = N_X$, the uniqueness of M_j implies that $\phi_1 \amalg \cdots \amalg \phi_N$ generates a unique cobordism class

$$M_1 \amalg \cdots \amalg M_N : X_1 \amalg \cdots \amalg X_N \to S^N.$$

Note that here we equipped the connected components of X with ordering, and that this ordering coincides with their labeling. We could call this *canonical* ordering. Now note that there exists a (canonical) cobordism class from X to $X_1 \amalg \cdots \amalg X_N$. Composing this with $M_1 \amalg \cdots \amalg M_N$ we obtain a cobordism class from X to S^N. This resulting cobordism class is again an isomorphism, and for any choice of labeling the connected components of X, this isomorphism should be unique. As this cobordism class is an isomorphism, it also has an inverse, and this inverse will help us finding a cobordism class from S^N to X. So it does not matter if X plays the role of an in-boundary or an out-boundary; in both cases X can represented by S^N.

We should take some care however with this approach, so a short comment should be added here. We could already have $X := X_1 \amalg X_2$ as an object, where both X_1 and X_2 are connected. Then we do not need to insert the cobordism class from X to $X_1 \amalg X_2$. But, another example is when Y_1 and Y_2 are ordered connected components of Y, but Y itself happens to be $Y_2 \amalg Y_1$. Then we still need to insert the cobordism class from Y to $Y_1 \amalg Y_2$. We can interpret this as *reordering* the connected components of X or Y, in case they are already ordered.

Of course, if also partial ordering is regarded, then nothing really changes. Here we mean if $X = X_1 \amalg X_2$, but if X_1 and X_2 are still not connected themselves. In this case we are dealing with *partial ordering*. Thus describing X (or Y) with this approach comes down to *overwriting* the (partial) ordering which eventually already exists. But from now on we will not really worry about this.

Diffeomorphic 1-manifolds. In general, if for a pair of 1-manifolds X and Y a diffeomorphism $\phi : X \to Y$ exists, then there also exists an OPD $\phi' : X \to Y$, and this OPD generates an invertible cobordism from X to Y. Then the manifolds X and Y also have the same number of connected components.

We claim without proof that the reverse implication is also true. If there exists an invertible cobordism from X to Y, then there also exists a diffeomorphism from X to Y. Thus saying that X and Y are diffeomorphic is equivalent to saying that there exists an isomorphism from X to Y, regarded as objects in **2Cob**.

Thus, if X and Y do *not* have the same number of connected components, then there exists no isomorphism between them either. A special case is when $X = S^m$ and $Y = S^n$. If $m \neq n$, then X and Y do not have the same number of connected components, thus there exists no isomorphism between them. As a result we can say that for any skeleton $\mathcal{C}'$ of **2Cob** and for any pair of distinct objects X and Y in $\mathcal{C}'$, the corresponding manifolds also have distinct numbers of connected components.

2cob, a skeleton of 2Cob. Let M be a cobordism from X to Y, and let $m = N_X$ and $n = N_Y$ be the numbers of connected components of X and Y. Then we can choose an ordering of the connected components of X and Y. These orderings induce OPDs $\phi : X \to S^m$ and $\psi : S^n \to Y$, and these OPDs in turn induce cobordism classes $[M_\phi]$ and $[M_\psi]$. Then we can split up M into three parts: $[M] = [M_\psi]\tilde{M}[M_\phi]$, where $\tilde{M}$ is a cobordisms class from S^m to S^n, induced by M itself of course. We should note that $[M_\phi]$ and $[M_\psi]$ are isomorphisms, thus we can also write $\tilde{M} = [M_\psi]^{-1}[M][M_\phi]^{-1} = [M_{\psi^{-1}}][M][M_{\phi^{-1}}]$.

We already know that for any object X there exists an isomorphism from X to S^m, where $m = N_X$. So we can say the collection of objects S^k (with k arbitrary) forms a skeleton of **2Cob**, and we will use the notation **2cob** to refer to this skeleton. The only remaining isomorphisms are those from any object S^k to itself. A large class of objects and arrows in **2Cob** will not appear in **2cob**. For example, the orientation reversing cobordism from S^1 to $\overline{S^1}$ is an isomorphism between two distinct objects, so it will not appear as an arrow in **2cob**. The object $\overline{S^1}$ will also not appear as an object in **2cob**, as S^1 is already an object in **2cob**, and there exists an isomorphism from S^1 to $\overline{S^1}$.

Porting over the structure from 2Cob to 2cob. As stated earlier, **2Cob** is just shorthand notation for $(\mathbf{2Cob}, \square, 1, \tau)$, or $(\mathbf{2Cob}, \amalg, \varnothing, \tau)$, so we are already dealing with a (strict) symmetric monoidal category.

Now we can try turning **2cob** into a symmetric monoidal category also. To do this we can shortly look at candidate operators. In other words, we will try to find operators $\square'$, $1'$ and τ' so that $(\mathbf{2cob}, \square', 1', \tau')$ is a symmetric monoidal category. The claim will simply be that $\square' = \square = \amalg$, $1' = 1 = \varnothing$ and $\tau' = \tau$, so that also $(\mathbf{2cob}, \amalg, \varnothing, \tau)$ is a symmetric monoidal category. Now we will elaborate on that.

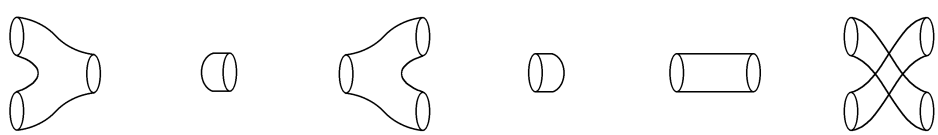
Let us first have a look at porting over the monoidal structure from **2Cob** to **2cob**. For k and l arbitrary, the oriented manifolds S^k and S^l are objects in **2cob**. If we are dealing with the nonstrict monoidal category **2Cob**, then we say there is a canonical isomorphism $S^k \amalg S^l \simeq S^{k+l}$, and S^{k+l} is also an object in **2cob**. In the strict case this isomorphism will of course be an identity: then we will just write $S^k \amalg S^l = S^{k+l}$. Also the disjoint union of two arrows in **2cob** will be another arrow in **2cob**, as easily shown, so **2cob** is closed under taking disjoint unions of objects and arrows. The object S^0 can be identified with the empty set, so, using formal symbols, this is all in accordance with the commuting diagrams in (3.1). See also the example of (3.12). So we directly see that $(\mathbf{2cob}, \amalg, \varnothing)$ can be regarded as a strict monoidal category.

Now we can have a look at porting over the *symmetric* structure. Let $\tau_{X,Y}$ be the twist cobordism class in **2Cob**, regarded as an arrow from $X \amalg Y$ to $Y \amalg X$. We should be aware that only some of these arrows are also lying in **2cob**: the arrow $\tau_{X,Y}$ only lies in **2cob** if $X \amalg Y = Y \amalg X$. This reflects the fact that the only isomorphisms in a skeleton, are arrows from an object to itself. The twist arrows remaining in **2cob** are $\tau_{k,l} : S^k \amalg S^l \to S^l \amalg S^k$. As **2cob** carries the same monoidal structure as **2Cob** we can say the arrows $\tau_{k,l}$ are also twist arrows with respect to **2cob**. Thus we can say $(\mathbf{2cob}, \amalg, \varnothing, \tau)$ is a symmetric monoidal category.

About the ordering of connected components of any object. Not every arbitrary object X in **2Cob** is equipped with an ordering or labeling of its connected components. On the other hand, every object in **2cob** is automatically equipped with such an ordering. We can however discuss some of the properties of these objects and their ordering within the context of Chapter 3: we can discuss any projection functor $P : \mathbf{2Cob} \to \mathbf{2cob}$. As we are already dealing with **2Cob** and **2cob** being strict monoidal categories, we could as well say that P is a strict monoidal functor.

We should recall that there is no canonical way of choosing this functor, but suppose we already made a choice. This means that for every object X in **2Cob**, having N connected components, an isomorphism $\iota_X : X \to S^N$ is chosen. This means that, after choosing P, we also chose an ordering of the connected components of every object in **2Cob**. But, this is only mentioned as a sidenote; the functor P was not really needed here, for porting over the monoidal structure. From now on we will mainly mention the skeleton **2cob**, not **2Cob**, so we will not mention any relation between objects (or arrows) in **2cob** and objects (or arrows) in **2Cob**.

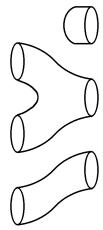
Generators and relations of 2cob. Every arrow $f : S^m \to S^n$ in **2cob**, regarded as a cobordism class, can be split up horizontally and vertically into six very simple basic arrows:



$\mu : S^2 \to S^1$ $\quad \eta : \varnothing \to S^1$ $\quad \delta : S^1 \to S^2$ $\quad \epsilon : S^1 \to \varnothing$ $\quad \mathrm{Id} : S^1 \to S^1$ $\quad \tau : S^2 \to S^2$ (8.2)

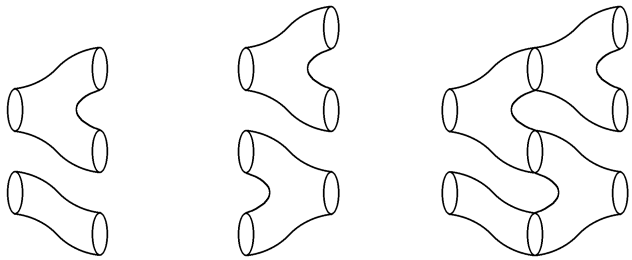
We will call these the *generators* of **2cob**. The arrow Id, also called a *cylinder* is nothing more than the identity arrow of S^1, so it satisfies $\mathrm{Id}^2 = \mathrm{Id}$. If we take arbitrary m-fold disjoint products $\mathrm{Id} \amalg \cdots \amalg \mathrm{Id}$, then we see that we obtain the identity arrows Id_m of all the other objects S^m. We can always write $f = \mathrm{Id}_n f = f \mathrm{Id}_m$, for every arrow $f : S^m \to S^n$, as expected. See also (7.3). Note that $\tau : S^2 \to S^2$ equals $\tau_{1,1} : S^1 \amalg S^1 \to S^1 \amalg S^1$, and we will call it the *twist generator*. Also note that τ is its own inverse: $\tau^2 = \mathrm{Id}_2 = \mathrm{Id} \amalg \mathrm{Id}$.

Every arrow f can be constructed by composing these generators horizontally and vertically. For example, a vertical composition of the arrows Id, μ and η yields another arrow, say $f := \mathrm{Id} \amalg \mu \amalg \eta$:



$f = \mathrm{Id} \amalg \mu \amalg \eta$

If f is an arrow from S^m to S^n, and if g is an arrow from S^n to S^p, then we can write gf as an arrow from S^m to S^p, which we will call the horizontal composition of f and g. For example, suppose we have the following two arrows: $f : S^2 \to S^3$, $f := \mathrm{Id} \amalg \delta$ and $g : S^3 \to S^3$, $g := \mu \amalg \delta$. Then the horizontal composition of these gives $gf = (\mu \amalg \delta)(\mathrm{Id} \amalg \delta)$:



$f = \mathrm{Id} \amalg \delta$ $\qquad$ $g = \mu \amalg \delta$ $\qquad$ $gf = (\mu \amalg \delta)(\mathrm{Id} \amalg \delta)$

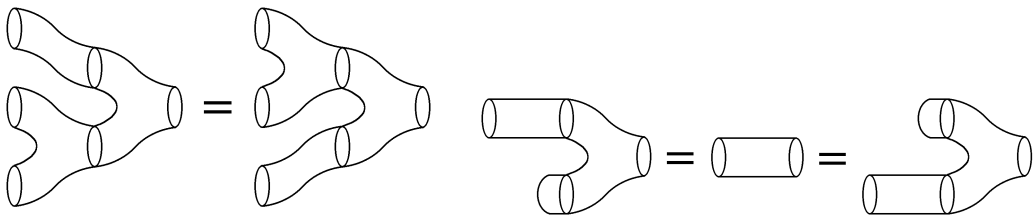
We should note that the connected components of any object are ordered. This ordering is induced by the disjoint product structure *and* by the fact that *every* object in the *skeleton* **2cob** is written as the disjoint product of circles. The pictures we use, and which can also be found in [7], reflect this; knowing that the connected components of any object are ordered *justifies* the way we depict the arrows. We will risk confusion if we use the same way of depicting arrows in **2cob** for depicting arrows in the category **2Cob**.

If M is a cobordism, lying in the cobordism class f, then we can always define a surjective special Morse function $m : M \to I \subset \mathbb{R}$. Recall that every connected component of any level set $m^{-1}(v)$ contains at most one critical point of m, and that we can split up I into closed subsets I_j, each of which containing at most one critical value, lying in its interior (or in ∂I). Using this m we can split up M into parts $M_j := m^{-1}(I_j)$. However, splitting up should also satisfy another condition: we can split up further until all parts of M_j containing more complex twists are split up into twist generators. Then we redefine I_j, according to the final splitting. Finally we can redefine parts $M_j := m^{-1}(I_j)$, and each part can be expressed as a disjoint product of generators. Note that this splitting will still satisfy the condition that each I_j contains at most one critical value. We cannot say with certainty that this would all be possible if m is not a special Morse function.

We should make clear that we use a Morse function to split up a cobordism, but we use a whole class of equivalent Morse functions to split up the corresponding cobordism class f. We say two Morse functions m and m' are *equivalent* if the results of splitting up f with respect to m and m' are the same.

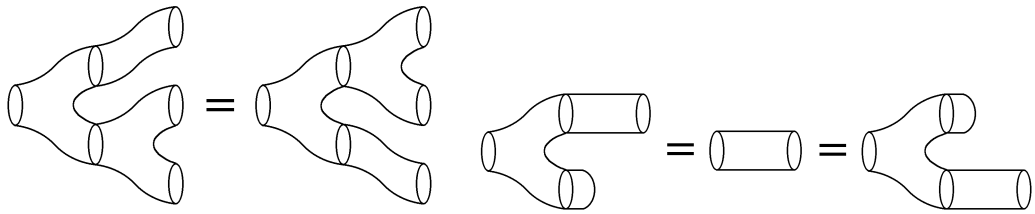
Now we can discuss some relations. Most of these relations can be explained by using special Morse functions. For example, assume $f : S^3 \to S^1$ is an arrow which can be split up into $f = \mu(\mu \amalg \mathrm{Id})$ with respect to one Morse function m. Just by choosing a different Morse function m', not equivalent to m, we can as well split up f into $f = \mu(\mathrm{Id} \amalg \mu)$. This yields the relation $\mu(\mu \amalg \mathrm{Id}) = \mu(\mathrm{Id} \amalg \mu)$. Similarly, we can

split up the cylinder Id into $\mu(\eta \amalg \mathrm{Id})$ or $\mu(\mathrm{Id} \amalg \eta)$, with respect to different Morse functions. Thus, using Morse functions, we obtain some relations, and these relations are similar to those of a monoid, see (4.3). The following diagrams should express these two relations:



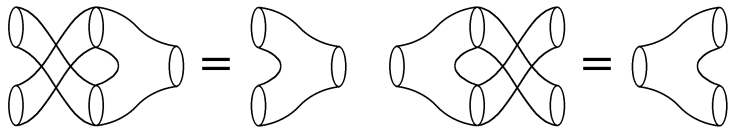
$\mu(\mu \amalg \mathrm{Id}) = \mu(\mathrm{Id} \amalg \mu)$ $\qquad$ $\mu(\eta \amalg \mathrm{Id}) = \mathrm{Id} = \mu(\mathrm{Id} \amalg \eta)$

Similarly, we can find relations similar to those of a comonoid, see (4.4), as expressed by the following diagrams:



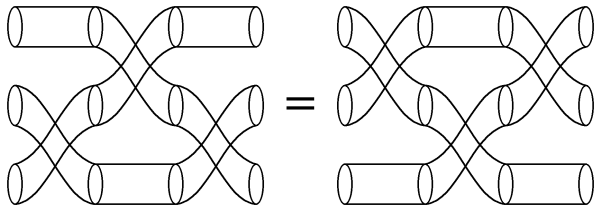
$(\delta \amalg \mathrm{Id})\delta = (\mathrm{Id} \amalg \delta)\delta$ $\qquad$ $(\epsilon \amalg \mathrm{Id})\delta = \mathrm{Id} = (\mathrm{Id} \amalg \epsilon)\delta$

There are some other relations however which cannot really be explained by using special Morse functions. For example, the properties of a general twist cobordism cannot be detected by using a (special) Morse function, but of course the information of interest is hidden in the corresponding cobordism class itself. The class of equivalent injections ι, telling us how the connected components of X, $\partial_+ M$, Y and $\partial_- M$ are related, carries this information. So, many times we can directly see whether two such cobordisms are equivalent, and the following examples of splitting up such cobordisms will reflect this. For example, the following diagrams express relations similar to those of commutative monoids and cocommutative comonoids, see (4.5):



$\mu\tau = \mu$ $\qquad$ $\tau\delta = \delta$

The naturality of the twist generator is another example of such a relation, see (4.6):

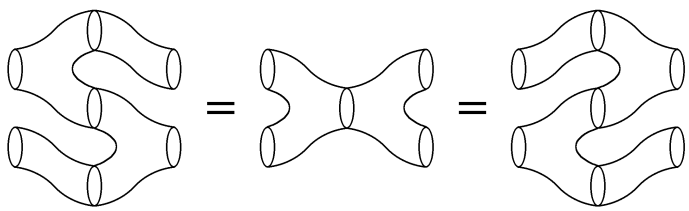


$(\tau \amalg \mathrm{Id})(\mathrm{Id} \amalg \tau)(\tau \amalg \mathrm{Id}) = (\mathrm{Id} \amalg \tau)(\tau \amalg \mathrm{Id})(\mathrm{Id} \amalg \tau)$

There are numerous other relations similar to this one: see (2.14) for the naturality of the twist in general.

Using the naturality of the twist generator τ, we can explain how to express a general twist arrow $\tau_{k,l} : S^k \amalg S^l \to S^l \amalg S^k$ as a horizontal composition of disjoint products of τ and Id. These general twist arrows can we used to generate arbitrary permutation cobordisms. As an example, let $\phi : S^4 \to S^4$ be the OPD sending every point in connected component k to a point in connected component $5-k$, and let $f : S^4 \to S^4$ be the arrow generated by ϕ. Then we can write $f = (\mathrm{Id} \amalg \tau \amalg \mathrm{Id})(\tau \amalg \tau)(\mathrm{Id} \amalg \tau \amalg \mathrm{Id})(\tau \amalg \tau)$.

Finally, we will present a relation similar to the Frobenius relation, see (4.7), and this relation can again be explained by using special Morse functions:



$$(\mu \amalg \mathrm{Id})(\mathrm{Id} \amalg \delta) = \delta\mu = (\mathrm{Id} \amalg \mu)(\delta \amalg \mathrm{Id})$$

The relations introduced so far, also called the relations of **2cob**, can be regarded as a minimal set of relations, generating all other possible relations. These relations also tell us that the circle, the object S^1, can be regarded as a commutative Frobenius object in **2cob**. We know that this object S^1 generates all other objects S^m in **2cob**. We also know that every arrow in **2cob** can be written as a horizontal and vertical composition of the six generators. These six generators are the identity arrow Id and the five basic arrows μ, η, δ, ϵ and τ. The conclusion will be that **2cob** is a free symmetric monoidal category on a commutative Frobenius object. We can again apply (4.14) and (4.15), and then we can say that S^1 is *the* commutative Frobenius object generating all other commutative Frobenius objects S^m in **2cob**.

The snake relation. As (4.9) and (4.10) show, any Frobenius object satisfies the snake relation. As we know, the circle can be regarded as a Frobenius object in **2cob**, so it also satisfies the snake relation. On the other hand, the cylinder generated by any object in **nCob** in general satisfies the snake decomposition, see (6.5). The snake relation and the snake decomposition are not quite the same, but they are related. The snake relation is a result of the Frobenius relation, but it can also be regarded as a slightly modified special case of the snake decomposition. In **2cob** we can split up the cylinder Id into four parts. One of these parts can be regarded as an arrow $\beta : S^1 \amalg S^1 \to \varnothing$, satisfying $\beta = \epsilon\mu$, and another of these parts can be regarded as an arrow $\gamma : \varnothing \to S^1 \amalg S^1$, satisfying $\gamma = \delta\eta$. However, we should note that the snake relation does not apply in **nCob** for all n. To be more precise, it does not apply in **1Cob**, the category discussed in Chapter 9. Let p_+ be a positively oriented point, and let p_- be a negatively oriented point, both regarded as objects in **1Cob**. Arrows like $\beta_+ : p_+ \amalg p_+ \to \varnothing$ or $\beta_- : p_- \amalg p_- \to \varnothing$ are simply impossible in **1Cob**, so it is impossible to find a relation in **1Cob** similar to the snake relation in **2cob**. On the other hand, arrows like $\beta : p_+ \amalg p_- \to \varnothing$ *are* possible in **1Cob**, so at least the snake decomposition still holds in **1Cob**.

Cobordisms in 2cob with some connected components being closed. Let f, g and h be arrows in **2cob**, and assume g and h can be represented by closed cobordisms. Then we know that we have the identities $f \amalg g = g \amalg f$ and $gh = hg = g \amalg h = h \amalg g$, as already explained in (7.5). Of course this means that the pictures corresponding to the complex arrows $f \amalg g$ and $g \amalg f$, looking different at first sight, should be regarded as identical. The same applies to the pictures corresponding to each of the complex arrows gh, hg, $g \amalg h$ and $h \amalg g$. We could call this an *artifact* of the $\amalg$-operator. As a consequence we might wonder, at first sight, whether we can interpret the already used pictures correctly. But, there is no real problem, just because it is mainly the ordering of the connected components of the objects that matters. The structure of the arrows does not matter that much, so this artifact does not really contradict anything.

The normal form of a connected cobordism in 2cob. Every connected cobordism M can be described as a surface of genus g, with n open disks missing. If k of these disks contribute to the in-boundary, and if l of these disks contribute to the out-boundary, then $k + l = n$. Then M can be described as a genus g cobordism from S^k to S^l. Of course we assume M itself is not empty. According to a theorem, which we will not explicitly prove here, the corresponding cobordism class can always be decomposed into three parts, say $M = M_3M_2M_1$, according to a standard scheme. We will call this scheme the *normal form* of a connected surface, and to prove this theorem we can use the relations of **2cob**. The first part M_1, also called the *in-part*, will be an arrow from S^k to S^1 of the following form:

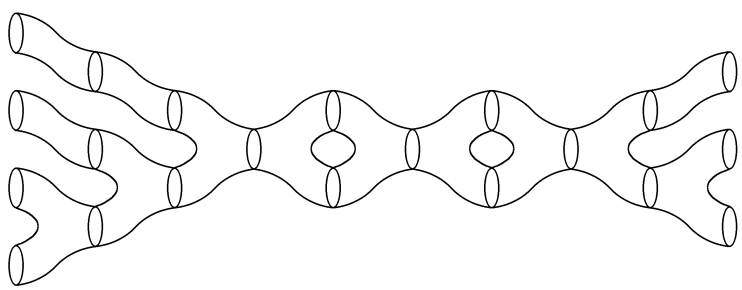
$$M_1 = \mu(\mu \amalg \mathrm{Id})(\mu \amalg \mathrm{Id} \amalg \mathrm{Id}) \cdots (\mu \amalg \mathrm{Id} \amalg \cdots \amalg \mathrm{Id}) = \mu(\mu \amalg \mathrm{Id})(\mu \amalg \mathrm{Id}_2) \cdots (\mu \amalg \mathrm{Id}_{k-2}).$$

The middle part M_2, also called the *topological part*, will be an arrow from S^1 to S^1, of the form $M_2 = (\mu\delta)^g$. This part can always be interpreted as a genus g surface with two disks missing. The last part M_3, also called the *out-part*, will be an arrow from S^1 to S^l of the following form:

$$M_3 = (\delta \amalg \mathrm{Id} \amalg \cdots \amalg \mathrm{Id}) \cdots (\delta \amalg \mathrm{Id} \amalg \mathrm{Id})(\delta \amalg \mathrm{Id})\delta = (\delta \amalg \mathrm{Id}_{l-2}) \cdots (\delta \amalg \mathrm{Id}_2)(\delta \amalg \mathrm{Id})\delta.$$

Of course, if $k = 0$ then we assume $M_1 = \eta$ and if $k = 1$ then we assume $M_1 = \mathrm{Id}$. Similarly, if $l = 0$ then we assume $M_3 = \epsilon$ and if $l = 1$ then we assume $M_3 = \mathrm{Id}$. Finally, if $g = 0$ then we assume $M_2 = \mathrm{Id}$.

As an example, we can have a look at the normal form of the cobordism (class) $M : S^4 \to S^3$ of genus two:



$$(\delta \amalg \mathrm{Id})\delta\mu\delta\mu\delta\mu(\mu \amalg \mathrm{Id})(\mu \amalg \mathrm{Id} \amalg \mathrm{Id})$$

It is striking that the generator τ is not involved in this context of connected cobordisms and expressing them using the normal form. But, as we *are* dealing with connected cobordisms, we should realize that any twist arrow, which eventually appears, will be cancelled by the relations $\mu\tau = \mu$ and $\tau\delta = \delta$. Of course, if we already decomposed a connected cobordism into generators, and assume a twist generator is involved by accident, then we can always use naturality of τ to move it left or right, until it meets a generator μ or δ. Just realize that both connected components of τ are somehow connected to each other, by a path within the connected cobordism, so the fact that the cobordism is connected assures us that such a generator μ or δ can always be found. Such a generator will cancel the twist generator.

Thus it is enough to use the generators μ, η, δ, ϵ and Id, and we should realize that using (special) Morse functions is enough for finding out how to split up any connected cobordism into these five generators.

General cobordisms. Let $M : S^j \to S^k$ be a general cobordism class in **2cob**. Then M can be considered as the result of joining multiple connected cobordisms (or in fact cobordism *classes*) M_i. Each of these M_i, when viewed as independent cobordism classes, can be decomposed into generators, according to the normal form. However, the domain (and codomain) of M should in general be considered as a mixed up composition of the domains (and codomains) of M_i, not just simply as the disjoint product of these domains (and codomains). For example, if $M = (\tau \amalg \mathrm{Id})(\mu \amalg \delta)(\mathrm{Id} \amalg \tau)$ then it is an arrow from S^3 to S^3. The

connected components of M can be written as $M_1 = \mu$ and $M_2 = \delta$, but we cannot say M equals $M_1 \amalg M_2$ or $M_2 \amalg M_1$. On the other hand we can write $M = (\tau \amalg \mathrm{Id})(M_1 \amalg M_2)(\mathrm{Id} \amalg \tau) = (\mathrm{Id} \amalg \tau)(M_2 \amalg M_1)(\tau \amalg \mathrm{Id})$.

In general, if $M : S^j \to S^k$ has m connected components M_i, we can always write

$$M = T_2(M_1 \amalg \cdots \amalg M_m)T_1.$$

The arrows $T_1 : S^j \to S^j$ and $T_2 : S^k \to S^k$ can be regarded as general permutation cobordisms, so they can be regarded as being a horizontal and vertical composition of the generators Id and τ. Recall that the arrows M_i, when written in normal form, do not contain *any* twist generators.

If $N : S^k \to S^l$ is another general cobordism, with n connected components and written as

$$N = U_2(N_1 \amalg \cdots \amalg N_n)U_1,$$

where $U_1 : S^k \to S^k$ and $U_2 : S^l \to S^l$ are general permutation cobordisms, then

$$P := NM = U_2(N_1 \amalg \cdots \amalg N_n)U_1T_2(M_1 \amalg \cdots \amalg M_m)T_1$$

is an arrow from S^j to S^l. Using the naturality of the general twist we can move the middle part U_1T_2 to the left or to the right, so that T_1, U_2, or some generator μ or δ absorbs the involved twist generators. Then P can be rewritten as

$$P = V_2(P_1 \amalg \cdots P_p)V_1,$$

where $V_1 : S^j \to S^j$ and $V_2 : S^l \to S^l$ are again general permutation cobordisms.

We conclude by observing that the only permutation cobordisms appearing, are those which do a permutation of those connected components of S^j and S^k which are *not* connected via M itself.

9 The category of 1-cobordisms

In this chapter we will study the category **1Cob** of cobordism classes of dimension 1. We will also study **1cob**, a skeleton of **1Cob**. But first we will look at the category **1Cob**$'$, as an intermediate step, which we will call a *stripped version* of **1Cob**. This category is a full subcategory of **1Cob** (but *not* a skeleton), and it is still closed under the ordinary disjoint union $\amalg$, so we can copy the symmetric monoidal structure from **1Cob** to **1Cob**$'$ without any difficulties. Here we can already study some generators and relations, without much confusion. We have, for example, relations like the naturality of the twist and the snake decomposition. Using some fuzzy language we could say that **1Cob**$'$ is *nearly* a skeleton of **1Cob**.

But finally we will focus on **1cob**. We will also discuss **1cob** when we will study 1-dimensional topological quantum field theories in Section 10.5. Before doing so, it would be nice if we can also introduce **1cob** with symmetric monoidal structure. As **1Cob** is already a symmetric monoidal category, we would like to find a way of porting over this structure to **1cob**. However, the monoidal structure of **1Cob** cannot just be copied to **1cob**. It needs to be modified. The monoidal structure can be slightly altered in this case, and this chapter will present a special proposal for this altered monoidal structure. The result will be that also **1cob** can be regarded as a monoidal category (even a strict one). After that, we will find out how to copy the symmetric structure from **1Cob** to **1cob**. Then we can rewrite the generators and relations we already found for **1Cob**$'$. Finally we could say that **1cob** (but also **1Cob**$'$) is a free symmetric monoidal category on a *dualizable object*. This basic dualizable object (p_+) generates all other dualizable objects in **1cob**.

9.1 Introducing 1Cob, 1Cob$'$ and 1cob

1Cob, the category of oriented cobordisms in dimension 1. Any object X in **1Cob**, fully written as $(\mathbf{1Cob}, \amalg, \varnothing, \tau)$ and regarded as a symmetric monoidal category, is a compact oriented manifold of dimension 0, and has N_X connected components. Any manifold of dimension 0 can be assumed to have no boundary, so they are automatically closed. Using somewhat less abstract language, any object is a finite collection of oriented points. We define N_X^+ as the number of positively oriented points in X and N_X^- as the number of negatively oriented points in X. These numbers should, of course, satisfy $N_X^+ + N_X^- = N_X$. We then also say that X has (N_X^+, N_X^-) connected components.

Any arrow in **1Cob**, from X to Y, is a class of equivalent cobordisms of dimension 1, from X to Y. Any connected component X_j of X is an oriented point, thus we could say that any connected component M_j of a 1-cobordism M, with boundary ∂M_j not empty, has the topology of an oriented line with two ends.

If ∂M_j is empty, then M_j itself is represented by a closed connected 1-manifold. The only closed connected 1-manifold existing is the circle. There exist OPDs from S^1 to $\overline{S^1}$ and we know $\partial S^1 = \partial \overline{S^1} = \varnothing$. Now see (7.1) applied to the special case of X and Y both being empty. Then S^1 and $\overline{S^1}$ lie in the same cobordism class, thus the arrow corresponding to M_j will be the standard circle *without orientation*. From now on, if we mention any 0-manifold or any connected 1-manifold with boundary, we will assume it is oriented.

We claim without proof that, contrary to the situation of **2Cob**, *not* for any arbitrary pair of objects X and Y there exists at least one cobordism from X to Y. We define the *signature* of each object X as $\Sigma(X) := N_X^+ - N_X^-$. We claim without proof that if and only if $\Sigma(X) = \Sigma(Y)$, then there exists at least one cobordism from X to Y. As a consequence, there exist no cobordisms from X to Y if $N_Y - N_X$ is an odd number.

If for example $N_X = N_Y = 0$, then $N_X^+ - N_X^- = N_Y^+ - N_Y^- = 0$, thus a (closed) cobordism M, with m connected components, exists from X to Y. Then any connected component M_j is closed, so M can be regarded as a collection of m circles without orientation.

Basic objects in 1Cob. If we choose an arbitrary point p without orientation, then p_+ and p_- are oriented points. We can use these two points as basic objects in **1Cob**. Recall that there are no OPDs possible between p_+ and p_-, thus no cobordisms exist from p_+ to p_-. Of course we already know that $\Sigma(p_+) = 1$ and $\Sigma(p_-) = -1$, which implies that no arrows exist from p_+ to p_-.

We can take disjoint powers of p_+ and p_-, for example $p_+^k := p_+ \amalg \cdots \amalg p_+$. We can also take arbitrary disjoint products of p_+ and p_-, for example $p_- \amalg p_+ \amalg p_- \amalg p_-$. There are 2^n possible choices of such disjoint

products of n oriented points, and these are also objects in **1Cob**. We could look at a stripped version of **1Cob**, only containing these objects, written as $\mathbf{1Cob}'$. Then $\mathbf{1Cob}'$ is a full subcategory of **1Cob**. For any object X in **1Cob** having $(k,l) := (N_X^+, N_X^-)$ connected components, we can say an isomorphism exists from X to an object X' in $\mathbf{1Cob}'$. As the connected components of X' in $\mathbf{1Cob}'$ are already ordered, we can say that each such an isomorphism also induces an ordering of the connected components of X.

Some other examples of cobordisms which are not possible in 1Cob. Knowing that the rule $\Sigma(X) = \Sigma(Y)$ must be satisfied, we can say that there exist no cobordisms in **1Cob** like the following:

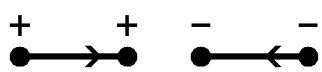
$$M_1 : p_+ \to p_+ \amalg p_+ \amalg p_+, \quad M_2 : p_+ \amalg p_+ \to \varnothing, \quad M_3 : p_+ \to p_- \amalg p_+, \quad M_4 : p_+ \to \varnothing.$$

As mentioned before, we can say that the snake *relation*, a relation valid in **2Cob**, will not be applicable in **1Cob**, knowing that cobordisms like M_1 not being possible in **1Cob** is mainly responsible for this. But, it will get clear later that we can at least do snake decompositions.

Properties of the category $\mathbf{1Cob}'$. In $\mathbf{1Cob}'$ we can simply take ordinary disjoint products of objects and arrows, and the resulting objects and arrows are again lying in $\mathbf{1Cob}'$. Or, to say differently, $\mathbf{1Cob}'$ is closed under taking disjoint unions of objects and arrows. Also the empty set $\varnothing = p_+^0 = p_-^0$ is an object in $\mathbf{1Cob}'$, so we can say $\mathbf{1Cob}'$ carries the same monoidal structure as **1Cob**. Thus $(\mathbf{1Cob}', \amalg, \varnothing)$ can also be regarded as a strict monoidal category.

Any twist arrow in **1Cob**, also lying in $\mathbf{1Cob}'$, is also properly functioning as a twist arrow in $\mathbf{1Cob}'$. So we can directly say that $\mathbf{1Cob}'$ carries the same symmetric monoidal structure as **1Cob**. Then also $\mathbf{1Cob}' = (\mathbf{1Cob}', \amalg, \varnothing, \tau)$ is a symmetric monoidal category, thus it is a full symmetric monoidal subcategory of **1Cob**.

Arrows and generators in $\mathbf{1Cob}'$. The basic cylinders in $\mathbf{1Cob}'$ are the identity arrows $\mathrm{Id}_+ : p_+ \to p_+$ and $\mathrm{Id}_- : p_- \to p_-$. By using disjoint products of these basic cylinders, we can construct general cylinders. The following diagrams should express these two arrows:



Id_+ $\qquad$ Id_-

The *arrows* drawn on the lines, not to be regarded as arrows in a category, indicate the orientation of the 1-manifold in question.

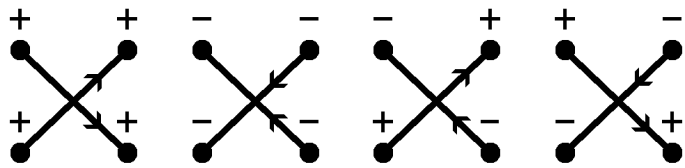
We also have some basic twists

$$\begin{aligned} \tau_+ &: p_+ \amalg p_+ \to p_+ \amalg p_+, & \tau_- &: p_- \amalg p_- \to p_- \amalg p_-, \\ \tau_{+-} &: p_+ \amalg p_- \to p_- \amalg p_+, & \tau_{-+} &: p_- \amalg p_+ \to p_+ \amalg p_-, \end{aligned}$$

satisfying

$$\begin{aligned} \tau_+^2 &= \mathrm{Id}_{++} = \mathrm{Id}_+ \amalg \mathrm{Id}_+, & \tau_-^2 &= \mathrm{Id}_{--} = \mathrm{Id}_- \amalg \mathrm{Id}_-, \\ \tau_{-+}\tau_{+-} &= \mathrm{Id}_{+-} = \mathrm{Id}_+ \amalg \mathrm{Id}_-, & \tau_{+-}\tau_{-+} &= \mathrm{Id}_{-+} = \mathrm{Id}_- \amalg \mathrm{Id}_+ . \end{aligned}$$

Note that $\tau_{-+} = \tau_{+-}^{-1}$ and $\tau_{+-} = \tau_{-+}^{-1}$. The following diagrams should express these four arrows:



τ_+ τ_- τ_{+-} τ_{-+}

By using disjoint products and horizontal compositions of these four arrows and identity arrows we can construct general twists and permutation arrows. Note that we cannot say that $\mathbf{1Cob}'$ is a skeleton of $\mathbf{1Cob}$, as arrows like τ_{+-} and τ_{-+} are still allowed, which means that $\mathbf{1Cob}'$ still contains isomorphisms between distinct objects.

We have another four arrows:

$$\beta : p_+ \amalg p_- \to \varnothing, \quad \gamma : \varnothing \to p_+ \amalg p_-, \quad \bar{\beta} : p_- \amalg p_+ \to \varnothing \quad , \bar{\gamma} : \varnothing \to p_- \amalg p_+.$$

The following diagrams should express these four arrows:

β γ $\bar{\beta}$ $\bar{\gamma}$

Two of these can be expressed as compositions of some of the other arrows:

$$\bar{\beta} = \beta\tau_{-+} \quad , \quad \bar{\gamma} = \tau_{+-}\gamma.$$

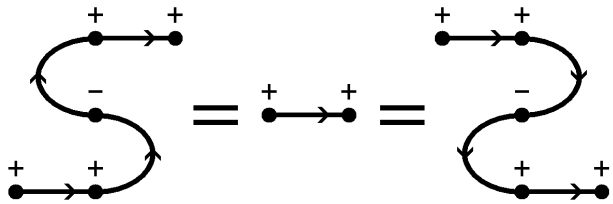
Note that this implies that

$$\bar{\beta}\bar{\gamma} = \beta\tau_{-+}\tau_{+-}\gamma = \beta\gamma. \tag{9.1}$$

Note that the arrow $\beta\gamma : \varnothing \to \varnothing$ represents a connected cobordism without boundary, thus it represents the circle. Also note that identity (9.1) reflects the fact that the arrow representing the circle does not carry any information about its orientation. Only after splitting up a circle into two parts with boundary we see that each part will be oriented again.

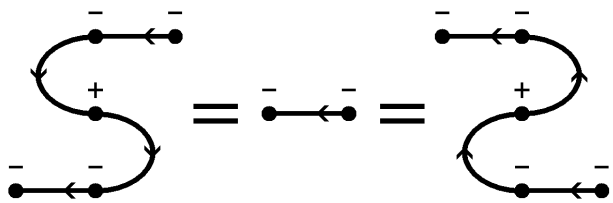
So finally we have eight independent arrows: Id_+, Id_-, τ_+, τ_-, τ_{+-}, τ_{-+}, β and γ. We claim that every other arrow in $\mathbf{1Cob}'$ can be written as a horizontal and vertical composition of these arrows. We could say these arrows are generators of $\mathbf{1Cob}'$.

Some relations in $\mathbf{1Cob}'$. There are a few relations which apply to the generators of $\mathbf{1Cob}'$. The relation of the snake decomposition holds in any $\mathbf{nCob}$. For example, we can apply the snake decomposition to the cylinders Id_+ and Id_-. For Id_+ we have the following relation:



$$(\beta \amalg \mathrm{Id}_+)(\mathrm{Id}_+ \amalg \bar{\gamma}) = \mathrm{Id}_+ = (\mathrm{Id}_+ \amalg \bar{\beta})(\gamma \amalg \mathrm{Id}_+) \tag{9.2}$$

And for Id_- we have the following relation:



(9.3)

$$(\bar{\beta} \amalg \mathrm{Id}_-)(\mathrm{Id}_- \amalg \gamma) = \mathrm{Id}_- = (\mathrm{Id}_- \amalg \beta)(\bar{\gamma} \amalg \mathrm{Id}_-)$$

Of course the naturality of the twist implies other relations.

We could express the relation between the objects p_+ and p_- as follows. We could say p_- is the object dual to p_+, and the existence of the generators β and γ and the snake decomposition support this. So, just as we called **2cob** a free symmetric monoidal category on a *commutative Frobenius object*, we could say that $\mathbf{1Cob}'$ is a free symmetric monoidal category on a *dualizable object*. (A similar statement can be found in [9].) For that we can use the arrow $\beta : p_+ \amalg p_- \to \varnothing$ and its dual arrow, which is $\bar{\gamma} : p_- \amalg p_+ \to \varnothing$ in this case. Thus $(p_+, p_-, \beta, \bar{\gamma})$ can be regarded as a dualizable object, generating all other dualizable objects in $\mathbf{1Cob}'$. (Note that $(p_+, p_-, \beta, \bar{\gamma})$ induces another dualizable object $(p_-, p_+, \bar{\beta}, \gamma)$.) Note that the snake decomposition implies the *zig-zag identities*, see (2.21), for these objects.

Splitting up a cobordism in $\mathbf{1Cob}'$ into generators. As we know, we can use a special Morse function for splitting up any cobordism into disjoint products of generators. We are a bit lucky in this case, as any level set of a Morse function on a 1-manifold is a finite collection of points. Then we can always say that especially all critical points are mutually separate, so *every* Morse function on a 1-manifold is automatically a special Morse function. Thus, after picking an arbitrary Morse function, we can always split up any cobordism into generators, at least if it reaches its global minimum at the in-boundary, and its global maximum at the out-boundary. The relation of the snake decomposition can again be explained by using Morse functions, as already explained in Section 6.3.

Again, relations coming from the naturality of the twist cannot be explained by using Morse functions. Only the twist parts cannot be detected by using any Morse function, but again we can use the information hidden in the class of equivalent injections ι telling us how the connected components of X, $\partial_+ M$, Y and $\partial_- M$ are related. As already pointed out before, any arrow, or cobordism class, contains this information.

Thus, if f is a cobordism class, and if M is a cobordism lying in f, then we can define a surjective Morse function $m : M \to I \subset \mathbb{R}$. Then we can split up I, using this m, into closed subsets I_j, each of which containing at most one critical value, lying in its interior. Using this m we can again split up M into parts $M_j := m^{-1}(I_j)$. Again, splitting up should also satisfy another condition: we can split up further until all parts of M_j containing more complex twists are split up into twist generators. Then we redefine I_j, according to the final splitting. Finally we can again redefine parts $M_j := m^{-1}(I_j)$, each expressed as a disjoint product of generators.

Invertible cobordisms. If we have arbitrary objects X and Y in **1Cob**, and if we found an orientation preserving diffeomorphism (or just an orientation preserving *map*) $\psi : X \to Y$, then this OPD induces a unique cobordism class, and it generates an isomorphism in **1Cob**. The reverse is also true: if an isomorphism exists from X to Y, then there exists and OPD $\psi : X \to Y$. The fact that β and γ are not invertible reflects this. To conclude, we can say that an arrow $f : X \to Y$ is invertible if and only if $(N_X^+, N_X^-) = (N_Y^+, N_Y^-)$.

The basic isomorphisms in $\mathbf{1Cob}'$ are Id_+, Id_-, τ_+, τ_-, τ_{+-} and τ_{-+}. Only the last two are isomorphisms between distinct objects. If $\mathbf{1Cob}'$ has $\mathcal{C}'$ as an arbitrary skeleton, which is also a skeleton of **1Cob** itself, then we can say that $\mathcal{C}'$ does not contain the arrows τ_{+-} and τ_{-+}. For any $\mathcal{C}'$ and for any pair of distinct objects X and Y in $\mathcal{C}'$ we can say that $(N_X^+, N_X^-) \neq (N_Y^+, N_Y^-)$.

1cob, a skeleton of 1Cob. We can choose a standard ordering for any object X in $\mathbf{1Cob}'$ with (N_X^+, N_X^-) connected components. There exist isomorphisms from X to $p_+^k \amalg p_-^l$, where $k := N_X^+$ and $l := N_X^-$. The collection of these objects of the form $p_+^k \amalg p_-^l$ (with k and l arbitrary) can be used as a skeleton of $\mathbf{1Cob}'$ and $\mathbf{1Cob}$. We will use the notation $\mathbf{1cob}$ to refer to this skeleton, and we can say that $\mathbf{1cob}$ is also a full subcategory of $\mathbf{1Cob}'$. Note that the empty set, regarded as the unit object in $\mathbf{1Cob}$, is also an object in $\mathbf{1cob}$, thus we can introduce it as the candidate unit object in $\mathbf{1cob}$. Now we will try to find operators $\square'$, $1'$ and τ' so that also $(\mathbf{1cob}, \square', 1', \tau')$ is a symmetric monoidal category. First we will prove that $(\mathbf{1cob}, \square', 1')$ can be regarded as a *strict* monoidal category. As we know, $1' = 1 = \varnothing$, so we will write $(\mathbf{1cob}, \square', 1, \tau')$ from now on.

9.2 Porting over the structure from 1Cob to 1cob

We know that $\mathbf{1Cob}$ is shorthand notation for $(\mathbf{1Cob}, \amalg, \varnothing, \tau)$, so we are again dealing with a symmetric monoidal category. Also $\mathbf{1Cob}' = (\mathbf{1Cob}', \amalg, \varnothing, \tau)$ is a symmetric monoidal category. Then we can try turning $\mathbf{1cob}$ into a symmetric monoidal category also, but this step will not be totally trivial, as was the case for $\mathbf{2cob}$.

Taking the graded disjoint union of objects. If we just take an ordinary disjoint product of two arbitrary objects $p_+^k \amalg p_-^l$ and $p_+^m \amalg p_-^n$ in $\mathbf{1cob}$, then we obtain $p_+^k \amalg p_-^l \amalg p_+^m \amalg p_-^n$, which is *not* an object in $\mathbf{1cob}$, except in some cases. Thus we cannot say that $\mathbf{1cob}$ is closed under the ordinary disjoint product $\amalg$. Thus, we need to define $\amalg'$, regarded as a candidate operator on $\mathbf{1cob}$, differently. We first need a projection functor $P : \mathbf{1Cob} \to \mathbf{1cob}$ (or $P' : \mathbf{1Cob}' \to \mathbf{1cob}$) to do this. So we define $\mu' := P\mu(I \times I)$, where $I : \mathbf{1cob} \to \mathbf{1Cob}$ is the canonical injection functor. If X is an object in $\mathbf{1Cob}$ with (k, l) connected components, then we define $X' = P_0(X) := p_+^k \amalg p_-^l$.

For any two objects $X' := p_+^k \amalg p_-^l$ and $Y' := p_+^m \amalg p_-^n$ in $\mathbf{1cob}$, we will write $X' \amalg' Y' = (X' \amalg Y')' = P_0(X' \amalg Y')$, if we apply (3.2) to $\amalg$. We can do something similar in case of arrows, see (3.3).

Now we will define $\amalg'$ explicitly for objects. Let $X'_+ := p_+^k$ and let $X'_- := p_-^l$, then $X' = X'_+ \amalg X'_-$. Then we define:

$$X' \amalg' Y' = (X'_+ \amalg X'_-) \amalg' (Y'_+ \amalg Y'_-) = P_0(X'_+ \amalg X'_- \amalg Y'_+ \amalg Y'_-) := X'_+ \amalg Y'_+ \amalg X'_- \amalg Y'_-. \tag{9.4}$$

Or written differently:

$$X' \amalg' Y' = (p_+^k \amalg p_-^l) \amalg' (p_+^m \amalg p_-^n) = p_+^k \amalg p_+^m \amalg p_-^l \amalg p_-^n = p_+^{k+m} \amalg p_-^{l+n}.$$

But also:

$$Y' \amalg' X' = (p_+^m \amalg p_-^n) \amalg' (p_+^k \amalg p_-^l) = p_+^m \amalg p_+^k \amalg p_-^n \amalg p_-^l = p_+^{m+k} \amalg p_-^{n+l} = \cdots = X' \amalg' Y'.$$

Note that we already treat $\mathbf{1Cob}$ as a strict monoidal category, thus its associator is trivial, thus we can write

$$(X'_+ \amalg X'_-) \amalg' (Y'_+ \amalg Y'_-) = P_0((X'_+ \amalg X'_-) \amalg (Y'_+ \amalg Y'_-)) = P_0(X'_+ \amalg X'_- \amalg Y'_+ \amalg Y'_-),$$

which was used in (9.4). Also note that $X' \amalg' Y'$ is again an object in $\mathbf{1cob}$, thus we can write

$$(X' \amalg' Y')_+ = (X'_+ \amalg Y'_+ \amalg X'_- \amalg Y'_-)_+ = X'_+ \amalg Y'_+ = Y'_+ \amalg X'_+. \tag{9.5}$$

Similarly we can write $(X' \amalg' Y')_- = X'_- \amalg Y'_- = Y'_- \amalg X'_-$.

Note that for any object $X' = p_+^k \amalg p_-^l$ in $\mathbf{1cob}$ we have $P_0(X') = X'$, which implies the following:

$$p_+^k \amalg p_-^l = P_0(p_+^k \amalg p_-^l) = (p_+^k \amalg p_-^l)' = p_+^k \amalg' p_-^l = p_-^l \amalg' p_+^k.$$

We know that P_0 is already defined, but we are still free to choose P_1. Choosing P_1 is a matter of choosing isomorphisms $\iota_X : X \to p_+^k \amalg p_-^l$ for any object X in **1Cob**, with $(k,l) = (N_X^+, N_X^-)$ connected components. Of course $X' \amalg Y'$ is still an object in **1Cob**, for any two objects X' and Y' in **1cob**. We can make a special choice for P in case of disjoint products of any pair (X', Y'). We define $\iota_{X' \amalg Y'}$, which is an arrow from $X' \amalg Y'$ to $X' \amalg' Y' = P_0(X' \amalg Y')$, as

$$\iota_{X' \amalg Y'} : X'_+ \amalg X'_- \amalg Y'_+ \amalg Y'_- \to X'_+ \amalg Y'_+ \amalg X'_- \amalg Y'_- \quad , \quad \iota_{X' \amalg Y'} := \mathrm{Id}_{X'_+} \amalg \tau_{X'_-, Y'_+} \amalg \mathrm{Id}_{Y'_-} \,. \tag{9.6}$$

In general we will restrict ourselves to a projection functor P, related to natural isomorphisms ι_X which can be written as a (horizontal and vertical) composition of only three generators: Id_+, Id_- and τ_{-+}. Of course this is with respect to the full subcategory $\mathbf{1Cob}'$, instead of **1Cob** itself, and any vertical composition of these three generators is based on the ordinary disjoint product $\amalg$, not $\amalg'$. This restriction can be explained using somewhat less abstract language: the mutual ordering of the positively oriented points, viewed as a part of the total object X in $\mathbf{1Cob}'$, will not be changed by ι. The same applies to the mutual ordering of the negatively oriented points, viewed as a part of the total object X in $\mathbf{1Cob}'$. We could say that the positively oriented points will all be moved to the left, without destroying the mutual ordering.

If we take a look again at (9.6), then we see that the twist arrow $\tau_{X'_-, Y'_+}$ can indeed be written as a composition of these three generators, and that indeed the mutual ordering is not destroyed. This is why we will call the operator $\amalg'$ the *graded disjoint union*, or the graded disjoint product.

Taking the graded disjoint union of arrows. Now we can write down $\amalg'$ explicitly for arrows. Let $f' : A' \to B'$ and $g' : C' \to D'$ be arrows in **1cob**. Using (3.3) and (9.6) we can define $f' \amalg' g'$, regarded as an arrow from $A' \amalg' C'$ to $B' \amalg' D'$, as follows:

$$\begin{aligned} f' \amalg' g' &:= P_1(f' \amalg g') = (f' \amalg g')' = \iota_{B' \amalg D'}(f' \amalg g')\iota^{-1}_{A' \amalg C'} \\ &= (\mathrm{Id}_{B'_+} \amalg \tau_{B'_-, D'_+} \amalg \mathrm{Id}_{D'_-})(f' \amalg g')(\mathrm{Id}_{A'_+} \amalg \tau_{C'_+, A'_-} \amalg \mathrm{Id}_{C'_-}). \end{aligned} \tag{9.7}$$

Now we can conclude that **1cob** is closed under the graded disjoint union $\amalg'$. As a consequence we can write the following:

$$\mathrm{Id}_{X'} \amalg' \mathrm{Id}_{Y'} = P_1(\mathrm{Id}_{X'} \amalg \mathrm{Id}_{Y'}) = P_1(\mathrm{Id}_{X' \amalg Y'}) = \iota_{X' \amalg Y'} \, \mathrm{Id}_{X' \amalg Y'} \, \iota^{-1}_{X' \amalg Y'} = \mathrm{Id}_{X' \amalg' Y'} \,. \tag{9.8}$$

The identity (2.8), which already applies to $\amalg$, was also used here. So, $\amalg$ satisfying (2.8) implies that $\amalg'$ also satisfies (2.8).

Assume f' and g' can be written as $f' = f'_+ \amalg f'_-$ and $g' = g'_+ \amalg g'_-$, with arrows

$$f'_+ : A'_+ \to B'_+ \quad f'_- : A'_- \to B'_- \quad g'_+ : C'_+ \to D'_+ \quad g'_- : C'_- \to D'_-$$

and note that we can write $f' \amalg g' = (f'_+ \amalg f'_-) \amalg (g'_+ \amalg g'_-) = f'_+ \amalg f'_- \amalg g'_+ \amalg g'_-$. Then (9.7) and naturality of the twist τ will imply that

$$\begin{aligned} f' \amalg' g' &= (\mathrm{Id}_{B'_+} \amalg \tau_{B'_-, D'_+} \amalg \mathrm{Id}_{D'_-})(f'_+ \amalg f'_- \amalg g'_+ \amalg g'_-)(\mathrm{Id}_{A'_+} \amalg \tau_{C'_+, A'_-} \amalg \mathrm{Id}_{C'_-}) \\ &= f'_+ \amalg g'_+ \amalg f'_- \amalg g'_-. \end{aligned} \tag{9.9}$$

As a special example of (9.9) we can write:

$$\begin{aligned} \mathrm{Id}_{X'} \amalg' \mathrm{Id}_{Y'} &= \mathrm{Id}_{X'_+ \amalg X'_-} \amalg' \mathrm{Id}_{Y'_+ \amalg Y'_-} = (\mathrm{Id}_{X'_+} \amalg \mathrm{Id}_{X'_-}) \amalg' (\mathrm{Id}_{Y'_+} \amalg \mathrm{Id}_{Y'_-}) \\ &= \mathrm{Id}_{X'_+} \amalg \mathrm{Id}_{Y'_+} \amalg \mathrm{Id}_{X'_-} \amalg \mathrm{Id}_{Y'_-} = \mathrm{Id}_{X'_+ \amalg Y'_+ \amalg X'_- \amalg Y'_-} = \mathrm{Id}_{X' \amalg' Y'} \,. \end{aligned}$$

This is in harmony with the (universal) identity (9.8).

If $A' \amalg' C' = A' \amalg C'$ and $B' \amalg' D' = B' \amalg D'$, then also $\iota_{A' \amalg C'} = \mathrm{Id}_{A' \amalg C'}$ and $\iota_{B' \amalg D'} = \mathrm{Id}_{B' \amalg D'}$. Then we can say that

$$f' \amalg' g' = \iota_{B' \amalg D'}(f' \amalg g')\iota^{-1}_{A' \amalg C'} = \mathrm{Id}_{B' \amalg D'}(f' \amalg g') \, \mathrm{Id}_{A' \amalg C'} = f' \amalg g'. \tag{9.10}$$

We already mentioned that τ_{+-} and τ_{-+} are not arrows in **1cob**. They will be projected to identity arrows by P. If we write $+- = p_+ \amalg p_-$, then we have $\iota_{+-} = \mathrm{Id}_{+-} = \mathrm{Id}_+ \amalg \mathrm{Id}_-$ and $\iota_{-+} = \tau_{-+}$. Then, knowing that $P_0(p_+ \amalg p_-) = P_0(p_- \amalg p_+) = p_+ \amalg p_-$, we obtain:

$$P_1(\tau_{+-}) : P_0(p_+ \amalg p_-) \to P_0(p_- \amalg p_+) \quad , \quad P_1(\tau_{+-}) = \iota_{-+}\tau_{+-}\iota_{+-}^{-1} = \tau_{-+}\tau_{+-} = \mathrm{Id}_{+-} .$$

Similarly we have $P_1(\tau_{-+}) = \mathrm{Id}_{+-}$.

About the ordering of connected components of any object. Before choosing a projection functor $P : \mathbf{1Cob} \to \mathbf{1cob}$, not every arbitrary object X in **1Cob** is equipped with an ordering or labeling of its connected components. On the other hand, we already know that every object in **1cob** is automatically equipped with an ordering of its connected components. After choosing a projection functor $P : \mathbf{1Cob} \to \mathbf{1cob}$ also an isomorphism ι_X exists for all objects X in **1Cob**. These isomorphisms induce ordering of the connected components of each X.

Porting over the monoidal structure from 1Cob to 1cob. Now a candidate operator $\amalg'$ is defined, but we still do not know exactly what type of category we are dealing with. According to the theory discussed in Chapter 3, we can say that **1cob** is at least a (not necessarily strict) monoidal category, but before porting over the symmetric structure, we will first check that we are dealing with a *strict* monoidal category. If **1cob** is not a strict monoidal category, then we can strictify it, but this will alter the originally induced structure. However, we can do some explicit checks, and in fact we are already dealing with a monoidal category which is strict *a priori*. But, these explicit checks can be rather laborious. Anyway, we will try to do these checks here.

A first step will be checking the strict behaviour of the unit object (and arrow). Let X' be an object in **1cob**, then $\varnothing$ is the candidate unit object for **1cob**:

$$\varnothing \amalg' X' = (\varnothing \amalg X')' = (X')' = X' = (X' \amalg \varnothing)' = X' \amalg' \varnothing.$$

So, indeed $\varnothing$ behaves like a (strict) unit object in **1cob**, with respect to $\amalg'$. This check was rather easy, and we can do a similar thing in case of arrows.

A following step might be checking the strict behaviour of the operator $\amalg'$, with respect to objects. We can write $X' \amalg' Y' = X'_+ \amalg Y'_+ \amalg X'_- \amalg Y'_-$, thus the associator α' of $\amalg'$ might be trivial if we can say that $(X' \amalg' Y') \amalg' Z' = X' \amalg' (Y' \amalg' Z')$, for all objects X', Y' and Z' in **1cob**. This is indeed the case:

$$\begin{aligned}(X' \amalg' Y') \amalg' Z' &= (X'_+ \amalg Y'_+ \amalg X'_- \amalg Y'_-) \amalg' (Z'_+ \amalg Z'_-) = X'_+ \amalg Y'_+ \amalg Z'_+ \amalg X'_- \amalg Y'_- \amalg Z'_- \\ &= (X'_+ \amalg X'_-) \amalg' (Y'_+ \amalg Z'_+ \amalg Y'_- \amalg Z'_-) = X' \amalg' (Y' \amalg' Z').\end{aligned}$$

We need to do another check before we can say that $\amalg'$ behaves like a strict operator. We did not check yet whether its behaviour is strict with respect to arrows. We cannot do a similar trick in case of arrows, but we can do it in a different way. Let $f' : A' \to B'$, $g' : C' \to D'$ and $h' : E' \to F'$ be an arbitrary triple of arrows in **1cob**. Then we can say that α' will be trivial if we can say that $(f' \amalg' g') \amalg' h' = f' \amalg' (g' \amalg' h')$, for all arrows f', g' and h'. To check this we will rewrite the triple products of arrows:

$$\begin{aligned}(f' \amalg' g') \amalg' h' &= \iota_{(B'\amalg' D')\amalg F'}((f' \amalg' g') \amalg h')\iota^{-1}_{(A'\amalg' C')\amalg E'} \\ &= \iota_{(B'\amalg' D')\amalg F'}(\iota_{B'\amalg D'}(f' \amalg g')\iota^{-1}_{A'\amalg C'} \amalg h')\iota^{-1}_{(A'\amalg' C')\amalg E'} \\ &= \iota_{(B'\amalg' D')\amalg F'}(\iota_{B'\amalg D'} \amalg \mathrm{Id}_{F'})((f' \amalg g') \amalg h')(\iota^{-1}_{A'\amalg C'} \amalg \mathrm{Id}_{E'})\iota^{-1}_{(A'\amalg' C')\amalg E'} \\ f' \amalg' (g' \amalg' h') &= \iota_{B'\amalg(D'\amalg' F')}(f' \amalg (g' \amalg' h'))\iota^{-1}_{A'\amalg(C'\amalg' E')} \\ &= \iota_{B'\amalg(D'\amalg' F')}(f' \amalg \iota_{D'\amalg F'}(g' \amalg h')\iota^{-1}_{C'\amalg E'})\iota^{-1}_{A'\amalg(C'\amalg' E')} \\ &= \iota_{B'\amalg(D'\amalg' F')}(\mathrm{Id}_{B'} \amalg \iota_{D'\amalg F'})(f' \amalg (g' \amalg h'))(\mathrm{Id}_{A'} \amalg \iota^{-1}_{C'\amalg E'})\iota^{-1}_{A'\amalg(C'\amalg' E')}\end{aligned}$$

We already know that $(f' \amalg g') \amalg h' = f' \amalg (g' \amalg h')$. Also note that, for example

$$(\iota^{-1}_{A'\amalg C'} \amalg \mathrm{Id}_{E'})\iota^{-1}_{(A'\amalg' C')\amalg E'} = (\iota_{(A'\amalg' C')\amalg E'}(\iota_{A'\amalg C'} \amalg \mathrm{Id}_{E'}))^{-1}.$$

Then all we need to check is whether

$$\iota_{(X'\amalg' Y')\amalg Z'}(\iota_{X'\amalg Y'} \amalg \mathrm{Id}_{Z'}) = \iota_{X'\amalg(Y'\amalg' Z')}(\mathrm{Id}_{X'} \amalg \iota_{Y'\amalg Z'}), \tag{9.11}$$

for any triple of objects X', Y' and Z'. Then we can conclude that $(f' \amalg' g') \amalg' h' = f' \amalg' (g' \amalg' h')$. Indeed, we know that α is trivial, as **1Cob** is already strict. Then (3.7) shows us that α' will also be trivial if (9.11) holds.

We can explicitly write down the arrows $\iota_{X'\amalg Y'}$, in terms of identities and twists, to prove (9.11). We should note that we *need* the arrows $\tau_{X',Y'}$, indicating the symmetric structure of **1Cob**, to do this explicit check. So, we already explicitly know the behaviour of the arrows $\iota_{X'\amalg Y'}$. In the general case we cannot do it this way. In the context of this subject it is still a mystery how to do it in general.

Using the explicit definition (9.6) of $\iota_{X'\amalg Y'}$, but also relations (2.9), (2.15), (2.16) and (9.5), we can rewrite (9.11). Here we have some intermediate steps:

$$\begin{aligned}
\iota_{(X'\amalg' Y')\amalg Z'} &= \mathrm{Id}_{(X'\amalg' Y')_+} \amalg \tau_{(X'\amalg' Y')_-,Z'_+} \amalg \mathrm{Id}_{Z'_-} = \mathrm{Id}_{X'_+\amalg Y'_+} \amalg \tau_{X'_-\amalg Y'_-,Z'_+} \amalg \mathrm{Id}_{Z'_-} \\
&= \mathrm{Id}_{X'_+} \amalg \mathrm{Id}_{Y'_+} \amalg (\tau_{X'_-,Z'_+} \amalg \mathrm{Id}_{Y'_-})(\mathrm{Id}_{X'_-} \amalg \tau_{Y'_-,Z'_+}) \amalg \mathrm{Id}_{Z'_-} \\
&= \mathrm{Id}_{X'_+} \amalg (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_-,Z'_+} \amalg \mathrm{Id}_{Y'_-})(\mathrm{Id}_{Y'_+} \amalg \mathrm{Id}_{X'_-} \amalg \tau_{Y'_-,Z'_+}) \amalg \mathrm{Id}_{Z'_-} \\
\iota_{X'\amalg Y'} \amalg \mathrm{Id}_{Z'} &= \mathrm{Id}_{X'_+} \amalg \tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Y'_-} \amalg \mathrm{Id}_{Z'_+} \amalg \mathrm{Id}_{Z'_-} \\
\iota_{X'\amalg(Y'\amalg' Z')} &= \mathrm{Id}_{X'_+} \amalg \tau_{X'_-,(Y'\amalg' Z')_+} \amalg \mathrm{Id}_{(Y'\amalg' Z')_-} = \mathrm{Id}_{X'_+} \amalg \tau_{X'_-,Y'_+\amalg Z'_+} \amalg \mathrm{Id}_{Y'_-\amalg Z'_-} \\
&= \mathrm{Id}_{X'_+} \amalg (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_-,Z'_+})(\tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Z'_+}) \amalg \mathrm{Id}_{Y'_-} \amalg \mathrm{Id}_{Z'_-} \\
&= \mathrm{Id}_{X'_+} \amalg (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_-,Z'_+} \amalg \mathrm{Id}_{Y'_-})(\tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Z'_+} \amalg \mathrm{Id}_{Y'_-}) \amalg \mathrm{Id}_{Z'_-} \\
\mathrm{Id}_{X'} \amalg \iota_{Y'\amalg Z'} &= \mathrm{Id}_{X'_+} \amalg \mathrm{Id}_{X'_-} \amalg \mathrm{Id}_{Y'_+} \amalg \tau_{Y'_-,Z'_+} \amalg \mathrm{Id}_{Z'_-}
\end{aligned}$$

Composing these arrows gives us:

$$\begin{aligned}
&\iota_{(X'\amalg' Y')\amalg Z'}(\iota_{X'\amalg Y'} \amalg \mathrm{Id}_{Z'}) \\
&= (\mathrm{Id}_{X'_+} \amalg (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_-,Z'_+} \amalg \mathrm{Id}_{Y'_-})(\mathrm{Id}_{Y'_+} \amalg \mathrm{Id}_{X'_-} \amalg \tau_{Y'_-,Z'_+}) \amalg \mathrm{Id}_{Z'_-})(\mathrm{Id}_{X'_+} \amalg \tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Y'_-} \amalg \mathrm{Id}_{Z'_+} \amalg \mathrm{Id}_{Z'_-}) \\
&= \mathrm{Id}_{X'_+} \amalg (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_-,Z'_+} \amalg \mathrm{Id}_{Y'_-})(\mathrm{Id}_{Y'_+} \amalg \mathrm{Id}_{X'_-} \amalg \tau_{Y'_-,Z'_+})(\tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Y'_-} \amalg \mathrm{Id}_{Z'_+}) \amalg \mathrm{Id}_{Z'_-} \\
&= \mathrm{Id}_{X'_+} \amalg (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_-,Z'_+} \amalg \mathrm{Id}_{Y'_-})(\tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Z'_+} \amalg \mathrm{Id}_{Y'_-})(\mathrm{Id}_{X'_-} \amalg \mathrm{Id}_{Y'_+} \amalg \tau_{Y'_-,Z'_+}) \amalg \mathrm{Id}_{Z'_-} \\
&= (\mathrm{Id}_{X'_+} \amalg (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_-,Z'_+} \amalg \mathrm{Id}_{Y'_-})(\tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Z'_+} \amalg \mathrm{Id}_{Y'_-}) \amalg \mathrm{Id}_{Z'_-})(\mathrm{Id}_{X'_+} \amalg \mathrm{Id}_{X'_-} \amalg \mathrm{Id}_{Y'_+} \amalg \tau_{Y'_-,Z'_+} \amalg \mathrm{Id}_{Z'_-}) \\
&= \iota_{X'\amalg(Y'\amalg' Z')}(\mathrm{Id}_{X'} \amalg \iota_{Y'\amalg Z'})
\end{aligned}$$

The third equality is obtained after applying relation (2.14). This finally proves (9.11), which in turn proves associativity of arrows in **1cob**, using $\amalg'$. Thus also α' is a trivial natural transformation, turning $(\mathbf{1cob}, \amalg', \varnothing)$ into a strict monoidal category. Then we can try to add symmetric structure to **1cob**.

Porting over the symmetric structure from 1Cob to 1cob. We know that $(\mathbf{1Cob}, \amalg, \varnothing, \tau)$ is already a symmetric monoidal category, thus the arrows $\tau_{X,Y}$ already satisfy the three rules mentioned in Section 2.3. See for example (2.14) and (2.15). On the other hand we know that $\amalg' \neq \amalg$, contrary to the situation of **2Cob** and **2cob**, so we need to find different twist arrows $\tau'_{X',Y'}$ for **1cob**. The arrows $\tau_{X,Y}$ satisfy the rules for turning **1Cob** into a symmetric monoidal category, but we need to define $\tau'_{X',Y'}$ differently so that these arrows satisfy the rules for turning **1cob** into a symmetric monoidal category.

If X' and Y' is a pair of objects in **1cob**, then the arrow $\tau_{X',Y'}$ from $X' \amalg Y'$ to $Y' \amalg X'$ can be written as follows:

$$\begin{aligned}
&\tau_{X',Y'} : X'_+ \amalg X'_- \amalg Y'_+ \amalg Y'_- \to Y'_+ \amalg Y'_- \amalg X'_+ \amalg X'_- \\
&\tau_{X',Y'} = (\mathrm{Id}_{Y'_+} \amalg \tau_{X'_+,Y'_-} \amalg \mathrm{Id}_{X'_-})(\tau_{X'_+,Y'_+} \amalg \tau_{X'_-,Y'_-})(\mathrm{Id}_{X'_+} \amalg \tau_{X'_-,Y'_+} \amalg \mathrm{Id}_{Y'_-})
\end{aligned}$$

Then we can define τ', regarded as the candidate twist arrow on **1cob**:

$$\begin{aligned}\tau'_{X',Y'} &:= P_1(\tau_{X',Y'}) = \iota_{Y'\amalg X'}\tau_{X',Y'}\iota^{-1}_{X'\amalg Y'} \\ &= (\mathrm{Id}_{Y'_+}\amalg\tau_{Y'_-,X'_+}\amalg\mathrm{Id}_{X'_-})\tau_{X',Y'}(\mathrm{Id}_{X'_+}\amalg\tau_{Y'_+,X'_-}\amalg\mathrm{Id}_{Y'_-}) \\ &= \tau_{X'_+,Y'_+}\amalg\tau_{X'_-,Y'_-} \end{aligned} \tag{9.12}$$

We can say that $\tau_{X'_+,Y'_+}$ is purely generated by τ_+ (and Id_+), and that $\tau_{X'_-,Y'_-}$ is purely generated by τ_- (and Id_-). Note that $\tau'_{X',Y'}$ are also natural isomorphisms, as τ' itself is a composition of other natural transformations:

$$\tau' = \mathrm{Id}_P * \tau * \mathrm{Id}_{I\times I}\,.$$

We should say that this is just an explicit choice for a *candidate* natural transformation, to be used to turn $(\mathbf{1cob}, \amalg', \varnothing)$ into a symmetric monoidal category. Within the material discussed here we cannot say that this method will also work in general. Now we can check if the three relations mentioned in Section 2.3, applied to τ', hold:

1. For any two arrows $f' : A' \to B'$ and $g' : C' \to D'$ we have:

$$\begin{aligned}(g'\amalg' f')\tau'_{A',C'} &= P_1(g'\amalg f')P_1(\tau_{A',C'}) = P_1((g'\amalg f')\tau_{A',C'}) \\ &= P_1(\tau_{B',D'}(f'\amalg g')) = P_1(\tau_{B',D'})P_1(f'\amalg g') = \tau'_{B',D'}(f'\amalg' g').\end{aligned}$$

Thus **1Cob** satisfying (2.14) implies that also **1cob** satisfies (2.14).

2. For any two objects X' and Y' we have:

$$\tau'_{Y',X'}\tau'_{X',Y'} = P_1(\tau_{Y',X'})P_1(\tau_{X',Y'}) = P_1(\tau_{Y',X'}\tau_{X',Y'}) = P_1(\mathrm{Id}_{X'\amalg Y'}) = \mathrm{Id}_{X'\amalg' Y'}$$

The last equality is a result of (9.8).

3. For any triple of objects X', Y' and Z' we have:

$$\begin{aligned}\tau'_{X',Y'\amalg' Z'} &= \tau_{X'_+,(Y'\amalg' Z')_+}\amalg\tau_{X'_-,(Y'\amalg' Z')_-} = \tau_{X'_+,Y'_+\amalg Z'_+}\amalg\tau_{X'_-,Y'_-\amalg Z'_-} \\ &= (\mathrm{Id}_{Y'_+}\amalg\tau_{X'_+,Z'_+})(\tau_{X'_+,Y'_+}\amalg\mathrm{Id}_{Z'_+})\amalg(\mathrm{Id}_{Y'_-}\amalg\tau_{X'_-,Z'_-})(\tau_{X'_-,Y'_-}\amalg\mathrm{Id}_{Z'_-}) \\ &= (\mathrm{Id}_{Y'_+}\amalg\tau_{X'_+,Z'_+}\amalg\mathrm{Id}_{Y'_-}\amalg\tau_{X'_-,Z'_-})(\tau_{X'_+,Y'_+}\amalg\mathrm{Id}_{Z'_+}\amalg\tau_{X'_-,Y'_-}\amalg\mathrm{Id}_{Z'_-}) \\ &= ((\mathrm{Id}_{Y'_+}\amalg\mathrm{Id}_{Y'_-})\amalg'(\tau_{X'_+,Z'_+}\amalg\tau_{X'_-,Z'_-}))((\tau_{X'_+,Y'_+}\amalg\tau_{X'_-,Y'_-})\amalg'(\mathrm{Id}_{Z'_+}\amalg\mathrm{Id}_{Z'_-})) \\ &= (\mathrm{Id}_{Y'}\amalg'\tau'_{X',Z'})(\tau'_{X',Y'}\amalg'\mathrm{Id}_{Z'})\end{aligned}$$

The first equality is a result of (9.12) and the fifth equality is a result of (9.9). Thus **1Cob** satisfying (2.15) implies that also **1cob** satisfies (2.15).

Thus τ satisfying the three rules implies that also τ' satisfies the three rules. Thus we can say that $(\mathbf{1cob}, \amalg', \varnothing, \tau')$ is a symmetric monoidal category.

A remark. To conclude, we can now finally say that, in this specific case, the fact that **1Cob** is a symmetric monoidal category implies that **1cob** is also a symmetric monoidal category, but we should be aware that we needed explicit definitions and checks for this. We could even think that we needed suitable choices for these definitions. For example, we used explicit definitions for P, ι and τ'. Especially note that the symmetric structure of **1Cob** was already needed for turning **1cob** into a strict monoidal category.

In the general case, as discussed in Chapter 3, we do not need explicit definitions of P and ι, at least not for copying the monoidal structure from a category $\mathcal{C}$ to a skeleton $\mathcal{C}'$ of $\mathcal{C}$. In that case even $\mathcal{C}$ being nonstrict was no problem, and a symmetric structure was not needed yet. However, then we cannot be sure if we are dealing with a skeleton which is a strict monoidal category a priori. So, in general this will

be a mystery, but we could always strictify this skeleton afterwards, if desired. On the other hand, for the approach introduced here we were already assuming **1Cob** to be strict.

So both approaches, the formal one of Chapter 3 and the explicit one of this chapter, each have their own advantages and disadvantages. At least we know that we do not really need special restrictions on P to be sure that the skeleton can be regarded as a (not necessarily strict) monoidal category. So we would not need specific properties of the mentioned categories at all. However, in this chapter the specific properties of **1Cob** and ι *were* needed for the explicit checks. We even made a special choice for ι, just to make sure we obtain the desired result, and we can say this is a restriction. This restriction was used to make sure that **1cob** itself will a priori also be strict, as a monoidal category. Especially note that strictifications and redefinitions are not needed then, thus adding simplified symmetric structure is directly possible. At this moment we (still) cannot say that the explicit definitions and checks, of τ' and its properties, can be used in the general case to make sure that the skeleton we are dealing with is also a symmetric monoidal category.

Of course, for the next chapter, we would like to know explicitly how the strict symmetric monoidal structure of **1cob** is defined, so the explicit porting over of the structure was at least needed for this. We can say that Chapter 3 was needed to check whether the monoidal structure can be ported correctly in general, and whether or not it all depends on the specific definitions and properties of $\mathcal{C}$, a skeleton $\mathcal{C}'$, P and ι. Indeed, in general the monoidal structure can be ported, and no specific information of $\mathcal{C}$ is needed for this. In some cases we only need to do one extra step, if desired, which is strictifying the skeleton in question.

9.3 Generators and relations of 1cob

Generators of 1cob. Every arrow $f' : p_+^k \amalg p_-^l \to p_+^m \amalg p_-^n$ in **1cob** can be split up horizontally and vertically into the six basic arrows Id_+, Id_-, τ_+, τ_-, β and γ. These can be regarded as a minimal set of *generators* of **1cob**. Then τ_+ and τ_- are the *twist generators*. Thus, again, every arrow f' in **1cob** can be constructed by composing these generators horizontally and vertically.

The connected components of any object are ordered. However, the ordering of the positively and negatively oriented points is separate, which is induced by the properties of the graded disjoint union. We should also note that $\bar{\beta}$ and $\bar{\gamma}$ are not arrows in **1cob**, as the object $p_- \amalg p_+$ and the arrows τ_{+-} and τ_{-+} would be involved then.

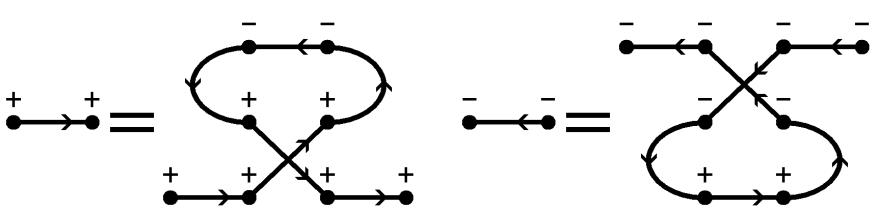
Let f' and g' be two arrows in **1cob**. As already explained by relation (9.10), we can for example write $f' \amalg' g' = f' \amalg g'$ in some special cases, if the domain and codomain of $f' \amalg g'$ are already objects in **1cob**. For example:

$$\mathrm{Id}_+ \amalg' \mathrm{Id}_- = \mathrm{Id}_+ \amalg \mathrm{Id}_-, \quad \mathrm{Id}_+ \amalg' \beta \amalg' \mathrm{Id}_- = \mathrm{Id}_+ \amalg \beta \amalg \mathrm{Id}_-, \quad \tau_+ \amalg' \tau_- = \tau_+ \amalg \tau_-.$$

Relations of 1cob. Now we can discuss the relations of **1cob**. Note that the snake decomposition in its standard form will be invalid here, as objects like $p_+ \amalg p_- \amalg p_+$ and arrows like $\bar{\beta}$ are involved then. However, we can modify the snake decomposition. We know that in $\mathbf{1Cob}'$ we can write $\mathrm{Id}_+ = (\beta \amalg \mathrm{Id}_+)(\mathrm{Id}_+ \amalg \bar{\gamma})$. Then we can write

$$\begin{aligned} \mathrm{Id}_+ &= P_1(\mathrm{Id}_+) = P_1((\beta \amalg \mathrm{Id}_+)(\mathrm{Id}_+ \amalg \bar{\gamma})) = P_1(\beta \amalg \mathrm{Id}_+) P_1(\mathrm{Id}_+ \amalg \bar{\gamma}) \\ &= \cdots = (\mathrm{Id}_+ \amalg \beta)(\tau_+ \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \gamma). \end{aligned}$$

We can rewrite Id_- in a similar way, and the following two diagrams will express these relations:



(9.13)

$\mathrm{Id}_+ = (\mathrm{Id}_+ \amalg \beta)(\tau_+ \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \gamma)$ $\quad$ $\mathrm{Id}_- = (\beta \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \tau_-)(\gamma \amalg \mathrm{Id}_-)$

These relations are in fact equivalent to relations (9.2) and (9.3), which apply to $\mathbf{1Cob}'$. We can call these relations examples of the *modified snake decomposition*. We could also say that the zig-zag identities (2.21) hold *with respect to* $\amalg'$:

$$\begin{aligned}
\mathrm{Id}_+ &= (\beta \amalg' \mathrm{Id}_+)(\mathrm{Id}_+ \amalg' \gamma) = (\mathrm{Id}_+ \amalg \beta)(\tau_+ \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \gamma) = (\mathrm{Id}_+ \amalg' \beta)(\gamma \amalg' \mathrm{Id}_+),\\
\mathrm{Id}_- &= (\beta \amalg' \mathrm{Id}_-)(\mathrm{Id}_- \amalg' \gamma) = (\beta \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \tau_-)(\gamma \amalg \mathrm{Id}_-) = (\mathrm{Id}_- \amalg' \beta)(\gamma \amalg' \mathrm{Id}_-).
\end{aligned}$$

Of course this identity also holds in $\mathbf{1Cob}'$, so it should really be considered as a rewrite only. Note that we used that

$$\beta = P_1(\beta) = P_1(\bar{\beta}) \quad , \quad \gamma = P_1(\bar{\gamma}) = P_1(\gamma). \tag{9.14}$$

We can say that $\gamma : p_- \amalg' p_+ \to \varnothing$ (or $\gamma : p_+ \amalg' p_- \to \varnothing$) is the arrow dual to $\beta : p_+ \amalg' p_- \to \varnothing$ (or $\beta : p_- \amalg' p_+ \to \varnothing$).

The other relations of **1cob** remaining, are those coming from the naturality of the twist, see (2.14). Note that we cannot say that any commutative object exists in **1cob**, so there is no other way to cancel twist generators. We know that $\mathbf{1Cob}'$ is a symmetric monoidal category on a dualizable object. We can say that also **1cob** is a symmetric monoidal category on a dualizable object. This time $(p_+, p_-, \beta, \gamma)$ can be regarded as a dualizable object, generating all other dualizable objects in **1cob**, and it induces another dualizable object $(p_-, p_+, \beta, \gamma)$.

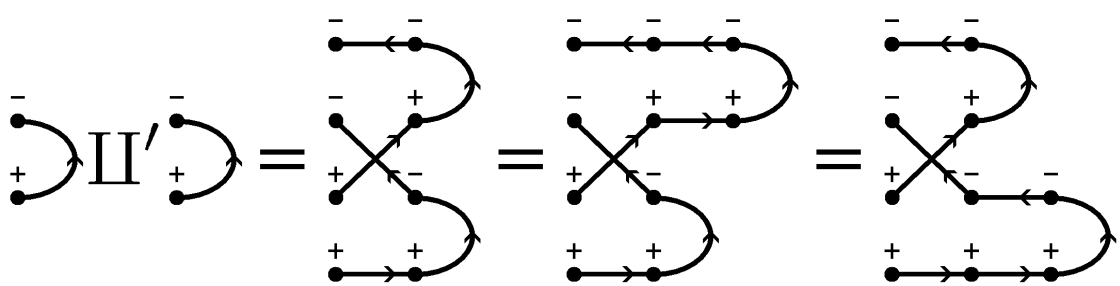
Relations of 1cob: An example. As an example of applying the relations, we will discuss the graded disjoint product of β with itself. We know that $\beta : p_+ \amalg p_- \to \varnothing$ is an arrow in **1cob**, so we can write:

$$\beta \amalg' \beta = (\beta \amalg \beta)' = \iota_{\varnothing}(\beta \amalg \beta)\iota^{-1}_{+-+-} = (\beta \amalg \beta)(\mathrm{Id}_+ \amalg \tau_{+-} \amalg \mathrm{Id}_-),$$

which is an arrow from $p_+ \amalg p_+ \amalg p_- \amalg p_-$ to $\varnothing$. This is still not written as a composition of the generators of **1cob** itself, as $\beta \amalg \beta$ and τ_{+-} are no valid arrows in **1cob**, so we will try to rewrite it. The first step will be:

$$\begin{aligned}
\beta \amalg' \beta &= (\beta \amalg \beta)(\mathrm{Id}_+ \amalg \tau_{+-} \amalg \mathrm{Id}_-)\\
&= \beta(\beta \amalg \mathrm{Id}_+ \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \tau_{+-} \amalg \mathrm{Id}_-)\\
&= \beta(\mathrm{Id}_+ \amalg \mathrm{Id}_- \amalg \beta)(\mathrm{Id}_+ \amalg \tau_{+-} \amalg \mathrm{Id}_-).
\end{aligned}$$

The following diagram expresses this:

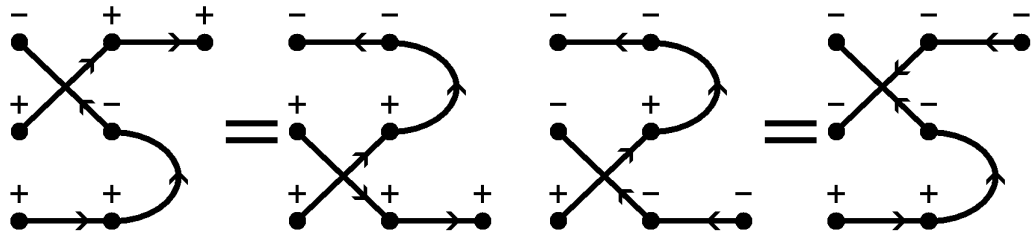


(9.15)

Now, using naturality of the twist, see (2.14), we can do a following step of rewriting. For example, the following two relations are a result of (2.14), together with (2.18), applied to $\mathbf{1Cob}'$:

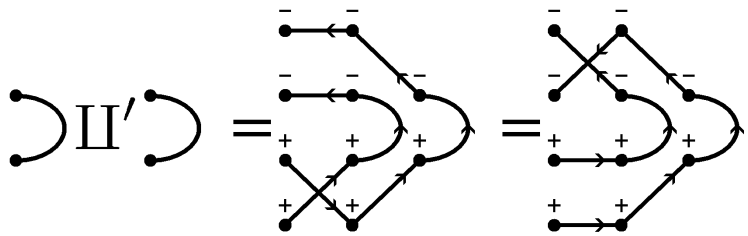
$$\begin{aligned}
\beta \amalg \mathrm{Id}_+ = \tau_{\varnothing, p_+}(\beta \amalg \mathrm{Id}_+) &= (\mathrm{Id}_+ \amalg \beta)\tau_{p_+ \amalg p_-, p_+} = (\mathrm{Id}_+ \amalg \beta)(\tau_+ \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \tau_{-+}),\\
\mathrm{Id}_- \amalg \beta = \tau_{p_-, \varnothing}(\mathrm{Id}_- \amalg \beta) &= (\beta \amalg \mathrm{Id}_-)\tau_{p_-, p_+ \amalg p_-} = (\beta \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \tau_-)(\tau_{-+} \amalg \mathrm{Id}_-).
\end{aligned}$$

As a consequence, we can say:



$(\beta \amalg \mathrm{Id}_+)(\mathrm{Id}_+ \amalg \tau_{+-}) = (\mathrm{Id}_+ \amalg \beta)(\tau_+ \amalg \mathrm{Id}_-) \quad (\mathrm{Id}_- \amalg \beta)(\tau_{+-} \amalg \mathrm{Id}_-) = (\beta \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \tau_-)$

These specific relations can be inserted into diagram (9.15), whereupon we will finally obtain the following relation:



$\beta \amalg' \beta = \beta(\mathrm{Id}_+ \amalg \beta \amalg \mathrm{Id}_-)(\tau_+ \amalg \mathrm{Id}_- \amalg \mathrm{Id}_-) = \beta(\mathrm{Id}_+ \amalg \beta \amalg \mathrm{Id}_-)(\mathrm{Id}_+ \amalg \mathrm{Id}_+ \amalg \tau_-)$

This diagram helps us expressing $\beta \amalg' \beta$ as a composition of generators of **1cob**, and in the meantime the second equality expresses a specific relation, derived from (2.14).

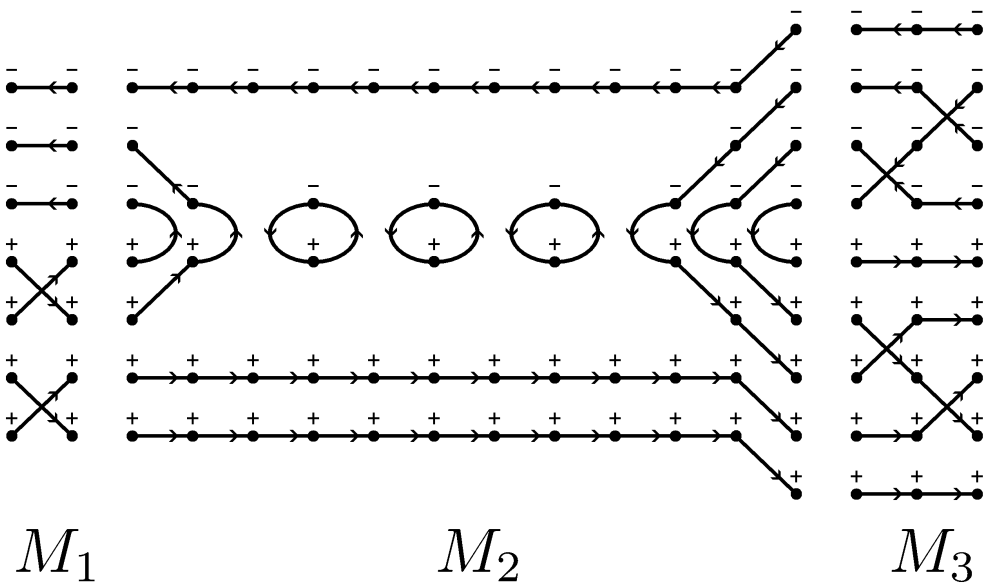
The generalized normal form of a cobordism in 1cob. Every cobordism class M in **1cob** can be described as a collection of oriented lines with boundary and circles without orientation. Let $X = p_+^k \amalg p_-^l$ and $Y = p_+^m \amalg p_-^n$ be objects in **1cob**, and assume $k - l = m - n$, so that at least one arrow exists from X to Y. Now let $M : X \to Y$ be such an arrow in **1cob**. Just like we did in Chapter 8, we claim without proof that any cobordism class M can be decomposed into three parts, say $M = M_3 M_2 M_1$, according to some standard scheme. We will call this scheme the *generalized normal form* of an arrow. This scheme is rather different when compared to the normal form of connected cobordisms in **2cob**.

The first part M_1 and the last part M_3, also called the *in-twist part* and the *out-twist part*, will be isomorphisms. These are arrows $M_1 : X \to X$ and $M_3 : Y \to Y$. These arrows are compositions of the four generators Id_+, Id_-, τ_+ and τ_- only. Note that we can say that

$$M_1 = M_1^+ \amalg' M_1^- = M_1^+ \amalg M_1^- \quad , \quad M_3 = M_3^+ \amalg' M_3^- = M_3^+ \amalg M_3^- .$$

Then M_1^+ and M_3^+ are compositions of Id_+ and τ_+, and M_1^- and M_3^- are compositions of Id_- and τ_-. The middle part $M_2 : X \to Y$ is a composition of the four generators Id_+, Id_-, β and γ only. It can be regarded as a horizontal composition of parts, each being written as a graded disjoint product of multiple arrows of type Id_+ first, then either a β or a γ, and finally multiple arrows of type Id_-. The splitting will not be unique, because of the naturality of the twist.

As an example, we can have a look at the normal form of a cobordism class $M : p_+^4 \amalg p_-^3 \to p_+^5 \amalg p_-^4$:



(9.16)

Using the naturality of the twist, we can rewrite any composition of multiple arrows into the generalized normal form. Let M and N be two arrows, already written in the generalized normal form, and assume NM exists, then we can move many of the twist parts from the middle to the left or to the right, using the naturality of the twist. The remaining twists are part of a *loop*, and the relations in (9.13) can be used for removing these loops immediately. Thus we can rewrite NM into the generalized normal form. Rewriting a vertical composition of multiple arrows into the generalized normal form is only a matter of inserting some extra twist arrows.

Any middle part M_2 can be indicated by five positive integers (zero included), say (k, l, m, n, p), and this information is sufficient. These numbers tell us that M_2 contains:

- k cylinders with positive orientation, thus $\mathrm{Id}_+^k = \mathrm{Id}_+ \amalg \cdots \amalg \mathrm{Id}_+$,
- l half circles from $p_+ \amalg p_-$ to $\varnothing$, thus β,
- m circles,
- n half circles from $\varnothing$ to $p_+ \amalg p_-$, thus γ,
- p cylinders with negative orientation, thus $\mathrm{Id}_-^p = \mathrm{Id}_- \amalg \cdots \amalg \mathrm{Id}_-$.

For example, the middle part M_2 in diagram (9.16) obtains the numbering $(k, l, m, n, p) = (2, 2, 3, 3, 1)$. Note that M is an isomorphism if and only if $l = m = n = 0$. We directly see that l, m and n will not contribute to the countings $N_X^+ - N_X^-$ and $N_Y^+ - N_Y^-$. We also see that $k = N_X^+ = N_Y^+$ and $p = N_X^- = N_Y^-$, so that

$$\Sigma(X) = N_X^+ - N_X^- = k - p = N_Y^+ - N_Y^- = \Sigma(Y).$$

The twist parts M_1 and M_3 can each be indicated by two permutations, one for the positively oriented points and one for the negatively oriented points, contained in X or Y. So we have four permutations in total, each being an element of one of the symmetric groups. Actually it depends on how points in the boundary are connected, inside of M_2, which of these combinations of four permutations will be considered as identical. So, in many cases we do *not* need the full symmetric groups for indicating the twist parts. Any arrow in **1cob** can be written in generalized normal form, indicated by four permutations and five positive numbers.

Cobordisms in 1cob with some connected components being closed. As already mentioned in Section 7.3, see (7.5), and in Chapter 8, we have some relations when closed cobordisms are concerned. A closed connected component M_j of any arrow M in $\mathbf{1Cob}'$ can be placed anywhere in a picture. As a reminder, any such M_j can only be a circle without orientation, so a cobordism with m closed connected components simply contains m circles. In a picture of any arrow in **1cob**, in generalized normal form, we will however, by convention, draw the circles as a horizontal string of arrows $\beta\gamma$.

10 Topological Quantum Field Theories

Finally we will discuss topological quantum field theories. First we will discuss three definitions, looking very different at first sight.

In Section 10.1 we will start with the definition of TQFTs which looks very straightforward, nearly trivial. It is the definition by J. Kock (see [7], 2003). The concepts of symmetric monoidal categories and functors, as introduced in earlier chapters here, will mainly be discussed. In this section we will also shortly discuss what is so topological about topological quantum field theories.

In Section 10.2 we will present the definition of TQFTs by M. Atiyah (see [4], 1988). This definition looks totally different at first sight, and no categories and functors are explicitly mentioned. The axioms presented there are not really explicitly focussed on cobordisms. We will compare this definition to the first definition, by Kock.

In Section 10.3 we will present the definition of TQFTs by C. Blanchet & M. Turaev (see [8], 2005). At first sight this definition seems to be a mixture of the definitions by Kock and Atiyah. The axioms presented there are focussed on cobordisms. Categories and functors will also be discussed, be it briefly and only in an alternative definition, not in the main definition. At the end of Section 10.3 we will compare this definition to the definitions by Kock and Atiyah. The results can be regarded as the main conclusion of this thesis, but not as the *only* conclusion. A short list of key points of this conclusion can be found in Chapter 11.

In Section 10.4 we will discuss 2D-TQFTs and present another result from [7]: the category of 2-dimensional topological quantum field theories is equivalent to the category of commutative Frobenius algebras. In this context we will restrict to **2cob**, skeleton of **2Cob**, as the source category for any 2D-TQFT.

In Section 10.5 we will discuss 1D-TQFTs. We could say that the category of 1-dimensional topological quantum field theories is equivalent to the category of dualizable vector spaces. In Chapter 9 we already concluded that **1cob** can also be regarded as a symmetric monoidal category, so in this context we can also restrict to **1cob**, skeleton of **1Cob**, as the source category for any 1D-TQFT. We can also restrict to $\mathbf{1Cob}'$ instead.

In this chapter we will use an artificial symbol: we say that $\varnothing_{(n)}$ is the empty n-dimensional manifold. Sometimes we will rewrite it as $\varnothing_{(n)} = \varnothing_{(n-1)} \times I$, where $I = [0, 1]$.

10.1 Topological Quantum Field Theories as described by Kock

Main definition. The definition of topological quantum field theories by Kock ([7]) looks very straightforward. Let **nCob** be the symmetric monoidal category of cobordism classes of dimension n, as introduced in Section 7.3, and let $\mathbf{Vect}_{\Bbbk}$ be the symmetric monoidal category of vector spaces (and linear maps) over some ground field $\Bbbk$, see (2.19). An n-dimensional topological quantum field theory, or nD-TQFT, over $\Bbbk$ is nothing more than a symmetric monoidal functor

$$\mathcal{A} : (\mathbf{nCob}, \amalg, \varnothing, \tau) \to (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau). \tag{10.1}$$

We will call the image of **nCob** under an nD-TQFT a *linear representation* of **nCob**. The empty set $\varnothing_{(n-1)}$, together with its cylinder $\varnothing_{(n)} = \varnothing_{(n-1)} \times I$, can be regarded as the *neutral object* of **nCob**. A TQFT will map $\varnothing_{(n-1)}$ to $\Bbbk$ and $\varnothing_{(n)}$ to the identity map of $\Bbbk$.

Alternative definition. An n-dimensional topological quantum field theory, over some ground field $\Bbbk$, is a rule $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ which maps each closed oriented manifold X of dimension $n-1$ to a vector space $\mathcal{A}_0(X)$, and maps each oriented cobordism (class) M, from X to Y, to a linear map $\mathcal{A}_1(M) : \mathcal{A}_0(X) \to \mathcal{A}_0(Y)$.

This rule $\mathcal{A}$ should satisfy the following six axioms:

K1. Two cobordisms M and N have the same image if they are equivalent:

$$[M] = [N] \Rightarrow \mathcal{A}_1(M) = \mathcal{A}_1(N).$$

K2. For any manifold X of dimension $n-1$, the cylinder $X \times I$ $(= X \times I_{++})$, which is a cobordism from X to itself, will be sent to the identity map of $\mathcal{A}_0(X)$.

K3. If M can be written as a decomposition $M = M_2 M_1$, then

$$\mathcal{A}_1(M) = \mathcal{A}_1(M_2 M_1) = \mathcal{A}_1(M_2)\mathcal{A}_1(M_1).$$

K4. If $X = X_1 \amalg X_2$ then

$$\mathcal{A}_0(X) = \mathcal{A}_0(X_1 \amalg X_2) = \mathcal{A}_0(X_1) \otimes \mathcal{A}_0(X_2).$$

Similarly, if M is a cobordism from $X = X_1 \amalg X_2$ to $Y = Y_1 \amalg Y_2$, and if there exist cobordisms M_1 from X_1 to Y_1 and M_2 from X_2 to Y_2, such that $M = M_1 \amalg M_2$, then

$$\mathcal{A}_1(M) = \mathcal{A}_1(M_1 \amalg M_2) = \mathcal{A}_1(M_1) \otimes \mathcal{A}_1(M_2).$$

K5. The empty manifold will be sent to the ground field $\Bbbk$:

$$X = \varnothing_{(n-1)} \Rightarrow \mathcal{A}_0(X) = \Bbbk.$$

K6. If $T_{X,Y}$ is the canonical (abstract) twist cobordism from $X \amalg Y$ to $Y \amalg X$ and if $\tau_{V,W}$ is the canonical twist map from $V \otimes W$ to $W \otimes V$, then $\mathcal{A}_1(T_{X,Y}) = \tau_{\mathcal{A}_0(X),\mathcal{A}_0(Y)}$.

The equalities in these axioms should be regarded as strict, not up to isomorphism or something else. We note that axiom K5, combined with axiom K2, implies that the empty cobordism, which can be viewed as the cylinder $\varnothing_{(n)}$, will be sent to the identity map of $\Bbbk$. Also note that axiom K6, which was not explicitly mentioned by Kock, is a consequence of axiom K4, thus we could say axiom K6 is rather trivial.

Axiom K1 says that the rule $\mathcal{A}$ does not fully depend on the cobordism, only on the cobordism *class*. This is in harmony with describing the collection of cobordism classes as the arrows in the source category. Axiom K3 can also be rewritten in terms of cobordism classes instead. We can split up cobordism classes, but we can also compose them. Then axioms K2 and K3 show that we are dealing with a functor. Axioms K4 and K5 show that we are dealing with a monoidal functor, and axiom K6 shows that we are dealing with a symmetric monoidal functor. Then we can say that the main definition (10.1) is just a summarized version of this alternative definition.

A proposition. Now we can discuss a consequence of these axioms, combined with the snake decomposition (6.5). For example, see (9.2) for the snake decomposition of the identity Id_+ in **1Cob** (**1Cob**$'$). In **nCob** we can rewrite this to

$$\mathrm{Id}_X = (\beta_X \amalg \mathrm{Id}_X)(\mathrm{Id}_X \amalg \gamma_X), \tag{10.2}$$

for any arbitrary X equipped with arrows $\beta_X : X \amalg \overline{X} \to \varnothing$ and $\gamma_X : \varnothing \to \overline{X} \amalg X$. If $\mathcal{A}$ is a (non-trivial) TQFT, then (10.2) can be mapped to a relation in $\mathbf{Vect}_\Bbbk$. We define the vector spaces $V := \mathcal{A}_0(X)$ and $W := \mathcal{A}_0(\overline{X})$, and the linear maps $\beta : V \otimes W \to \Bbbk$, $\beta := \mathcal{A}_1(\beta_X)$ and $\gamma : \Bbbk \to W \otimes V$, $\gamma := \mathcal{A}_1(\gamma_X)$ and write:

$$\begin{aligned} \mathrm{Id}_V &= \mathcal{A}_1(\mathrm{Id}_X) = \mathcal{A}_1((\beta_X \amalg \mathrm{Id}_X)(\mathrm{Id}_X \amalg \gamma_X)) = \mathcal{A}_1(\beta_X \amalg \mathrm{Id}_X)\mathcal{A}_1(\mathrm{Id}_X \amalg \gamma_X) \\ &= (\mathcal{A}_1(\beta_X) \otimes \mathcal{A}_1(\mathrm{Id}_X))(\mathcal{A}_1(\mathrm{Id}_X) \otimes \mathcal{A}_1(\gamma_X)) = (\beta \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes \gamma). \end{aligned} \tag{10.3}$$

As can be read in [7] we can now draw a conclusion from this. The fact that relation (10.3) holds, *is precisely to say that the pairing β is nondegenerate.*

The proposition will be that if $\mathcal{A}$ is an nD-TQFT and if X is a closed manifold (of dimension $n-1$), then the image vector space $V := \mathcal{A}_0(X)$ comes equipped with a nondegenerate pairing with $W := \mathcal{A}_0(\overline{X})$. This nondegenerate pairing induces a canonical identification of W with V^*, the vector space dual to V. This can be rewritten as another axiom

$$X \mapsto V = \mathcal{A}_0(X) \quad \Rightarrow \quad \overline{X} \mapsto V^* = \mathcal{A}_0(\overline{X}), \tag{10.4}$$

but we should note that this rewritten axiom does not exactly fit anymore in the context of Kock's axioms, as a restriction is involved here. We conclude that this axiom is a consequence of the main definition of TQFTs and the snake decomposition, which is a relation which holds in the source category **nCob**. We will soon return to this conclusion, when we will discuss TQFTs as described by Atiyah: (10.4) fits best in the context of Atiyah's axioms. A corollary of (10.4) is that a TQFT will map any object in **nCob** to a vector space of *finite dimension*.

A question. We could ask what is so topological about topological quantum field theories. We could say that a quantum field theory is topological if definition (10.1) is sufficient to describe the theory completely. In this case the theory will not depend on anything else, for example a metric, defined on the manifolds and cobordisms in question. We could also say a theory is topological if it only depends on the topology, direction and orientation of the manifolds in question.

We could say that a TQFT is at least topological with respect to the arrows in **nCob**. A cobordism class mainly contains the topological information of the cobordisms involved.

A TQFT is *not* topological with respect to the objects in **nCob**, which may seem a bit remarkable. If X and Y are distinct objects, then there is no rule in general saying that $\mathcal{A}_0(X) = \mathcal{A}_0(Y)$ in specific cases. Even if an orientation preserving diffeomorphism $\phi : X \to Y$ exists, then we still cannot say that $\mathcal{A}_0(X)$ and $\mathcal{A}_0(Y)$ should be *identical*, only isomorphic.

Topological invariants. If M is a cobordism class corresponding to a closed cobordism, then we see that M is a cobordism from the empty set to itself, thus we can write $M : \varnothing \to \varnothing$. Any arbitrary TQFT $\mathcal{A}$ will map this to $\mathcal{A}_1(M) : \mathcal{A}_0(\varnothing) \to \mathcal{A}_0(\varnothing)$, or $\mathcal{A}_1(M) : \Bbbk \to \Bbbk$. This is a linear map from the field $\Bbbk$ to itself, thus we can say that $\mathcal{A}_1(M)$ can be described as a multiplication with an element $c_M \in \Bbbk$. This constant c_M can be regarded as a *topological invariant* of any closed cobordism lying in the cobordism class M.

Note that $\Bbbk$ itself is commutative and that $\Bbbk \otimes \Bbbk = \Bbbk$, implied by the relation $\Bbbk \otimes V = V = V \otimes \Bbbk$. If M can be written as $M = NP$, and if N and P are also closed, then $c_M = c_N c_P = c_P c_N$. If M can be written as $M = N \amalg P$, and if again both N and P are closed, then $c_M = c_N \otimes c_P = c_N c_P = c_P c_N = c_N \otimes c_P$. Thus we can write:

$$\mathcal{A}_1(N \amalg P) = \mathcal{A}_1(P \amalg N) = \mathcal{A}_1(NP) = \mathcal{A}_1(PN).$$

Of course this is no coincidence, as this should already follow from the relations closed cobordisms themselves should satisfy, see (7.5). We conclude that the relations presented here, induced by the properties of $\Bbbk$, are in harmony with the relations expressed in (7.5), as expected.

10.2 Topological Quantum Field Theories as described by Atiyah

Main definition. The definition of topological quantum field theories by Atiyah ([4]) looks very different, compared to the definition by Kock. In some relations the word *functorial* is mentioned, but in general we will not encounter notions like categories and functors. The notion of cobordisms is used but cobordism classes are not explicitly mentioned. We note that Atiyah mentions ground rings Λ and finitely generated Λ-modules, but we will restrict to vector spaces, which are special examples of Λ-modules. We also note that an n-dimensional topological quantum field theory as described by Atiyah, is equivalent to an $(n+1)$-dimensional topological quantum field theory as described by Kock. We will slightly rewrite the definitions and axioms of Atiyah: an nD-TQFT corresponds to an $(n-1)$D-TQFT in [4], and we will mention vector spaces over $\Bbbk$ instead of Λ-modules.

An n-dimensional topological quantum field theory, over some ground field $\Bbbk$, is a rule Z consisting of the following data:

- A vector space $Z(X)$ of finite dimension over $\Bbbk$, associated to each oriented closed smooth manifold X of dimension $n-1$.
- An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth manifold (with boundary) M of dimension n.

These data are subject to the following axioms:

A1. Z is *functorial* with respect to orientation preserving diffeomorphisms of X and M.

A2. Z is *involutory*, thus $Z(\overline{X}) = Z(X)^*$, where $\overline{X}$ is X with opposite orientation and $Z(X)^*$ denotes the vector space dual to $Z(X)$.

A3. Z is *multiplicative*.

A4. Z is subject to the *non-triviality axiom*.

More detailed definitions and some remarks. We will now elaborate on the precise meaning of these axioms:

- Axiom A1 means that an orientation preserving diffeomorphism (OPD) $f : X \to Y$ induces an isomorphism $Z(f) : Z(X) \to Z(Y)$ and that if $g : Y \to Z$ is another OPD then $Z(gf) = Z(g)Z(f)$. (We should also note that Z will map identity OPDs to identity linear maps.) Also, if f extends to an OPD from M to N, with $\partial M = X$ and $\partial N = Y$, then $Z(f)$ takes $Z(M)$ to $Z(N)$.
- Axiom A2 already explains itself, so it should be clear.
- Axiom A3 can be split up further into three sub-axioms:

  A3.1 If the closed oriented $(n-1)$-manifold X can be written as $X = X_1 \amalg X_2$, then

  $$Z(X) = Z(X_1 \amalg X_2) = Z(X_1) \otimes Z(X_2).$$

  A3.2 If we have n-manifolds M_1 and M_2, with boundary $\partial M_1 = X_1 \amalg Y$ and $\partial M_2 = \overline{Y} \amalg X_2$, and if $M = M_1 \amalg_Y M_2$ is the manifold obtained by gluing together the common Y-component, then we require:

  $$Z(M) = \langle Z(M_1), Z(M_2) \rangle,$$

  where $\langle Z(M_1), Z(M_2) \rangle$ denotes the natural pairing

  $$Z(X_1) \otimes Z(Y) \otimes Z(\overline{Y}) \otimes Z(X_2) \to Z(X_1) \otimes Z(X_2).$$

  A3.3 If we apply A3.2 to the special case $Y = \varnothing_{(n-1)}$, so that we can write $M = M_1 \amalg M_2$, then A3.2 reduces to the obvious multiplicative requirement

  $$Z(M) = Z(M_1 \amalg M_2) = Z(M_1) \otimes Z(M_2).$$

  Note that if $X_1 = X_2 = \varnothing_{(n-1)} = X_1 \amalg X_2$, and if we write $V := Z(\varnothing_{(n-1)})$, then axiom A3.1 implies that $V = V \otimes V$. Also note that if $M_1 = M_2 = \varnothing_{(n)} = M_1 \amalg M_2$, and if we write $f := Z(\varnothing_{(n)}) \in V$, then axiom A3.3 implies that $f = f \otimes f$. (Here we write $\partial \varnothing_{(n)} = \varnothing_{(n-1)}$.) These equations do not yet imply that V and f are non-trivial.

  Axiom A3.2 can be reformulated. First we could rewrite any boundary ∂M: we can choose X_1 and X_2 such that $\partial M = \overline{X}_1 \amalg X_2$. Then axiom A3.1, combined with axiom A2, implies that

  $$Z(\partial M) = Z(\overline{X}_1 \amalg X_2) = Z(\overline{X}_1) \otimes Z(X_2) = Z(X_1)^* \otimes Z(X_2) = \mathrm{Hom}(Z(X_1), Z(X_2)).$$

  Then we can reformulate M, with boundary ∂M, as a cobordism $\tilde{M}$ with in-boundary $\partial_+ \tilde{M} = X_1$ and out-boundary $\partial_- \tilde{M} = X_2$. Thus $\partial \tilde{M} = \partial_+ \tilde{M} \cup \partial_- \tilde{M} \simeq X_1 \amalg X_2$. (Note that the same convention introduced in Section 6.2 can be used here for the orientation of in- and out-boundaries.) The TQFT Z will map this cobordism to the linear map

  $$Z(M) : Z(X_1) \to Z(X_2).$$

Then we can write $Z(M) = \mathcal{A}_1(\tilde{M})$. (Here $\mathcal{A}$ means a TQFT according to Kock's definitions, see (10.1).) Now assume we have n-manifolds M_1 and M_2 with boundary $\partial M_1 = \overline{X}_1 \amalg Y$ and $\partial M_2 = \overline{Y} \amalg X_2$, and $M = M_1 \amalg_Y M_2$. (We write $\overline{X}_1$ instead of X_1 just for convenience.) Then these can be rewritten as cobordisms $\tilde{M}_1$ from X_1 to Y and $\tilde{M}_2$ from Y to X_2. Then axiom A3.2 can be reformulated as

$$Z(M) = \mathcal{A}_1(\tilde{M}) = \mathcal{A}_1(\tilde{M}_2)\mathcal{A}_1(\tilde{M}_1) = Z(M_2)Z(M_1). \tag{10.5}$$

From now on we will write M instead of $\tilde{M}$, but we should realize that if an n-manifold M with boundary, discussed in the context of Atiyah's axioms, is rewritten as a *cobordism* M from X to Y, then the in-boundary of the cobordism M has reversed orientation compared to the original n-manifold M.

- Also axiom A4 can be split up further into three sub-axioms:

  A4.1 We would like $\varnothing_{(n-1)}$ to be mapped to the non-trivial solution of the equation $V \otimes V = V$ (where V is a vector space over $\Bbbk$):
  $$Z(\varnothing_{(n-1)}) = \Bbbk.$$

  A4.2 We would like $\varnothing_{(n)}$ to be mapped to the non-trivial solution of the equation $f^2 = f$ (where $f \in \Bbbk$):
  $$Z(\varnothing_{(n)}) = 1_{\Bbbk}.$$

  A4.3 For any $(n-1)$-manifold X, the cylinder $X \times I$ will be mapped to $Z(X \times I)$. We will assume that $Z(X \times I)$ is an identity map.

  The manifold $X \times I$ in axiom A4.3 should already be considered as a cobordism. In [4] we can read that $\sigma := Z(X \times I)$ should be an element of $\mathrm{End}(Z(X))$, and that this σ should be regarded as an idempotent operator: $\sigma^2 = \sigma$. Any such σ would be suitable in any TQFT as, according to [4], its image will still satisfy all the axioms, but we simply choose $\sigma = 1$ so that the theory does not degenerate into a (partially) trivial one.

Some other comments.

- When discussing axiom A4.3 we should remark that it is not totally clear a priori why we can say that σ is idempotent, but hopyfully this can be more clearly understood if we take axiom A1 into account. Gluing together two copies M_1 and M_2 of the cylinder $X \times I$ gives us another copy M of the same cylinder. But from the context of [4] we cannot clearly conclude why they should be the same, or at least give the same image under a TQFT. (Does M have twice the length of M_1 or M_2, or not? Is their length of any importance anyway?) In [4] we can also find the statement $Z(X \times I) \in \mathrm{End}(Z(X))$, so apparently M_1, M_2 and M are regarded as cobordisms from X to itself. Then at least their images $Z(M_1)$, $Z(M_2)$ and $Z(M)$ should be elements of the same vector space: $Z(\overline{X} \amalg X)$. Then we can apply axiom A1. If we use the trivial OPD $f_X : X \to X$ ($f(p) = p$ for all $p \in X$), then this f_X induces a trivial OPD g_X from $\overline{X} \amalg X$ to itself, and this OPD can in turn be lifted to OPDs F_j from M to M_j. Then $Z(F_j)$ should take $Z(M)$ to $Z(M_j)$, but knowing that g_X is trivial this implies that $Z(M) = Z(M_1) = Z(M_2) = \sigma$. Combining this with (10.5) implies that $\sigma^2 = \sigma$.

  However, it is still not clear why we can write $Z(X \times I) \in \mathrm{End}(Z(X))$ in the strict sense. Atiyah at least does not make clear what $\amalg$ exactly means, and why we could write $\partial(X \times I) = \overline{X} \amalg X$ as a strict equality. Of course we could say it is an equality up to, for example, an isomorphism.

  Also note that Atiyah *mentions* the notion of disjoint union, but in [4] he uses the symbol $\cup$, not $\amalg$, which should mean ordinary union as usual. However, then we cannot say that axiom A3.1 has any exact meaning, as in general we can write $X \cup Y = Y \cup X$, but *not* $Z(X) \otimes Z(Y) = Z(Y) \otimes Z(X)$. Thus, knowing that [4] could cause some confusion, we could still assume that the standard disjoint union $\amalg$ was mentioned here.

- We see that all manifolds (possibly with boundary) involved here are regarded as oriented, but for example ∂M and M can each have their own orientation, and these orientations are independent. This also applies to each of the connected components of M and ∂M. In this context we do not decide yet which part of ∂M is an in-boundary or an out-boundary.

- As we will soon discuss, $(n-1)$-manifolds are related to objects in **nCob** in Kock's context, and n-manifolds with boundary are related to arrows then. Note that axiom A2 only mentions the effect of reversing the orientation of the $(n-1)$-manifolds. The effect of reversing the orientation of the n-manifolds is not described by axiom A2 or any other basic axiom. However, an extra axiom can be used to describe this effect. We will discuss this later.

- We also note that axiom A2 is in harmony with the idea that $\overline{\varnothing} = \varnothing$ and that $\Bbbk^* \simeq \Bbbk$: $Z(\overline{\varnothing}) = Z(\varnothing)^* = \Bbbk^* \simeq \Bbbk = Z(\varnothing)$.

- Note that if $X = X_1 \amalg X_2$, then the ordinary orientation reversal will give us $\overline{X} = \overline{(X_1 \amalg X_2)} = \overline{X}_1 \amalg \overline{X}_2$. Then axiom A2, combined with axiom A3.1, will give us:

$$\begin{aligned} Z(\overline{X}) &= Z(X)^* = (Z(X_1 \amalg X_2))^* = (Z(X_1) \otimes Z(X_2))^*, \\ Z(\overline{X}) &= Z(\overline{X}_1 \amalg \overline{X}_2) = Z(\overline{X}_1) \otimes Z(\overline{X}_2) = Z(X_1)^* \otimes Z(X_2)^*. \end{aligned}$$

  This yields an exact identity $(V \otimes W)^* = V^* \otimes W^*$. However, if V and W are finite-dimensional, then there only exists a natural isomorphism between $(V \otimes W)^*$ and $V^* \otimes W^*$. This raises the question whether we should really understand axioms A2 and A3.1 as identities, *or* just as natural isomorphisms.

Comparing Atiyah's definitions to Kock's definitions. Now we can discuss the connection between Atiyah's axioms and definitions and Kock's axioms and definitions.

- There seems to be a connection between axiom A1 and the definition and properties of **nCob** itself. The first thing to notice is the relation between different n-manifolds M and N with the same boundary X, and the relation it induces between $Z(X)$, $Z(M)$ and $Z(N)$. Suppose $\partial M = \partial N = X$, and let $f : X \to X$ be the identity OPD. This should of course induce the identity map $Z(f) : Z(X) \to Z(X)$. If f can be extended to an OPD $\tilde{f} : M \to N$, then $Z(f)$ takes $Z(M)$ to $Z(N)$. Knowing that $Z(f)$ is an identity map, this should induce that $Z(M) = Z(N)$. This should remind us of (a special case of) relation (7.1). Now we write $X = \overline{X}_1 \amalg X_2$ for convenience. Suppose $X_1 = \partial_+ M = \partial_+ N$ and $X_2 = \partial_- M = \partial_- N$, and suppose that the injections are all trivial. Then we can say that M and N can be smoothly deformed into each other, with $Z(M) = Z(N)$ remaining constant. In the context of cobordisms we can say that the TQFT Z is insensible to smooth deformations. This seems to support the idea of restricting to cobordism *classes* instead. Knowing that in Kock's context the idea of gluing cobordism classes is a more natural concept, compared to gluing cobordisms, we do not need to worry about subtleties of composing cobordisms from now on. We conclude that axiom A1 is connected to axiom K1, so that we can restrict to cobordism classes, which we will do from now on.

- In general there seems to be a connection between axiom A1 and generating cylinders as an intermediate step. We could then say that axiom K3, especially applied to these cylinders, implies axiom A1. In the context of Kock we could start with an OPD $f : X \to Y$. Then, according to the properties of **nCob**, f induces a general cylinder C_f from X to Y, regarded as a cobordism class, thus as an arrow. We already know that this arrow is an isomorphism. Now assume $\mathcal{A}$ is a TQFT satisfying Kock's axioms, and that $Z(X) = \mathcal{A}_0(X)$ for all (objects!) X. (Then Z also satisfies axioms A3.1 and A4.1.) Then $\mathcal{A}$ will map C_f to $\mathcal{A}_1(C_f) : Z(X) \to Z(Y)$, which should again be an isomorphism. Then we can simply define $Z(f) := \mathcal{A}_1(C_f)$, thus $Z(f) : Z(X) \to Z(Y)$ is indeed an isomorphism. If now f can be lifted to an OPD $\tilde{f} : M \to N$, where $\partial M = \partial_- M = X$ and $\partial N = \partial_- N = Y$, so that M and N can be represented by cobordism classes $[M] : \varnothing \to X$ and $[N] : \varnothing \to Y$, then we can assume that $C_f[M] = [N]$, so that $Z(f)$ will map $Z(M)$ to $Z(N)$. If we now assume that we have another OPD

$g : Y \to Z$, then this g induces another arrow C_g. The functorial property of $\mathcal{A}$, see axiom K3, yields that indeed

$$Z(gf) = \mathcal{A}_1(C_{gf}) = \mathcal{A}_1(C_g C_f) = \mathcal{A}_1(C_g)\mathcal{A}_1(C_f) = Z(g)Z(f).$$

Thus Z is indeed functorial with respect to OPDs. Rewriting this into the context of Atiyah, with $\partial C_f \simeq \overline{X} \amalg Y$, gives us an equivalent statement: $Z(f) \in Z(\partial C_f) \simeq Z(X)^* \otimes Z(Y)$. We conclude that if $\mathcal{A}$ is a TQFT according to Kock's axioms, then Z satisfies at least axiom A1.

- Axiom A2 can be directly related to the 'extra axiom' described by (10.4). We can say that this axiom is caused by properties (see the snake decomposition) of the source category **nCob** itself, as already worked out. However, in Kock's context we have more freedom in choosing an image W of $\overline{X}$, if X is already mapped to V. Then the strict relation $W = V^*$, presented here, can be replaced by a weaker relation $W \simeq V^*$.

- Axiom A3.2, viewed from the context of cobordisms and related to the ordinary (horizontal) composition of arrows in **nCob**, can be directly related to axiom K3. Axiom A4.3 can be directly related to axiom K2. Thus axioms A3.2 and A4.3 together are related to the properties of any ordinary functor.

- Axioms A3.1, A3.3, A4.1 and A4.2 can be related to the monoidal properties of a TQFT-functor. Axiom A3.1 equals the rule for vertical composition of objects in **nCob**, and axiom A3.3 is related to the rule for vertical composition of arrows in **nCob**. Thus axioms A3.1 and A3.3 are related to axiom K4. Axioms A4.1 and A4.2 are related to axiom K5 and its consequences. Especially axiom A4.2 is related to the combination of axioms K5 and K2.

- In the context of Atiyah there is no mention of symmetric structure and twist cobordisms, thus there is no explicit connection with axiom K6. But as already discussed, axiom K6 is rather trivial, thus it is not really needed.

We conclude that Atiyah's definition and Kock's definition are equivalent. Only their point of view differs.

Closed cobordisms. In mathematics there is always a risk of overlooking things. Why should we think that the definitions and axioms of Kock and Atiyah are *really* equivalent? In many situations we can say that these different approaches give the same result, but there are some exceptional situations when we are studying closed cobordisms, and in these situations we are not sure yet. At least reading [4] could cause some confusion when we focus on the details about closed cobordisms. There we can read that if M is a closed cobordism, and if the extra *hermitian axiom* is satisfied, then $Z(M)$ will change to its conjugate $\overline{Z(M)}$ under orientation reversal ($M \mapsto \overline{M}$). We know that if M is a closed cobordism, then $Z(\partial M) = Z(\varnothing) = \Bbbk$. For example if $\Bbbk = \mathbb{C}$, then we can say that $\overline{Z(M)}$ is simply the complex conjugate of $Z(M) \in \Bbbk$. In [4] we can read that the basic axioms A1,A2,A3 and A4 do not yet give a relation between $Z(M)$ and $Z(\overline{M})$ for closed manifolds M, but also that the relation $Z(\overline{M}) = \overline{Z(M)}$ implies that numerical invariant $Z(M)$ might help us detecting the orientation of M. Unfortunately this seems to be a mistake. To remove this mistake we will introduce the proposition that in many cases $Z(M)$ equals its own conjugate.

If we study Kock's approach, then we directly see that for any closed cobordism M an OPD $\phi : M \to \overline{M}$ exists, at least if $\dim(M) \in \{1, 2\}$. (For $\dim(M) \geq 3$ we are not sure.) As $\partial M = \varnothing$ we can say that ϕ has no restrictions, thus it automatically satisfies relation (7.1). Then we can say that the cobordism classes $[M]$ and $[\overline{M}]$ are equal. (As an example we have $\beta\gamma = \bar{\beta}\bar{\gamma}$ in **1Cob**, see (9.1).) Then of course also $\mathcal{A}_1([M]) = \mathcal{A}_1([\overline{M}])$. Using Atiyah's approach we can say that such a map ϕ can always be interpreted as a lift of the canonical OPD $I_\varnothing : \varnothing \to \varnothing$, see (6.1). Then we can apply axiom A1 and use the identity map $Z(I_\varnothing) : \Bbbk \to \Bbbk$. Then we should come to the same conclusion: $Z(M) = Z(\overline{M})$. This should be the easiest way.

Of course we could say that $I_\varnothing$ is a rather artificial map, thus it is not really an OPD, but then we can still use a different approach and apply axiom A3 instead. For example, if M_1 is a closed cobordism and if M_2 is an arbitrary one, so that $M := M_1 \amalg M_2$, then we can write a relation. If $X = \partial M_2$ then we can write that $Z(\partial M) = Z(\partial M_1) \otimes Z(\partial M_2) = \Bbbk \otimes Z(X) \simeq Z(X)$. Now we can write $k := Z(M_1) \in \Bbbk$. Then

also $Z(M) = Z(M_1 \amalg M_2) = Z(M_1) \otimes Z(M_2) = kZ(M_2)$. Now define $N := \overline{M}_1 \amalg M_2$, and assume that $Z(\overline{M}_1) = \overline{k}$, then $Z(N)$ will be mapped to $\overline{k}Z(M_2)$. On the other hand, we know that $\partial M = \partial N = X$, thus the trivial OPD $f_X : X \to X$ can always be lifted to the OPD $\phi : M \to N$. Then also $Z(M) = Z(N)$. To conclude: then $Z(M) = kZ(M_2) = \overline{k}Z(M_2) = Z(N)$. As this should apply for *all* choices for M_2, this should imply that $\overline{k} = k$. Thus $k = Z(M_1)$ is its own (complex) conjugate. This result is in harmony with the result of Kock's approach.

At least if $\dim(M) \in \{1, 2\}$ we conclude that the proposition of saying that $\overline{k} = k$, for any $k = Z(M)$, with $\partial M = \varnothing$, is just a direct consequence of Atiyah's basic axioms and the fact that an OPD $\phi : M \to \overline{M}$ exists. Hopefully this removes the paradox we meet in the literature. Thus we can also safely conclude that in many cases $Z(M)$ does *not* depend on the orientation of M, also in case of a TQFT satisfying the extra hermitian axiom.

We should also note that it is easy to get confused by reading the parts concerning closed cobordisms in [4]. We could as well assume that there is no mistake there and start to wonder why Kock's axioms and Atiyah's axioms differ (only) at this point. We know that Kock starts with the definition of cobordism classes, which first mainly focusses on the cobordisms, before focussing on the boundaries: two cobordisms M and N are considered equivalent if an OPD $\phi : M \to N$ exists such that relation (7.1) holds. We also know that Atiyah starts with axiom A1, but this axiom first focusses on the boundaries, not on the cobordisms themselves. This might imply that we are really dealing with a subtle difference in describing equivalent cobordisms. Luckily this does not seem to be the case.

Of course we cannot say that for every cobordism M with $\dim(M) \geq 3$ an OPD $\phi : M \to \overline{M}$ exists. In this case there is no restriction on $Z(M)$ either: it is not necessarily its own conjugate. Then all we know is that $Z(\overline{M}) = \overline{Z(M)}$. Thus, if M does not admit such an OPD ϕ, then we can indeed say that it *might* perhaps be possible to detect its orientation.

10.3 Topological Quantum Field Theories as described by Blanchet & Turaev

Main definition. The definition of topological quantum field theories by Blanchet & Turaev ([8]) looks a bit like a mixture of the definitions by Kock and Atiyah, but it also has an appearance of its own. The main definition itself does not explicitly mention categories and functors, but a short alternative description is also presented, indeed mentioning them. The notion of cobordism is mainly used, but also here cobordism classes are not explicitly mentioned. All manifolds and cobordisms discussed here are supposed to be smooth and oriented.

An n-dimensional topological quantum field theory V, over some ground field $\Bbbk$, assigns to every closed $(n-1)$-dimensional manifold X a finite-dimensional vector space $V_0(X)$ over $\Bbbk$ and assigns to every n-dimensional cobordism (M, X, Y) a $\Bbbk$-linear map

$$V_1(M) = V_1(M, X, Y) : V_0(X) \to V_0(Y).$$

A cobordism (M, X, Y) from X to Y is a compact manifold M with boundary. There is a diffeomorphism from $\overline{X} \amalg Y$ to ∂M. A TQFT must satisfy the following axioms.

BT1. Any orientation preserving diffeomorphism (OPD) of closed $(n-1)$-dimensional manifolds $f : X \to X'$ induces an isomorphism $f_\sharp : V_0(X) \to V_0(X')$. For an OPD g between the cobordisms (M, X, Y) and (M', X', Y'), the following diagram is commutative.

$$\begin{tikzcd} V_0(X) \arrow[r, "(g|_X)_\sharp"] \arrow[d, "V_1(M)"'] & V_0(X') \arrow[d, "V_1(M')"] \\ V_0(Y) \arrow[r, "(g|_Y)_\sharp"] & V_0(Y') \end{tikzcd} \tag{10.6}$$

BT2. If a cobordism (W, X, Z) is obtained by gluing two cobordisms (M, X, Y) and (M', Y', Z) along an OPD $f : Y \to Y'$, then the following diagram is commutative.

$$\begin{tikzcd} V_0(X) \arrow[r, "V_1(W)"] \arrow[d, "V_1(M)"'] & V_0(Z) \\ V_0(Y) \arrow[r, "f_\sharp"] & V_0(Y') \arrow[u, "V_1(M')"'] \end{tikzcd}$$

BT3. For any $(n-1)$-dimensional manifold X, the linear map

$$V_1([0,1] \times X) : V_0(X) \to V_0(X)$$

is identity. (We assume that $[0,1]$ has standard orientation.)

BT4. There are natural isomorphisms

$$V_0(X \amalg Y) \simeq V_0(X) \otimes V_0(Y) \quad , \quad V_0(\varnothing) \simeq \Bbbk$$

such that the following diagrams are commutative.

$$\begin{tikzcd} V_0((X \amalg Y) \amalg Z) \arrow[r, "\simeq"] \arrow[d] & (V_0(X) \otimes V_0(Y)) \otimes V_0(Z) \arrow[d] \\ V_0(X \amalg (Y \amalg Z)) \arrow[r, "\simeq"] & V_0(X) \otimes (V_0(Y) \otimes V_0(Z)) \end{tikzcd} \qquad \begin{tikzcd} V_0(X \amalg \varnothing) \arrow[r, "\simeq"] \arrow[d] & V_0(X) \otimes \Bbbk \arrow[d] \\ V_0(X) \arrow[r, "="] & V_0(X) \end{tikzcd}$$

The vertical maps are respectively the ones induced by the canonical OPDs, and the standard isomorphisms of vector spaces.

BT5. The isomorphism

$$V_0(X \amalg Y) \simeq V_0(Y \amalg X)$$

induced by the canonical OPDs corresponds to the standard isomorphism of vector spaces

$$V_0(X) \otimes V_0(Y) \simeq V_0(Y) \otimes V_0(X).$$

Comparing the definitions by Blanchet & Turaev to Kock's and Atiyah's definitions. Now we can discuss the connections between the axioms and definitions by Blanchet & Turaev and Kock's and Atiyah's axioms and definitions.

- We first note that we assume the $\sharp$-symbol satisfies the following equalities (for all f, g and X):

  $$(gf)_\sharp = g_\sharp f_\sharp \quad , \quad (\mathrm{Id}_X)_\sharp = \mathrm{Id}_{V_0(X)} .$$

  (Of course this is no surprise as the $\sharp$-symbol is often used for denoting a functor.) This property is equivalent to the first part of axiom A1.

- Note that there is a diffeomorphism, say ρ, from $\overline{X} \amalg Y$ to ∂M. We cannot find any a priori comments in [8] about the behaviour of this ρ. Thus we can say that ρ is not necessarily orientation preserving. However, mentioning $\overline{X} \amalg Y$ seems to fit best in the context of Atiyah's axioms, so we could assume that this ρ is meant to be an OPD. (If $\overline{X} \amalg Y = \partial M$ then ρ should be the trivial OPD.) It all depends on the convention used for the orientation of ∂M. In this case we say that $\partial M = \rho(\overline{X} \amalg Y)$ takes over

the orientation of $\overline{X} \amalg Y$. We will use the symbol $\partial_A M$ instead, and the subscript A means that we are dealing with the boundary of an n-manifold in the context of Atiyah.

Now we can redefine the orientation of ∂M as follows: the orientation of $\rho(\overline{X})$ will be reversed. Something similar is discussed in Section 10.2, in the more detailed definition of axiom A3. There is a relation between the orientation of the n-manifold M with boundary, in the context of Atiyah, and the orientation of (the boundary of) the cobordism $\tilde{M}$. After redefining the orientation of ∂M we can rewrite ρ as an OPD $\iota = \iota_{M,X,Y}$ from $X \amalg Y$ to ∂M. In this case we say $\partial M = \iota(X \amalg Y)$ takes over the orientation of $X \amalg Y$. From now on we will assume that ι, just like ρ, is an OPD. (The maps ρ and ι are actually the same if we regard them as diffeomorphisms only.) We also note that $\partial_A M \neq \partial M$ if orientation is considered.

Note that using ι is equivalent to Kock's definition of a cobordism: ι maps X to $\partial_+ M$, which is the in-boundary of M, and Y to $\partial_- M$, which is the out-boundary of M.

- Note that maps like $g|_X$ (see axiom BT1) should be regarded as a pullback of the map $g|_{\partial M}$ to a map from $X \amalg Y$ to $X' \amalg Y'$, of course after restricting to X. We can do something similar to $g|_Y$. If $\iota = \iota_{M,X,Y}$ and $\iota' = \iota_{M',X',Y'}$ are OPDs, then the following diagram commutes:

$$\begin{tikzcd} X \amalg Y \arrow[r, "\iota"] \arrow[d, "g|_{X \amalg Y}"'] & \partial M \arrow[d, "g|_{\partial M}"] \\ X' \amalg Y' \arrow[r, "\iota'"'] & \partial M' \end{tikzcd}$$

  Here we already assume that $g|_X$ maps X to X' and that $g|_Y$ maps Y to Y', thus we can also write $g|_{X \amalg Y} = g|_X \amalg g|_Y$. We also note that $g|_{\partial M} : \partial M \to \partial M'$ is just the natural restriction of the map $g : M \to M'$. Saying that g is an OPD from (M, X, Y) to (M', X', Y') means that $g|_X$, $g|_Y$ and g restricted to the internal part of M, are OPDs.

- Now we note that axiom BT1 can be regarded as axiom A1, reformulated into the context of cobordisms. We assume that $\partial M = X \amalg Y$ and that $\partial M' = X' \amalg Y'$, so that ι and ι' are trivial OPDs and $g|_{X \amalg Y} = g|_{\partial M}$. Then we note that any OPD $\phi : X \amalg Y \to X' \amalg Y'$, satisfying $X' = \phi(X)$ and $Y' = \phi(Y)$, can be rewritten as an OPD $\phi_A : \overline{X} \amalg Y \to \overline{X}' \amalg Y'$. Then the boundaries of M and M' can be redefined as $\partial_A M = \overline{X} \amalg Y$ and $\partial_A M' = \overline{X}' \amalg Y'$. Then we obtain an OPD $(g|_{\partial M})_A = (g|_{X \amalg Y})_A = g|_{\overline{X} \amalg Y} = g|_{\partial_A M}$ from $\partial_A M$ to $\partial_A M'$. Now define $f := g|_{\overline{X} \amalg Y}$. Especially note that g itself can still be regarded as a lift of this f. Now we can rewrite diagram (10.6). First we rewrite maps like $(g|_X)_\sharp : V_0(X) \to V_0(X')$ as $Z(g|_X) : Z(X) \to Z(X')$. A linear map like $V_1(M) : V_0(X) \to V_0(Y)$ will be an element $Z(M)$ of $Z(X)^* \otimes Z(Y) = Z(\partial_A M)$. Then we can define $Z(f)$ such that it will map $Z(M)$ to $Z(M') = Z(g|_Y) Z(M) Z((g|_X)^{-1})$. This shows that axiom BT1 implies axiom A1. The reverse implication also holds (at least for special cases of trivial ι). We conclude that axioms A1 and BT1 are equivalent.

- If $X' = X$ and $Y' = Y$ and if the cobordisms M and M' from X to Y lie in the same cobordism class, then we may assume that $g|_{X \amalg Y}$ reduces to an identity map. We then rewrite $\iota' = \iota_{M',X,Y}$ and the following diagram commutes:

$$\begin{tikzcd} X \amalg Y \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \partial M \arrow[d, "g|_{\partial M}"] \\ & \partial M' \end{tikzcd}$$

This diagram is equivalent to diagram (7.1), and $g|_{X \amalg Y} = \mathrm{Id}_{X \amalg Y}$ implies that diagram (10.6) can be reduced to the identity $V_1(M) = V_1(M')$. This again shows that V_1 *only* depends on the cobordism *class*, thus we can say that we can restrict to cobordism classes only: $V_1(M) = V_1([M])$. In other words, we can say that V effectively maps from the category **nCob** to the category $\mathbf{Vect}_{\Bbbk}$. We already know that we can directly compose two cobordism classes $[M]$ and $[M']$, corresponding to cobordisms (M, X, Y) and (M', Y, Z), thus we can also glue M and M'.

- Axioms BT2 and BT3 are equivalent to the ordinary properties of a functor. If we have the trivial case $Y' = Y$ and $f = \mathrm{Id}_Y$, then we also have $f_\sharp = \mathrm{Id}_{V_0(Y)}$, so that we can write

$$V_1(W) = V_1([W]) = V_1([M'][M]) = V_1([M'])V_1([M]) = V_1(M')V_1(M)$$

  as a result of axiom BT2. This is the ordinary property of any functor. If $Y' \neq Y$ or if f is not trivial, then axiom BT2 reflects an extra property of the source category: any OPD $f : X \to X'$ generates a cylinder from X to X', which in turn generates an isomorphism C_f in **nCob**, and V will map this C_f to $f_\sharp$. We say that axiom BT3 is equivalent to axiom K2, which in turn is equivalent to axiom A4.3, and that axiom BT2 is similar to axiom K3, which in turn is equivalent to axiom A3.2. Thus V is indeed a functor, from **nCob** to $\mathbf{Vect}_{\Bbbk}$.

- We already mentioned that the collection of axioms of Blanchet & Turaev has an appearance of its own. We could say that axiom BT4 looks similar to assuming that V is a monoidal functor. However, there are no exact equalities mentioned here, only natural isomorphisms. Apparently also $(X \amalg Y) \amalg Z$ and $X \amalg (Y \amalg Z)$ are considered to be distinct objects, so we can assume that in this context, **nCob** and $\mathbf{Vect}_{\Bbbk}$ are regarded as nonstrict monoidal categories, and V as a nonstrict monoidal functor. To be more precise, we assume V can be regarded as a *strong* monoidal functor. If we also take account of axiom BT5, then **nCob** and $\mathbf{Vect}_{\Bbbk}$ seem to be nonstrict symmetric monoidal categories, and V seems to be a nonstrict (or strong) symmetric monoidal functor.

  On the other hand, the axioms of both Kock and Atiyah are in the context of strict monoidal categories and strict monoidal functors. Then the symmetric structure is added to the already strictified categories $(\mathbf{nCob}, \amalg, \varnothing)$ and $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$ before mentioning any strict symmetric monoidal functor. Thus, if we strictify **nCob** and $\mathbf{Vect}_{\Bbbk}$ before writing down the axioms of Blanchet & Turaev, then all the natural isomorphisms will turn into natural identities, so that we can ignore the two commuting diagrams in the description of axiom BT4.

- We note that disjoint products of cobordisms, and their image under a TQFT, are not explicitly mentioned in axiom BT4. At least not as the second part of axiom K4 shows us. However, we are dealing with a natural isomorphism from $V_0(X \amalg Y)$ to $V_0(X) \otimes V_0(Y)$. Assume we are dealing with the strict monoidal categories $(\mathbf{nCob}, \amalg, \varnothing)$ and $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$ so that the natural isomorphism will in fact be a natural identity. Then we can say that also $V_1(M \amalg N) = V_1(M) \otimes V_1(N)$, for any pair of cobordism classes M and N. In this case we can say that axiom BT4 is equivalent to axioms A3.1, A3.3 and A4.1 together, which in turn are equivalent to axioms K4 and K5 together. Thus axiom BT4 describes the properties of a monoidal functor.

- We know that axiom A4.2 is related to axioms K5 and K2. Similarly we can say that axiom A4.2 is related to axioms BT3 and BT4. Thus, if $\varnothing_{(n)}$ is the empty cobordism, then axioms BT3 and BT4 imply that $V_1(\varnothing_{(n)}) = \mathrm{Id}_{\Bbbk}$.

- Axiom BT5 is equivalent to axiom K6, describing the property of a symmetric monoidal functor.

- Note that the equality $V_0(\overline{X}) = V_0(X)^*$, which is equivalent to axiom A2, is also mentioned in [8], but it is not introduced as one of the axioms. We know that it is also not introduced as an axiom in [7], but there it is related to the snake decomposition, which is related to a property of the category **nCob** itself.

A conclusion. We conclude that the definitions by Blanchet & Turaev are equivalent to Atiyah's and Kock's definitions, even if they look very different at first sight. We could say that this conclusion is not totally unexpected. We found relations between the three different collections of definitions and axioms, and these relations are not that trivial.

10.4 Topological Quantum Field Theories in dimension 2

In this section we will present topological quantum field theories in dimension 2, or 2D-TQFTs. We will use the definition of TQFTs by Kock, and we will also present the category of these 2D-TQFTs.

Main definition. A 2D-TQFT, over some ground field $\Bbbk$, is a symmetric monoidal functor $\mathcal{A}$ from the symmetric monoidal category **2Cob** to the symmetric monoidal category $\mathbf{Vect}_{\Bbbk}$:

$$\mathcal{A} : (\mathbf{2Cob}, \amalg, \varnothing, \tau) \to (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau).$$

In Chapter 8 we discussed that the circle S^1 can be interpreted as a commutative Frobenius object in **2cob**, skeleton of **2Cob**. This object generates all other objects S^m in **2cob**, as defined in (8.1), and these can also be turned into commutative Frobenius objects if we apply (4.14) and (4.15). Thus **2cob** is a free symmetric monoidal category on a commutative Frobenius object, say $S^1 = (S^1, \mu, \eta, \delta, \epsilon)$. (See (8.2) for a definition of the arrows μ, η, δ and ϵ.) Then we can say that also the image $\mathcal{A}(\mathbf{2cob})$ is a free symmetric monoidal category on a commutative Frobenius object, say

$$A = (A, \mu_A, \eta_A, \delta_A, \epsilon_A) = (\mathcal{A}_0(S^1), \mathcal{A}_1(\mu), \mathcal{A}_1(\eta), \mathcal{A}_1(\delta), \mathcal{A}_1(\epsilon)). \tag{10.7}$$

Then A is a commutative Frobenius object in $\mathbf{Vect}_{\Bbbk}$, thus it is a commutative Frobenius algebra over $\Bbbk$. (Note that the arrows Id and τ corresponding to S^1 will always be mapped to the arrows Id_A and τ_A in $\mathbf{Vect}_{\Bbbk}$, by *any* TQFT which maps S^1 to A.)

We know that any object in **2cob** can be written as S^k, for some k, and its image can be written as

$$A^k := \mathcal{A}_0(S^k) = \mathcal{A}(S^1) \otimes \cdots \otimes \mathcal{A}(S^1) = A \otimes \cdots \otimes A.$$

Any arrow f in **2cob** can be written as a composition of the six generators of **2cob**, and $\mathcal{A}_1(f)$ can be written as a composition of the six generators Id_A, τ_A, μ_A, η_A, δ_A and ϵ_A. To conclude, (10.7) completely defines $\mathcal{A}$, thus we can say there is a one-to-one correspondence between the collection of TQFTs $\mathcal{A} : \mathbf{2cob} \to \mathbf{Vect}_{\Bbbk}$ and the collection of commutative Frobenius algebras.

The category of 2-dimensional topological quantum field theories. We define

$$\mathbf{2D\text{-}TQFT}_{\Bbbk} := \mathbf{SymmMonCat}(\mathbf{2Cob}, \mathbf{Vect}_{\Bbbk})$$

as the symmetric monoidal functor category of linear representations of **2Cob**. Any 2D-TQFT, first regarded as a functor from **2Cob** to $\mathbf{Vect}_{\Bbbk}$, will now be regarded as an object in $\mathbf{2D\text{-}TQFT}_{\Bbbk}$, and the arrows are monoidal natural transformations between them.

Now we define

$$\mathbf{2d\text{-}TQFT}_{\Bbbk} := \mathbf{SymmMonCat}(\mathbf{2cob}, \mathbf{Vect}_{\Bbbk})$$

as the symmetric monoidal functor category of linear representations of **2cob**, skeleton of **2Cob**. We could say this is a *skeletal version* of $\mathbf{2D\text{-}TQFT}_{\Bbbk}$.

There are relations between these categories $\mathbf{2D\text{-}TQFT}_{\Bbbk}$ and $\mathbf{2d\text{-}TQFT}_{\Bbbk}$, and the category $\mathbf{cFA}_{\Bbbk}$ of commutative Frobenius algebras. In Section 4.3 we already introduced $\mathbf{cFA}_{\Bbbk}$ as the category of commutative Frobenius objects in $\mathbf{Vect}_{\Bbbk}$, see (4.19). Then we write $\mathbf{cFA}_{\Bbbk} = \mathbf{cFrob}(\mathbf{Vect}_{\Bbbk})$. The one-to-one correspondence between the collection of TQFTs $\mathcal{A} : \mathbf{2cob} \to \mathbf{Vect}_{\Bbbk}$ and the collection of commutative Frobenius algebras can be regarded as a one-to-one correspondence between the collection of objects in $\mathbf{2d\text{-}TQFT}_{\Bbbk}$ and the collection of objects in $\mathbf{cFA}_{\Bbbk}$.

What about the arrows? Let for example α be a monoidal natural transformation between TQFTs $\mathcal{A}$ and $\mathcal{A}'$, depicted by

$$\begin{tikzcd} \mathbf{2cob} \arrow[r, bend left, "\mathcal{A}"] \arrow[r, bend right, "\mathcal{A}'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Vect}_{\Bbbk} \end{tikzcd}$$

Then S^1 (together with the four generators μ, η, δ and ϵ) will be mapped to $A := \mathcal{A}_0(S^1)$ and $A' := \mathcal{A}'_0(S^1)$, and we define $\alpha_1 : A \to A'$ as the natural arrow corresponding to S^1. It suffices to check the natural properties of α_1 with respect to the generators, to be sure that this α_1 indeed generates a monoidal natural transformation, an arrow in $\mathbf{2d\text{-}TQFT}_{\Bbbk}$. We already know that A and A' (together with the images of the four generators) are commutative Frobenius algebras. Then α_1 precisely corresponds to a Frobenius algebra homomorphism, an arrow in $\mathbf{cFA}_{\Bbbk}$. For example, the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}_0(S^2) = \mathcal{A}_0(S^1) \otimes \mathcal{A}_0(S^1) = A \otimes A \arrow[r, "\mu_A = \mathcal{A}_1(\mu)"] \arrow[d, "\alpha_2 = \alpha_1 \otimes \alpha_1"'] & A = \mathcal{A}_0(S^1) \arrow[d, "\alpha_1"] \\ \mathcal{A}'_0(S^2) = \mathcal{A}'_0(S^1) \otimes \mathcal{A}'_0(S^1) = A' \otimes A' \arrow[r, "\mu_{A'} = \mathcal{A}'_1(\mu)"'] & A' = \mathcal{A}'_0(S^1) \end{tikzcd}$$

This diagram is similar to the left part of the left diagram in (4.13).

Kock's main conclusion. Now we can say that there is also a one-to-one correspondence between the collection of arrows in $\mathbf{2d\text{-}TQFT}_{\Bbbk}$ and the collection of arrows in $\mathbf{cFA}_{\Bbbk}$. Then we can say that

- the category $\mathbf{2d\text{-}TQFT}_{\Bbbk}$ is *isomorphic* to the category $\mathbf{cFA}_{\Bbbk}$.

Similarly we can say that

- the category $\mathbf{2D\text{-}TQFT}_{\Bbbk}$ is *equivalent* to the category $\mathbf{cFA}_{\Bbbk}$.

Thus saying that $\mathbf{2d\text{-}TQFT}_{\Bbbk}$ and $\mathbf{cFA}_{\Bbbk}$ are isomorphic should mean that for any commutative Frobenius algebra A there exists a unique $\mathcal{A}$ in $\mathbf{2d\text{-}TQFT}_{\Bbbk}$ such that $A = \mathcal{A}_0(S^1)$, and that for any Frobenius algebra homomorphism $f : A \to A'$ there exists a unique monoidal natural transformation $\alpha : \mathcal{A} \to \mathcal{A}'$. In general, a monoidal functor is determined completely by its values on the generators of the source category. The source category $\mathbf{2cob}$ of any $\mathcal{A}$ in $\mathbf{2d\text{-}TQFT}_{\Bbbk}$ has only six generators, so we only need the image of these six generators to determine $\mathcal{A}$ completely. (Of course two of these six images are trivial: Id and τ will always be mapped to Id_A and τ_A respectively.) Thus such a TQFT can be described more easily if we restrict to $\mathbf{2cob}$ instead of $\mathbf{2Cob}$, as its source category, and this description corresponds exactly to that of a commutative Frobenius algebra. This can be regarded as the main conclusion of Kock's book [7].

The monoidal category of 2-dimensional topological quantum field theories. In Section 4.3 we also discussed that $\mathbf{cFA}_{\Bbbk}$ itself can again be regarded as a monoidal category, say $(\mathbf{cFA}_{\Bbbk}, \otimes, \Bbbk)$. This implies that at least also $\mathbf{2d\text{-}TQFT}_{\Bbbk}$ can be regarded as a monoidal category. This is in harmony with the idea that a pair of 2D-TQFTs $\mathcal{A}$ and $\mathcal{A}'$ can be used to construct a new one, say $\mathcal{A}''$: if

$$(A, \mu_A, \eta_A, \delta_A, \epsilon_A) = (\mathcal{A}_0(S^1), \mathcal{A}_1(\mu), \mathcal{A}_1(\eta)\mathcal{A}_1(\delta)\mathcal{A}_1(\epsilon))$$

and if

$$(A', \mu_{A'}, \eta_{A'}, \delta_{A'}, \epsilon_{A'}) = (\mathcal{A}'_0(S^1), \mathcal{A}'_1(\mu), \mathcal{A}'_1(\eta)\mathcal{A}'_1(\delta)\mathcal{A}'_1(\epsilon)),$$

then we can define

$$(\mathcal{A}_0''(S^1), \mathcal{A}_1''(\mu), \mathcal{A}_1''(\eta)\mathcal{A}_1''(\delta)\mathcal{A}_1''(\epsilon)) := (A \otimes A', \mu_{A\otimes A'}, \eta_{A\otimes A'}, \delta_{A\otimes A'}, \epsilon_{A\otimes A'}).$$

Here we used that two commutative Frobenius algebras A and B will induce another commutative Frobenius algebra $A \otimes B$, see (4.18).

The extended version of Kock's main conclusion. In many physical applications the theory of 2-dimensional topological quantum field theories cannot really be described as a theory of functors from **2Cob** to $\mathbf{Vect}_{\Bbbk}$. The original formulation of a TQFT $\mathcal{A} : \mathbf{2Cob} \to \mathbf{Vect}_{\Bbbk}$ does not allow supersymmetric theories. But instead of mapping from $(\mathbf{2Cob}, \amalg, \varnothing, \tau)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau)$ we can map to another symmetric monoidal category $(\mathbf{grVect}_{\Bbbk}, \otimes, \Bbbk, \kappa)$. This is the category of *graded vector spaces*, which is equipped with (for example) a different collection κ of twist arrows. Any TQFT $\mathcal{A} : \mathbf{2Cob} \to \mathbf{grVect}_{\Bbbk}$ will allow supersymmetric theories. The commutative Frobenius objects in $\mathbf{grVect}_{\Bbbk}$ are now called *graded-commutative* Frobenius algebras. The cohomology ring of a compact oriented manifold is an example of a graded-commutative Frobenius algebra.

There seems to be no problem if we replace $\mathbf{Vect}_{\Bbbk}$ with an arbitrary symmetric monoidal category $(\mathcal{C}, \Box, 1, \xi)$. (Similarly we can allow this in general for any nD-TQFT.) The extended version of Kock's main conclusion tells us that any symmetric monoidal functor

$$\mathcal{A} : (\mathbf{2Cob}, \amalg, \varnothing, \tau) \to (\mathcal{C}, \Box, 1, \xi)$$

can be interpreted as a 2-dimensional topological quantum field theory, and that the category of 2D-TQFTs over this $\mathcal{C}$, will be *equivalent* to the category of commutative Frobenius objects in $\mathcal{C}$:

$$\mathbf{2D\text{-}TQFT}_{\mathcal{C}} := \mathbf{SymmMonCat}(\mathbf{2Cob}, \mathcal{C}) \simeq^{(\text{equiv})} \mathbf{cFrob}(\mathcal{C}).$$

We will suggest that we can also say that the skeletal version of the category of 2D-TQFTs over this $\mathcal{C}$ will be *isomorphic* to the category of commutative Frobenius objects in $\mathcal{C}$:

$$\mathbf{2d\text{-}TQFT}_{\mathcal{C}} := \mathbf{SymmMonCat}(\mathbf{2cob}, \mathcal{C}) \simeq^{(\text{iso})} \mathbf{cFrob}(\mathcal{C}).$$

10.5 Topological Quantum Field Theories in dimension 1

In this section we will present topological quantum field theories in dimension 1, or 1D-TQFTs. We will again use the definition of TQFTs by Kock and present the category of 1D-TQFTs.

Main definition. A 1D-TQFT, over some ground field $\Bbbk$, is a symmetric monoidal functor $\mathcal{A}$ from the symmetric monoidal category **1Cob** to the symmetric monoidal category $\mathbf{Vect}_{\Bbbk}$:

$$\mathcal{A} : (\mathbf{1Cob}, \amalg, \varnothing, \tau) \to (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau).$$

In Chapter 9 we discussed that the positively oriented point p_+ (and of course also the negatively oriented point p_-) can be interpreted as a dualizable object in $\mathbf{1Cob}'$, subcategory of **1Cob**, and in **1cob**, skeleton of **1Cob**. These two objects p_+ and p_- generate all other objects in $\mathbf{1Cob}'$ and in **1cob**. If X is an object in **1cob**, then it can always be written as $X = p_+^k \amalg p_-^l$, for some integers k and l. Any such X is again dualizable, thus $\mathbf{1Cob}'$ and **1cob** are free symmetric monoidal categories on a dualizable object.

Any object X in $\mathbf{1Cob}'$ will be a dualizable object, thus its image $V_X := \mathcal{A}_0(X)$ will also be a dualizable object in $\mathbf{Vect}_{\Bbbk}$, or a dualizable vector space. We can also say that V_X is a finite-dimensional vector space. We know that for any pair of finite-dimensional vector spaces V and W also $V \otimes W$ is a finite-dimensional vector space.

Especially the dualizable object $(p_+, p_-, \beta, \bar{\gamma})$ will be mapped to a dualizable vector space

$$(V, W, \kappa, \lambda) := (\mathcal{A}_0(p_+), \mathcal{A}_0(p_-), \mathcal{A}_1(\beta), \mathcal{A}_1(\bar{\gamma})), \tag{10.8}$$

and (again) we can say that this object completely describes the TQFT $\mathcal{A} : \mathbf{1Cob}' \to \mathbf{Vect}_{\Bbbk}$. The other generators Id_+, Id_-, τ_+, τ_-, τ_{+-} and τ_{-+} of $\mathbf{1Cob}'$ will be mapped to standard arrows in $\mathbf{Vect}_{\Bbbk}$. Their image will be the same for any TQFT. To conclude, we can say there is a one-to-one correspondence between the collection of TQFTs $\mathcal{A} : \mathbf{1Cob}' \to \mathbf{Vect}_{\Bbbk}$ and the collection of dualizable vector spaces.

We know that $\mathbf{1Cob}'$ is still a symmetric monoidal subcategory of $\mathbf{1Cob}$, but $\mathbf{1cob}$ is not. The source category of any 1D-TQFT can be restricted to $\mathbf{1Cob}'$, but not precisely to $\mathbf{1cob}$. It is possible to restrict any TQFT $\mathcal{A} : \mathbf{1Cob} \to \mathbf{Vect}_{\Bbbk}$ to $\mathbf{1cob}$, but then it will not precisely be a symmetric monoidal functor. We need to find another category $\mathcal{C}$ such that $\mathcal{A} : \mathbf{1cob} \to \mathcal{C}$ becomes a symmetric monoidal functor. Using arguments related to the extended version of Kock's main conclusion, we can then say that this $\mathcal{A}$ is a TQFT.

The category of signed vector spaces. Thus we need to find another target category $\mathcal{C}$ to replace $\mathbf{Vect}_{\Bbbk}$. Let $(\mathcal{C}, \Box, 1, \tau)$ be an arbitrary symmetric monoidal category. Then we can introduce the *signed* symmetric monoidal category (or just the *signed version* of the symmetric monoidal category $\mathcal{C}$)

$$S\mathcal{C} = (S\mathcal{C}, \Box', 1', \tau').$$

We can say that $S\mathcal{C}$ itself is also a symmetric monoidal category. The collection of objects $S\mathcal{C}_0$ of this category will satisfy $S\mathcal{C}_0 = \mathcal{C}_0 \times \mathcal{C}_0$, thus any pair of objects A and B in $\mathcal{C}$ gives us an object (A, B) in $S\mathcal{C}$. Then we define $(A, B)\Box'(C, D) := (A\Box C, B\Box D)$ for any pair of objects (A, B) and (C, D) in $S\mathcal{C}$. The unit object will be written as $1' = (1, 1)$. For any object A in $\mathcal{C}$ we can say that $A_+ = (A, 1)$ and $A_- = (1, A)$ are (different) objects in $S\mathcal{C}$. Thus we can say that now we can attach a sign to any object in $\mathcal{C}$ so that we have an object in $S\mathcal{C}$. There are also mixed objects: we can say that there is a one-to-one correspondence between (A, B) and $A_+\Box B_-$. For example, $1' = (1, 1) \simeq 1_+\Box 1_- \simeq 1$. Thus from now on we will write $1' = 1$. Note that we can always write

$$(A, B) = (A, 1)\Box'(1, B) = (1, B)\Box'(A, 1). \tag{10.9}$$

Every pair of arrows $g : A \to C$ and $h : B \to D$ in $\mathcal{C}$ induces an arrow $f := (g, h) : (A, B) \to (C, D)$ in $S\mathcal{C}$. For example the twist arrow will be written as $\tau_{(A,B),(C,D)} = (\tau_{A,C}, \tau_{B,D})$, but not every arrow $f : (A, B) \to (C, D)$ in $S\mathcal{C}$ can be written as such a decomposition. However, if $f : A \to B$ is an arrow in $\mathcal{C}$, and if A and B can be written as $A = A_1\Box A_2$ and $B = B_1\Box B_2$, then we will assume this uniquely induces an arrow $\tilde{f} : (A_1, A_2) \to (B_1, B_2)$. We assume that every arrow in $S\mathcal{C}$ can at least be written this way. Now if $f = g\Box h = g'\Box h'$, for some arrows $g, g' : A \to C$ and $h, h' : B \to D$, then we assume that also $\tilde{f} = (g, h) = (g', h')$. (This means that we are dealing with an equivalence class.) As a consequence, if we have for example $\tilde{f} : (A, 1) \to (B, 1)$, then we can uniquely represent $\tilde{f}$ by (f, Id_1).

For every pair of arrows $f : A_1\Box A_2 \to B_1\Box B_2$ and $g : C_1\Box C_2 \to D_1\Box D_2$ we can say that $\tilde{f}\Box'\tilde{g}$ is an arrow in $S\mathcal{C}$ from $(A_1\Box C_1, A_2\Box C_2)$ to $(B_1\Box D_1, B_2\Box D_2)$. Then $\tilde{f}\Box'\tilde{g}$ can be represented by an arrow

$$h : (A_1\Box C_1)_+\Box(A_2\Box C_2)_- \to (B_1\Box D_1)_+\Box(B_2\Box D_2)_-$$

between signed objects, which can also be interpreted as an ordinary arrow in $\mathcal{C}$:

$$\begin{aligned} &h : A_1\Box C_1\Box A_2\Box C_2 \to B_1\Box D_1\Box B_2\Box D_2, \\ &h = (\mathrm{Id}_{B_1} \Box \tau_{B_2,D_1} \Box \mathrm{Id}_{D_2})(f\Box g)(\mathrm{Id}_{A_1} \Box \tau_{C_1,A_2} \Box \mathrm{Id}_{C_2}). \end{aligned}$$

This should of course remind us of the relation between $\amalg$ and $\amalg'$, applied to arrows in $\mathbf{1cob}$, see (9.7). (Note that $\mathbf{1cob}$ itself should not exactly be regarded as a signed version of another symmetric monoidal category.)

Now we could for example say that $\mathcal{C} = (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau)$. Then $S\mathcal{C}$ will be written as $(\mathbf{SVect}_{\Bbbk}, \otimes', \Bbbk, \tau')$. We will call this the *category of signed vector spaces*. For any pair of objects $(A, B) \simeq A_+ \otimes B_-$ and $(C, D) \simeq C_+ \otimes D_-$ in $\mathbf{SVect}_{\Bbbk}$ we will indeed write

$$\begin{aligned} (A, B) \otimes' (C, D) &\simeq (A_+ \otimes B_-) \otimes' (C_+ \otimes D_-) = A_+ \otimes C_+ \otimes B_- \otimes D_- \\ &= (A \otimes C)_+ \otimes (B \otimes D)_- \simeq (A \otimes C, B \otimes D). \end{aligned}$$

Now we can assume that symmetric monoidal functors $\mathcal{A}' : (\mathbf{1cob}, \amalg', \varnothing, \tau') \to (\mathbf{SVect}_{\Bbbk}, \otimes', \Bbbk, \tau')$ are possible. Our convention will be that any $\mathcal{A}'$ will map p_+ to some $(V, \Bbbk)$ and p_- to some $(\Bbbk, W)$. Any arbitrary object $p_+^k \amalg p_-^l$ in $\mathbf{1cob}$ will then be mapped to $(V^k, W^l) = (V \otimes \cdots \otimes V, W \otimes \cdots \otimes W) \simeq V_+^k \otimes W_-^l$.

Any TQFT $\mathcal{A} : \mathbf{1Cob}' \to \mathbf{Vect}_{\Bbbk}$ can be rewritten as a TQFT $\mathcal{A}' : \mathbf{1cob} \to \mathbf{SVect}_{\Bbbk}$. Then (10.8) can be rewritten as

$$((V, \Bbbk), (\Bbbk, W), \kappa', \lambda') := (\mathcal{A}'_0(p_+), \mathcal{A}'_0(p_-), \mathcal{A}'_1(\beta), \mathcal{A}'_1(\gamma)).$$

This can be regarded as an object in

$$\mathbf{DSVS}_{\Bbbk} := \mathbf{DO}(\mathbf{SVect}_{\Bbbk}, \otimes', \Bbbk, \tau'),$$

which is the *category of dualizable signed vector spaces*. (See Section 2.4 for the definition of $\mathbf{DO}(\mathcal{C})$, the category of dualizable objects in an arbitrary symmetric monoidal category $\mathcal{C}$.)

Note that here κ' is an arrow from (V, W) to $\Bbbk$ and λ' is an arrow from $\Bbbk$ to (V, W). As discussed in Section 9.3 we can write $p_+ \amalg' p_- = p_- \amalg' p_+$. Then the same relation applies to their image:

$$\begin{aligned}(V, W) &= (V, \Bbbk) \otimes' (\Bbbk, W) = \mathcal{A}'_0(p_+) \otimes' \mathcal{A}'_0(p_-) = \mathcal{A}'_0(p_+ \amalg' p_-) = \mathcal{A}'_0(p_- \amalg' p_+) \\ &= \mathcal{A}'_0(p_-) \otimes' \mathcal{A}'_0(p_+) = (\Bbbk, W) \otimes' (V, \Bbbk) = (V, W).\end{aligned}$$

This is in harmony with (10.9) and (9.14). Just like we mentioned in Section 9.3 that $(p_+, p_-, \beta, \gamma)$ induces another dualizable object $(p_-, p_+, \beta, \gamma)$ in $\mathbf{1cob}$, we can say that $((V, \Bbbk), (\Bbbk, W), \kappa', \lambda')$ induces another dualizable object $((\Bbbk, W), (V, \Bbbk), \kappa', \lambda')$ in $\mathbf{SVect}_{\Bbbk}$, which can also be regarded as an object in $\mathbf{DSVS}_{\Bbbk}$.

In general there are also objects in $\mathbf{DSVS}_{\Bbbk}$ which cannot be written as $((A, \Bbbk), (\Bbbk, B), \cdots)$. We will define $\mathbf{DSVS}^+_{\Bbbk}$ as the subcategory of $\mathbf{DSVS}_{\Bbbk}$, only containing objects that *can* be written like that. In general we should write $\mathbf{DSVS}_{\Bbbk} = (\mathbf{DSVS}_{\Bbbk}, \otimes', \Bbbk)$, but we can rewrite $\mathbf{DSVS}^+_{\Bbbk} = (\mathbf{DSVS}^+_{\Bbbk}, \otimes', \Bbbk)$ as $(\mathbf{DSVS}^+_{\Bbbk}, \otimes, \Bbbk)$. There is a relation between this category $\mathbf{DSVS}^+_{\Bbbk}$ and the category

$$\mathbf{DVS}_{\Bbbk} := \mathbf{DO}(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau)$$

of dualizable vector spaces, as introduced in (2.23). We can propose that there is an isomorphism

$$(\mathbf{DVS}_{\Bbbk}, \otimes, \Bbbk) \simeq^{(\text{iso})} (\mathbf{DSVS}^+_{\Bbbk}, \otimes, \Bbbk) = (\mathbf{DSVS}^+_{\Bbbk}, \otimes', \Bbbk) \subset (\mathbf{DSVS}_{\Bbbk}, \otimes', \Bbbk). \tag{10.10}$$

We can replace $\mathbf{Vect}_{\Bbbk}$ with an arbitrary symmetric monoidal category $(\mathcal{C}, \Box, 1, \xi)$. Then we can also replace $\mathbf{SVect}_{\Bbbk}$ with $(S\mathcal{C}, \Box', 1, \xi')$ and $\mathbf{DVS}_{\Bbbk}$ with $\mathbf{DO}(\mathcal{C}) = \mathbf{DO}(\mathcal{C}, \Box, 1, \xi)$. Then (10.10) can be rewritten as

$$(\mathbf{DO}(\mathcal{C}), \Box, 1) \simeq^{(\text{iso})} (\mathbf{DO}_+(S\mathcal{C}), \Box, 1) = (\mathbf{DO}_+(S\mathcal{C}), \Box', 1) \subset (\mathbf{DO}(S\mathcal{C}), \Box', 1). \tag{10.11}$$

Reduced symmetric monoidal functor categories. If $(\mathcal{C}, \Box, 1, \xi)$ is an arbitrary symmetric monoidal category, and if $\mathcal{D}$ is a symmetric monoidal functor category defined by

$$\mathcal{D} := \mathbf{SymmMonCat}(\mathbf{1Cob}', \mathcal{C}),$$

then we will define

$$\mathcal{D}' := \mathbf{SymmMonCat}_+(\mathbf{1cob}, S\mathcal{C}),$$

as a *reduced symmetric monoidal functor category*. The collection of objects in this category $\mathcal{D}'$ will *only* contain functors $\mathcal{A}'$ sending p_+ to $\mathcal{A}'_0(p_+) = (A, 1)$ and p_- to $\mathcal{A}'_0(p_-) = (1, B)$, for some objects A and B in $\mathcal{C}$.

The category of 1-dimensional topological quantum field theories. We define

$$\begin{aligned}&\mathbf{1D\text{-}TQFT}_{\Bbbk} := \mathbf{SymmMonCat}(\mathbf{1Cob}, \mathbf{Vect}_{\Bbbk}), \\ &\mathbf{1D'\text{-}TQFT}_{\Bbbk} := \mathbf{SymmMonCat}(\mathbf{1Cob}', \mathbf{Vect}_{\Bbbk}), \\ &\mathbf{1d\text{-}TQFT}_{\Bbbk} := \mathbf{SymmMonCat}_+(\mathbf{1cob}, \mathbf{SVect}_{\Bbbk})\end{aligned}$$

as the symmetric monoidal functor categories of linear representations of **1Cob**, **1Cob**$'$ and **1cob**. Any 1D-TQFT, first regarded as a functor from **1Cob** to $\mathbf{Vect}_{\Bbbk}$, will be regarded as an object in $\mathbf{1D\text{-}TQFT}_{\Bbbk}$. The arrows are (again) monoidal natural transformations between them.

There are relations between these categories $\mathbf{1D\text{-}TQFT}_{\Bbbk}$, $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ and $\mathbf{1d\text{-}TQFT}_{\Bbbk}$, and the category $\mathbf{DVS}_{\Bbbk}$. Again we can have a look at the arrows. Let for example α be a monoidal natural transformation between TQFTs $\mathcal{A}$ and $\mathcal{A}'$, depicted by

$$\begin{tikzcd} \mathbf{1Cob}' \arrow[r, bend left, "\mathcal{A}"] \arrow[r, bend right, "\mathcal{A}'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Vect}_{\Bbbk} \end{tikzcd}$$

Then p_+ and p_- (together with the two generators β and $\bar{\gamma}$) will be mapped to $V := \mathcal{A}_0(p_+)$, $W := \mathcal{A}_0(p_-)$, $V' := \mathcal{A}'_0(p_+)$ and $W' := \mathcal{A}'_0(p_-)$, and we define $\alpha_+ : V \to V'$ and $\alpha_- : W \to W'$ as the natural arrows corresponding to p_+ and p_-. We can propose that it suffices to check the natural properties of these α_+ and α_- with respect to the generators, to be sure that they indeed generate a monoidal natural transformation, an arrow in $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$. Any pair of arrows α_+ and α_- will satisfy naturality with respect to the generators Id_+, Id_-, τ_+, τ_-, τ_{+-} and τ_{-+} of **1Cob**$'$. We already know that $(V, W, \cdots)$ and $(V', W', \cdots)$ are dualizable vector spaces. Then (α_+, α_-) precisely corresponds to a dualizable homomorphism, an arrow in $\mathbf{DVS}_{\Bbbk}$. Again we can say there is a one-to-one correspondence between the collection of arrows in $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ and the collection of arrows in $\mathbf{DVS}_{\Bbbk}$.

A rewrite of Kock's main conclusion. Now we can introduce the following relations, and we can say that they are similar to Kock's main conclusion. We can say that

- the category $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ is *isomorphic* to the category $\mathbf{DVS}_{\Bbbk}$.

A similar statement can be found in [8]. There we can read that 1D-TQFTs are in one-to-one correspondence with finite-dimensional vector spaces. Of course any finite-dimensional vector space V is dualizable, but this one-to-one correspondence only holds if the TQFTs map from **1Cob**$'$ instead of **1Cob**, and if we also specify the information (V, W, κ, λ) for each V. Note that there are many possibilities for specifying the rest of the information. For example if $W' \neq W$, then $(V, W, \cdots)$ and $(V, W', \cdots)$ should be associated to different TQFTs.

Similarly we can say that

- the category $\mathbf{1D\text{-}TQFT}_{\Bbbk}$ is *equivalent* to the category $\mathbf{DVS}_{\Bbbk}$.

Saying that $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ and $\mathbf{DVS}_{\Bbbk}$ are isomorphic should mean that for any dualizable vector space V there exists a unique $\mathcal{A}$ in $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ such that $V = (V, W, \kappa, \lambda) = (\mathcal{A}_0(p_+), \mathcal{A}_0(p_-), \mathcal{A}_1(\beta), \mathcal{A}_1(\bar{\gamma}))$, and that for any dualizable homomorphism

$$(f_V, f_W) : (\mathcal{A}_0(p_+), \mathcal{A}_0(p_-), \mathcal{A}_1(\beta), \mathcal{A}_1(\bar{\gamma})) \to (\mathcal{A}'_0(p_+), \mathcal{A}'_0(p_-), \mathcal{A}'_1(\beta), \mathcal{A}'_1(\bar{\gamma}))$$

there exists a unique monoidal natural transformation $\alpha : \mathcal{A} \to \mathcal{A}'$.

Now knowing that any TQFT $\mathcal{A} : \mathbf{1Cob}' \to \mathbf{Vect}_{\Bbbk}$ induces a TQFT $\mathcal{A}' : \mathbf{1cob} \to \mathbf{SVect}_{\Bbbk}$, we can say that any object $\mathcal{A}$ in $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ (or in $\mathbf{1D\text{-}TQFT}_{\Bbbk}$) induces an object $\mathcal{A}'$ in $\mathbf{1d\text{-}TQFT}_{\Bbbk}$, defined by

$$\mathbf{1d\text{-}TQFT}_{\Bbbk} := \mathbf{SymmMonCat}_+(\mathbf{1cob}, \mathbf{SVect}_{\Bbbk}) \simeq^{(\mathrm{iso})} \mathbf{DSVS}^+_{\Bbbk} \subset \mathbf{DSVS}_{\Bbbk}\,.$$

Assuming that also $\mathbf{DSVS}^+_{\Bbbk} \simeq^{(\mathrm{iso})} \mathbf{DVS}_{\Bbbk}$, as (10.10) implies, we can say this is in harmony with the relation $\mathbf{1D'\text{-}TQFT}_{\Bbbk} \simeq^{(\mathrm{iso})} \mathbf{1d\text{-}TQFT}_{\Bbbk}$. Thus we can say that

- also the category $\mathbf{1d\text{-}TQFT}_{\Bbbk}$ is *isomorphic* to the category $\mathbf{DVS}_{\Bbbk}$.

The monoidal category of 1-dimensional topological quantum field theories. In Section 2.4 we also discussed that $\mathbf{DVS}_{\Bbbk}$ itself can again be regarded as a monoidal category, say $(\mathbf{DVS}_{\Bbbk}, \otimes, \Bbbk)$. This implies that at least also $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ can be regarded as a monoidal category. This is in harmony with the idea that a pair of 1D-TQFTs $\mathcal{A}$ and $\mathcal{A}'$ can be used to construct a new one, say $\mathcal{A}''$:

$$(\mathcal{A}''_0(p_+), \mathcal{A}''_0(p_-), \mathcal{A}''_1(\beta), \mathcal{A}''_1(\bar{\gamma})) := (\mathcal{A}_0(p_+), \mathcal{A}_0(p_-), \mathcal{A}_1(\beta), \mathcal{A}_1(\bar{\gamma}))\Box(\mathcal{A}'_0(p_+), \mathcal{A}'_0(p_-), \mathcal{A}'_1(\beta), \mathcal{A}'_1(\bar{\gamma})).$$

See (2.22) for the definition of a $\Box$-product of two dualizable objects.

A rewrite of the extended version of Kock's main conclusion. As we did in the previous section about 2D-TQFTs, we assume that, instead of mapping from $\mathbf{1Cob}$ to $\mathbf{Vect}_{\Bbbk}$, we can map to another arbitrary symmetric monoidal category $\mathcal{C}$. For example, we can again define supersymmetric topological quantum field theories $\mathcal{A} : \mathbf{1Cob} \to \mathbf{grVect}_{\Bbbk}$. A rewrite of the extended version of Kock's main conclusion tells us that any symmetric monoidal functor

$$\mathcal{A} : (\mathbf{1Cob}, \amalg, \varnothing, \tau) \to (\mathcal{C}, \Box, 1, \xi)$$

can be interpreted as a 1-dimensional topological quantum field theory. Then we will suggest that the category of 1D-TQFTs over this $\mathcal{C}$ will be *equivalent* to the category of dualizable objects in $\mathcal{C}$:

$$\mathbf{1D\text{-}TQFT}_{\mathcal{C}} := \mathbf{SymmMonCat}(\mathbf{1Cob}, \mathcal{C}) \simeq^{(\text{equiv})} \mathbf{DO}(\mathcal{C}).$$

A similar statement can be found in [9]. There we can read that $\mathbf{1D\text{-}TQFT}_{\mathcal{C}}$ is equivalent to the *groupoid* of dualizable objects in $\mathcal{C}$ and isomorphisms between them. However, there was no explanation of what isomorphisms are involved here exactly, and there was no mention of dualizable homomorphisms.

We will also suggest that the category of 1D-TQFTs from $\mathbf{1Cob}'$ to $\mathcal{C}$ will be *isomorphic* to the category of dualizable objects in $\mathcal{C}$:

$$\mathbf{1D'\text{-}TQFT}_{\mathcal{C}} := \mathbf{SymmMonCat}(\mathbf{1Cob}', \mathcal{C}) \simeq^{(\text{iso})} \mathbf{DO}(\mathcal{C}).$$

Now knowing that any TQFT $\mathcal{A} : \mathbf{1Cob}' \to \mathcal{C}$ induces a TQFT $\mathcal{A}' : \mathbf{1cob} \to S\mathcal{C}$, we can say that any object $\mathcal{A}$ in $\mathbf{1D'\text{-}TQFT}_{\mathcal{C}}$ (or in $\mathbf{1D\text{-}TQFT}_{\mathcal{C}}$) induces an object $\mathcal{A}'$ in $\mathbf{1d\text{-}TQFT}_{\mathcal{C}}$, defined by

$$\mathbf{1d\text{-}TQFT}_{\mathcal{C}} := \mathbf{SymmMonCat}_+(\mathbf{1cob}, S\mathcal{C}) \simeq^{(\text{iso})} \mathbf{DO}_+(S\mathcal{C}) \subset \mathbf{DO}(S\mathcal{C}).$$

Assuming that also $\mathbf{DO}_+(S\mathcal{C}) \simeq^{(\text{iso})} \mathbf{DO}(\mathcal{C})$, as (10.11) implies, we can say this is in harmony with the relation $\mathbf{1D'\text{-}TQFT}_{\mathcal{C}} \simeq^{(\text{iso})} \mathbf{1d\text{-}TQFT}_{\mathcal{C}}$.

11 Conclusion

Now we can look back on the central questions of this thesis and on the answers.

- In Section 10.1, 10.2 and 10.3 we discussed how the three different definitions of topological quantum field theories, as mentioned in [7] (Kock), [4] (Atiyah) and [8] (Blanchet & Turaev), are related. We can enumerate a list of key points:

  - In [4] there is no explicit mention of cobordisms and their in- and out-boundaries. However, a translation is possible by redefining the orientation of the boundary (and its connected components).
  - Only in [7] the concept of cobordism classes (and categories of them) is presented. However, we can explain how TQFTs only depend on these cobordism classes in the context of [4] and [8].
  - Only in [4] there is a restriction to TQFTs mapping $\overline{X}$ to the canonical dual V^*, if X is mapped to V. In this context we can still say that a TQFT can also be described as a symmetric monoidal functor from **nCob** to $\mathbf{Vect}_{\Bbbk}$, but we cannot say that the category of *these* TQFTs in, for example, dimension two is equivalent to the category of commutative Frobenius algebras. In the context of [7] and [8] there is more freedom.
  - The axioms of [8] fit best in the context of *nonstrict* symmetric monoidal categories and strong monoidal functors, whereas the axioms of [7] are written in the context of *strict* symmetric monoidal categories.

  We can say that the axioms and definition of TQFTs in the context of [7] *look* rather trivial and straightforward, but of course it takes some effort to define the abstract notions (like **nCob** and $\mathbf{Vect}_{\Bbbk}$) and to test whether it really coincides with the original definitions.

- In Section 10.5 we discussed how a topological quantum field theory $\mathcal{A} : \mathbf{1Cob} \to \mathbf{Vect}_{\Bbbk}$ induces a topological quantum field theory $\mathcal{A}' : \mathbf{1cob} \to \mathbf{SVect}_{\Bbbk}$. The category $\mathbf{SVect}_{\Bbbk}$ of signed vector spaces was introduced, which is the signed version of $\mathbf{Vect}_{\Bbbk}$. In general a TQFT $\mathcal{A} : \mathbf{1Cob} \to \mathcal{C}$ induces a TQFT $\mathcal{A}' : \mathbf{1cob} \to S\mathcal{C}$, where $\mathcal{C}$ is an arbitrary symmetric monoidal category and $S\mathcal{C}$ is the signed version of $\mathcal{C}$.

- In Section 10.5 we also discussed the category of 1-dimensional topological quantum field theories. To be more precise, we discussed three different ones, each of which has a different source category. These categories are $\mathbf{1D\text{-}TQFT}_{\Bbbk}$, $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ and $\mathbf{1d\text{-}TQFT}_{\Bbbk}$. The source categories are **1Cob** (the full category of 1-cobordisms), $\mathbf{1Cob}'$ (the minimal full symmetric monoidal subcategory of **1Cob**) and **1cob** (skeleton of both **1Cob** and $\mathbf{1Cob}'$), respectively. We can say that $\mathbf{1D\text{-}TQFT}_{\Bbbk}$ is *equivalent* to $\mathbf{DVS}_{\Bbbk}$, the category of dualizable vector spaces, and that $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ and $\mathbf{1d\text{-}TQFT}_{\Bbbk}$ are *isomorphic* to $\mathbf{DVS}_{\Bbbk}$. The destination category $\mathbf{Vect}_{\Bbbk}$ (and $\mathbf{SVect}_{\Bbbk}$) can be replaced by an arbitrary symmetric monoidal category $\mathcal{C}$ (and $S\mathcal{C}$). Then we can say that $\mathbf{1D\text{-}TQFT}_{\mathcal{C}}$ is *equivalent* to $\mathbf{DO}(\mathcal{C})$, the category of dualizable objects in $\mathcal{C}$, and that $\mathbf{1D'\text{-}TQFT}_{\mathcal{C}}$ and $\mathbf{1d\text{-}TQFT}_{\mathcal{C}}$ are *isomorphic* to $\mathbf{DO}(\mathcal{C})$.

- In Chapter 9 we discussed how to port over the symmetric monoidal structure from **1Cob** to its skeleton **1cob**. The explicit checks show that the graded disjoint union $\amalg'$ (together with the modified twist τ') can be used as a symmetric monoidal structure on **1cob**, and that this structure is already strict.

- In Chapter 3 we discussed the general techniques for porting over the monoidal structure from an arbitrary monoidal category to one of its skeletons. If $(\mathcal{C}, \mu, \eta, \alpha, \beta, \gamma)$ is a nonstrict monoidal category, if $\mathcal{C}'$ is a skeleton of $\mathcal{C}$, and if $P : \mathcal{C} \to \mathcal{C}'$ is an arbitrary projection functor, then P will help us porting over the monoidal structure from $\mathcal{C}$ to $\mathcal{C}'$ so that we can construct another nonstrict monoidal category $(\mathcal{C}', \mu', \eta', \alpha', \beta', \gamma')$. For example, we define $\mu' := P\mu(I \times I)$ and $\eta' := P\eta$. An explicit definition of P is not needed for checking out whether porting over all the structure will work out correctly.

A retrospective view of the restriction of 1D-TQFTs to 1cob, the skeleton of 1Cob. The original reason for restricting to **1cob**, skeleton of **1Cob**, as the source category of a 1D-TQFT, was for finding out how the theory of 1D-TQFTs (and the category of them) would simplify. As we can read in [7] we can certainly say that restricting to **2cob**, skeleton of **2Cob**, simplified the description of 2D-TQFTs, and the essential information is still there.

However, there is no law telling us that we *must* restrict to a skeleton for simplifying the theory. To keep it simple in general, we can as well restrict ourselves to $\mathbf{nCob}'$, a minimal full symmetric monoidal subcategory of **nCob**. Of course we see that, for example, **2cob** is also a minimal full symmetric monoidal subcategory of **2Cob**, so we can directly write $\mathbf{2Cob}' := \mathbf{2cob}$. In case of **1Cob**, $\mathbf{1Cob}'$ and **1cob** this is not possible.

We know that there are six generators for **1cob**: Id_+, Id_-, τ_+, τ_-, β and γ. We know that **1cob** is a free symmetric monoidal category on a dualizable object, and that only β and γ carry the essential structure. The same applies to $\mathbf{1Cob}'$, which has only two extra generators: τ_{+-} and τ_{-+}. No new information will be involved if we study TQFTs from $\mathbf{1Cob}'$ to $\mathbf{Vect}_{\Bbbk}$. This explains why $\mathbf{1D'\text{-}TQFT}_{\Bbbk}$ and $\mathbf{1d\text{-}TQFT}_{\Bbbk}$, the categories of TQFTs with source category $\mathbf{1Cob}'$ and **1cob** respectively, are isomorphic.

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